mathematics

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Archimedean Copulas-Based Estimation under One-Parameter Distributions in Coherent Systems

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Abstract: In the present work we provide a signature-based framework for delivering the estimated mean lifetime along with the variance of the continuous distribution of a coherent system consisting of exchangeable components. The dependency of the components is modelled by the aid of well-known Archimedean multivariate copulas. The estimated results are calculated under two different copulas, namely the so-called Frank copula and the Joe copula. A numerical experimentation is carried out for illustrating the proposed procedure under all possible coherent systems with three components.

Keywords: moment estimator; Frank copula; Joe copula; maximal signatures; exchangeable components

MSC: 62F10; 62H12; 62N05

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Citation: Triantafyllou, I.S. Archimedean Copulas-Based Estimation under One-Parameter Distributions in Coherent Systems. *Mathematics* **2024**, *12*, 334. https://doi.org/10.3390/math12020334

Academic Editors: Arne Johannssen and Nataliya Chukhrova

Received: 31 December 2023
Revised: 15 January 2024
Accepted: 18 January 2024
Published: 19 January 2024

1. Introduction

In the field of Statistical Reliability Modeling, several studies have been carried out under the assumption that the components of the underlying structures are independent. However, this condition is not always fulfilled in real-life problems. Therefore, it is of some research interest to investigate reliability systems consisting of exchangeable components, namely components which are identically distributed but are (possibly) dependent to each other. For instance, [1] delivered a signature-based analysis of m-consecutive k-out-of-n: F systems with exchangeable components. Moreover, [2] proved that the lifetime of any coherent system with dependent components can be expressed as generalized mixture of series (or parallel) subsystem lifetime distributions.

For a reliability coherent system with n exchangeable components, the dependency between them can be well modelled by the aid of appropriately chosen copulas. It is noteworthy that copulas have been proved to be a useful tool for studying the joint distribution of the random lifetimes of the components of a reliability model (see, e.g., [3–5]). For example, a copula-based approach can be applied in order to evaluate the reliability characteristics such as availability, reliability, and mean time to failure of a coherent system [4,5].

In addition, the parameter's estimation for the common continuous distribution of the components of the underlying reliability system is of high importance. Having at hand a point or interval estimation of the distributional parameter, one may readily deduce several results and conclusions concerning the behavior not only of the components, but also of the whole structure (see, e.g., [4]).

In the present paper, we provide a theoretical framework for providing the estimated mean lifetime (along with its variance) for a reliability structure and also for establishing the moment estimator of the parameter of the common continuous distribution of its components. However, the present work focuses on the reliability study of coherent systems with exchangeable components. More precisely, it aims to draw conclusions about their expected lifetimes and the respective variances. Thereof, we provide just a short discussion about the parameter's estimation of the common continuous distribution of its components at the end of Section 3. All necessary notions and formulae about the copulas which shall be used later on are presented in Section 2. A short introduction referring to the maximal signature of a coherent reliability system is also provided.

In Section 3, the proposed procedure is described in detail. The main results of the paper refer to the lifetime of a reliability system having exchangeable components and the parameter's estimation of the underlying components' distribution. This goal is fulfilled by the aid of explicit expressions, which are introduced and proved under different Archimedean copulas-based models. More precisely, the Frank copula model and the Joe copula model are considered and studied in some detail. It is evident that the shapes of these two copulas are quite similar, but the Frank Archimedean copula lacks an asymptote at $-\infty$, whereas the Joe Archimedean copula does not. In addition, it is known that Joe Archimedean copulas have an exponential functional form, while Frank Archimedean copulas have a logarithmic functional form. Among other reasons, we chose to consider the specific models due to their wide applicability in several fields. For instance, one may refer to the utilization of Frank Archimedean copulas for studying the linked risk factors, while Joe Archimedean copulas are frequently employed to model negatively related risk factors. For more details about the Archimedean copulas, the interested reader is referred to [6–11] and the references therein.

In Section 4, an extensive numerical experimentation is carried out and the implementation of the proposed estimation procedure is illustrated. In order to provide adequate numerical evidence about the ability of the proposed technique to estimate the desired quantities, all possible coherent systems with three components are taken into account. In addition, Monte Carlo simulations are also realized for studying the distribution of the resulting moment estimator. Finally, the Discussion section summarizes the contribution of the present paper, while some practical concluding remarks are also highlighted therein.

2. General Notions and Notations

In this section, we present the necessary notions and notations for establishing the proposed estimation procedure. In what follows, some basic results referring to the copula models and maximal signatures shall be discussed in order to pave the way for delivering the main results of the paper in the next section.

Let us first consider a coherent system consisting of n exchangeable components with common continuous distribution function F. The dependence between the components could be readily modelled by the aid of appropriate Archimedean copulas. A sufficient incentive to choose Archimedean copulas over other types of copulas is their simple form and ease with which they can be constructed. Moreover, the great variety of families of copulas which belong to this class gives the Archimedean copulas a central role and great applicability.

Generally speaking, the copulas are useful tools for determining the joint distribution of random variables, since they are functions that join (or couple) multivariate distribution functions to their one-dimensional marginal distribution functions. For recent advances and applications of the Archimedean copulas, one may refer to [12–17].

The main findings of the present work refer to the expected lifetime of a coherent system consisting of n exchangeable components under the assumption that the dependence of the components is modelled by the aid of specific copula models. In most of real-life engineering systems, such as transportation systems, communication networks, aerospace systems, healthcare delivery systems, and manufacturing processes, the dependence among the components is inevitable due to the common random production and operating environments.

Let us first denote by $T_1, T_2, \ldots, T_n$, the lifetimes of the components of underlying reliability structure. If $G_j(t_j) = P(T_j \leq t_j)$ corresponds to the cumulative distribution function of the variable $T_j, j = 1, 2, \ldots, n$, then $H(t_1, \ldots, t_n) = P(T_1 \leq t_1, T_2 \leq t_2, \ldots, T_n \leq t_n)$ is simply the joint distribution function of the lifetimes of the components of the system.

Each vector $(t_1, t_2, \ldots, t_n)$ of real numbers leads to a point $(G_1(t_1), G_2(t_2), \ldots, G_n(t_n))$ in the unit region $[0,\ 1]^n$, while these ordered coordinates correspond to a number $H(t_1, \ldots, t_n)$ in [0, 1]. The aforementioned correspondence, which assigns the value of the joint distribution function to each ordered vector of values of the individual distribution functions, is actually the copula function C. Generally speaking, note that the probability density function of the copulas can be derived from the corresponding cumulative density function by the aid of appropriate derivatives of the copula function.

Due to the exchangeability of the components of the underlying reliability structure, the following holds true

$$G_j(t) = G(t),\ j = 1, 2, \ldots, n. \tag{1}$$

Therefore, if $C(u_1, \ldots, u_n)$ is the copula function related to $H(t_1, \ldots, t_n)$, we deduce that

$$H(t_1, \ldots, t_n) = C(G(t_1), G(t_2), \ldots, G(t_n)). \tag{2}$$

An Archimedean copula behaves like a binary operation on the interval [0, 1]. In other words, the copula function C assigns to each pair (u, v) in [0, 1] a number $C(u, v)$ in [0, 1]. In addition, the function C is commutative, associative, and preserves order, e.g., $u_1 \leq u_2$ and $v_1 \leq v_2$ implies $C(u_1, v_1) \leq C(u_2, v_2)$.

Throughout the course of the present work, we shall consider two different copula functions for modeling the dependence of the components in the underlying reliability system. Both models implemented in the next lines are members of the well-known class of Archimedean copulas (see, e.g., [3,18,19] and references therein). Kindly note that under the assumption of exchangeability, these models have never been studied before for modeling the dependency of the components of a system. More precisely, we shall consider the following multivariate Archimedean copulas:

- The Frank family of n-copulas.

  The generator function of the bivariate Frank copulas is given by

$$\varphi_\theta(t) = -\ln\left[\frac{e^{-\theta t} - 1}{e^{-\theta} - 1}\right]. \tag{3}$$

For $\theta > 0$ and $n \geq 2$, the copula function of the multivariate Frank class of n-copulas is expressed as

$$C_\theta^n(u_1, u_2, \ldots, u_n) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u_1} - 1)\cdot(e^{-\theta u_2} - 1)\cdots(e^{-\theta u_n} - 1)}{(e^{-\theta} - 1)^{n-1}}\right). \tag{4}$$

- The Joe family of n-copulas.

  The generator function of the bivariate Joe copulas is given by

$$\varphi_\theta(t) = -\ln\left[1 - (1-t)^\theta\right],\ \theta \geq 1. \tag{5}$$

For $\theta \geq 1$ and $n \geq 2$, the copula function of the multivariate Joe class of n-copulas is written as

$$C_{*,\theta}^n(u_1, u_2, \ldots, u_n) = n - 1 - \left[\sum_{i=1}^{n}(1-u_i)^\theta - \prod_{i=1}^{n}(1-u_i)^\theta\right]^{1/\theta}. \tag{6}$$

If T corresponds to the lifetime of a coherent system with n exchangeable components and $T_{i:n}, i = 1,\ 2, \ldots, n$ is the i-th ordered component's lifetime, then the reliability function of the system is expressed as

$$P(T > t) = \sum_{i=1}^{n}\beta_i P(T_{i:i} > t) = \sum_{i=1}^{n}\beta_i(1 - P(T_1 \leq t, T_2 \leq t, \ldots, T_i \leq t)), \tag{7}$$

where $\beta_i,\ i = 1,\ 2, \ldots, n$ satisfy the condition $\sum_{i=1}^{n} \beta_i = 1$. Note that the vector $(\beta_1, \beta_2, \ldots, \beta_n)$ is the maximal signature of the coherent system (see, e.g., [2,20]). It is worth mentioning that, under specified reliability models, several general results have been proved in the literature for determining the β_i's. For instance, one may refer to the exact closed formulae for the maximal signatures of an m-consecutive-k-out-of-n: F system, which have been delivered in [1]. Throughout the lines of the next sections, we shall focus on specific models of coherent systems, such as parallel structures, series structures, or consecutive-type systems. For the latter ones, it is known that a consecutive k-out-of-n: F system is a structure made up of n components ordered sequentially and fails if and only if at least k consecutive components fail (see, e.g., [1]).

3. Main Results

In this section, we study the lifetime of a reliability structure consisting of n exchangeable components. The dependency between the components is modelled by the aid of two specific copula models. More precisely, the Frank and the Joe copulas are considered. Generally speaking, Frank Archimedean copulas are more sensitive to positive association than Joe Archimedean copulas in terms of association sensitivity.

The main result refers to the expected lifetime of such a structure, while the corresponding variance is determined. Based on these outcomes, the estimation of the parameter of the underlying components' distribution can also be achieved.

Let us next consider a reliability system consisting of n exchangeable components with a common continuous distribution G. We denote by $T_1, T_2, \ldots, T_n$ the lifetimes of the components, while $T = \varphi(T_1, T_2, \ldots, T_n)$ corresponds to the lifetime of the resulting structure.

The next proposition offers expressions for determining the expected lifetime and its corresponding variance for a system under the Frank copula-based dependency.

Proposition 1. *Let us consider a reliability system with n exchangeable and exponentially distributed components with parameter λ. If the dependency of the components is described by the Frank copula model, the following ensue,*

(i) *The expected lifetime of the system is given by*

$$E(T) = \sum_{i=1}^{n} \beta_i \cdot \frac{i}{\lambda} \times \int_0^\infty \frac{t \cdot e^{-t/\lambda - \theta \cdot (1-e^{-t/\lambda})} \cdot \left(e^{-\theta} - 1\right)^{1-i} \cdot (e^{-\theta \cdot (1-e^{-t/\lambda})} - 1)^{i-1}}{1 + \left(e^{-\theta} - 1\right)^{1-i} \cdot (e^{-\theta \cdot (1-e^{-t/\lambda})} - 1)^{i}} dt, \tag{8}$$

(ii) *The variance of the lifetime of the system is given by*

$$\begin{aligned} Var(T) = \sum_{i=1}^{n} \beta_i \cdot \tfrac{i}{\lambda} \cdot \int_0^\infty \frac{t^2 \cdot e^{-\frac{t}{\lambda} - \theta \cdot (1-e^{-\frac{t}{\lambda}})} \cdot \left(e^{-\theta} - 1\right)^{1-i} \cdot A_{\theta,\lambda}(t, i-1)}{1 + \left(e^{-\theta} - 1\right)^{1-i} \cdot A_{\theta,\lambda}(t, i)} dt \\ - \left(\sum_{i=1}^{n} \beta_i \cdot \tfrac{i}{\lambda} \cdot \int_0^\infty \frac{t \cdot e^{-t/\lambda - \theta \cdot (1-e^{-t/\lambda})} \cdot \left(e^{-\theta} - 1\right)^{1-i} \cdot \left(e^{-\theta \cdot (1-e^{-t/\lambda})} - 1\right)^{i-1}}{1 + \left(e^{-\theta} - 1\right)^{1-i} \cdot \left(e^{-\theta \cdot (1-e^{-t/\lambda})} - 1\right)^{i}} dt \right)^2. \end{aligned} \tag{9}$$

Proof. (i) Given that the components of the system are exponentially distributed with parameter λ, e.g., $G(t) = 1 - e^{-t/\lambda},\ t > 0$, the joint distribution function of their lifetimes under the Frank copula model can be written as (see (2) and (4))

$$H(t_1, \ldots, t_n) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta \cdot (1-e^{-t_1/\lambda})} - 1) \cdot (e^{-\theta \cdot (1-e^{-t_2/\lambda})} - 1) \cdots (e^{-\theta \cdot (1-e^{-t_n/\lambda})} - 1)}{(e^{-\theta} - 1)^{n-1}} \right). \tag{10}$$

It is known that the expected value of the lifetime T of a reliability system can be determined by the aid of the following formula:

$$E(T) = \int_0^\infty t dP(T \le t). \tag{11}$$

Recalling (7), the last expression can be rewritten as

$$E(T) = \sum_{i=1}^{n} \beta_i \int_0^\infty t dP(T_{i:i} \le t) = \sum_{i=1}^{n} \beta_i \cdot E(T_{i:i}) \tag{12}$$

where the vector $(\beta_1, \beta_2, \ldots, \beta_n)$ is the maximal signature of the system. Moreover, the event $\{T_{i:i} \le t\}$ practically means that the maximum of the components' lifetimes $T_1, T_2, \ldots, T_i$, $i = 1, 2, \ldots, n$, does not exceed the value t, while no restriction is stated for the $n - i$ remaining lifetimes. Therefore, we may readily deduce that

$$P(T_{i:i} \le t) = P(T_1 \le t, T_2 \le t, \ldots, T_i \le t) = C_\theta^n(\underbrace{1 - e^{-t/\lambda}, \ldots, 1 - e^{-t/\lambda}}_{i}, \underbrace{1, \ldots, 1}_{n-i}) \tag{13}$$

where the copula function C_θ^n is defined in (4). Combining Formulae (4) and (13), we conclude that

$$P(T_{i:i} \le t) = -\frac{1}{\theta} \ln\left(1 + \frac{(e^{-\theta \cdot (1 - e^{-t/\lambda})} - 1)^i}{(e^{-\theta} - 1)^{i-1}}\right) \tag{14}$$

and the expected value of the random variable $T_{i:i}$ is now determined as

$$E(T_{i:i}) = \int_0^\infty t dP(T_{i:i} \le t) = \frac{i}{\lambda} \cdot \int_0^\infty \frac{t \cdot e^{-t/\lambda - \theta \cdot (1 - e^{-t/\lambda})} \cdot \left(e^{-\theta} - 1\right)^{1-i} \cdot \left(e^{-\theta \cdot (1 - e^{-t/\lambda})} - 1\right)^{i-1}}{1 + \left(e^{-\theta} - 1\right)^{1-i} \cdot (e^{-\theta \cdot (1 - e^{-t/\lambda})} - 1)^i} dt, \tag{15}$$

The result we are chasing for is effortlessly derived by replacing the last expression in (12).

(ii) The variance of the system's lifetime T shall be determined by applying the well-known identity

$$Var(T) = E\left(T^2\right) - [E(T)]^2 \tag{16}$$

where the expected value of T is given by (8). In addition, the 2nd moment of T can be expressed as

$$E\left(T^2\right) = \int_0^\infty t^2 dP(T \le t) = \sum_{i=1}^{n} \beta_i \cdot E\left(T_{i:i}^2\right) = \sum_{i=1}^{n} \beta_i \cdot \int_0^\infty t^2 dP(T_{i:i} \le t). \tag{17}$$

Following a parallel argumentation with the one implemented at part (i), the integral expression in (18) leads, by the aid of (7) and (14), to the desired result. □

The following proposition offers expressions for determining the expected lifetime and its corresponding variance for a system under the Joe copula-based dependency.

Proposition 2. *Let us consider a reliability system with n exchangeable and exponentially distributed components with parameter λ. If the dependency of the components is described by the Joe copula model, the following ensue,*

(i) *The expected lifetime of the system is given by*

$$E(T) = \lambda \cdot \sum_{i=1}^{n} \beta_i \cdot i^{1/\theta}, \tag{18}$$

(ii) *The variance of the lifetime of the system is given by*

$$Var(T) = 2 \cdot \lambda^2 \cdot \sum_{i=1}^{n} \beta_i \cdot i^{1/\theta} - \left(\lambda \cdot \sum_{i=1}^{n} \beta_i \cdot i^{1/\theta} \right)^2. \tag{19}$$

Proof. (i) Given that the components of the system are exponentially distributed with parameter λ, e.g., $G(t) = 1 - e^{-t/\lambda},\ t > 0$, the joint distribution function of their lifetimes under the Joe copula model can be written as (see (2) and (6))

$$\begin{aligned} H(t_1, \ldots, t_n) &= n - 1 - \left[e^{-\theta \cdot t_1/\lambda} + \cdots + e^{-\theta \cdot t_n/\lambda} - e^{-\theta \cdot t_1/\lambda} \cdots e^{-\theta \cdot t_n/\lambda} \right]^{1/\theta} \\ &= n - 1 - \left(\sum_{j=1}^{n} e^{-\theta \cdot t_j/\lambda} - e^{-\theta \cdot \sum_{j=1}^{n} t_j/\lambda} \right)^{1/\theta} \end{aligned} \tag{20}$$

Since the copula function for the Joe model is given by (6), the following ensues (see also (13)),

$$P(T_{i:i} \le t) = n - 1 - \left(\sum_{j=1}^{i} e^{-\theta \cdot t/\lambda} \right)^{1/\theta}. \tag{21}$$

The expected value of the random variable $T_{i:i}$ is now determined as

$$E(T_{i:i}) = \int_0^\infty t dP(T_{i:i} \le t) = \frac{i^{1/\theta}}{\lambda} \cdot \int_0^\infty t \cdot e^{-t/\lambda} dt = \lambda \cdot i^{1/\theta}. \tag{22}$$

We next combine (12) and (22) and the desired result is straightforward.

(ii) The variance of the system's lifetime T shall be determined by applying the well-known identity (16). The 2nd moment of T can be expressed as

$$E\left(T^2\right) = \int_0^\infty t^2 dP(T \le t) = \sum_{i=1}^{n} \beta_i \cdot E\left(T_{i:i}^2\right) = \sum_{i=1}^{n} \beta_i \cdot \int_0^\infty t^2 dP(T_{i:i} \le t) = 2 \cdot \lambda^2 \cdot \sum_{i=1}^{n} \beta_i \cdot i^{1/\theta}. \tag{23}$$

The result we are chasing for is now immediately derived. □

It is evident that Propositions 1 and 2 of the present paper provide general results for any coherent system consisting of n exchangeable and exponentially distributed components. These results can be easily modified under different distributional assumptions for the components' lifetimes of the underlying structure.

It is noteworthy that the aforementioned results, which have been proved in Propositions 1 and 2, may contribute to deliver the estimation of the parameter of the common distribution of the components' lifetimes. According to the well-known moment estimation procedure, the theoretical moments provided by the previous propositions should be equated to the corresponding sample moments.

For instance, let us consider the same case with the one studied in Propositions 1 and 2, namely, we assume that the components of the system share a common exponential distribution with parameter λ. In order to estimate the distribution's mean λ, we need to determine the corresponding sample mean lifetime $\overline{T}$ of the resulting system. More precisely, if $E(T)$ denotes the 1st moment of system's lifetime T, then the desired estimation of parameter λ shall be delivered by solving the equation $E(T) = \overline{T}$ with respect to λ.

Since we study reliability systems having exchangeable components, whose dependency is modelled by an appropriately chosen Archimedean copula, the computation of the sample mean lifetime of the resulting system calls for a sampling procedure from the underlying copula. The challenge of efficiently sampling exchangeable Archimedean copulas has been already addressed in the literature (see, e.g., [9–11]).

One of the most powerful tools for sampling exchangeable Archimedean copulas is provided by the algorithm of Marshall and Olkin (algorithm *MO*, hereafter). According to the algorithmic procedure *MO*, we may simulate a random sample of size n from a specific Archimedean copula with generator ψ and continuous joint cumulative distribution function $H(t_1, \ldots, t_n)$ if we follow the next steps (see, e.g., [9]):

Step 1. Sample $V \sim F = \mathcal{L}S^{-1}(\psi)$, where $\mathcal{L}S^{-1}(\psi)$ denotes the Laplace-Stieltjes transform of ψ.

Step 2. Sample independent and identically distributed random variables $U_i, i = 1,\, 2, \ldots, n$ from the Uniform distribution in [0, 1], namely $U_i \sim U[0,\, 1],\ i = 1, 2 \ldots, n$.

Step 3. Determine $X_i = \psi(-\ln(U_i)/V),\ i = 1,\, 2, \ldots, n$.

Step 4. The random variables $G^{-1}(X_i), = 1,\, 2, \ldots, n$ constitute a sample from the exchangeable joint distribution function H, where G corresponds to the marginal cumulative distribution function of H.

4. Numerical Results

In the present section, we compute the expected mean lifetime and the corresponding variance for all possible coherent reliability structures with three exchangeable components, which are exponentially distributed components with parameter λ. The dependency between the components is modelled by either the Frank or the Joe copula. The numerical results and graphical illustrations that appeared in this section are all produced by the aid of the theoretical outcomes proved in the previous section of the present manuscript.

Let us denote by X_1, X_2, X_3 the lifetimes of the components of a reliability structure with three components. All possible reliability systems consisting of three exchangeable components are listed below (see, also [6]):

- **RS1**: Series system. The particular system fails if and only if at least one component fails. Thereof, the lifetime of a series system with three exchangeable components can be expressed as $S_1 = \min(X_1, X_2, X_3)$, while the corresponding maximal signature vector is given as $(\beta_1, \beta_2, \beta_3) = (3, -3,\, 1)$.
- **RS2**: Series-parallel system. The particular system fails if and only if either the 1st component fails or both the other two (e.g., the 2nd and the 3rd component) components fail. Thereof, the lifetime of a series-parallel system with three exchangeable components can be expressed as $S_2 = \min(X_1, \max(X_2, X_3))$, while the corresponding maximal signature vector is given as $(\beta_1, \beta_2, \beta_3) = (1,\, 1, -1)$.
- **RS3**: 2-out-of-3 system. The particular system fails if and only if at least two components fail. Thereof, the lifetime of a 2-out-of-3 system with three exchangeable components can be expressed as $S_3 = \max\limits_{1 \le i < j \le 3} \min(X_i, X_j)$, while the corresponding maximal signature vector is given as $(\beta_1, \beta_2, \beta_3) = (0,\, 3, -2)$.
- **RS4**: Parallel-Series system. The particular system fails if and only if the 1st component fails and one of the other two (e.g., either the 2nd or the 3rd component) components fail. Thereof, the lifetime of a parallel-series system with three exchangeable components can be expressed as $S_4 = \max(X_1, \min(X_2, X_3))$, while the corresponding maximal signature vector is given as $(\beta_1, \beta_2, \beta_3) = (0,\, 2, -1)$.
- **RS5**: Parallel system. The particular system fails if and only if all components fail. Thereof, the lifetime of a parallel system with three exchangeable components can be expressed as $S_5 = \min(X_1, X_2, X_3)$, while the corresponding maximal signature vector is given as $(\beta_1, \beta_2, \beta_3) = (0,\, 0,\, 1)$.

We first compute the expected lifetimes and the corresponding variances for the above-mentioned structures under the Frank and Joe copula-based dependence. The common distribution of all components is assumed to be the exponential with parameter λ. The numerical results provided at Table 1 have been produced by the aid of Propositions 1 and 2 (see Section 3 of the present manuscript).

Table 1. Expected lifetime and its variance of all possible coherent systems with three exchangeable components under Frank and Joe copula-based dependency.

		Frank Copula		Joe Copula	
System	$(\boldsymbol{\lambda},\boldsymbol{\theta})$	**Expected Lifetime**	**Variance**	**Expected Lifetime**	**Variance**
RS1		0.448826	0.219162	0.489410	0.739298
RS2		0.720882	0.438656	0.682163	0.898979
RS3	(1, 2)	0.856910	0.492892	0.778539	0.950955
RS4		1.136030	0.978296	1.096380	0.990712
RS5		1.694260	1.481690	1.732050	0.464102
RS1		0.897652	0.876646	0.978820	2.957190
RS2		1.441760	1.754630	1.364330	3.595920
RS3	(2, 2)	1.713820	1.971570	1.557080	3.803820
RS4		2.272050	3.913230	2.192750	3.962850
RS5		3.388530	5.926690	3.464100	1.856410
RS1		1.346480	1.972450	1.468230	6.653680
RS2		2.162650	3.947880	2.046490	8.090820
RS3	(3, 2)	2.570730	4.436030	2.335620	8.558600
RS4		3.408080	8.804730	3.289130	8.916400
RS5		5.082790	13.33510	5.196150	4.176910
RS1		1.795300	3.506600	1.957640	11.82880
RS2		2.883530	7.018480	2.728650	14.38370
RS3	(4, 2)	3.427640	7.886270	3.114160	15.21530
RS4		4.544110	15.65280	4.385510	15.85140
RS5		6.777060	23.70680	6.928200	7.535630
RS1		0.501188	0.267310	0.662486	0.886085
RS2		0.747671	0.485253	0.817671	0.966756
RS3	(1, 3)	0.870913	0.548659	0.895264	0.989030
RS4		1.123240	1.001210	1.077590	0.993979
RS5		1.627900	1.524290	1.442250	0.804415
RS1		1.002380	1.069230	1.324970	3.544340
RS2		1.495340	1.941020	1.635340	3.867030
RS3	(2, 3)	1.741830	2.194620	1.790530	3.956120
RS4		2.246480	4.004860	2.155190	3.975920
RS5		3.255800	6.097180	2.884500	3.217660
RS1		1.503560	2.405800	1.987460	7.974760
RS2		2.243010	7.162340	2.453010	8.700810
RS3	(3, 3)	2.612740	4.937920	2.685790	8.901270
RS4		3.369730	9.010870	3.232780	8.945810
RS5		4.883700	13.71860	4.326750	7.239740
RS1		2.004750	4.276790	2.649950	14.17740
RS2		2.990680	7.764080	3.270690	15.46810
RS3	(4, 3)	3.483650	8.778550	3.581060	15.82450
RS4		4.492970	16.01930	4.310370	15.90370
RS5		6.511600	24.38870	5.769000	12.87060

The upper (lower) entry of each cell corresponds to the mean lifetime (its variance) of the underlying structure.

Based on the numerical results provided in Table 1, we may readily deduce that the expected lifetime of reliability structure with exchangeable components under either Frank or Joe copula-based dependency increases,

- for fixed θ as the parameter λ increases
- for fixed λ as the parameter θ decreases

In addition, the numerical results displayed in Table 1, confirm, as it was expected,

- the superiority of RS5 against the structures RS1–RS4, as it has the largest expected lifetime among all systems taken into consideration under the same designs,

- the inferiority of RS1 against the structures RS2–RS5, as it has the smallest expected lifetime under the same designs.

It is also evident that the parallel-series system provides a more reliable structure compared to the 2-out-of-3 system in terms of expectation. At the same time, the 2-out-of-3 system seems to be better than the series-parallel system, which in turn overperforms the series system consisting of three exchangeable components.

Figures 1–4 provide some illustration for the behavior of the expected lifetime and the corresponding variance under the Frank copula-based dependency. More precisely, the mean lifetimes (along with their estimated variances) of systems RS2 and RS3 are depicted in Figures 1–4 for different values of the design parameter θ.

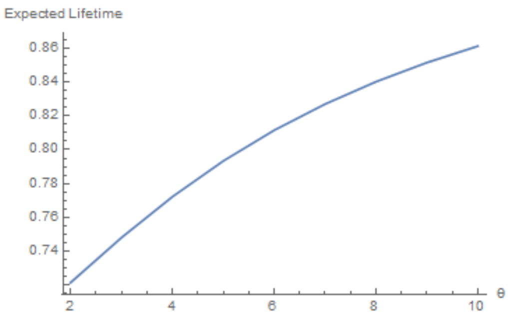

Figure 1. Expected lifetime of RS2 system versus parameter θ under Frank copula ($\lambda = 1$).

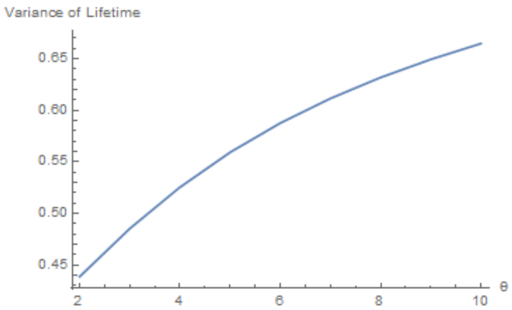

Figure 2. Variance of lifetime of RS2 system versus parameter θ under Frank copula ($\lambda = 1$).

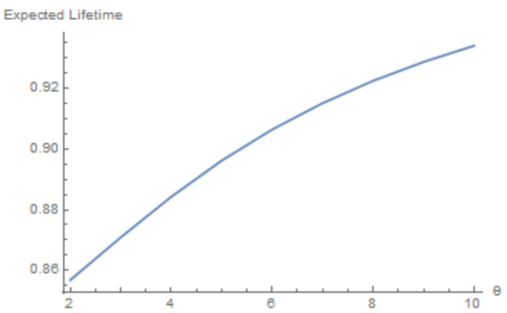

Figure 3. Expected lifetime of RS3 system versus parameter θ under Frank copula ($\lambda = 1$).

It is also of some interest to investigate the impact of parameter λ on the lifetime of the resulting reliability schemes. For this reason, we next construct some relative illustrations (see Figures 5–8), where the systems RS2 and RS3 have been taken into consideration once again.

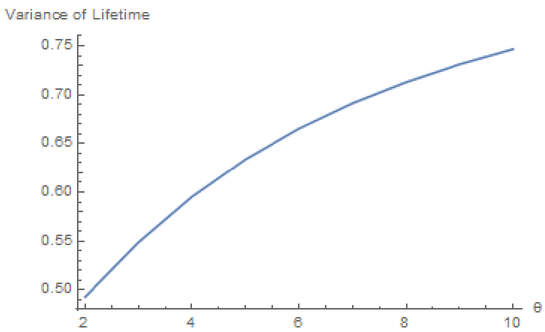

Figure 4. Variance of lifetime of RS3 system versus parameter θ under Frank copula ($\lambda = 1$).

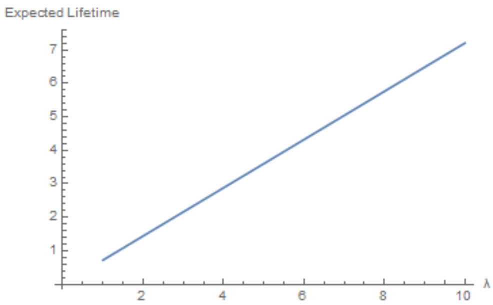

Figure 5. Expected lifetime of RS2 system versus parameter λ under Frank copula ($\theta = 2$).

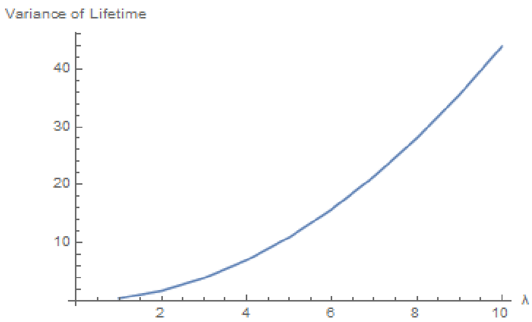

Figure 6. Variance of lifetime of RS2 system versus parameter λ under Frank copula ($\theta = 2$).

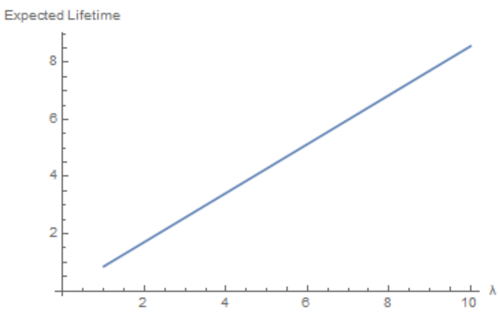

Figure 7. Expected lifetime of RS3 system versus parameter λ under Frank copula ($\theta = 2$).

Based on Figures 5–8, it is easily observed that the expected lifetime of the resulting reliability system increases in a linear way in terms of the parameter λ under the assumption that θ remains unchanged.

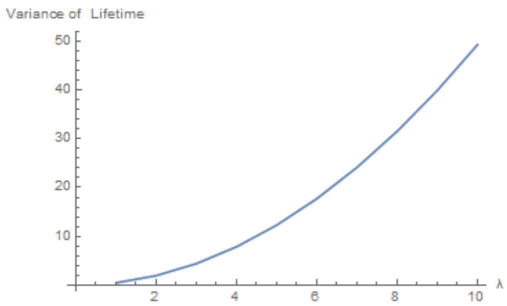

Figure 8. Variance of lifetime of RS3 system versus parameter λ under Frank copula ($\theta = 2$).

In addition, Figures 9–16 provide some illustration for the behavior of the expected lifetime and the corresponding variance under the Joe copula-based dependency. More precisely, we focus now on the RS1 and RS5 cases, and the respective mean lifetimes (along with their estimated variances) are displayed at Figures 9–12 for different values of the design parameter θ.

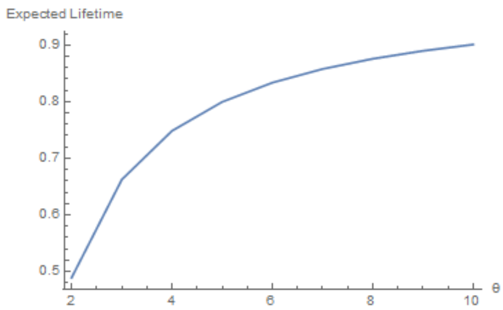

Figure 9. Expected lifetime of RS1 system versus parameter θ under Joe copula ($\lambda = 1$).

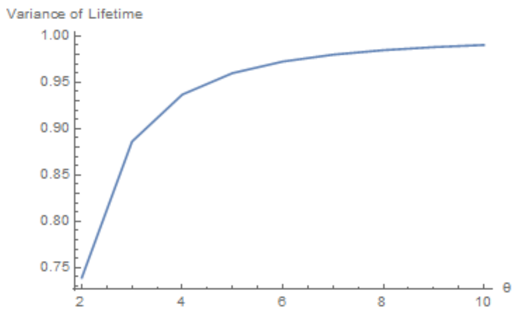

Figure 10. Variance of lifetime of RS1 system versus parameter θ under Joe copula ($\lambda = 1$).

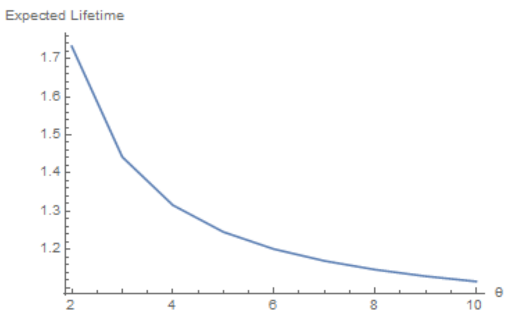

Figure 11. Expected lifetime of RS5 system versus parameter θ under Joe copula ($\lambda = 1$).

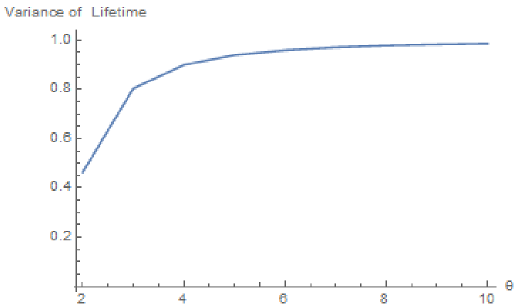

Figure 12. Variance of lifetime of RS5 system versus parameter θ under Joe copula ($\lambda = 1$).

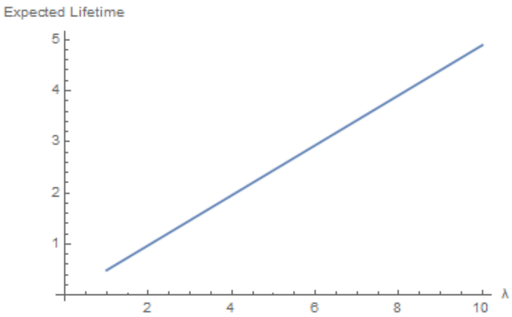

Figure 13. Expected lifetime of RS1 system versus parameter λ under Joe copula ($\theta = 2$).

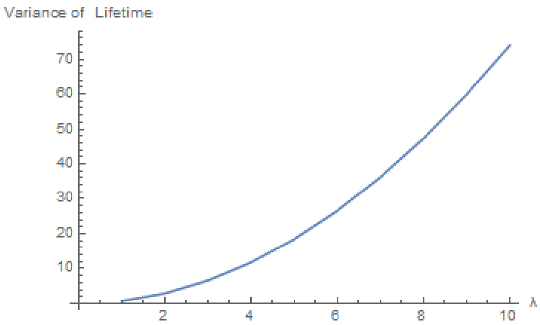

Figure 14. Variance of lifetime of RS1 system versus parameter λ under Joe copula ($\theta = 2$).

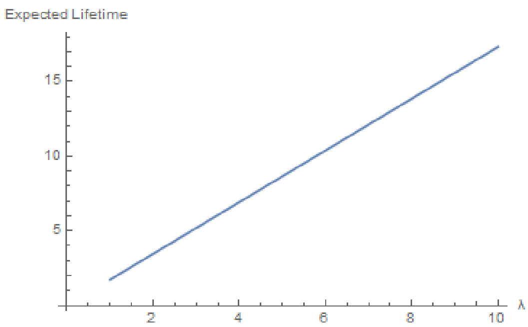

Figure 15. Expected lifetime of RS5 system versus parameter λ under Joe copula ($\theta = 2$).

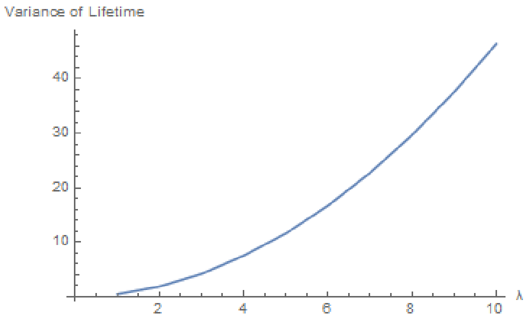

Figure 16. Variance of lifetime of RS5 system versus parameter λ under Joe copula ($\theta = 2$).

It goes without saying that systems RS1 and RS5 do not share a common behavior in terms of parameter θ under Joe copula-based dependency of their components. More specifically, we observe that,

- the expected lifetime of RS1 system becomes larger as θ increases, while
- the expected lifetime of RS5 system becomes larger as θ decreases.

The impact of parameter λ on the lifetime of the resulting reliability schemes is taken into consideration in Figures 13–16.

Based on Figures 13–16, it is deduced that the expected lifetime of the resulting reliability system increases in a linear way in terms of λ under the assumption that θ remains unchanged.

According to the numerical results provided previously, it is evident in both copulas that the expected lifetime of the underlying structure becomes larger as the corresponding parameter θ increases. More precisely, it seems that under the Frank copula the increase is more pronounced in comparison with the one observed under Joe copula. On the other hand, the numerical results do not point out that the Frank copula results in larger lifetimes than the Joe copula under the same reliability structure. In fact, there are cases where the Frank copula seems to formulate a structure with a larger expected lifetime than the corresponding one under Joe copula, while in other cases it holds the opposite conclusion. In other words, the numerical results seem to be quite robust under these two models.

5. Discussion

In the present article, a signature-based framework is provided for delivering the estimated mean lifetime (along with its variance) for a reliability structure with exchangeable components under the assumption that their dependency is modelled by the aid of well-known Archimedean copulas. The theoretical results contribute to the reliability study of such structures, while their usefulness can be also extended to the estimation of the parameter of the corresponding (common) distribution of their components. It is noteworthy that, since the present work deals with two specific copulas, the applicability of the results requires that the connection between the components of the underlying system can be described well by the copulas, which have been taken into consideration. It is noteworthy that the proposed framework does not strongly depend on the specific structure of the underlying reliability model. Therefore, similar steps could be followed in order to deliver respective results for any coherent systems with exchangeable components.

Moreover, the proposed framework seems to result in simple explicit integral expressions for the expected lifetime of the system and its corresponding variance. This conclusion seems to be evident under the assumption that the common distribution of the components has a quite simple form (as the exponential does). Therefore, in such cases, which are the most common in practice, the computational effort of the proposed methodology is manageable.

However, several bivariate copulas cannot (or at least are difficult to) be extended to multivariate models and therefore their applicability remains low. It is clear that this limitation deprives the implementation of the proposed approach to reliability structures consisting of more than two exchangeable components if the dependency among them is modeled by the aid of such copulas. For future research, it is of some interest to investigate different Archimedean copulas for modeling the dependency of the components of the underlying structure.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors wish to thank four anonymous referees for their useful comments and suggestions, which improved the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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