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Constraint Qualifications and Optimality Conditions for Nonsmooth Semidefinite Multiobjective Programming Problems with Mixed Constraints Using Convexificators

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Abstract: In this article, we investigate a class of non-smooth semidefinite multiobjective programming problems with inequality and equality constraints (in short, NSMPP). We establish the convex separation theorem for the space of symmetric matrices. Employing the properties of the convexificators, we establish Fritz John (in short, FJ)-type necessary optimality conditions for NSMPP. Subsequently, we introduce a generalized version of Abadie constraint qualification (in short, NSMPP-ACQ) for the considered problem, NSMPP. Employing NSMPP-ACQ, we establish strong Karush-Kuhn-Tucker (in short, KKT)-type necessary optimality conditions for NSMPP. Moreover, we establish sufficient optimality conditions for NSMPP under generalized convexity assumptions. In addition to this, we introduce the generalized versions of various other constraint qualifications, namely Kuhn-Tucker constraint qualification (in short, NSMPP-KTCQ), Zangwill constraint qualification (in short, NSMPP-ZCQ), basic constraint qualification (in short, NSMPP-BCQ), and Mangasarian-Fromovitz constraint qualification (in short, NSMPP-MFCQ), for the considered problem NSMPP and derive the interrelationships among them. Several illustrative examples are furnished to demonstrate the significance of the established results.

Keywords: semidefinite programming; multiobjective optimization; mixed constraints; constraint qualifications; optimality conditions; convexificators

MSC: 90C22; 90C29; 90C46

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1. Introduction

In optimization theory, multiobjective programming problems (in short, MOP) refer to a class of optimization problems that involve the simultaneous minimization of several conflicting objectives. MOP hold significant importance in practical optimization scenarios, such as business, economics, and various scientific and engineering fields (see, for instance, [1–3] and the references mentioned therein). For a more comprehensive overview and updated survey of multiobjective optimization; see [4–7] and the references mentioned therein.

In mathematical optimization problems, non-smooth phenomena occur frequently, resulting in the formulation of numerous types of generalized directional derivatives and subdifferentials (see, for instance, [8–10] and the references mentioned therein). Demyanov [11] introduced the concept of convexificator as an extension of the notions of lower concave and upper convex approximations. Demyanov and Jeyakumar [10] investigated convexificator for locally Lipschitz and positively homogeneous functions. Jeyakumar and Luc [12] presented non-compact convexificators and introduced various calculus rules for computing convexificators. Convexificators can be seen as weaker versions of the various well-known subdifferentials, such as Clarke [8], Michel-Penot [9], Ioffe-Morduchovich-Shao [13,14], and Treiman [15], as convexificators, in general, are closed-set and are not necessarily bounded or convex, unlike most known subdifferentials. For a locally Lipschitz function, generalized subdifferentials, such as [8,9,13–15] can be considered as convexificators, and these aforementioned subdifferentials may include the convex hull of a convexificator (see, for instance, [12]). Convexificators serve as an essential tool in establishing optimality results for non-smooth MOP. Luu [16,17] employed the concept of convexificators to formulate necessary optimality conditions for local Pareto and weak Pareto minimums in MOP that involve a combination of inequality, equality, and set constraints in the context of Banach spaces. Convexificators have been utilized to extend various results in the field of non-smooth analysis (see, for instance, [18–21] and the references mentioned therein).

In the mathematical theory of optimization, constraint qualifications (in short, CQ) play a very significant role in deriving Karush-Kuhn-Tucker (in short, KKT)-type necessary optimality criteria (see, for instance, [22]). Constraint qualifications were first introduced by Kuhn and Tucker [23] for non-linear programming problems. Maeda [24] presented several CQ for differentiable multiobjective programming problems and established interrelationships among them. Similar to the result obtained by Maeda [24], Preda and Chiţescu [25] extended the corresponding results within the context of semidifferentiable analysis. Jourani [26] investigated CQ for non-smooth single-objective programming problems with both inequality and equality constraints on Banach spaces. Li [27] explored constraint qualifications for non-smooth multiobjective programming problems using locally Lipschitz functions in Euclidean spaces and derived strong KKT conditions for such problems. Stein [28] studied Mangasarian-Fromovitz constraint qualifications and Abadie constraint qualifications for single-objective non-smooth programming problems. Gupta and Srivastava [29] explored CQ for a class of multiobjective programming problems characterized by locally Lipschitz objective functions and inequality and equality constraints. Using convexificators, Golestani and Nobakhtian [30] introduced various CQ for non-smooth multiobjective programming problems and established interrelationships among them. Several authors have introduced CQ for multiobjective programming problems under different assumptions (see, for instance, [31–34] and the references mentioned therein).

Non-linear semidefinite programming problems (in short, NSDP) are essentially a generalization of non-linear programming problems, where the vector variables are substituted with symmetric positive semidefinite matrices. Arising from various areas of modern research, semidefinite programming problems have numerous applications, for instance, combinatorial optimization [35], control theory [36], and eigenvalue optimization [37]. Under convexity assumptions for NSDP, Shapiro [38] established both first- and second-order necessary as well as sufficient optimality conditions. Forsgren [39] extended the results obtained by Shapiro [38] for non-convex semidefinite programming problems. Many researchers have widely discussed various algorithmic approaches for solving NSDP (see, for instance, [40,41] and the references mentioned therein). Furthermore, Yamashita and Yabe [42] developed numerical methods and discussed their convergence properties for NSDP. Employing convexificators, Golestani and Nobakhtian [43] introduced the generalized Abadie CQ for non-smooth semidefinite programming problems and established both necessary and sufficient optimality conditions for non-smooth NSDP. Lai et al. [44] employed convexificators and established both necessary as well as sufficient optimality conditions for non-smooth semidefinite MOP with vanishing constraints. Mishra et al. [45] derived optimality conditions and numerous duality theorems for non-smooth semidefinite MOP using convexificators. Recently, Upadhyay et al. [46] established optimality and duality results for non-smooth semidefinite multiobjective fractional programming problems.

It is worth mentioning that Golestani and Nobakhtian [43] introduced several CQ for non-smooth semidefinite single-objective programming problems. However, the constraint qualifications introduced for single-objective optimization problems cannot be employed for multiobjective optimization problems because they do not guarantee the positiveness of the Lagrange multipliers related to the different components of the objective function. Therefore, some of the components of the objective function do not contribute to determining the necessary optimality conditions. To the best of our knowledge, constraint qualifications and optimality conditions for non-smooth semidefinite multiobjective programming problems with mixed constraints have not been addressed. This article aims to bridge this research gap. In this article, motivated by the works [24,30,43], we investigate a class of NSMPP. We establish FJ-type necessary optimality conditions. We introduce NSMPP-ACQ and establish strong KKT-type necessary optimality conditions for NSMPP using convexificators. Under the assumptions of generalized convexity, we establish the sufficient optimality condition for NSMPP. Moreover, we introduce NSMPP-KTCQ, NSMPP-ZCQ, NSMPP-MFCQ, and NSMPP-BCQ for NSMPP and derive the interrelationships among them. The significance of these findings is demonstrated by the inclusion of several non-trivial illustrative examples.

The primary contributions and novel aspects of the present article are threefold. Firstly, we generalize the various constraint qualifications introduced in [30] from non-smooth multiobjective programming problems to a more general programming problem, NSMPP, as well as the corresponding results from the Euclidean space to the space of symmetric matrices. Moreover, the results derived in this article generalize the corresponding results derived in [34] from the Euclidean space to the space of symmetric matrices. Secondly, we generalize the constraint qualifications introduced in [43] from non-smooth semidefinite single-objective programming problems to a more general programming problem, NSMPP in the space of symmetric matrices. Thirdly, in view of the fact that convexificators are weaker versions of Clarke sudifferentials (see [12]), the results established in this article sharpen the corresponding results derived by Giorgi et al. [34]. To the best of our knowledge, this is the first time that the CQ and optimality conditions for NSMPP have been explored via convexificators.

The present article is structured as follows: In Section 2, we revisit some basic definitions and mathematical preliminaries. In Section 3, we introduce NSMPP-ACQ and derive the necessary and sufficient optimality conditions for NSMPP. In Section 4, we introduce various constraint qualifications for NSMPP and establish interrelationships among them. In Section 5, we conclude this article by summarizing the key findings and outlining potential avenues for future research.

2. Preliminaries

In the present article, the symbols $\mathbb{R}^n$ and $\mathbb{N}$ are used to represent the n-dimensional Euclidean space and the set consisting of all natural numbers, respectively. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\emptyset$ denote an empty set. The space of $n \times n$ symmetric matrices is denoted by $\mathbb{S}^n$. The set of all symmetric positive semidefinite matrices and symmetric positive definite matrices are denoted by $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$, respectively.

Let $p, q \in \mathbb{R}^n$. The following notations are used in the article:

$$p \prec q \iff p_k < q_k, \ \forall k \in \{1, \dots, n\},$$
$$p \preceq q \iff \begin{cases} p_k \leq q_k, \ \forall k \in \{1, \dots, n\}, \\ p_r < q_r, \ \text{for at least one } r \in \{1, \dots, n\}. \end{cases}$$

For $\mathscr{A}, \mathscr{B} \in \mathbb{S}^n$, we define the inner product between $\mathscr{A}$ and $\mathscr{B}$ as

$$\langle \mathscr{A}, \mathscr{B} \rangle = \text{trace}(\mathscr{A}\mathscr{B}).$$

The norm related to the inner product is referred to as the Frobenius norm, denoted by

$$\| \mathscr{A} \|_{\mathcal{F}} = \text{tr}(\mathscr{A}\mathscr{A})^{1/2} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

Let $\mathcal{B}$ be a nonempty subset of $\mathbb{S}^n$. We employ the symbols $\text{cl}\,\mathcal{B}$, $\text{co}\,\mathcal{B}$ and $\text{cone}\,\mathcal{B}$ to denote the closure of $\mathcal{B}$, convex hull of $\mathcal{B}$, and the convex cone (including the origin) generated by $\mathcal{B}$, respectively.

Now, we define the following sets, which will be useful in the subsequent sections:

$$\mathcal{B}^- := \{\mathcal{Z} \in \mathbb{S}^n : \langle \mathcal{Z}, \mathcal{M}\rangle \leq 0, \forall \mathcal{M} \in \mathcal{B}\},$$

$$\mathcal{B}^s := \{\mathcal{Z} \in \mathbb{S}^n : \langle \mathcal{Z}, \mathcal{M}\rangle < 0, \forall \mathcal{M} \in \mathcal{B}\}.$$

The following definition will be employed in the sequel.

Definition 1 ([43])**.** *Let $\mathcal{B}$ be a nonempty subset of $\mathbb{S}^n_+$ and $\mathcal{Z} \in \text{cl}\,\mathcal{B}$.*

- *The contingent cone $T(\mathcal{B}, \mathcal{Z})$ at $\mathcal{Z} \in \text{cl}\,\mathcal{B}$ is defined as*

$$T(\mathcal{B}, \mathcal{Z}) := \{\mathbb{V} \in \mathbb{S}^n : \exists \alpha_n \downarrow 0 \text{ and } \mathbb{V}_n \to \mathbb{V} \text{ such that } \mathcal{Z} + \alpha_n \mathbb{V}_n \in \mathcal{B}, \forall n \in \mathbb{N}\}.$$

- *The cone of feasible directions $F(\mathcal{B}, \mathcal{Z})$ at $\mathcal{Z} \in \text{cl}\,\mathcal{B}$ is defined as*

$$F(\mathcal{B}, \mathcal{Z}) := \{\mathbb{V} \in \mathbb{S}^n | \; \exists \kappa > 0 \text{ such that } \mathcal{Z} + \eta \mathbb{V} \in \mathcal{B}, \forall \eta \in (0, \kappa)\}.$$

- *The cone of attainable directions $A(\mathcal{B}, \mathcal{Z})$ at $\mathcal{Z} \in \text{cl}\,\mathcal{B}$ is defined as*

$$A(\mathcal{B}, \mathcal{Z}) := \Big\{\mathbb{V} \in \mathbb{S}^n | \; \exists \kappa > 0, \Theta : \mathbb{R} \to \mathbb{S}^n \text{ such that } \Theta(\eta) \in \mathcal{B}, \forall \eta \in (0, \kappa), \Theta(0) = \mathcal{Z}, \lim_{\eta \downarrow 0} \frac{\Theta(t) - \Theta(0)}{\eta} = \mathbb{V}\Big\}.$$

Remark 1. *It can be demonstrated that (see [43]):*

$$\text{cl}\,F(\mathcal{B}, \mathcal{Z}) \subseteq A(\mathcal{B}, \mathcal{Z}) \subseteq T(\mathcal{B}, \mathcal{Z}).$$

In the following definition, we introduce the notions of subgradient and subdifferential of a convex function for the space of symmetric matrices.

Definition 2. *Let $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$ be an extended real-valued convex function and $\mathcal{Z} \in \text{dom}(\Phi)$, where $dom(\Phi) := \{\mathcal{Z} \in \mathbb{S}^n : \Phi(\mathcal{Z}) \neq +\infty\}$. We say that $\xi \in \mathbb{S}^n$ is a subgradient of Φ at $\mathcal{Z}$ if for all $\mathcal{X} \in \text{dom}(\Phi)$ we have*

$$f(\mathcal{X}) \geqslant f(\mathcal{Z}) + \langle \xi, \mathcal{X} - \mathcal{Z}\rangle.$$

The set of all subgradients of Φ at $\mathcal{Z}$ is called the subdifferential of Φ at $\mathcal{Z}$ and is denoted by $\partial\Phi(\mathcal{Z})$.

Remark 2. *Definition 2 generalizes the definition of subgradient and subdifferential given in [47] from the Euclidean space to the space of symmetric matrices.*

The following definitions of lower and upper Dini derivatives, convexificators, upper semi-regular convexificator (in short, USRC), and generalized convexity for the space of symmetric matrices will be beneficial in the subsequent sections of the article.

Definition 3 ([43])**.** *Let $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$ be an extended real-valued function and $\mathcal{Z} \in dom(\Phi)$, where $dom(\Phi) := \{\mathcal{Z} \in \mathbb{S}^n : \Phi(\mathcal{Z}) \neq +\infty\}$. The lower and upper Dini derivatives of Φ at $\mathcal{Z}$ in the direction $\mathbb{V} \in \mathbb{S}^n$ are defined, respectively, by*

$$\Phi^-(\mathcal{Z}; \mathbb{V}) := \liminf_{\lambda \downarrow 0} \frac{\Phi(\mathcal{Z} + \lambda\mathbb{V}) - \Phi(\mathcal{Z})}{\lambda},$$

$$\Phi^{+}(\mathcal{Z};\mathbb{V}) := \limsup_{\lambda \downarrow 0} \frac{\Phi(\mathcal{Z}+\lambda\mathbb{V}) - \Phi(\mathcal{Z})}{\lambda}.$$

Definition 4 ([43]). *Let $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$ be an extended real-valued function. We say that Φ has an upper semi-regular convexificator (USRC), $\partial^*\Phi(\mathcal{Z}) \subset \mathbb{S}^n$ at $\mathcal{Z} \in dom(\Phi)$ if $\partial^*\Phi(\mathcal{Z})$ is a closed set and for every $\mathbb{V} \in \mathbb{S}^n$ we have*

$$\Phi^{+}(\mathcal{Z};\mathbb{V}) \leq \sup_{\zeta \in \partial^*\Phi(\mathcal{Z})} \langle \zeta, \ \mathbb{V} \rangle.$$

Definition 5 ([43]). *Let $\Phi : \mathbb{S}^n \to \overline{\mathbb{R}}$ be an extended real-valued function. Assume that $\mathcal{Z} \in \mathbb{S}^n$ is a point such that $\Phi(\mathcal{Z})$ is finite and Φ admits an upper semi-regular convexificator $\partial^*\Phi(\mathcal{Z})$ at $\mathcal{Z}$. Then*

- *Φ is ∂^*-pseudoconvex at $\mathcal{Z}$ if, for all $\mathcal{X} \in \mathbb{S}^n$,*

$$\Phi(\mathcal{X}) < \Phi(\mathcal{Z}) \implies \langle \zeta, \mathcal{X} - \mathcal{Z} \rangle < 0, \ \ \forall \zeta \in \partial^*\Phi(\mathcal{Z}).$$

- *Φ is ∂^*-quasiconvex at $\mathcal{Z}$ if, for all $\mathcal{X} \in \mathbb{S}^n$,*

$$\Phi(\mathcal{X}) \leq \Phi(\mathcal{Z}) \implies \langle \zeta, \mathcal{X} - \mathcal{Z} \rangle \leq 0, \ \ \forall \zeta \in \partial^*\Phi(\mathcal{Z}).$$

The subsequent lemmas are employed to derive the main results of the article.

Lemma 1 ([43]). *Let $\mathcal{C} \in \mathbb{S}^n$ such that $\langle \mathcal{C}, \ \mathcal{D} \rangle \geq 0$, $\forall \mathcal{D} \in \mathbb{S}^n_+$. Then $\mathcal{C} \in \mathbb{S}^n_+$.*

Lemma 2 ([48]). *Suppose $\mathcal{T}(\mathcal{Z}) := (\mathcal{T}_1(\mathcal{Z}), \cdots, \mathcal{T}_m(\mathcal{Z}))$ such that $\mathcal{T}_i : \mathbb{S}^n \to \mathbb{R}$, $i \in \{1, \cdots, m\}$ are convex functions. Then the system*

$$\begin{cases} \mathcal{T}(\mathcal{Z}) < 0, \\ \mathcal{Z} \in \mathbb{S}^n_{++}, \end{cases}$$

has no solution if and only if there exist $\nu_i \geq 0$, $i \in \{1, \cdots, m\}$ and $\mathtt{U} \in \mathbb{S}^n_+$, not all zero simultaneously such that

$$\sum_{i=1}^{m} \nu_i \mathcal{T}_i(\mathcal{Z}) - \langle \mathtt{U}, \ \mathcal{Z} \rangle \geq 0, \ \forall \mathcal{Z} \in \mathbb{S}^n.$$

The subsequent theorem is an adaptation of the Weierstrass Theorem for the space of symmetric matrices $\mathbb{S}^n$.

Theorem 1 ([49]). *Let us assume that $\mathcal{W}$ is a nonempty compact set in $\mathbb{S}^n$. Moreover, let $\Phi : \mathcal{W} \to \mathbb{R}$ be continuous on $\mathcal{W}$. Then for the problem $min\{\Phi(\mathcal{Z}) : \mathcal{Z} \in \mathcal{W}\}$, the set $\arg\min\{\Phi(\mathcal{Z}) : \mathcal{Z} \in \mathcal{W}\}$ is nonempty.*

The subsequent proposition will be beneficial in establishing the separation theorem for the space of symmetric matrices $\mathbb{S}^n$.

Proposition 1. *Let us suppose that $\mathcal{W}$ is a nonempty closed convex set in $\mathbb{S}^n$. Moreover, we assume that $\mathcal{Q} \notin \mathcal{W}$. Then there exists a unique point $\overline{\mathcal{P}} \in \mathcal{W}$ such that the distance between $\overline{\mathcal{P}}$ and $\mathcal{Q}$ is minimum. Furthermore, $\overline{\mathcal{P}}$ is at the minimum distance from $\mathcal{Q}$ if and only if*

$$\langle \mathcal{P} - \overline{\mathcal{P}}, \ \mathcal{Q} - \overline{\mathcal{P}} \rangle \leq 0, \ \forall \mathcal{P} \in \mathcal{W}.$$

Proof. Since $\mathcal{W} \neq \emptyset$, therefore there exists a point $\mathcal{P} \in \mathcal{W}$. Define,

$$\overline{\mathcal{W}} := \mathcal{W} \cap \{\mathcal{P} : \| \mathcal{Q} - \mathcal{P} \| \leq \| \mathcal{Q} - \mathcal{Z} \|\}.$$

Then the problem of finding the point nearest to the point $\mathcal{Q}$ is the same as finding

$$\inf\{\| \mathcal{Q} - \mathcal{P} \|: \mathcal{P} \in \overline{\mathcal{W}}\}.$$

By Theorem 1, there exists a point $\overline{\mathcal{P}} \in \mathcal{W}$ nearest to the point $\mathcal{Q}$. Let $\mathcal{P}'$ be another point in $\mathcal{W}$ such that $\| \mathcal{Q} - \overline{\mathcal{P}} \| = \| \mathcal{Q} - \mathcal{P}' \|$. Since $\mathcal{W}$ is a convex set,

$$\frac{\overline{\mathcal{P}} + \mathcal{P}'}{2} \in \mathcal{W}.$$

By the triangle inequality, we have

$$\left\| \mathcal{Q} - \frac{\overline{\mathcal{P}} + \mathcal{P}'}{2} \right\| \leq \frac{1}{2} \| \mathcal{Q} - \overline{\mathcal{P}} \| + \frac{1}{2} \| \mathcal{Q} - \mathcal{P}' \| . \tag{1}$$

From the given hypothesis, it follows that $\overline{\mathcal{P}}$ is the nearest point to $\mathcal{Q}$. Hence, strict inequality cannot hold in (1). Therefore,

$$\mathcal{Q} - \overline{\mathcal{P}} = \mu(\mathcal{Q} - \mathcal{P}'),\ |\mu| = 1.$$

We have $\mu \neq -1$ as $\mathcal{Q} \notin \mathcal{W}$. Therefore, $\mu = 1$. Hence, there exists a unique point that is at a minimum distance from $\mathcal{Q}$.

Moreover, let $\mathcal{P} \in \mathcal{W}$. Then

$$\| \mathcal{Q} - \mathcal{P} \|^2 = \| \mathcal{Q} - \overline{\mathcal{P}} + \overline{\mathcal{P}} - \mathcal{P} \|^2 = \| \mathcal{Q} - \overline{\mathcal{P}} \|^2 + \| \overline{\mathcal{P}} - \mathcal{P} \|^2 + 2\langle \mathcal{Q} - \overline{\mathcal{P}}, \overline{\mathcal{P}} - \mathcal{P} \rangle.$$

Therefore, for all $\mathcal{P} \in \mathcal{W}$,

$$\| \mathcal{Q} - \mathcal{P} \|^2 > \| \mathcal{Q} - \overline{\mathcal{P}} \|^2 .$$

Hence, $\overline{\mathcal{P}}$ is the point at the minimum distance from $\mathcal{Q}$.

Conversely, we assume that

$$\| \mathcal{Q} - \mathcal{P} \|^2 > \| \mathcal{Q} - \overline{\mathcal{P}} \|^2,\ \forall \mathcal{P} \in \mathcal{W}.$$

Since $\mathcal{W}$ is a convex set, therefore

$$\overline{\mathcal{P}} + \mu(\mathcal{P} - \overline{\mathcal{P}}) \in \mathcal{W},\ \text{for } 0 \leq \mu \leq 1.$$

Thus,

$$\| \mathcal{Q} - \overline{\mathcal{P}} - \mu(\mathcal{P} - \overline{\mathcal{P}}) \|^2 \geq \| \mathcal{Q} - \overline{\mathcal{P}} \|^2 . \tag{2}$$

Further,

$$\| \mathcal{Q} - \overline{\mathcal{P}} - \mu(\mathcal{P} - \overline{\mathcal{P}}) \|^2 = \| \mathcal{Q} - \overline{\mathcal{P}} \|^2 + \mu^2 \| \mathcal{P} - \overline{\mathcal{P}} \|^2 - 2\mu\langle \mathcal{Q} - \overline{\mathcal{P}},\ \mathcal{P} - \overline{\mathcal{P}} \rangle. \tag{3}$$

From (2) and (3), we get

$$2\langle \mathcal{Q} - \overline{\mathcal{P}},\ \mathcal{P} - \overline{\mathcal{P}} \rangle \leq \mu \| \mathcal{P} - \overline{\mathcal{P}} \|^2,\ 0 \leq \mu \leq 1.$$

Letting $\mu \to 0^+$, the result follows. □

Remark 3. *Proposition 1 generalizes Theorem 2.4.1 established in [49] from the Euclidean space to the space of symmetric matrices.*

Now, we prove the separation theorem for a closed convex set in $\mathbb{S}^n$ that will be employed in establishing strong KKT-type necessary optimality conditions for NSMPP.

Theorem 2 (Separation Theorem)**.** *Assume $\mathcal{W}$ to be a nonempty closed convex set in $\mathbb{S}^n$. Let $\mathcal{Q} \notin \mathcal{W}$. Then there exist $\mathbb{V} \in \mathbb{S}^n$ and a scalar β such that*

$$\langle \mathcal{P}, \mathbb{V} \rangle \leq \beta < \langle \mathcal{Q}, \mathbb{V} \rangle, \forall \mathcal{P} \in \mathcal{W}.$$

Proof. Let $\mathbb{V} = \mathcal{Q} - \overline{\mathcal{P}}$, $\mathcal{Q} \notin \mathcal{W}$ and $\beta = \langle \overline{\mathcal{P}}, \mathbb{V} \rangle$. Then

$$\langle \mathbb{V}, \mathcal{P} \rangle = \langle \mathcal{Q} - \overline{\mathcal{P}}, \mathcal{P} \rangle.$$

Since $\mathcal{Q} \notin \mathcal{W}$ and $\mathcal{W}$ is a nonempty closed convex set, therefore, by Proposition 1, there exists a unique point $\overline{\mathcal{P}} \in \mathcal{W}$ that is at minimum distance from $\mathcal{Q}$ satisfying

$$\begin{aligned} & \langle \mathcal{P} - \overline{\mathcal{P}}, \mathcal{Q} - \overline{\mathcal{P}} \rangle \leq 0, \forall \mathcal{P} \in \mathcal{W} \\ \Longrightarrow\ & \langle \mathcal{P} - \overline{\mathcal{P}}, \mathbb{V} \rangle \leq 0, \forall \mathcal{P} \in \mathcal{W} \\ \Longrightarrow\ & \langle \mathcal{P}, \mathbb{V} \rangle \leq \beta, \ \forall \mathcal{P} \in \mathcal{W}. \end{aligned}$$

Now,

$$\langle \mathcal{Q}, \mathbb{V} \rangle - \beta = \langle \mathcal{Q} - \overline{\mathcal{P}}, \mathbb{V} \rangle = \| \mathcal{Q} - \overline{\mathcal{P}} \|^2 > 0.$$

Hence, the proof is complete. □

Remark 4. *Theorem 2 generalizes Theorem 2.4.4 derived in [49] from the Euclidean space to the space of symmetric matrices.*

3. Optimality Conditions

In this section, we consider a non-smooth semidefinite multiobjective programming problem with mixed constraints NSMPP and establish FJ-type necessary optimality conditions. Moreover, we introduce NSMPP-ACQ for NSMPP. Employing NSMPP-ACQ, we derive the strong KKT-type necessary optimality conditions for NSMPP.

Let us consider the following non-smooth semidefinite multiobjective programming problem with both inequality and equality constraints:

$$\begin{aligned} \text{(NSMPP) Minimize } & \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \ldots, \Phi_m(\mathcal{Z})), \\ \text{subject to } & \Psi(\mathcal{Z}) = (\Psi_1(\mathcal{Z}), \ldots, \Psi_n(\mathcal{Z})) \leq 0, \\ & \Theta(\mathcal{Z}) = (\Theta_1(\mathcal{Z}), \ldots, \Theta_p(\mathcal{Z})) = 0, \\ & \mathcal{Z} \in \mathbb{S}^n_+, \end{aligned}$$

where $\Phi_i : \mathbb{S}^n \to \overline{\mathbb{R}}, \ i \in \mathbb{I} := \{1, \ldots, m\}$, $\Psi_j : \mathbb{S}^n \to \overline{\mathbb{R}}, \ j \in \mathbb{J} := \{1, \ldots, n\}$ and $\Theta_k : \mathbb{S}^n \to \overline{\mathbb{R}}, \ k \in \mathbb{K} = \{1, \ldots, p\}$ are extended real-valued functions. Moreover, we assume that each function $\Phi_i, \ i \in \mathbb{I}$, $\Psi_j, \ j \in \mathbb{J}$ and $\Theta_k, \ k \in \mathbb{K}$ admit bounded USRC. Let $\mathbb{J}(\overline{\mathcal{Z}}) := \{j \in \mathbb{J} \mid \Psi_j(\overline{\mathcal{Z}}) = 0\}$. We define the set of all feasible solutions $\mathscr{F}$ of NSMPP as

$$\mathscr{F} := \{\mathcal{Z} \in \mathbb{S}^n : \Psi_j(\mathcal{Z}) \leq 0, \ j \in \mathbb{J}, \ \Theta_k(\mathcal{Z}) = 0, \ k \in \mathbb{K}, \ \mathcal{Z} \in \mathbb{S}^n_+\}.$$

The following definitions of weak Pareto solutions and local weak Pareto solutions for NSMPP will be utilized in the subsequent sections of the article.

Definition 6 ([45])**.** *Let $\overline{\mathcal{Z}} \in \mathscr{F}$. Then $\overline{\mathcal{Z}}$ is said to be a weak Pareto solution of NSMPP if there does not exist $\mathcal{Z} \in \mathscr{F}$ such that $\Phi(\mathcal{Z}) \prec \Phi(\overline{\mathcal{Z}})$.*

Definition 7 ([45])**.** *Let $\overline{\mathcal{Z}} \in \mathscr{F}$. Then $\overline{\mathcal{Z}}$ is said to be a local weak Pareto solution of NSMPP if for any neighborhood $\mathscr{N}$ of $\overline{\mathcal{Z}}$ there does not exist $\mathcal{Z} \in \mathscr{N} \cap \mathscr{F}$ such that $\Phi(\mathcal{Z}) \prec \Phi(\overline{\mathcal{Z}})$.*

For convenience, we introduce the following notations that will be used throughout the subsequent sections of this article:

$$\begin{aligned}
&\mathcal{F} := \bigcup_{i\in\mathbb{I}} \operatorname{co}\partial^*\Phi_i(\mathcal{Z}),\\
&\mathcal{F}^i := \bigcup_{j\in\mathbb{I}\setminus\{i\}} \operatorname{co}\partial^*\Phi_j(\mathcal{Z}),\\
&\mathcal{G} := \bigcup_{j\in\mathbb{J}(\overline{\mathcal{Z}})} \operatorname{co}\partial^*\Psi_j(\mathcal{Z}),\\
&\mathcal{H} := \bigcup_{k\in\mathbb{K}} \operatorname{co}\partial^*\Theta_k(\mathcal{Z}) \cup \operatorname{co}\partial^*(-\Theta_k)(\mathcal{Z}),\\
&\Gamma(\mathcal{Z}) := (\mathcal{F}^i)^- \cap \mathcal{G}^- \cap \mathcal{H}^- \cap \mathbb{S}^n_+,\\
&\Lambda(\mathcal{Z}) := (\mathcal{F}^i)^s \cap \mathcal{G}^s \cap \mathcal{H}^s,\\
&\mathcal{S} = \{\mathcal{Z} \in \mathbb{S}^n_+ | \Psi(\mathcal{Z}) \le 0,\ \Theta(\mathcal{Z}) := 0\},\\
&\mathcal{S}^i := \{\mathcal{Z} \in \mathbb{S}^n_+ | \Phi_j(\mathcal{Z}) \le \Phi_j(\mathcal{Z}),\ \forall j \in \mathbb{I}\setminus\{i\},\ \Psi(\mathcal{Z}) \le 0,\ \Theta(\mathcal{Z}) = 0\}.
\end{aligned}$$

Now, using the properties of convexificators, we establish the FJ-type necessary optimality conditions for NSMPP.

Theorem 3. *Let us assume that $\overline{\mathcal{Z}}$ is a local weak Pareto solution of NSMPP. Moreover, $\Phi_i,\ i \in \mathbb{I}$, $\Psi_j,\ j \in \mathbb{J}(\overline{\mathcal{Z}}),\ \pm\Theta_k,\ k \in \mathbb{K}$ admit bounded USRC and each $\Psi_j,\ j \in \mathbb{J}\setminus\mathbb{J}(\overline{\mathcal{Z}})$ is continuous. Then there exist $\overline{\lambda}_i \ge 0, i \in \mathbb{I},\ \mu_j \ge 0,\ j \in \mathbb{J}(\overline{\mathcal{Z}}),\ \nu_k \ge 0,\ \tau_k \ge 0,\ k \in \mathbb{K}, \overline{\mho} \in \mathbb{S}^n_+$; not all can be zero simultaneously such that*

$$0 \in \sum_{i\in\mathbb{I}} \overline{\lambda}_i \operatorname{co}\partial^*\Phi_i(\overline{\mathcal{Z}}) + \sum_{j\in\mathbb{J}} \mu_j \operatorname{co}\partial^*\Psi_j(\overline{\mathcal{Z}}) + \sum_{k\in\mathbb{K}} [\nu_k \operatorname{co}\partial^*\Theta_k(\overline{\mathcal{Z}}) + \tau_k \operatorname{co}\partial^*(-\Theta_k)(\overline{\mathcal{Z}})] - \overline{\mho},$$

$$\langle \overline{\mho}, \overline{\mathcal{Z}} \rangle = 0, \quad \mu_j \Psi_j(\overline{\mathcal{Z}}) = 0,\ \forall j \in \mathbb{J}.$$

Proof. Let us define

$$\Delta := \Big[\mathcal{F}^s + \overline{\mathcal{Z}}\Big] \bigcap \Big[\mathcal{G}^s + \overline{\mathcal{Z}}\Big] \bigcap \Big[\mathcal{H}^s + \overline{\mathcal{Z}}\Big].$$

We claim that

$$\Delta \cap \mathbb{S}^n_{++} = \emptyset. \tag{4}$$

On the contrary, we suppose that $\mathcal{Z} \in \Delta \cap \mathbb{S}^n_{++}$. Since $\Phi_i,\ i \in \mathbb{I}$, $\Psi_j,\ j \in \mathbb{J}(\overline{\mathcal{Z}})$ and $\pm\Theta_k,\ k \in \mathbb{K}$ admit bounded USRC; we have

$$\begin{aligned}
\Phi_i^+(\overline{\mathcal{Z}}; \mathcal{Z} - \overline{\mathcal{Z}}) &< 0,\ \forall i \in \mathbb{I},\\
\Psi_j^+(\overline{\mathcal{Z}}; \mathcal{Z} - \overline{\mathcal{Z}}) &< 0,\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}),\\
\Theta_k^+(\overline{\mathcal{Z}}; \mathcal{Z} - \overline{\mathcal{Z}}) &< 0,\ \forall k \in \mathbb{K},\\
-\Theta_k^+(\overline{\mathcal{Z}}; \mathcal{Z} - \overline{\mathcal{Z}}) &< 0,\ \forall k \in \mathbb{K}.
\end{aligned}$$

Hence, there exists some $\delta_1 > 0$ such that for all $t \in (0, \delta_1)$; we have

$$\Phi_i(\overline{\mathcal{Z}} + t(\mathcal{Z} - \overline{\mathcal{Z}})) - \Phi_i(\overline{\mathcal{Z}}) < 0,\ \forall i \in \mathbb{I},$$

$$\Psi_j(\overline{\mathcal{Z}} + t(\mathcal{Z} - \overline{\mathcal{Z}})) < 0,\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}),$$

$$\Theta_k(\overline{\mathcal{Z}} + t(\mathcal{Z} - \overline{\mathcal{Z}})) - \Theta_k(\overline{\mathcal{Z}}) < 0,\ \forall k \in \mathbb{K},$$

$$(-\Theta_k)(\overline{\mathcal{Z}}+t(\mathcal{Z}-\overline{\mathcal{Z}}))-(-\Theta_k)(\overline{\mathcal{Z}})<0,\ \forall k\in\mathbb{K}.$$

By the continuity of $\Psi_j,\ j\in\mathbb{J}\backslash\mathbb{J}(\overline{\mathcal{Z}})$, there exists some $\delta_2>0$ such that

$$\Psi_j(\overline{\mathcal{Z}}+t(\mathcal{Z}-\overline{\mathcal{Z}}))<0,\ \forall j\in\mathbb{J}\backslash\mathbb{J}(\overline{\mathcal{Z}}),\ \forall t\in(0,\delta_2).$$

Using the convexity of $\mathbb{S}^n_+$, we get a contradiction to the assumption that $\overline{\mathcal{Z}}$ is a local weak Pareto solution of NSMPP. Let us denote

$$\begin{aligned}
\rho_i^{\Phi}(\mathcal{Z}) &:= \sup_{\xi_i^1\in\partial^*\Phi_i(\overline{\mathcal{Z}})}\langle\xi_i^1,\ \mathcal{Z}-\overline{\mathcal{Z}}\rangle,\ i\in\mathbb{I},\\
\rho_j^{\Psi}(\mathcal{Z}) &:= \sup_{\xi_j^2\in\partial^*\Psi_j(\overline{\mathcal{Z}})}\langle\xi_j^2,\ \mathcal{Z}-\overline{\mathcal{Z}}\rangle,\ j\in\mathbb{J}(\overline{\mathcal{Z}}),\\
\rho_k^{\Theta}(\mathcal{Z}) &:= \sup_{\xi_k^3\in\partial^*(\Theta_k)(\overline{\mathcal{Z}})}\langle\xi_k^3,\ \mathcal{Z}-\overline{\mathcal{Z}}\rangle,\ k\in\mathbb{K},\\
\rho_k^{-\Theta}(\mathcal{Z}) &:= \sup_{\xi_k^3\in\partial^*(-\Theta_k)(\overline{\mathcal{Z}})}\langle\xi_k^3,\ \mathcal{Z}-\overline{\mathcal{Z}}\rangle,\ k\in\mathbb{K}.
\end{aligned}$$

It is notable that $\rho_i^{\Phi}(.)$, $\rho_j^{\Psi}(.)$, and $\rho_k^{\Theta}(.)$ are convex functions. From (4), we deduce that the subsequent system does not possess a solution:

$$(\mathscr{K})\begin{cases}\rho_i^{\Phi}(\mathcal{Z})<0,\ \text{if } i\in\mathbb{I},\\ \rho_j^{\Psi}(\mathcal{Z})<0,\ \text{if } j\in\mathbb{J}(\overline{\mathcal{Z}}),\\ \rho_k^{\Theta}(\mathcal{Z})<0,\ \text{if } k\in\mathbb{K},\\ \rho_k^{-\Theta}(\mathcal{Z})<0,\ \text{if } k\in\mathbb{K},\\ \mathcal{Z}\in\mathbb{S}^n_{++}.\end{cases}$$

By Lemma 2, there exist non-negative multipliers $\overline{\lambda}_i\geq 0,\ i\in\mathbb{I},\ \mu_j\geq 0,\ j\in\mathbb{J}(\overline{\mathcal{Z}}),\nu_k\geq 0,$ $\tau_k\geq 0, k\in\mathbb{K}$ and $\overline{\mathrm{U}}\in\mathbb{S}^n_+$; not all can be zero simultaneously such that

$$\Pi(\mathcal{Z}):=\sum_{i\in\mathbb{I}}\overline{\lambda}_i\rho_i^{\Phi}(\mathcal{Z})+\sum_{j\in\mathbb{J}}\mu_j\rho_j^{\Psi}(\mathcal{Z})+\sum_{k\in\mathbb{K}}\nu_k\rho_k^{\Theta}(\mathcal{Z})+\sum_{k\in\mathbb{K}}\tau_k\rho_k^{-\Theta}(\mathcal{Z})-\langle\overline{\mathrm{U}},\ V\rangle\geq 0. \tag{5}$$

From (5), we have $\langle\overline{\mathrm{U}},\overline{\mathcal{Z}}\rangle\leq 0$. Since $\overline{\mathrm{U}},\ \overline{\mathcal{Z}}\in\mathbb{S}^n_+$, hence $\langle\overline{\mathrm{U}},\overline{\mathcal{Z}}\rangle=0$. Consequently, $\Pi(\mathcal{Z})$ is a convex function and $\Pi(\overline{\mathcal{Z}})=0$. Thus $0\in\partial\Pi(\overline{\mathcal{Z}})$, where ∂ represents the symbol of subdifferential in the context of convex analysis. Therefore, there exist $\overline{\lambda}_i\geq 0,\ i\in\mathbb{I}$, $\mu_j\geq 0,\ j\in\mathbb{J}(\overline{\mathcal{Z}}),\nu_k\geq 0,\ \tau_k\geq 0,\ k\in\mathbb{K}$ such that

$$\begin{aligned}
0\in\sum_{i\in\mathbb{I}}\overline{\lambda}_i\,\mathrm{co}\,\partial^*\Phi(\overline{\mathcal{Z}})\ &+\sum_{j\in\mathbb{J}(\overline{\mathcal{Z}})}\mu_j\,\mathrm{co}\,\partial^*\Psi_j(\overline{\mathcal{Z}})\\
&+\sum_{k\in\mathbb{K}}[\nu_k\,\mathrm{co}\,\partial^*\Theta_k(\overline{\mathcal{Z}})+\tau_k\,\mathrm{co}\,\partial^*(-\Theta_k)(\overline{\mathcal{Z}})]-\overline{\mathrm{U}},\\
&\langle\overline{\mathrm{U}},\ \overline{\mathcal{Z}}\rangle=0.
\end{aligned}$$

Taking $\mu_j=0$ for $j\in\mathbb{J}\backslash\mathbb{J}(\overline{\mathcal{Z}})$. This completes the proof. □

Remark 5. *For $m=1$ and $\mathbb{K}=\emptyset$, Theorem 3 extends Theorem 3.1 derived by Golestani and Nobakhtian [43] from non-smooth semidefinite single-objective programming problems to a more general programming problem, NSMPP.*

In the subsequent definition, we introduce generalized Abadie constraint qualification (NSMPP-ACQ) in the context of NSMPP, which will prove to be useful in deriving strong KKT-type necessary optimality conditions for local weak Pareto solutions of NSMPP.

Definition 8. *The Abadie constraint qualification NSMPP-ACQ is said to satisfy at $\overline{\mathcal{Z}} \in \mathcal{S}$, if for every $r \in \mathbb{I}$,*

$$D^r := \text{cone co}\,\mathcal{F}^r + \text{cone co}\,\mathcal{G} + \text{cone co}\,\mathcal{H} - \mathbb{S}^n_+.$$

is closed and

$$\Gamma(\overline{\mathcal{Z}}) \subset T(\mathcal{S}^r, \overline{\mathcal{Z}}).$$

Now, using the properties of convexificators, we establish the strong KKT-type necessary optimality conditions for NSMPP.

Theorem 4. *Let $\overline{\mathcal{Z}} \in \mathscr{F}$ be a local weak Pareto solution of NSMPP. Suppose that at $\overline{\mathcal{Z}}$, $\Phi_i,\ i \in \mathbb{I}$, $\Psi_j,\ j \in \mathbb{J}$ and $\pm\Theta_k,\ k \in \mathbb{K}$ admit bounded USRC. Moreover, assume that NSMPP-ACQ is satisfied at $\overline{\mathcal{Z}}$. Then there exist $\lambda_i > 0,\ i \in \mathbb{I}$, $\mu_j \geq 0,\ j \in \mathbb{J}$, $\nu_k \geq 0,\ \tau_k \geq 0,\ k \in \mathbb{K}$ and $\overline{\mathtt{U}} \in \mathbb{S}^n_+$ such that*

$$\begin{aligned} 0 \in \sum_{i \in \mathbb{I}} \lambda_i \,\text{co}\,\partial^* \Phi_i(\overline{\mathcal{Z}}) &+ \sum_{j \in \mathbb{J}} \mu_j \,\text{co}\,\partial^* \Psi_j(\overline{\mathcal{Z}}) \\ &+ \sum_{k \in \mathbb{K}} [\nu_k \,\text{co}\,\partial^* \Theta_k(\overline{\mathcal{Z}}) + \tau_k \,\text{co}\,\partial^*(-\Theta_k)(\overline{\mathcal{Z}})] - \overline{\mathtt{U}}, \end{aligned} \tag{6}$$

$$\langle \overline{\mathtt{U}}, \overline{\mathcal{Z}} \rangle = 0, \quad \mu_j \Psi_j(\overline{\mathcal{Z}}) = 0,\ \forall j \in \mathbb{J}. \tag{7}$$

Proof. To derive the above result, it is sufficient to show that for every $i \in \mathbb{I}$, the following inclusion relation holds:

$$0 \in \text{co}\,\partial^* \Phi_i(\overline{\mathcal{Z}}) + D^i. \tag{8}$$

On the contrary, let us assume that there exists $r \in \mathbb{I}$ such that

$$0 \notin \text{co}\,\partial^* \Phi_r(\overline{\mathcal{Z}}) + D^r.$$

Since $\Phi_r,\ r \in \mathbb{I}$ admits bounded USRC, thus $\text{co}\,\partial^* \Phi_r(\overline{\mathcal{Z}})$ is compact and convex set in $\mathbb{S}^n$. From the definition of NSMPP-ACQ, D^r is a closed convex set in $\mathbb{S}^n$. Hence, $\text{co}\,\partial^* \Phi_r(\overline{\mathcal{Z}}) + D^r$ is a closed convex set in $\mathbb{S}^n$. By employing the separation theorem, there exists $\mathbb{V} \in \mathbb{S}^n$ such that

$$\langle \xi + \zeta,\ \mathbb{V} \rangle < 0,\ \forall \xi \in \text{co}\,\partial^* \Phi_r(\overline{\mathcal{Z}}),\ \forall \zeta \in D^r.$$

Since zero is contained in every cone, we get

$$\langle \xi,\ \mathbb{V} \rangle < 0,\ \forall \xi \in \text{co}\,\partial^* \Phi_r(\overline{\mathcal{Z}}).$$

Thus,

$$\Phi_r^+(\overline{\mathcal{Z}}; \mathbb{V}) < 0.$$

Hence, $\exists \kappa > 0$ such that

$$\Phi_r(\overline{\mathcal{Z}} + t\mathbb{V}) < \Phi_r(\overline{\mathcal{Z}}),\ \forall t \in (0, \kappa). \tag{9}$$

Moreover, we deduce that

$$\langle \zeta,\ \mathbb{V} \rangle \leq 0,\ \forall\, \zeta \in D^r.$$

Consequently,

$$\langle \eta_i,\ \mathbb{V} \rangle \leq 0,\ \forall \eta_i \in \text{co}\,\partial^* \Phi_i(\overline{\mathcal{Z}}),\ \forall i \in \mathbb{I} \backslash \{r\}, \tag{10}$$

$$\langle \eta_j,\ \mathbb{V} \rangle \leq 0,\ \forall \eta_j \in \text{co}\,\partial^* \Psi_j(\overline{\mathcal{Z}}),\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}), \tag{11}$$

$$\langle \eta_k,\ \mathbb{V} \rangle \leq 0,\ \forall \eta_k \in \text{co}\,\partial^* \Theta_k(\overline{\mathcal{Z}}) \cup \text{co}\,\partial^*(-\Theta_k)(\overline{\mathcal{Z}}),\ \forall k \in \mathbb{K}, \tag{12}$$

$$\langle \overline{\mathtt{U}},\ \mathbb{V} \rangle \geq 0,\ \forall \overline{\mathtt{U}} \in \mathbb{S}^n_+. \tag{13}$$

From (10)–(13) and Lemma 1, we get

$$\mathbb{V} \in T(\mathcal{S}^r, \overline{\mathcal{Z}}).$$

Therefore, there exists $t_n \downarrow 0$ and $\mathbb{V}_n \to \mathbb{V}$ such that

$$\overline{\mathcal{Z}} + t_n \mathbb{V}_n \in \mathcal{S}^r.$$

Then for t small enough and n large enough, we have

$$\overline{\mathcal{Z}} + t\mathbb{V} \in \mathcal{S}^r. \tag{14}$$

From (9) and (14), we arrive at a contradiction with the local weak Pareto solution at $\overline{\mathcal{Z}}$. Hence, (8) holds. Therefore, there exist $\lambda_i > 0$, $\mu_j \geq 0$, $\nu_k \geq 0$ and $\tau_k \geq 0$ such that

$$\begin{aligned} 0 \in \sum_{i\in\mathbb{I}} \lambda_i \operatorname{co} \partial^* \Phi_i(\overline{\mathcal{Z}}) + \sum_{j\in\mathbb{J}} \mu_j \operatorname{co} \partial^* \Psi_j(\overline{\mathcal{Z}}) \\ + \sum_{k\in\mathbb{K}} [\nu_k \operatorname{co} \partial^* \Theta_k(\overline{\mathcal{Z}}) + \tau_k \operatorname{co} \partial^*(-\Theta_k)(\overline{\mathcal{Z}})] - \bar{\mathbb{U}}, \end{aligned}$$

$$\langle \bar{\mathbb{U}}, \overline{\mathcal{Z}} \rangle = 0, \quad \mu_j \Psi_j(\overline{\mathcal{Z}}) = 0, \ \forall j \in \mathbb{J}.$$

This completes the proof. □

Remark 6.

1. *Theorem 4 generalizes Theorem 3.2 established by Golestani and Nobakhtian [30] from the Euclidean space to the space of symmetric matrices.*
2. *For $m = 1$ and $\mathbb{K} = \emptyset$, Theorem 4 extends Theorem 3.3 derived by Golestani and Nobakhtian [43] from non-smooth semidefinite single-objective programming problems to a more general programming problem NSMPP.*
3. *Theorem 4 generalizes Theorem 4.1 established by Giorgi et al. [34] from the Euclidean space setting to the space of symmetric matrices in terms of convexificators.*

Now, in the subsequent example, utilizing NSMPP-ACQ, we examine the Lagrange multipliers of NSMPP to illustrate the significance of the Theorem 4.

Example 1. *We consider the following mathematical programming problem with mixed constraints given by*

$$\begin{aligned} \text{(P1) Minimize } \ & \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \Phi_2(\mathcal{Z})) := (|z_1|, |z_3|), \\ \text{subject to } \ & \Psi(\mathcal{Z}) := -2z_2 \leq 0, \\ & \Theta(\mathcal{Z}) := 2z_2 = 0, \\ & \mathcal{Z} \in \mathbb{S}^2_+, \end{aligned}$$

where $\Phi_i : \mathbb{S}^2 \to \overline{\mathbb{R}}$, $i \in \{1, 2\}$, $\Psi : \mathbb{S}^2 \to \overline{\mathbb{R}}$ and $\Theta : \mathbb{S}^2 \to \overline{\mathbb{R}}$. The set consisting of all feasible solutions of (P1) is given by

$$\mathscr{F} := \left\{ \mathcal{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2 : z_2 = 0, \ \mathcal{Z} \in \mathbb{S}^2_+ \right\}.$$

It is evident that $\overline{\mathcal{Z}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a local weak Pareto solution of (P1). Moreover, it can be verified that

$$\partial^* \Phi_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \quad \partial^* \Phi_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

$$\partial^*\Psi(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}, \qquad \partial^*\Theta(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\},$$
$$\partial^*(-\Theta)(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Therefore, we have

$$\operatorname{co}\partial^*\Phi_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in [-1,1] \right\},$$
$$\operatorname{co}\partial^*\Phi_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} : b \in [-1,1] \right\}.$$

It is evident that

$$\Gamma(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \subset T(\mathcal{S}^r, \overline{\mathcal{Z}}).$$

Since D^r is a closed set, NSMPP-ACQ is satisfied at $\overline{\mathcal{Z}}$. Moreover, there exist

$$\xi_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Phi_1(\overline{\mathcal{Z}}), \qquad \xi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \operatorname{co}\partial^*\Phi_2(\overline{\mathcal{Z}}),$$
$$\xi_3 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Psi(\overline{\mathcal{Z}}), \qquad \xi_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Theta(\overline{\mathcal{Z}}),$$
$$\xi_5 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in \operatorname{co}\partial^*(-\Theta)(\overline{\mathcal{Z}}),$$

such that for $\lambda_1 = 1,\ \lambda_2 = 1,\ \mu = 0,\ \nu = 1,\ \tau = 1$ and $\overline{\mathbb{U}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ Equations (6) and (7) hold true at $\overline{\mathcal{Z}}$. Hence, strong KKT-type necessary optimality conditions are satisfied at a local weak Pareto solution $\overline{\mathcal{Z}}$ of the considered problem (P1).

Now, in the subsequent theorem, under the assumptions of generalized convexity, we establish sufficient optimality conditions for a weak Pareto solution of NSMPP.

Theorem 5. *Let $\overline{\mathcal{Z}} \in \mathscr{F}$ satisfy the strong KKT-type necessary optimality conditions established in Theorem 4. Assume that $\Phi_i,\ i \in \mathbb{I}$, are ∂^*-pseudoconvex at $\overline{\mathcal{Z}}$ and $\Psi_j,\ j \in \mathbb{J},\ \pm\Theta_k,\ k \in \mathbb{K}$ are ∂^*-quasiconvex at $\overline{\mathcal{Z}}$. Then $\overline{\mathcal{Z}}$ is a weak Pareto solution of NSMPP.*

Proof. Since $\overline{\mathcal{Z}} \in \mathscr{F}$ satisfies the strong KKT-type necessary optimality conditions, there exist $\lambda_i > 0,\ i \in \mathbb{I}, \mu_j \geq 0,\ j \in \mathbb{J}, \nu_k \geq 0,\ \tau_k \geq 0,\ k \in \mathbb{K}$ and $\overline{\mathbb{U}} \in \mathbb{S}^n_+$ such that

$$\begin{aligned} 0 \in \sum_{i\in\mathbb{I}} \lambda_i \operatorname{co}\partial^*\Phi_i(\overline{\mathcal{Z}}) &+ \sum_{j\in\mathbb{J}} \mu_j \operatorname{co}\partial^*\Psi_j(\overline{\mathcal{Z}}) \\ &+ \sum_{k\in\mathbb{K}} [\nu_k \operatorname{co}\partial^*\Theta_k(\overline{\mathcal{Z}}) + \tau_k \operatorname{co}\partial^*(-\Theta_k)(\overline{\mathcal{Z}})] - \overline{\mathbb{U}}. \end{aligned} \tag{15}$$

Assume, on the contrary, that $\overline{\mathcal{Z}}$ is not a weak Pareto solution of NSMPP. Then there exists $Y \in \mathscr{F}$ such that $\Phi(Y) \prec \Phi(\overline{\mathcal{Z}})$. From the ∂^*-pseudoconvexity of $\Phi_i,\ i \in \mathbb{I}$ at $\overline{\mathcal{Z}}$, we have

$$\langle \xi_i^1, Y - \overline{\mathcal{Z}} \rangle < 0,\ \ \forall \xi_i^1 \in \partial^*\Phi_i(\overline{\mathcal{Z}}),\ \ \forall i \in \mathbb{I}. \tag{16}$$

At feasible point Y of NSMPP, we have

$$\Psi_j(Y) \leq \Psi_j(\overline{\mathcal{Z}}),\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}).$$

Therefore, from the ∂^*-quasiconvexity of $\Psi_j,\ j \in \mathbb{J}(\overline{\mathcal{Z}})$, we have

$$\langle \xi_j^2, Y - \overline{\mathcal{Z}} \rangle \leq 0,\ \ \forall\, \xi_j^2 \in \partial^*\Psi_j(\overline{\mathcal{Z}}),\ \ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}). \tag{17}$$

Similarly, we get

$$\langle \xi_k^3, Y - \overline{\mathcal{Z}} \rangle \leq 0, \ \ \forall \xi_k^3 \in \partial^* \Theta_k(\overline{\mathcal{Z}}), \ \ \forall k \in \mathbb{K}, \tag{18}$$

$$\langle \xi_k^4, Y - \overline{\mathcal{Z}} \rangle \leq 0, \ \ \forall \xi_k^4 \in \partial^* (-\Theta_k)(\overline{\mathcal{Z}}), \ \ \forall k \in \mathbb{K}. \tag{19}$$

Since $\overline{\mathrm{U}}, \ Y \in \mathbb{S}_+^n$, we get

$$\langle -\overline{\mathrm{U}}, Y - \overline{\mathcal{Z}} \rangle = -\langle \overline{\mathrm{U}}, Y \rangle + \langle \overline{\mathrm{U}}, \overline{\mathcal{Z}} \rangle \leq 0. \tag{20}$$

From (16)–(20), there exist $\lambda_i > 0, i \in \mathbb{I}, \ \mu_j \geq 0, \ j \in \mathbb{J}(\overline{\mathcal{Z}}), \ \nu_k \geq 0, \ \tau_k \geq 0, \ k \in \mathbb{K}$ such that

$$\left\langle \sum_{i \in \mathbb{I}} \lambda_i \xi_i^1 + \sum_{j \in \mathbb{J}(\overline{\mathcal{Z}})} \mu_i \xi_j^2 + \sum_{k \in \mathbb{K}} [\nu_k^{\Theta} \xi_k^3 + \tau_k^{\Theta} \xi_k^4] - \overline{\mathrm{U}}, Y - \overline{\mathcal{Z}} \right\rangle < 0. \tag{21}$$

Hence, we arrive at a contradiction with (15). This completes the proof. □

Remark 7. *For $m = 1$ and $\mathbb{K} = \emptyset$, Theorem 4 and Theorem 5 reduce to Theorem 3.2 and Theorem 3.5, respectively, derived in [43] from non-smooth semidefinite single-objective programming problems to a more general programming problem NSMPP.*

To illustrate the results established in Theorem 5, we furnish a non-trivial illustrative example as follows:

Example 2. *We investigate the following mathematical programming problem incorporating mixed constraints given by*

$$\begin{aligned} \text{(P2) Minimize } \ & \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \Phi_2(\mathcal{Z})) := (z_2, z_3), \\ \text{subject to } \ & \Psi(\mathcal{Z}) := -|z_3| \leq 0, \\ & \Theta(\mathcal{Z}) := z_2 = 0, \\ & \mathcal{Z} \in \mathbb{S}_+^2, \end{aligned}$$

where $\Phi_i : \mathbb{S}^2 \to \overline{\mathbb{R}}, \ i \in \{1, 2\}, \Psi : \mathbb{S}^2 \to \overline{\mathbb{R}}$ and $\Theta : \mathbb{S}^2 \to \overline{\mathbb{R}}$. The set of all feasible solutions of (P2) is given by

$$\mathscr{F} := \left\{ \mathcal{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2 : -|z_3| \leq 0, \ z_2 = 0, \ \mathcal{Z} \in \mathbb{S}_+^2 \right\}.$$

Since there exist $\lambda_1 = 1, \ \lambda_2 = 1, \ \mu = 1, \ \nu = 1, \ \tau = 2$ and $\overline{\mathrm{U}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ such that at $\overline{\mathcal{Z}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ the strong KKT-type necessary optimality conditions are satisfied. Moreover, $\Phi_i, \ i \in \{1, 2\}$ are ∂^*-pseudoconvex and $\Psi, \ \pm\Theta$ are ∂^*-quasiconvex at $\overline{\mathcal{Z}}$. Hence, conditions of Theorem 5 are satisfied at $\overline{\mathcal{Z}}$.

4. Constraint Qualifications

In this section, we introduce generalized versions of some well-known constraint qualifications for the considered problem, NSMPP. Moreover, we establish the interrelationship among the various constraint qualifications introduced in this section.

Definition 9. *The generalized Kuhn-Tucker constraint qualification NSMPP-KTCQ is satisfied at $\overline{\mathcal{Z}} \in \mathcal{S}$ if for every $r \in \mathbb{I}$, the set D^r is closed and*

$$\Gamma(\overline{\mathcal{Z}}) \subset A(\mathcal{S}^r, \overline{\mathcal{Z}}).$$

Remark 8. *Definition 9 extends the Definition 4.1 defined in [30] from the non-smooth multiobjective programming problems in the Euclidean space to a more general programming problem NSMPP.*

Definition 10. *The Zangwill constraint qualification NSMPP-ZCQ is satisfied at $\overline{\mathcal{Z}} \in \mathcal{S}$ if for every $r \in \mathbb{I}$, the set D^r is closed and*

$$\Gamma(\overline{\mathcal{Z}}) \subset cl\ D(\mathcal{S}^r, \overline{\mathcal{Z}}).$$

Remark 9. *Definition 10 extends the Definition 4.2 defined in [30] from the non-smooth multiobjective programming problems in the Euclidean space to a more general programming problem, NSMPP.*

In the following proposition, we establish the interrelationship among NSMPP-ZCQ, NSMPP-KTCQ, and NSMPP-ACQ.

Proposition 2. *Suppose that $\overline{\mathcal{Z}}$ is a feasible solution of NSMPP. Moreover, assume that NSMPP-ZCQ is satisfied at $\overline{\mathcal{Z}}$. Then*

$$\text{NSMPP-ZCQ} \implies \text{NSMPP-KTCQ} \implies \text{NSMPP-ACQ}.$$

Proof. Since

$$cl\ F(\mathcal{S}^r, \overline{\mathcal{Z}}) \subseteq A(\mathcal{S}^r, \overline{\mathcal{Z}}) \subseteq T(\mathcal{S}^r, \overline{\mathcal{Z}}). \tag{22}$$

Therefore, from (22), we get the desired result. □

Remark 10. *The above three cones may exhibit strict containment; therefore, the reverse implication may not hold true.*

Definition 11. *The Mangasarian-Fromovitz constraint qualification NSMPP-MFCQ is satisfied at $\overline{\mathcal{Z}} \in \mathcal{S}$ if for every $r \in \mathbb{I}$,*

$$\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++} \neq \emptyset.$$

Remark 11. *Definition 11 extends the Definition 4.4 defined in [30] from the non-smooth multiobjective programming problems in the Euclidean space to a more general programming problem NSMPP.*

Definition 12. *The basic constraint qualification NSMPP-BCQ is satisfied at $\overline{\mathcal{Z}} \in \mathcal{S}$ if for every $\mathtt{U} \in \mathbb{S}^n_+$,*

$$\mathtt{U} \notin \operatorname{co} \Delta(\overline{\mathcal{Z}}),$$

where

$$\Delta(\overline{\mathcal{Z}}) := \left(\bigcup_{i \in \mathbb{I} \setminus \{r\}} \operatorname{co} \partial^* \Phi_i(\mathcal{Z}) \right) \bigcup \left(\bigcup_{j \in \mathbb{J}(\overline{\mathcal{Z}})} \operatorname{co} \partial^* \Psi_j(\mathcal{Z}) \right)$$
$$\bigcup \left(\bigcup_{s \in \mathbb{S}} \operatorname{co} \partial^* \Theta_k(\mathcal{Z}) \right) \bigcup \left(\bigcup_{s \in \mathbb{S}} \operatorname{co} \partial^* (-\Theta_k)(\mathcal{Z}) \right).$$

Remark 12. *Definition 12 extends the Definition 4.5 defined in [30] from the non-smooth multiobjective programming problems in the Euclidean space to a more general programming problem NSMPP.*

In the following proposition, we establish that the NSMPP-BCQ implies NSMPP-MFCQ.

Proposition 3. *Assume that Φ_i, $i \in \mathbb{I}$, Ψ_j, $j \in \mathbb{J}(\overline{\mathcal{Z}})$ and $\pm\Theta_k$, $k \in \mathbb{K}$ admit bounded USRC. Then, NSMPP-BCQ implies NSMPP-MFCQ.*

Proof. Let us assume that NSMPP-BCQ is satisfied at $\overline{\mathcal{Z}}$. Then

$$\mho \notin \operatorname{co}\Delta(\overline{\mathcal{Z}}).$$

Moreover, $\operatorname{co}(\Delta(\overline{\mathcal{Z}}))$ being a compact convex set and $\mathbb{S}^n_+$ is a closed convex set, employing the separation theorem, there exist scalar β and a nonzero $\mathbb{V} \in \mathbb{S}^n$ such that

$$\langle \zeta, \mathbb{V} \rangle < \beta < \langle \eta, \mathbb{V} \rangle, \forall \zeta \in \operatorname{co}(\Delta(\overline{\mathcal{Z}})), \ \forall \eta \in \mathbb{S}^n_+. \tag{23}$$

Since $0 \in \mathbb{S}^n_+$, we get

$$\langle \zeta, \mathbb{V} \rangle < 0, \ \forall \zeta \in \operatorname{co}(\Delta(\overline{\mathcal{Z}})).$$

Thus,

$$\langle \zeta_i^1, \mathbb{V} \rangle < 0, \ \forall\, \zeta_i^1 \in \operatorname{co}\partial^*\Phi_i(\overline{\mathcal{Z}}), \ \ \forall i \in \mathbb{I}\backslash\{r\},$$
$$\langle \zeta_j^2, \mathbb{V} \rangle < 0, \ \forall\, \zeta_j^2 \in \operatorname{co}\partial^*\Psi_j(\overline{\mathcal{Z}}), \ \ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}),$$
$$\langle \zeta_k^3, \mathbb{V} \rangle < 0, \ \forall\, \zeta_k^3 \in \operatorname{co}\partial^*\Theta_k(\overline{\mathcal{Z}}), \ \ \forall k \in \mathbb{K},$$
$$\langle \zeta_k^4, \mathbb{V} \rangle < 0, \ \forall\, \zeta_k^4 \in \operatorname{co}\partial^*(-\Theta_k)(\overline{\mathcal{Z}}), \ \ \forall k \in \mathbb{K}.$$

Therefore,

$$\mathbb{V} \in \Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++}. \tag{24}$$

Since $\mathbb{S}^n_+$ is a cone, from (23), we get

$$\langle \eta, \mathbb{V} \rangle \geq 0, \ \forall \eta \in \mathbb{S}^n_+. \tag{25}$$

From Lemma 1 and (25), we get $\mathbb{V} \in \mathbb{S}^n_+$. Since $\operatorname{cl}\mathbb{S}^n_{++} = \mathbb{S}^n_+$. If $V \in \mathbb{S}^n_{++}$, NSMPP-MFCQ holds from (24). Alternatively, considering the density of $\mathbb{S}^n_{++}$ and the openness of the right-hand side of (24), we have $\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++} \neq \emptyset$. Hence, NSMPP-MFCQ holds at $\overline{\mathcal{Z}}$. □

The following example illustrates the result established in Proposition 3, that at a feasible point of a multiobjective programming problem with mixed constraints, NSMPP-BCQ implies NSMPP-MFCQ.

Example 3. *We consider the following mathematical programming problem given by*

$$\begin{aligned} &\text{(P3) Minimize } \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \Phi_2(\mathcal{Z})) := (-z_1, -z_3), \\ &\quad\text{subject to } \Psi(\mathcal{Z}) := 2z_2 \leq 0, \\ &\quad\quad \mathcal{Z} \in \mathbb{S}^2_+, \end{aligned}$$

where $\Phi_i : \mathbb{S}^2 \to \overline{\mathbb{R}}, \ i \in \{1,2\}, \Psi : \mathbb{S}^2 \to \overline{\mathbb{R}}$ *and* $\Theta : \mathbb{S}^2 \to \overline{\mathbb{R}}$*. The set of all feasible solutions of (P3) is given by*

$$\mathscr{F} := \left\{ \mathcal{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2 : 2z_2 \leq 0, \ \mathcal{Z} \in \mathbb{S}^2_+ \right\}.$$

It is evident that $\overline{\mathcal{Z}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *is a feasible solution of (P3). Moreover, it can be verified that*

$$\partial^*\Phi_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \quad \partial^*\Phi_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\},$$
$$\partial^*\Psi(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

It is evident that for all $\mho \in \mathbb{S}^2_+$,

$$\mho \notin \operatorname{co}\Delta(\overline{\mathcal{Z}}).$$

Hence, NSMPP-BCQ holds at $\overline{\mathcal{Z}}$. Furthermore,

$$\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++} \neq \emptyset,$$

which implies that NSMPP-MFCQ holds at $\overline{\mathcal{Z}}$.

The following proposition states that if both the objective functions and the active constraint functions admit bounded USRC and the active constraints are continuous, then NSMPP-MFCQ implies NSMPP-ACQ.

Proposition 4. *Let $\overline{\mathcal{Z}}$ be a feasible solution of NSMPP. Suppose that Φ_i, $i \in \mathbb{I}$, Ψ_j, $j \in \mathbb{J}(\overline{\mathcal{Z}})$ and $\pm\Theta_k$, $k \in \mathbb{K}$ admit bounded USRC and $\Psi_j, j \in \mathbb{J}\backslash\mathbb{J}(\overline{\mathcal{Z}})$ are continuous. Moreover, if for each $r \in \mathbb{I}$, the set D^r is closed and NSMPP-MFCQ holds at $\overline{\mathcal{Z}}$, then NSMPP-ACQ also holds at $\overline{\mathcal{Z}}$.*

Proof. Let NSMPP-MFCQ hold at $\overline{\mathcal{Z}}$. Therefore, there exists $\mathbb{V} \in \mathbb{S}^n_+$ such that $\mathbb{V} \in \Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++}$. From the assumptions, both the objective and the active constraint functions admit bounded USRC. Hence, we have

$$\Phi_i^+(\overline{\mathcal{Z}};\mathbb{V}) < 0,\ \forall i \in \mathbb{I}\backslash\{r\}, \tag{26}$$

$$\Psi_j^+(\overline{\mathcal{Z}};\mathbb{V}) < 0,\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}), \tag{27}$$

$$\Theta_k^+(\overline{\mathcal{Z}};\mathbb{V}) < 0,\ \forall k \in \mathbb{K}. \tag{28}$$

$$(-\Theta_k)^+(\overline{\mathcal{Z}};\mathbb{V}) < 0,\ \forall k \in \mathbb{K}. \tag{29}$$

$\mathbb{S}^n_+$ being a convex cone, there exists $\kappa > 0$ such that

$$\Psi_j(\overline{\mathcal{Z}} + \lambda\mathbb{V}) < 0,\ j \in \mathbb{J}(\overline{\mathcal{Z}}),\ \ \overline{\mathcal{Z}} + \lambda\mathbb{V} \in \mathbb{S}^n_+,\ \ \forall \lambda \in (0,\kappa).$$

From (26)–(29), we have

$$\Phi_i(\overline{\mathcal{Z}} + t\mathbb{V}) - \Phi_i(\overline{\mathcal{Z}}) < 0,\ \forall i \in \mathbb{I}\backslash\{r\},$$
$$\Psi_j(\overline{\mathcal{Z}} + t\mathbb{V}) < 0,\ \forall j \in \mathbb{J}(\overline{\mathcal{Z}}),$$
$$\Theta_k(\overline{\mathcal{Z}} + t\mathbb{V}) - \Theta_k(\overline{\mathcal{Z}}) < 0,\ \forall k \in \mathbb{K},$$
$$(-\Theta_k)(\overline{\mathcal{Z}} + t\mathbb{V}) - (-\Theta_k)(\overline{\mathcal{Z}}) < 0,\ \forall k \in \mathbb{K}.$$

As assumed, $\Psi_j,\ j \in \mathbb{J}\backslash\mathbb{J}(\overline{\mathcal{Z}})$ is continuous; it follows that $\overline{\mathcal{Z}} + t\mathbb{V} \in \mathcal{S}^r$. Thus, $\mathbb{V} \in T(\mathcal{S}^r, \overline{\mathcal{Z}})$. Hence,

$$\begin{aligned}
\Gamma(\overline{\mathcal{Z}}) &= (\mathcal{F}^r)^- \cap \mathcal{G}^- \cap \mathcal{H}^- \cap \mathbb{S}^n_+ \\
&= \mathrm{cl}(\mathcal{F}^r)^s \cap \mathrm{cl}\,\mathcal{G}^s \cap \mathrm{cl}\,\mathcal{H}^s \cap \mathrm{cl}\,\mathbb{S}^n_{++} \\
&= \mathrm{cl}((\mathcal{F}^r)^s \cap \mathcal{G}^s \cap \mathcal{H}^s \cap \mathbb{S}^n_{++}) \\
&= cl(\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^n_{++}) \\
&\subset \mathrm{cl}\,T(\mathcal{S}^r, \overline{\mathcal{Z}}) \\
&= T(\mathcal{S}^r, \overline{\mathcal{Z}}).
\end{aligned}$$

This completes the proof. □

Remark 13. *If $m = 1$ and $\mathbb{K} = \emptyset$, then Proposition 4 established in this article reduces to Proposition 3.3 derived by Golestani and Nobakhtian [43].*

The subsequent example illustrates that NSMPP-ACQ does not necessarily imply NSMPP-MFCQ.

Example 4. *We investigate the following mathematical programming problem given by*

$$\begin{aligned} \textit{(P4) Minimize } & \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \Phi_2(\mathcal{Z})) := (|z_1|, |z_3|), \\ \textit{subject to } & \Psi(\mathcal{Z}) := -2z_2 \leq 0, \\ & \Theta(\mathcal{Z}) := z_2 = 0, \\ & \mathcal{Z} \in \mathbb{S}^2_+, \end{aligned}$$

where $\Phi_i : \mathbb{S}^2 \to \overline{\mathbb{R}},\ i \in \{1,2\}$, $\Psi : \mathbb{S}^2 \to \overline{\mathbb{R}}$ *and* $\Theta : \mathbb{S}^2 \to \overline{\mathbb{R}}$. *The set of all feasible solutions of (P4) is given by*

$$\mathscr{F} := \left\{ \mathcal{Z} = \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix} \in \mathbb{S}^2 : z_2 = 0,\ \mathcal{Z} \in \mathbb{S}^2_+ \right\}.$$

It is evident that

$$\left(\bigcup_{i \in \mathbb{I} \setminus \{2\}} \operatorname{co} \partial^* \Phi_i(\overline{\mathcal{Z}}) \right)^s = \emptyset,$$

$$\left(\bigcup_{i \in \mathbb{I} \setminus \{1\}} \operatorname{co} \partial^* \Phi_i(\overline{\mathcal{Z}}) \right)^s = \emptyset.$$

Therefore, we have $\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^2_{++} = \emptyset$. *However, from Example 1, it is evident that NSMPP-ACQ is satisfied at* $\overline{\mathcal{Z}}$. *Hence, NSMPP-ACQ is satisfied at* $\overline{\mathcal{Z}}$ *but not NSMPP-MFCQ.*

Through the following example, employing NSMPP-ACQ, we verify the strong KKT-type necessary optimality conditions for NSMPP. Moreover, the following example illustrates that NSMPP-ACQ does not necessarily imply NSMPP-MFCQ.

Example 5. *We consider the following mathematical programming problem with mixed constraints given by*

$$\begin{aligned} \textit{(P5) Minimize } & \Phi(\mathcal{Z}) = (\Phi_1(\mathcal{Z}), \Phi_2(\mathcal{Z}), \Phi_3(\mathcal{Z})) := (|z_{11}|e^{z_{11}}, e^{|z_{22}|}, |z_{33} - 1|), \\ \textit{subject to } & \Psi_1(\mathcal{Z}) := z_{22}(z_{22} - 1) \leq 0, \\ & \Psi_2(\mathcal{Z}) := -z_{33}^2 - |z_{33}| \leq 0, \\ & \Theta(\mathcal{Z}) := z_{13}^2 + z_{23}^2 + |z_{12}| = 0, \\ & \mathcal{Z} \in \mathbb{S}^3_+, \end{aligned}$$

where $\Phi_i : \mathbb{S}^3 \to \overline{\mathbb{R}},\ i \in \{1,2,3\}$, $\Psi_j : \mathbb{S}^3 \to \overline{\mathbb{R}},\ j \in \{1,2\}$ *and* $\Theta : \mathbb{S}^3 \to \overline{\mathbb{R}}$. *The set consisting of all feasible solutions of (P1) is given by*

$$\mathscr{F} := \left\{ \mathcal{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{13} & z_{23} & z_{33} \end{bmatrix} \in \mathbb{S}^3 : z_{22}(z_{22} - 1) \leq 0,\ -z_{33}^2 - |z_{33}| \leq 0, \right.$$
$$\left. z_{13}^2 + z_{23}^2 + |z_{12}| = 0,\ \mathcal{Z} \in \mathbb{S}^3_+ \right\}.$$

It is evident that $\overline{\mathcal{Z}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is a local weak Pareto solution of (P5). Moreover, it can be verified that*

$$\partial^*\Phi_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \qquad \partial^*\Phi_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*\Phi_3(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \qquad \partial^*\Psi_1(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*\Psi_2(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \qquad \partial^*\Theta(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

$$\partial^*(-\Theta)(\overline{\mathcal{Z}}) = \left\{ \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

It can be verified that D^r is closed set and $\Gamma(\overline{\mathcal{Z}}) \subset T(\mathcal{S}^r, \overline{\mathcal{Z}})$. Hence, NSMPP-ACQ is satisfied at $\overline{\mathcal{Z}}$. Furthermore, there exist

$$\eta_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Phi_1(\overline{\mathcal{Z}}), \qquad \eta_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Phi_2(\overline{\mathcal{Z}}),$$

$$\eta_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \operatorname{co}\partial^*\Phi_3(\overline{\mathcal{Z}}), \qquad \eta_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Psi_1(\overline{\mathcal{Z}}),$$

$$\eta_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \operatorname{co}\partial^*\Psi_2(\overline{\mathcal{Z}}), \quad \eta_6 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*(-\Theta)(\overline{\mathcal{Z}}),$$

$$\eta_7 = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \operatorname{co}\partial^*\Theta(\overline{\mathcal{Z}}),$$

such that for $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 1\ \mu_1 = 1,\ \mu_2 = 1,\ \nu = 2,\ \tau = 2$ and $\overline{\mathrm{U}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Equations (6) and (7) hold true at $\overline{\mathcal{Z}}$. Hence, strong KKT-type necessary optimality conditions are satisfied at a local weak Pareto solution $\overline{\mathcal{Z}}$ of the considered problem (P5).

In addition to this, it can be verified that

$$\Lambda(\overline{\mathcal{Z}}) \cap \mathbb{S}^2_{++} = \emptyset.$$

Thus, NSMPP-ACQ is satisfied at $\mathcal{Z}$, but not NSMPP-MFCQ. Hence, NSMPP-ACQ does not necessarily imply NSMPP-MFCQ.

The following Figure 1 summarizes the above results and illustrates the interrelationships among the different constraint qualifications introduced for NSMPP.

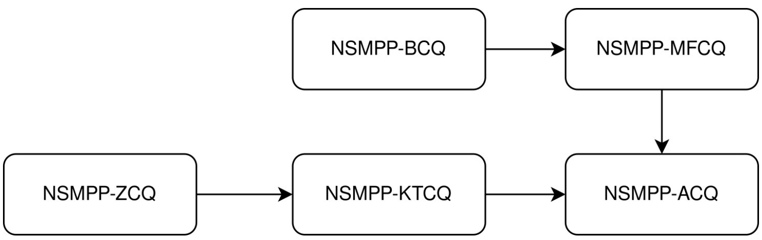

Figure 1. Relationship among the different constraint qualifications.

5. Conclusions and Future Directions

In this article, we explored a class of non-smooth semidefinite multiobjective programming problems with mixed constraints (NSMPP). We have established the separation theorem for the space of symmetric matrices. We have established FJ-type necessary optimality conditions for NSMPP. Moreover, we have introduced NSMPP-ACQ for NSMPP in terms of convexificator and employed it to establish strong KKT-type necessary optimality conditions for a local weak Pareto solution of NSMPP. Furthermore, we have introduced NSMPP-ZCQ, NSMPP-KTCQ, NSMPP-MFCQ, and NSMPP-BCQ for NSMPP and established interrelationships among them. Several non-trivial examples are furnished, illustrating the significance of the established results.

The constraint qualifications and optimality conditions established in this article extend several well-known results existing in the literature for non-smooth multiobjective programming problems to a more general programming problem, NSMPP, in terms of convexificators. In particular, we have generalized the various constraint qualifications introduced in [30] from non-smooth multiobjective problems to a more general programming problem, NSMPP. Moreover, the results derived in this article generalize the corresponding results derived in [34] from the Euclidean space to the space of symmetric matrices in terms of convexificators. Furthermore, we have generalized the results established in [43] from non-smooth single-objective programming problems to a more general programming problem, NSMPP.

It is worthwhile to mention that inequality and equality constraints are assumed to be finite in the considered problem, NSMPP. Therefore, the results established in this article cannot be applied to the class of problems involving infinite inequality or equality constraints. This may be considered a limitation of this article. We intend to address this limitation in our future course of study.

The results established in this article open up several possibilities for future research. For instance, in view of the work presented by Ardali et al. [50,51], it would be intriguing to introduce constraint qualifications and to establish optimality conditions for non-smooth multiobjective semidefinite programming problems with equilibrium constraints. In addition to this, it would be interesting to solve the NSMPP employing the augmented Lagrangian method and its splitting method proposed by Bai et al. [52].

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