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A Note on a Min–Max Method for a Singular Kirchhoff Problem of Fractional Type

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Abstract: In the present work, we study a fractional elliptic Kirchhoff-type problem that has a singular term. More precisely, we start by proving some properties related to the energy functional associated with the studied problem. Then, we use the variational method combined with the min–max method to prove that the energy functional reaches its global minimum. Finally, since the energy functional has a singularity, we use the implicit function theorem to show that the point where the minimum is reached is a weak solution for the main problem. To illustrate our main result, we give an example at the end of this paper.

Keywords: variational methods; singular equations; existence of solutions; min–max method

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1. Introduction

In this study, we prove some existing results related to the following fractional and singular problem:

$$(Q_\mu)\begin{cases} M(A_p w)\mathrm{L}^p_{\mathbf{K}}(w)(\xi) = \frac{h(\xi)}{|w(\xi)|^{\delta(\xi)}} + \mu f(\xi, w(\xi)) & \text{in } \Omega, \\ w = 0, \quad \text{on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\mu > 0$, Ω is a bounded domain of $\mathbb{R}^N$, δ is a continuous function on $\overline{\Omega}$, and $p : \mathbb{R}^N \times \mathbb{R}^N \to (1, \infty)$ is a continuous function, which satisfies

$$1 < p^- \leq p^+ < \infty, \tag{1}$$

$$p(\xi, \eta) = p(\eta, \xi), \ \forall\ (\xi, \eta) \in \overline{\Omega} \times \overline{\Omega}, \tag{2}$$

and

$$p((\xi, \eta) - (y, y)) = p(\xi, \eta), \ \ \forall\ (\xi, \eta, y) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N, \tag{3}$$

where

$$p^- = \inf_{\overline{\Omega} \times \overline{\Omega}} p(\xi, \eta), \text{ and } \ p^+ = \sup_{\overline{\Omega} \times \overline{\Omega}} p(\xi, \eta).$$

The operators A_p and $\mathrm{L}^p_{\mathbf{K}}$ are defined, respectively, by

$$A_p(w) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(\xi) - w(\eta)|^{p(\xi, \eta)}}{p(\xi, \eta)} \mathbf{K}(\xi, \eta)\, d\xi d\eta, \tag{4}$$

and

$$\mathrm{L}^p_{\mathbf{K}} w(\xi) = C \int_{\mathbb{R}^N} |w(\xi) - w(\eta)|^{p(\xi, \eta) - 2}(w(\xi) - w(\eta))\mathbf{K}(\xi, \eta)\, d\eta, \ \forall\ \xi \in \mathbb{R}^N, \tag{5}$$

for some normalized constant C. Finally, we assume that the function $\mathbf{K}$ is positive measurable on $\mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega$ and satisfies the following properties:

(B1) $\mathbf{K}(\xi,\eta) = \mathbf{K}(\eta,\xi)$ for any $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$.

(B2) There exist $k_0 > 0$ and $s \in (0,1)$, such that for all $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\xi \neq \eta$, we have

$$\mathbf{K}(\xi,\eta) \geq k_0|\xi-\eta|^{-(N+sp(\xi,\eta))}.$$

(B3) $\varphi\mathbf{K} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $\varphi(\xi,\eta) = \min(|\xi-\eta|^{p(\xi,\eta)}, 1)$.

We note that in the particular case when $\mathbf{K}(\xi,\eta) = \frac{1}{|\xi-\eta|^{N+sq(\xi,\eta)}}$, the operator $\mathrm{L}^p_{\mathbf{K}}$ is reduced to the fractional $p(.,.)$-Laplacian operator $(-\Delta_{p(.,.)})^s$, which is studied by many researchers (we cite, for example, the works [1–4]). A logical consequence is that every application of $(-\Delta_{q(.,.)})^s$ is also an application of $\mathrm{L}^p_{\mathbf{K}}$; so, we can find several applications of our problem in many fields like fluids, mechanics and image processing (see, for example, the reference [5]). As researchers delve deeper into understanding intricate systems and phenomena, the general non-local integro-differential operator continues to be indispensable, driving advancements in both theoretical frameworks and practical applications.

Significant attention has been directed toward investigating challenges associated with these operators. Specifically, in the literature, there are too many problems of Kirchhoff type involving variable exponents that we refer interested readers to the papers [2,6–19] and others cited therein.

For problems with singular terms arising from $\mathrm{L}^p_{\mathbf{K}}$, occurrences are quite rare and we are possibly among the first to address them through this paper. Concerning other operators like the $p(.)$-Laplacian operator, there are many published papers involving singular nonlinearities in addition; we cite, for example [9,18,20–23]. In particular, in [18], the authors considered the following problem:

$$\begin{cases} -G\left(\int_\Omega \frac{1}{q(\xi)}|\nabla u|^{q(\xi)}d\xi\right)\Delta_{q(.)}u(\xi) = h(\xi)u^{-\gamma(\xi)} - \lambda g(\xi,u(\xi)) & \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $G \in C((0,\infty),(0,\infty))$, $g \in C^1(\Omega \times \mathbb{R})$, q and γ are continuous on Ω. Under suitable assumptions and using some variational techniques combined with the min–max method, the authors proved that problem (6) has a nontrivial solution.

Recently, Ben Ali et al. [24] considered the following fractional problem:

$$\begin{cases} G\left(\int_{\Omega\times\Omega} \frac{|w(\xi)-w(\eta)|^{q(\xi,\eta)}}{q(\xi,\eta)|\xi-\eta|^{N+sq(\xi,\eta)}}d\xi d\eta\right)(-\Delta)^s_{q(.,.)}w(\xi) = \lambda h(\xi,w(\xi)) - |w(\xi)|^{p(\xi)-2}w(\xi) & \text{in } \Omega, \\ w = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{7}$$

Under appropriate hypotheses and by combining Ekland's variational principle with the mountain pass theorem, the authors proved the existence of two nontrivial solutions for problem (7).

Azroul et al. [25] considered the following problem:

$$(P_K)\begin{cases} \mathrm{L}^p_{\mathbf{K}}u(\xi) = f(\xi,u(\xi)) & \text{in } \Omega, \\ u = 0, \quad \text{on } \mathbb{R}^N\setminus\Omega. \end{cases}$$

Under certain conditions, the authors showed that the problem (P_K) has a unique weak solution, and this is proven by the means of the Minty–Browder Theorem.

In this work, we continue to investigate a fractional problem of Kirchhoff type. We note that Kirchhoff-type problems often refer to problems in mathematical physics related to Kirchhoff's laws or equations. These problems typically arise in contexts like electrical circuits, heat conduction, or wave propagation. These types of problems are introduced by Kirchhoff [14], and more precisely, the author studied the following equation:

$$\delta \frac{\partial^2 \psi}{\partial u^2} - \left(\frac{t_0}{s} + \frac{\Delta}{2h} \int_0^h |\frac{\partial \psi}{\partial u}|^2 \, du \right) \frac{\partial^2 \psi}{\partial v^2},$$

where h represents the length of a cross and s represents the area of a cross-section; the initial axial extension is denoted by Δ and ψ denotes the lateral displacement at x and y. The novelty in our study is that the Kirchhoff function M is in a more general class of functions. The presence of the singular term also implies that the functional energy is not regular, so the direct variational method cannot be applied. We note that the presence of the non-local operator $\mathrm{L}^p_{\mathbf{K}}$ generalizes other operators in the literature. To prove the existence of solutions, we present several results and notations in Section 2, after which we present and prove the main result of this paper in Section 3. We finish our work by presenting an illustrative example.

2. Preliminaries

In this section, we introduce the functional framework within which we will examine our principal result. To accomplish this, we present the essential characteristics of variable exponent spaces. Interested readers can find further properties in [25–28] and the associated literature.

Next, we denoted by $C_+(\Omega)$ the sets of all continuous functions q such that $\inf_{\xi \in \overline{\Omega}} q(\xi) > 1$, and for a fixed function $w \in C_+(\Omega)$, we define

$$w^- = \inf_{\xi \in \overline{\Omega}} w(\xi) \text{ and } w^+ = \sup_{\xi \in \overline{\Omega}} w(\xi).$$

Let $q \in C_+(\Omega)$. We define the space $L^{q(.)}(\Omega)$ as the set of all measurable functions w such that $\int_\Omega |w(\xi)|^{q(\xi)} d\xi < \infty$, and we equip it with the following norm:

$$|w|_{q(.)} = \inf \left\{ \mu > 0 : \int_\Omega |\frac{w(\xi)}{\mu}|^{q(\xi)} d\xi \leq 1 \right\}.$$

We recall that $L^{q(.)}(\Omega)$ is a Banach space, moreover, it is separable and reflexive if and only if

$$1 < q^- \leq q^+ < \infty.$$

Also, the Hölder inequality holds in this space.

Put

$$\rho_{q(.)}(w) = \int_\Omega |w(\xi)|^{q(\xi)} d\xi.$$

Proposition 1. *For all $w \in L^{q(.)}(\Omega)$, we have the following:*

(1) Both $|w|_{q(.)}$ and $\rho_{q(.)}(w)$ are less than one, or both greater than one, or both equal to one.

(2) $\min(|w|^{q^-}_{q(.)}, |w|^{q^+}_{q(.)}) \leq \rho_{q(.)}(w) \leq \max(|w|^{q^-}_{q(.)}, |w|^{q^+}_{q(.)})$.

Also, we have the following interesting proposition.

Proposition 2. *Let m be a measurable function in $L^\infty(\mathbb{R}^N)$, and let q be a measurable function, such that for any $\xi \in \mathbb{R}^N$, we have*

$$1 \leq q(\xi) m(\xi) \leq \infty.$$

If $0 \neq w \in L^{q(.)}(\mathbb{R}^N)$*, then, we obtain*

$$\min(|w|^{q^-}_{m(.)q(.)}, |w|^{q^+}_{m(.)q(.)}) \leq ||w|^{m(.)}|_{q(.)} \leq \max(|w|^{q^-}_{m(.)q(.)}, |w|^{q^+}_{m(.)q(.)}).$$

Now, for a function p satisfying Equations (1)–(3), we define the Sobolev space $W^{s,\overline{p}(.),p(.,.)}(\Omega)$ by

$$W^{s,\overline{p}(.),p(.,.)}(\Omega) = \left\{ w \in L^{\overline{p}(.)}(\Omega), \int_{\Omega\times\Omega} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{s^{p(\xi,\eta)}|\xi - \eta|^{N+sp(\xi,\eta)}} \, d\xi d\eta < \infty, \ \ s > 0 \right\},$$

and we endow it with the following norm:

$$||w||_{W^{s,\overline{p}(.),p(.,.)}(\Omega)} = |w|_{\overline{p}(.)} + [w]_{s,p(.,.)},$$

where $\overline{p}(\xi) = p(\xi, \xi)$, and

$$[w]_{s,p(.,.)} = \inf\left\{ s > 0 : \int_{\Omega\times\Omega} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{s^{p(\xi,\eta)}|\xi - \eta|^{N+sp(\xi,\eta)}} \, d\xi d\eta \leq 1 \right\}.$$

We denote by

$$\mathbb{B} = \left\{ w : \mathbb{R}^N \longrightarrow \ \mathbb{R} \text{ measurable} : w_{|\Omega} \in L^{\overline{p}(\xi)}(\Omega) \right\}.$$

Next, we define the space

$$W^{\mathbf{K},p(.,.)}(\Omega) = \left\{ w \in \mathbb{B} : \ \text{with} \ \int_{\Omega\times\Omega} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{s^{p(\xi,\eta)}} \mathbf{K}(\xi, \eta) \, d\xi d\eta < \infty, \ \text{for some } s > 0 \right\},$$

equipped with the norm

$$||w||_{W^{\mathbf{K},p(.,.)}(\Omega)} = ||w||_{\overline{p}(.)} + [w]_{\mathbf{K},p(.,.)},$$

where

$$[w]_{\mathbf{K},p} = \inf\left\{ s > 0 : \int_{\Omega\times\Omega} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{s^{p(\xi,\eta)}} \mathbf{K}(\xi, \eta) \, d\xi d\eta \leq 1 \right\}.$$

Hereafter, we will work on the following space:

$$E := W_0^{\mathbf{K},p(.,.)}(\Omega) = \left\{ w \in W^{\mathbf{K},p(.,.)}(\Omega), \ \ w(\xi) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

with the equivalent norm

$$||.|| = [.]_{\mathbf{K},p}.$$

We recall from [25] that this space contains $C_0^\infty(\Omega)$; moreover, the following propositions hold.

Proposition 3. *The space* $(E, ||.||)$ *is a uniformly convex Banach space. Moreover, it is reflexive and separable.*

For $w \in E$, we define the functional

$$\sigma_{\mathbf{K},p}(w) = \int_{\Omega\times\Omega} |w(\xi) - w(\eta)|^{p(\xi,y)} \mathbf{K}(\xi, \eta) \, d\xi d\eta.$$

Proposition 4 ([25])**.** *For any* $w \in E$*, we have*

$$\min\left(||w||^{p^-}, ||w||^{p^+}\right) \leq \sigma_{\mathbf{K},p}(w) \leq \max\left(||w||^{p^-}, ||w||^{p^+}\right).$$

Proposition 5 ([25]). *Let w, $w_n \in E$, $n \in \mathbb{N}$, and then, we have*

$$\lim_{n\to\infty} ||w_n - w|| = 0 \iff \lim_{n\to\infty} \sigma_{\mathbf{K},p}(w_n - w) = 0.$$

Finally, we recall again from [25] that if for any $\xi \in \overline{\Omega}$, we have

$$1 < \alpha(\xi) < p_s^*(\xi) := \frac{Np(\xi,\xi)}{N - sp(\xi,\xi)},$$

and if $\mathbf{K}$ satisfies conditions $(B1)$–$(B3)$, then we have a continuous and compact embedding from E into $L^{\alpha(.)}(\Omega)$; in particular, there is $C_\alpha > 0$, satisfying

$$|w|_{\alpha(.)} \le C_\alpha ||w||.$$

Definition 1. *By a solution of problem (Q_μ), we mean a function $w \in E$ for which for any $\varphi \in E$, we have*

$$\begin{aligned} & M(A_P(w)) \int_{\Omega\times\Omega} |w(\xi) - w(\eta)|^{p(\xi,\eta)-2}(w(\xi) - w(\eta))(\varphi(\xi) - \varphi(\eta))\mathbf{K}(\xi,\eta)\, d\xi d\eta \\ = & \int_\Omega h(\xi)|w(\xi)|^{-\delta(\xi)}\varphi(\xi)\, d\xi + \mu \int_\Omega f(\xi, w(\xi))\varphi(\xi) d\xi. \end{aligned}$$

To prove the existence of solutions for problem (Q_μ), we associate it with the following singular functional $I_\mu : E \to \mathbb{R}$, which is defined by

$$I_\mu(w) = \widehat{M}(A_p(w)) - \int_\Omega \frac{h(\xi)}{1-\delta(\xi)}|w(\xi)|^{1-\delta(\xi)} d\xi - \mu \int_\Omega F(\xi, w(\xi)) d\xi,$$

where

$$F(\xi,t) = \int_0^t f(\xi,s) ds, \text{ and } \widehat{M}(t) = \int_0^t M(s) ds.$$

It is worth noting that I_μ is well defined but it is non-differentiable. So, we use the min–max method to prove our existing result.

3. Main Result and Its Proof

In this section, we give and demonstrate the main result of this paper. To this aim, we assume the following hypotheses:

(N_0) The function M is continuous and positive in $\mathbb{R}$; moreover, there exist $a > 1$ and $\nu > 1$, such that

$$\frac{1}{a}t^{\nu-1} \le M(t) \le a t^{\nu-1}, \ \forall\, t \ge 0.$$

(N_1) For all $(\xi,\eta) \in \overline{\Omega} \times \overline{\Omega}$, we have

$$\nu p(\xi,\xi) < p_s^*(\xi), \quad sp(\xi,\eta) < N, \text{ and } \nu p(\xi,\xi) < \frac{N}{s}.$$

(N_2) A function h is positive almost everywhere in Ω, such that

$$h \in L^{\frac{\theta(.)}{\theta(.)+\delta(.)-1}}(\Omega),$$

for some $1 < \theta(\xi) < p_s^*(\xi)$.

(H_1) There exist $B, \alpha \in C_+(\overline{\Omega})$ and $\psi \in L^{B(\xi)}(\Omega)$, such that, for all $(\xi,\eta) \in \overline{\Omega} \times \mathbb{R}$, we have

$$f(\xi,\eta) \le c\psi(\xi)|\eta|^{\alpha(\xi)-2}\eta, \ \text{ for some } c > 0,$$

and

$$\alpha(\xi) < p(\xi,\xi) < \eta p(\xi,\xi) < \frac{N}{s} < B(\xi).$$

(H_2) There exists $\Omega' \subset\subset \Omega$, $|\Omega'| > 0$, for which we have

$$f(\xi,\eta) \geq 0, \ \forall \xi \in \Omega', \text{ and all } \ \eta \in \mathbb{R}.$$

Now, we state the main result of this work.

Theorem 1. *Assume that hypotheses (N_0)–(N_2) and (H_1)–(H_2) hold. If the kernel $\mathbf{K}$ satisfies hypotheses $(B1)$–$(B3)$, then, for each $\mu > 0$, the problem Q_μ admits a nontrivial solution.*

To prove Theorem 1, we will prove several Lemmas. The first one concerns the coercivity of the associated functional energy. In particular, we prove the following Lemma:

Lemma 1. *If the hypotheses (N_0)–(N_2) and (H_1) hold, then I_μ becomes coercive in E.*

Proof. Let $w \in E$ with $\|w\| > 1$. Then, by hypothesis (N_0), Equation (1) and Proposition 4, we obtain

$$\begin{aligned}
\widehat{M}(A_p(w)) &= \widehat{M}\bigg(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{p(\xi,\eta)} \mathbf{K}(\xi,\eta)\, d\xi d\eta \bigg) \\
&= \widehat{M}\bigg(\int_{\Omega \times \Omega} \frac{|w(\xi) - w(\eta)|^{p(\xi,\eta)}}{p(\xi,\eta)} \mathbf{K}(\xi,\eta)\, d\xi d\eta \bigg) \\
&\geq \frac{1}{a\nu} \bigg(\int_{\Omega \times \Omega} |w(\xi) - w(\eta)|^{p(\xi,\eta)} \mathbf{K}(\xi,\eta)\, d\xi d\eta \bigg)^{\nu} \\
&\geq \frac{1}{p^+ a\nu} \sigma^{\nu}_{\mathbf{K},p}(w) \\
&\geq \frac{1}{p^+ a\nu} ||w||^{\nu p^-}.
\end{aligned} \tag{8}$$

Now, from hypothesis (N_2) and the Hölder inequality, we conclude that

$$\begin{aligned}
\int_\Omega \frac{h(\xi)}{1-\delta(\xi)} (w(\xi))^{1-\delta(\xi)} d\xi &\leq \frac{1}{1-\delta^+} \int_\Omega h(\xi) (w(\xi))^{1-\delta(\xi)} d\xi \\
&\leq \frac{1}{1-\delta^+} |h|_{\frac{\theta(.)}{\theta(.)+\delta(.)-1}} ||w|^{1-\delta(.)}|_{\frac{\theta(.)}{1-\delta(.)}}.
\end{aligned} \tag{9}$$

So, by combining Proposition 1 and the compact embedding with the fact that $1 < \theta(\xi) < q_s^*$, we conclude the existence of $c_1 > 0$ such that

$$\begin{aligned}
\int_\Omega \frac{h(\xi)}{1-\delta(\xi)} (w(\xi))^{1-\delta(\xi)} d\xi &\leq \frac{1}{1-\delta^+} |h|_{\frac{\theta(.)}{\theta(.)+\delta(.)-1}} \max(|w|^{1-\delta^+}_{\theta(.)}, |w|^{1-\delta^-}_{\theta(.)}) \\
&\leq \frac{c_1}{1-\delta^+} |h|_{\frac{\theta(.)}{\theta(.)+\delta(.)-1}} ||w||^{1-\delta^-}.
\end{aligned} \tag{10}$$

Next, from (H_1), the Hölder inequality and the compact embedding, we obtain

$$\begin{aligned}
\int_\Omega F(\xi, w(\xi)) d\xi &\leq c \int_\Omega \psi(\xi) |w(\xi)|^{\alpha(\xi)} d\xi \\
&\leq c|\psi|_{B(.)} ||w|^{\alpha(.)}|_{B'(.)} \\
&\leq c|\psi|_{B(.)} \max(|w|^{\alpha^+}_{B'(.)\alpha(.)}, |w|^{\alpha^-}_{B'(.)\alpha(.)}).
\end{aligned}$$

Using (H_1), we have

$$p_s^*(\xi) - \alpha(\xi)B'(\xi) = \frac{NB(\xi)[p(\xi,\xi) - \alpha(\xi)] + p(\xi,\xi)[B(\xi)\alpha(\xi)s - N]}{(B(\xi)-1)(N - sp(\xi,\xi))} > 0, \tag{11}$$

where

$$\frac{1}{B(\xi)} + \frac{1}{B'(\xi)} = 1.$$

In accordance with the compact embedding, we obtain the existence of $c_2 > 0$, satisfying

$$\int_\Omega F(\xi, w(\xi))d\xi \le c_2|\psi|_{B(.)}||w||^{\alpha^+}. \tag{12}$$

Finally, by combining Equations (8) and (10) with Equation (12), we obtain

$$I_\mu(w) \ge \frac{1}{p^+ a\eta}||w||^{\nu p^-} - \frac{c_1}{1-\delta^+}|h|_{\frac{\theta(.)}{\theta(.)+\delta(.)-1}}||w||^{1-\delta^-} - c_2|\psi|_{B(.)}||w||^{\alpha^+}. \tag{13}$$

Since $1 - \delta^- < \alpha^+ < \nu p^-$, we can see that $I_\mu(w) \to \infty$ as $||w|| \to \infty$. This ends the proof. □

Lemma 2. *If the hypothesis (H_2) is satisfied, then we obtain a non-negative function $0 \neq u$ in E, for which, for a sufficiently small $t > 0$, we have $I_\mu(tu) < 0$.*

Proof. We begin by fixing a function u in $C_0^\infty(\Omega)$ with $supp(u) \subset \Omega' \subset\subset \Omega$. We assume further that $u = 1$ in some subset $\Omega_1 \subset supp(u)$ and $0 \le u \le 1$ in Ω.

Now, let $t \in (0,1)$. Then, from hypothesis (N_0) and Proposition 4, we obtain

$$\begin{aligned}
\widehat{M}(A_p(tu)) &\le \frac{a}{\nu}A_p^\nu(tu) \\
&\le \frac{a}{\nu}\left(\int_{\Omega\times\Omega} t^{p(\xi,\eta)}\frac{|u(\xi)-u(\eta)|^{p(\xi,\eta)}}{p(\xi,\eta)}\mathbf{K}(\xi,\eta)\,d\xi d\eta\right)^\nu \\
&\le \frac{a}{\nu p^-}t^{\nu p^-}\sigma_{\mathbf{K},p}^\nu(u) \\
&\le \frac{a}{\nu p^-}t^{\nu p^-}||u||^{\nu p^-}.
\end{aligned} \tag{14}$$

So, using Equation (14) and the fact that F is a non-negative function, we obtain

$$\begin{aligned}
I_\mu(tu) &\le \frac{a}{\nu p^-}t^{\nu p^-}||u||^{\nu p^-} - \int_\Omega \frac{h(\eta)}{1-\delta(\eta)}|tu|^{1-\delta(\eta)}d\eta \\
&\le \frac{a}{\nu p^-}t^{\nu p^-}||u||^{\nu p^-} - t^{1-\delta^-}\frac{1}{1-\delta^-}\int_\Omega h(\eta)|u|^{1-\delta(\eta)}d\eta \\
&\le t^{1-\delta^-}\left(t^{\nu p^- + \delta^- - 1}\frac{a}{\nu p^-}||u||^{\nu p^-} - \frac{1}{1-\delta^-}\int_\Omega h(\eta)|u|^{1-\delta(\eta)}d\eta\right).
\end{aligned}$$

Since $\frac{a}{\nu p^-}||u||^{\nu p^-} > 0$ and $\nu p^- + \delta^- - 1 > 0$, we have

$$I_\mu(tu) < 0 \;\text{ for }\; t < min(1, S),$$

where

$$S = \left(\frac{\frac{1}{1-\delta^-}\int_\Omega h(\xi)|u|^{1-\delta(\xi)}d\xi}{\frac{a}{\nu p^-}||u||^{\nu p^-}}\right)^{\frac{1}{\nu p^- + \delta^- - 1}}.$$

□

In the sequel, we put

$$m_\mu = \inf_{w \in E} I_\mu(w). \tag{15}$$

Lemma 3. *Assume that assertions* (N_1)–(N_2) *and* (H_1)–(H_2) *hold. If, in addition,* **K** *satisfies hypotheses* $(B1)$–$(B3)$*, then there exists* $w_* \in E$*, satisfying*

$$I_\mu(w_*) = m_\mu < 0.$$

Proof. Let $\{w_n\}$ be a sequence in E that satisfies

$$I_\mu(w_n) \to m_\mu. \tag{16}$$

We claim that $\{w_n\}$ is bounded in E. Indeed, if this is not true, then we have

$$||w_n|| \to \infty \text{ as } n \to \infty.$$

The coercivity of the functional I_μ implies that

$$I_\mu(w_n) \to \infty \text{ as } ||w_n|| \to \infty,$$

which contradicts with Equation (16).

Now, the reflexivity of the space E implies that we can find a sub-sequence (still denoted by $\{w_n\}$) and a function w_* in E, such that

$$\begin{cases} w_n \rightharpoonup w_* \;\; weakly \;\; in \;\; E, \\ w_n \to w_* \;\; strongly \;\; in \;\; L^{\alpha(.)}(\Omega), \;\; 1 \leq \alpha(\xi) < p_s^*(\xi), \\ w_n \to w_* \;\; a.e \;\; in \;\; \Omega. \end{cases} \tag{17}$$

Next, our purpose is to prove that

$$I_\mu(w) = \widehat{M}(A_p(w)) - \int_\Omega \frac{h(\xi)}{1-\delta(\xi)} |w(\xi)|^{1-\delta(\xi)} d\xi - \mu \int_\Omega F(\xi, w(\xi)) d\xi$$

is weakly lower semi-continuous on E.

By the continuity of the function $\widehat{M}$, the fact that $w_n \to u_*$ a.e in Ω, and using Fatou's lemma, we deduce that

$$\widehat{M}(A_p(w_*)) \leq \lim_{n\to\infty} inf \widehat{M}(A_p(w_n)). \tag{18}$$

Now, it is shown in [29] (Theorem 2.3) that if $h \in L^{\zeta(.)}(\Omega)$ with

$$0 < \frac{\zeta(1-\delta)}{1-\zeta} < p_s^*, \tag{19}$$

then, we have

$$\lim_{n\to\infty} \int_\Omega h(\eta) |w_n(\eta)|^{1-\delta(\eta)} d\eta = \int_\Omega h(\eta) |w_*(\eta)|^{1-\delta(\eta)} d\eta. \tag{20}$$

A simple calculation shows that $\zeta = \frac{\theta(.)}{\theta(.)+\delta(.)-1}$ satisfies Equation (19). So, Equation (20) holds.

Next, we will demonstrate that

$$\lim_{n\to\infty} \int_\Omega F(\eta, w_n(\eta)) d\eta = \int_\Omega F(\eta, w_*(\eta)) d\eta. \tag{21}$$

For this purpose, let $\varepsilon > 0$; then, from hypothesis (H_1), we can find $c_\varepsilon > 0$, such that

$$|F(\eta, w_n(\eta))| \leq \frac{c_\varepsilon}{\alpha^-}|\psi(\eta)||w_n|^{\alpha(\eta).}$$

From Equation (11), we have $B^{'}(\xi)\alpha(\xi) < p_s^*$. So, from the fact that $w_n \rightharpoonup w_*$ in E, and from the compact embedding, we deduce the existence of a sub-sequence still denoted by $\{w_n\}$ that converges strongly in $L^{B^{'}(.)\alpha(.)}(\Omega)$. Thus, $w_{n \to} w_*$ a.e in Ω. Moreover, we have

$$|w_n(\xi)| \leq g(\xi), \text{ for some } g \in L^{\alpha(\xi)B^{'}(\xi)}.$$

Therefore, we obtain

$$|F(\xi, w_n(\xi))| \leq \frac{c_\varepsilon}{\alpha^-}|\psi(\xi)||g(\xi)|^{\alpha(\xi)}.$$

Hence, using the Hölder's inequality, one has

$$\int_\Omega |F(\xi, w_n(\xi))| d\xi \leq \frac{c_\varepsilon}{\alpha^-}|\psi|_{B(.)}|g|^{\alpha(.)}_{S'(.)}.$$

So, if we combine Proposition 2 with the Lebesgue-dominated convergence, we obtain the result of Equation (21).

Finally, Equations (18), (20) and (21) yield to the weakly lower semi-continuity of the functional I_δ. So, we deduce

$$I_\delta(w_*) \leq \lim_{n\to\infty} inf I_\delta(w_n) = m_\delta.$$

On the other hand, from (3), we have

$$I_\delta(w_*) \geq m_\delta.$$

We conclude that

$$I_\delta(w_*) = m_\delta.$$

□

Proof of Theorem 1. From Lemma 3, we deduce that the function w_* is a global minimum for the functional I_δ. So, for all $\phi \in E$ and all $t > 0$, we have

$$I_\delta(w_* + t\phi) - I_\delta(w_*) \geq 0.$$

By dividing the last inequality by t and by letting t tend to zero, we obtain

$$M(A_p(w_*)) \int_{\Omega\times\Omega} |w_*(\xi) - w_*(\eta)|^{p(\xi,\eta)-2}(w_*(\xi) - w_*(\eta))(\phi(\xi) - \phi(\eta))\mathbf{K}(\xi,\eta)\, d\xi d\eta$$
$$- \int_\Omega h(\xi)|w_*|^{-\delta(\xi)}\phi(\xi)d\xi - \mu \int_\Omega f(\xi, w_*(\xi))\phi(\xi)d\xi \geq 0.$$

As ϕ is arbitrary in E, we have the flexibility to substitute ϕ with $-\phi$ in the last inequality with the equality still preserved. Then, w_* is a weak solution to the problem (Q_δ). Furthermore, condition $I_\delta(w_*) < 0$ indicates that w_* is nontrivial. □

4. An Example

This section provides an example that improves the main result of this paper.

Example 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$. We consider the following problem:*

$$(Q_\delta)\begin{cases} (A_p(w))^{\nu-1}(-\Delta_{p(.,.)})^s(w) = \frac{h(\xi)}{|w(\xi)|^{\delta(\xi)}} + \mu a(\xi)|w(\xi)|^{\alpha(\xi)-2}w(\xi) & in\ \Omega, \\ w = 0, \quad on\ \mathbb{R}^N \backslash \Omega. \end{cases}$$

It is easy to state that the problem (Q_δ) corresponds to the case when $M(t) = t^{\nu-1}$, which clearly satisfies hypothesis (N_1). On the other hand, the operator $(-\Delta_{p(.,.)})$ corresponds to the case when

$$\mathbf{K}(\xi,\eta) = \frac{1}{|\xi-\eta|^{N+sq(\xi,\eta)}}.$$

It is not difficult to prove that the last kernel satisfies conditions $(B1)$–$(B3)$. Also, problem (Q_δ) corresponds to the case

$$f(t,\xi) = a(\xi)|\xi|^{\alpha(\xi)-2}\xi,$$

with $a \in L^{B(.)}(\Omega)$. So, clearly, f satisfies condition (H_2) and the first part of condition (H_1). Assume further that α satisfies the second part of condition (N_1). Finally, if ν and p satisfy condition (N_1) and Equations (1)–(3), and, if the function h satisfies hypothesis (N_2), then Theorem 1 can be applied, ensuring that the problem (Q_δ) admits a non-trivial solution.

5. Conclusions

In this paper, we studied a singular elliptic problem of Kirchhoff type. We transformed the study from that of the existence of a weak solution to the question of the existence of extremum points of the associated functional energy. Therefore, we have proven some properties of this functional and that it reached its global minimum at a point in an appropriate function space. To prove that this point is a weak solution for the studied problem, we have used the implicit function theorem; this is due to the singular term. We note that in some special cases of the kernel $L^p_{\mathbf{K}}$ and the Kirchhoff function M, we obtained the same results as those in the literature. We hope to generalize this study to problems involving the $p(x,y)$-Laplacian operator.

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