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Pre-Compactness of Sets and Compactness of Commutators for Riesz Potential in Global Morrey-Type Spaces

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Abstract: In this paper, we establish sufficient conditions for the pre-compactness of sets in the global Morrey-type spaces $GM^{w(\cdot)}_{p\theta}$. Our main result is the compactness of the commutators of the Riesz potential $[b, I_\alpha]$ in global Morrey-type spaces from $GM^{w_1(\cdot)}_{p_1\theta_1}$ to $GM^{w_2(\cdot)}_{p_2\theta_2}$. We also present new sufficient conditions for the commutator $[b, I_\alpha]$ to be bounded from $GM^{w_1(\cdot)}_{p_1\theta_1}$ to $GM^{w_2(\cdot)}_{p_2\theta_2}$. In the proof of the theorem regarding the compactness of the commutator for the Riesz potential, we primarily utilize the boundedness condition for the commutator for the Riesz potential $[b, I_\alpha]$ in global Morrey-type spaces $GM^{w(\cdot)}_{p\theta}$, and the sufficient conditions derived from the theorem on pre-compactness of sets in global Morrey-type spaces $GM^{w(\cdot)}_{p\theta}$.

Keywords: commutator; Riesz potential; compactness; global Morrey space; VMO

MSC: 42B20; 42B25

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1. Introduction

Morrey spaces $M^\lambda_p(\mathbb{R}^n)$ were introduced by C. Morrey [1] in 1938 during his studies of quasilinear elliptic differential equations. In this paper, we consider the global Morrey-type spaces $GM^{w(\cdot)}_{p\theta}(\mathbb{R}^n)$. Morrey spaces and generalized Morrey spaces are special cases of these spaces. We obtain sufficient conditions for the pre-compactness of sets in global Morrey-type spaces. These results are analogous to the well-known Fréchet–Kolmogorov theorems on the pre-compactness of sets in a Lebesgue space. The Fréchet–Kolmogorov theorem for the pre-compactness of sets in a Lebesgue space, in terms of average functions, contains three conditions that are necessary and sufficient. To prove a similar result for global Morrey-type spaces, we derive four conditions for pre-compactness of sets in terms of averaging functions. We also consider an example of a set of functions for which not all specified conditions are necessary. Consequently, the question of finding necessary and sufficient conditions in global Morrey spaces remains open. The conditions for the pre-compactness of sets in global Morrey-type spaces , in terms of the difference of functions, are similar to the corresponding conditions in the Fréchet–Kolmogorov theorem. This result is further utilized to prove the compactness of the commutator for the Riesz potential in the global Morrey-type spaces under consideration. Moreover, we preliminarily prove a theorem on the boundedness for the commutator of the Riesz potential in global Morrey-type spaces.

The first significant result regarding the commutator for the Riesz transforms was presented by Coifman et al. [2], which characterizes the boundedness of commutators for the Riesz transforms on the Lebesgue space $L_p(\mathbb{R}^n)$, with $p \in (1, \infty)$, via the well-known space $BMO(\mathbb{R}^n)$. The commutators of various operators play key roles in harmonic analysis (see, for instance, [3]), partial differential equations (see, for instance, [4]), and quasiregular mappings (see, for instance, [5]).

The compactness of the commutator for the Riesz potential on the Morrey spaces M_p^λ was considered in [6], while, in generalized Morrey spaces $M_p^{w(\cdot)}$, it was addressed in [7]. The compactness of the multi-commutators on the Morrey spaces with non-doubling measures was studied in [8]. Compactness characterizations of commutators on ball Banach function spaces was discussed in [9]. Additionally, the pre-compactness of sets on the Morrey spaces and on variable exponent Morrey spaces was examined in [10–12].

The boundedness of the Riesz potential on the Morrey spaces was investigated by S. Spanne, J. Peetre [13], and D. Adams [14]. T. Mizuhara [15], E. Nakai [16], and V.S. Guliyev [17] generalized the results of D. Adams and obtained sufficient conditions for the boundedness of I_α on the generalized Morrey spaces. The boundedness of the Riesz potential in local and global Morrey-type spaces was considered in [18,19].

Boundedness of the commutator for the Riesz potential on the Morrey spaces and on the generalized Morrey spaces was examined in [20,21], respectively.

The main goal of this paper is to find conditions for the pre-compactness of sets in the global Morrey-type spaces and to determine sufficient conditions for the compactness of the commutator for the Riesz potential $[b, I_\alpha]$ in the global Morrey-type spaces. Specifically, we aim to find conditions on the parameters p, q, α and the functions w_1 and w_2 that ensure the compactness of the operators $[b, I_\alpha]$ from $GM_{p_1\theta_1}^{w_1(\cdot)}$ to $GM_{p_2\theta_2}^{w_2(\cdot)}$.

This paper is organized as follows. In Section 2, we present definitions and preliminaries. To do this, we will establish some auxiliary lemmas. In Section 3, we present results on the pre-compactness of sets in terms of averaging function on global Morrey-type spaces. In Section 4, we present results on the pre-compactness of sets in terms of uniform equi-continuity on global Morrey-type spaces. In Section 5, we give sufficient conditions for the compactness of the commutator for the Riesz potential $[b, I_\alpha]$ on the global Morrey-type space $GM_{p\theta}^{w(\cdot)}$. Finally, we present conclusions in Section 6.

We make some conventions on notation. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_p, are dependent on the subscript p. We denote $f \lesssim g$ if $f \le Cg$ and $B \approx D$ if there exists C_1, C_2, such that $C_1 B \le D \le C_2 B$. By $C(\mathbb{R})$ we denote the set of all continuous bounded functions on $\mathbb{R}$ with the uniform norm, by χ_A we denote the characteristic function of a set $A \subset \mathbb{R}^n$, and by ${}^cA$ we denote the complement of A.

2. Definitions and Preliminaries

In this section, we recall some definitions of various function spaces, along with their properties, and establish some auxiliary lemmas.

For a Lebesgue measurable set $E \subset \mathbb{R}^n$ and $0 < p \le \infty$, $L_p(E)$ denotes the standard Lebesgue spaces of all functions f, Lebesgue measurable on E, for which

$$\|f\|_{L_p(E)} = \begin{cases} \left(\int\limits_E |f(y)|^p dy \right)^{\frac{1}{p}} < \infty, & \text{if} \quad 1 \le p < \infty, \\ ess\sup |f(y)| < \infty, & \text{if} \quad p = \infty. \end{cases}$$

Throughout the paper, $I := (a, b) \subseteq (0, \infty)$. By $\mathfrak{M}(I)$ we denote the set of all measurable functions on I. The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f \in \mathfrak{M}(I)$ that are non-negative on I, while $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$ denote the subsets of those functions that are non-increasing and non-decreasing on I, respectively. When $I = (0, \infty)$, we simply write $\mathfrak{M}\downarrow$ and $\mathfrak{M}\uparrow$ instead of $\mathfrak{M}^+(I;\downarrow)$ and $\mathfrak{M}^+(I;\uparrow)$, respectively. The family of all weight functions (also called just weights) on I, that is, globally integrable non-negative functions on $(0, \infty)$, is given by $W(I)$. For $p \in (0, \infty]$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\|\cdot\|_{L_{p,w}(I)}$ on $\mathfrak{M}(I)$ by

$$\|f\|_{L_{p,w}(I)} = \begin{cases} \left(\int_I |f(x)|^p w(x)dx\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ ess\sup_{(0;\infty)} |f(x)|w(x), & \text{if } p = \infty. \end{cases}$$

For $0 \leq \lambda \leq \frac{n}{p}$, $0 < p \leq \infty$, the Morrey spaces $M_p^\lambda \equiv M_p^\lambda(\mathbb{R}^n)$ are defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$, for which

$$\|f\|_{M_p^\lambda(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda} \left(\int\limits_{B(x,r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ of radius $r > 0$.

Recently, boundedness and compactness of various operators in Morrey-type spaces have been actively studied ([7,10]).

Note that

$$\|f\|_{M_p^0(\mathbb{R}^n)} \equiv \|f\|_{L_p(\mathbb{R}^n)}, \|f\|_{M_p^{\frac{n}{p}}(\mathbb{R}^n)} \equiv \|f\|_{L_\infty(\mathbb{R}^n)}.$$

If $\lambda < 0$, $\lambda > \frac{n}{p}$, the space $M_p^\lambda(\mathbb{R}^n)$ is trivial, i.e., consists only of functions equivalent to zero on $\mathbb{R}^n$.

Definition 1. *Let $0 < p, \theta \leq \infty$, and let w be a non-negative measurable function on $(0, \infty)$. We denote by $GM_{p\theta}^{w(\cdot)} \equiv GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ the global Morrey-type space, the space of all functions $f \in L_p^{loc}(R^n)$ with finite quasi-norm*

$$\|f\|_{GM_{p\theta}^{w(\cdot)}} = \sup_{x\in\mathbb{R}^n} \left\| w(r)\|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)}.$$

If $\theta = \infty$, the space $GM_{p\infty}^{w(\cdot)} \equiv M_p^{w(\cdot)}$ is called the generalized Morrey space. The space $M_p^{w(\cdot)}$ coincides with the Morrey space M_p^λ if $w(r) = r^{-\lambda}$, where $0 \leq \lambda \leq \frac{n}{p}$.

Definition 2. *Let $0 < p, \theta \leq \infty$. We denote by $\Omega_{p\theta}$ the set of all functions, w, which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such, that for some $t > 0$ (hence, for all t>0),*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty, \quad \|w(r)r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty. \tag{1}$$

If condition $||w(r)||_{L_\theta(t,\infty)} < \infty$ is replaced by $||w(r)||_{L_\theta(0,\infty)} < \infty$, we say that $w \in \widetilde{\Omega}_{p\theta}$.

The space $GM_{p\theta}^{w(\cdot)}$ is non-trivial, that is consists not only of functions, equivalent to 0 on $\mathbb{R}^n$, if and only if $w \in \Omega_{p\theta}$ [22].

Definition 3. *Let $\epsilon \in (0, \infty)$, $\mathcal{F}$ be a subset of a quasi-normed space X, and $\mathcal{G} \subset \mathcal{F}$. Then $\mathcal{G}$ is called an ϵ-net of $\mathcal{F}$ if, for any $f \in \mathcal{F}$, there exists a $g \in \mathcal{G}$, such that $\|f - g\|_X < \epsilon$. Moreover, if $\mathcal{G}$ is an ϵ-net of $\mathcal{F}$ and the cardinality of $\mathcal{G}$ is finite, then $\mathcal{G}$ is called a finite ϵ-net of $\mathcal{F}$. Furthermore, $\mathcal{F}$ is said to be totally bounded if, for any $\epsilon \in (0, \infty)$, there exists a finite ϵ-net. In addition, $\mathcal{F}$ is said to be relatively compact (pre-compact) if the closure in X of $\mathcal{F}$ is compact.*

From the Hausdorff theorem (see, for instance, [23] p. 13), it follows that a subset F of a Banach space X, is relatively compact if and only if F is totally bounded due to the completeness of X.

We recall the well-known Fréchet–Kolmogorov theorem on the pre-compactness of set $S \subset L_p(\mathbb{R}^n)$ in terms of uniform equi-continuity.

Theorem 1 ([23]). *A set $S \subset L_p(\mathbb{R}^n)$, where $1 \le p < \infty$, is pre-compact if and only if*

$$\sup_{f \in S} \|f\|_{L_p(\mathbb{R}^n)} < \infty, \tag{2}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \sup_{|h| \le \delta} \|f(\cdot + h) - f(\cdot)\|_{L_p(\mathbb{R}^n)} = 0 \tag{3}$$

and

$$\lim_{R \to \infty} \sup_{f \in S} \|f\|_{L_p({}^c B(0,R))} = 0, \tag{4}$$

where ${}^c B(0,R)$ is the complement of the ball $B(0,R)$.

We denote by $A_\delta f$ the Steklov averaging function: for any $\delta > 0$ and $f \in L_1^{loc}(\mathbb{R}^n)$

$$(A_\delta f)(x) = \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} f(y) dy.$$

Remark 1. *Condition (3) can be replaced by the following condition*

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0.$$

That is, the following Fréchet–Kolmogorov theorem on the pre-compactness of sets in $L_p(\mathbb{R}^n)$ in terms of averaging functions is true.

Theorem 2 ([23]). *A set $S \subset L_p(\mathbb{R}^n)$, where $1 \le p < \infty$, is pre-compact if and only if*

$$\sup_{f \in S} \|f\|_{L_p(\mathbb{R}^n)} < \infty, \tag{5}$$

$$\lim_{\delta \to 0^+} \sup_{f \in S} \|A_\delta f - f\|_{L_p(\mathbb{R}^n)} = 0, \tag{6}$$

and

$$\lim_{R \to \infty} \sup_{f \in S} \|f\|_{L_p({}^c B(0,R))} = 0. \tag{7}$$

It is clear that

$$(A_\delta f)(x) = \int_{B(0,\delta)} \psi_\delta(y) f(x-y) dy = (\psi_\delta * f)(x), \; x \in \mathbb{R}^n, \tag{8}$$

where $\psi_\delta(x) = \frac{\chi_{B(0,\delta)}(x)}{|B(0,\delta)|}$.

Let $f \in L_1^{loc}(\mathbb{R}^n)$. The Riesz potential $I_\alpha(f)$ is defined by

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n, \quad 0 < \alpha < n.$$

The Riesz potential I_α plays an important role in the harmonic analysis and in the theory of operators.

For a function $b \in L_1^{loc}(\mathbb{R}^n)$, let M_b denote the multiplication operator $M_b f = bf$, where f is a Lebesgue measurable function. Then the commutator of I_α and M_b is defined by

$$([b, I_\alpha]f)(x) = ((M_b I_\alpha - I_\alpha M_b)f)(x) = \int\limits_{\mathbb{R}^n} \frac{[b(x) - b(y)]f(y)}{|x-y|^{n-\alpha}} dy.$$

A function $b \in L_{loc}(\mathbb{R}^n)$ is said to be in $BMO(\mathbb{R}^n)$, if

$$\|b\|_* = \sup_{x \in \mathbb{R}^n, \quad r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |b(y) - (A_r b)(y)| dy < \infty.$$

By $VMO(\mathbb{R}^n)$ we denote the BMO- closure of the space $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ is the set of all $C^\infty(\mathbb{R}^n)$ functions with compact support.

Remark 2. *In what follows we shall essentially use the following statement, proved in [24]: let $1 \le p \le \theta \le \infty$, then for any $w \in \Omega_{p\theta}$, $\delta > 0$ and $f \in GM_{p\theta}^{w(\cdot)}$*

$$\|A_\delta f\|_{GM_{p\theta}^{w(\cdot)}} \le \|f\|_{GM_{p\theta}^{w(\cdot)}}. \tag{9}$$

Lemma 1. *Let $1 \le p \le \infty$. Then for any $R > 0$, $\delta > 0$, and $f \in L_p(B(0, R+\delta))$*

$$||A_\delta f||_{L_p(B(0,R))} \le ||f||_{L_p(B(0,R+\delta))}. \tag{10}$$

Lemma 2. *Let $1 \le p \le \infty$. Then for any $0 < \delta < R_1 < R_2 < \infty$ and $f \in L_p(B(0, R_2+\delta) \setminus B(0, R_1 - \delta))$*

$$||A_\delta f||_{L_p(B(0,R_2)\setminus B(0,R_1))} \le ||f||_{L_p(B(0,R_2+\delta)\setminus B(0,R_1-\delta))}. \tag{11}$$

Lemmas 1 and 2 are particular cases of the following variant of Young's inequality [24] for convolutions

$$\left\| \int\limits_G k(y) f(x-y) dy \right\|_{L_p(H)} \le \|k\|_{L_1(G)} \|f\|_{L_p(G-H)},$$

(see Formula (8)), with $k = \psi_\delta$, $G = B(0,\delta)$, and $H = B(0,R)$ in Lemma 1, and $G = B(0,\delta)$, $H = B(0,R_2) \setminus B(0,R_1)$ in Lemma 2.

Lemma 3. *Let $1 \le p \le \infty$, $0 < \theta \le \infty$, $w \in \Omega_{p\theta}$ and $||w||_{L_\theta(R,\infty)} > 0$ for some $R > 0$. Then for any $\delta > 0$ and $f \in GM_{p\theta}^{w(\cdot)}$*

$$||A_\delta f||_{L_p(B(0,R))} \le C_1 ||f \chi_{B(0,R+\delta)}||_{GM_{p\theta}^{w(\cdot)}},$$

where $C_1 = \|w\|^{-1}_{L_\theta(R,\infty)}$.

Proof. According to Lemma 2

$$\|A_\delta f\|_{L_p(B(0,R))} \le \|f\|_{L_p(B(0,R+\delta))} = \frac{\|w\|_{L_\theta(R+\delta,\infty)} \|f\|_{L_p(B(0,R+\delta))}}{\|w\|_{L_\theta(R+\delta,\infty)}}$$

$$= \frac{\left\| w(r) \|f\|_{L_p(B(0,R+\delta))} \right\|_{L_\theta(R+\delta,\infty)}}{\|w\|_{L_\theta(R+\delta,\infty)}} = \frac{\left\| w(r) \left\| f\chi_{(B(0,R+\delta))} \right\|_{L_p(B(0,r))} \right\|_{L_\theta(R+\delta,\infty)}}{\|w\|_{L_\theta(R+\delta,\infty)}}$$

$$\leq \frac{1}{\|w\|_{L_\theta(R,\infty)}} \Big\| w(r) \Big\| f\chi_{B(0,R+\delta)} \Big\|_{L_p(B(0,r))} \Big\|_{L_\theta(0,\infty)} = C_1 \Big\| f\chi_{B(0,R+\delta)} \Big\|_{GM^{w(\cdot)}_{p\theta}}.$$

□

Lemma 4. *Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then for any $0 < R_1 < R_2 < \infty, \delta > 0$ and for any functions $f, \varphi \in GM^{w(\cdot)}_{p\theta}$*

$$||f-\varphi||_{GM^{w(\cdot)}_{p\theta}} \leq ||f\chi_{B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}} + ||\varphi\chi_{B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}}$$
$$+||(A_\delta f - f)\chi_{B(0,R_2)\backslash B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}} + ||(A_\delta f - A_\delta \varphi)\chi_{B(0,R_2)\backslash B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}} \quad (12)$$
$$+||(A_\delta \varphi - \varphi)\chi_{B(0,R_2)\backslash B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}} + ||f\chi_{{}^cB(0,R_2)}||_{GM^{w(\cdot)}_{p\theta}} + ||\varphi\chi_{{}^cB(0,R_2)}||_{GM^{w(\cdot)}_{p\theta}}.$$

Proof. By adding and subtracting the corresponding summands and applying the Minkowski inequality, we obtain that

$$||f-\varphi||_{GM^{w(\cdot)}_{p\theta}} = ||(f-\varphi)\chi_{B(0,R_1)} + (f-\varphi)\chi_{B(0,R_2)\backslash B(0,R_1)} + (f-\varphi)\chi_{{}^cB(R_2,\infty)}||_{GM^{w(\cdot)}_{p\theta}}$$

$$= ||f\chi_{B(0,R_1)} - \varphi\chi_{B(0,R_1)} + f\chi_{{}^cB(R_2,\infty)} - \varphi\chi_{{}^cB(R_2,\infty)} - (A_\delta f - f)\chi_{B(0,R_2)\backslash B(0,R_1)}$$

$$+(A_\delta\varphi - \varphi)\chi_{B(0,R_2)\backslash B(0,R_1)} + (A_\delta f - A_\delta\varphi)\chi_{B(0,R_2)\backslash B(0,R_1)}||_{GM^{w(\cdot)}_{p\theta}}$$

$$\leq \Big\|f\chi_{B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} + \Big\|\varphi\chi_{B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} + \Big\|f\chi_{{}^cB(R_2,\infty)}\Big\|_{GM^{w(\cdot)}_{p\theta}}$$

$$+\Big\|\varphi\chi_{{}^cB(R_2,\infty)}\Big\|_{GM^{w(\cdot)}_{p\theta}} + \Big\|(A_\delta f - f)\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}}$$

$$+\Big\|(A_\delta\varphi - \varphi)\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} + \Big\|(A_\delta f - A_\delta\varphi)\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}}.$$

□

Lemma 5. *Let $0 < p \leq \infty$, $0 < \theta \leq \infty$, $w \in \widetilde{\Omega}_{p\theta}$. Then for any $0 < R_1 < R_2 < \infty$ and for any function $f \in GM^{w(\cdot)}_{p\theta}$*

$$\Big\|f\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} \leq \|w\|_{L_\theta(0,\infty)}\|f\|_{L_p(B(0,R_2)\backslash B(0,R_1))}. \quad (13)$$

Proof. Since

$$\Big\|f\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} = \sup_{x\in R^n}\Big\|w(r)\Big\|f\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{L_p(B(x,r))}\Big\|_{L_\theta(0,\infty)},$$

and

$$\Big\|f\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{L_p(B(x,r))} = \|f\|_{L_p(B(x,r)\cap(B(0,R_2)\backslash B(0,R_1)))} \leq \|f\|_{L_p(B(0,R_2)\backslash B(0,R_1))},$$

we have

$$\Big\|f\chi_{B(0,R_2)\backslash B(0,R_1)}\Big\|_{GM^{w(\cdot)}_{p\theta}} \leq \|w\|_{L_\theta(0,\infty)}\|f\|_{L_p(B(0,R_2)\backslash B(0,R_1))}.$$

□

3. Pre-Compactness of Sets in the Global Morrey-Type Spaces in Terms of Averaging Function

In this section, we provide sufficient conditions for the pre-compactness of sets in the global Morrey-type spaces $GM_{p\theta}^{w(\cdot)}$ in terms of averaging function.

Theorem 3. *Let $1 \le p \le \theta \le \infty, w \in \widetilde{\Omega}_{p\theta}$, $||w||_{L_\theta(t,\infty)} > 0$ for any $t > 0$, and a set $S \subset GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ satisfy the following conditions:*

$$\sup_{f\in S} \|f\|_{GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)} < \infty, \tag{14}$$

$$\lim_{R_1\to 0^+} \sup_{f\in S} \left\| f\chi_{B(0,R_1)} \right\|_{GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)} = 0, \tag{15}$$

for any $0 < R_1 < R_2 < \infty$

$$\lim_{\delta\to 0^+} \sup_{f\in S} \|A_\delta f - f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = 0, \tag{16}$$

and

$$\lim_{R_2\to \infty} \sup_{f\in S} \left\| f\chi_{{}^cB(0,R_2)} \right\|_{GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)} = 0. \tag{17}$$

Then the set $S \subset GM_{p\theta}^{w(\cdot)}(\mathbb{R}^n)$ is pre-compact in $GM_{p\theta}^{w(\cdot)}$.

Proof of Theorem 3. Let $S \subset GM_{p\theta}^{w(\cdot)}$ and suppose that conditions (14)–(17) are satisfied.

Step 1. For any $0 < \delta < R_1 < R_2 < \infty$, the set $S_\delta = \{A_\delta f : f \in S\}$ is pre-compact in $L_p(B(0,R_2) \setminus B(0,R_1))$.

Note that

$$||f||_{GM_{p\theta}^{w(\cdot)}} \ge \left\| w(r)||f||_{L_p(B(0,r))} \right\|_{L_\theta(R_2,\infty)} \ge \|w(r)\|_{L_\theta(R_2,\infty)} \|f\|_{L_p(B(0,R_2)\setminus B(0,R_1))},$$

hence,

$$\|f\|_{L_p(B(0,R_2)\setminus B(0,R_1))} \le \|w(r)\|^{-1}_{L_\theta(R_2,\infty)} ||f||_{GM_{p\theta}^{w(\cdot)}}. \tag{18}$$

Therefore, according to inequalities (18), (9) and condition (14) for $g \in S_\delta$, we have

$$\sup_{g\in S_\delta} \|g\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = \sup_{f\in S} \|A_\delta f\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$\le \|w(r)\|^{-1}_{L_\theta(R_2,\infty)} \sup_{f\in S} \|A_\delta f\|_{GM_{p\theta}^{w(\cdot)}} \le \|w(r)\|^{-1}_{L_\theta(R_2,\infty)} \sup_{f\in S} \|f\|_{GM_{p\theta}^{w(\cdot)}} < \infty. \tag{19}$$

By inequality (11) and by condition (16) we obtain

$$\lim_{\tau\to 0^+} \sup_{g\in S_\delta} \|A_\tau g - g\|_{L_p(B(0,R_2)\setminus B(0,R_1))} = \lim_{\tau\to 0^+} \sup_{f\in S} \|A_\tau A_\delta f - A_\delta f\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$= \lim_{\tau\to 0^+} \sup_{f\in S} \|A_\delta (A_\tau f - f)\|_{L_p(B(0,R_2)\setminus B(0,R_1))} \le \lim_{\tau\to 0^+} \sup_{f\in S} \|A_\tau f - f\|_{L_p(B(0,R_2+\delta)\setminus B(0,R_1-\delta))} = 0.$$

Hence, by Theorem 2, it follows that the set S_δ is pre-compact in $L_p(B(0,R_2) \setminus B(0,R_1))$ or, equivalently, totally bounded in $L_p(B(0,R_2) \setminus B(0,R_1))$.

Step 2. The set S is totally bounded in $GM_{p\theta}^{w(\cdot)}$.

Let us show that the set S is a pre-compact set in $GM_{p\theta}^{w(\cdot)}$. By inequalities (12) and (13) for any $f, \varphi \in S$, we have

$$||f-\varphi||_{GM_{p\theta}^{w(\cdot)}} \leq 2\sup_{g\in S}||g\chi_{B(0,R_1)}||_{GM_{p\theta}^{w(\cdot)}} + 2\sup_{g\in S}||(A_\delta g - g)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{GM_{p\theta}^{w(\cdot)}}$$

$$+2\sup_{g\in S}||g\chi_{^cB(0,R_2)}||_{GM_{p\theta}^{w(\cdot)}} + \|w\|_{L_{\theta(R_1,\infty)}}\|A_\delta f - A_\delta\varphi\|_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

$$= E_1 + E_2 + E_3 + E_4.$$

Let $\varepsilon > 0$. Using condition (15), we can choose a radius $R_1 = R_1(\varepsilon)$ of the ball $B(0, R_1)$, such that

$$E_1 = 2\sup_{g\in S}||g\chi_{B(0,R_1)}||_{GM_{p\theta}^{w(\cdot)}} < \frac{\varepsilon}{4}.$$

By condition (17), we can choose $R_2 = R_2(\varepsilon)$, such that

$$E_3 = 2\sup_{g\in S}||g\chi_{^cB(0,R_2)}||_{GM_{p\theta}^{w(\cdot)}} < \frac{\varepsilon}{4}.$$

By Lemma 5 and by condition (16), we can choose $\delta = \delta(\varepsilon)$, such that

$$E_2 = 2\sup_{g\in S}||(A_\delta g - g)\chi_{B(0,R_2)\setminus B(0,R_1)}||_{GM_{p\theta}^{w(\cdot)}}$$

$$\leq 2||w||^{-1}_{L_\theta(R_1,\infty)}\sup_{g\in S}\|A_\delta g - g\|_{L_p(B(0,R_2)\setminus B(0,R_1))} < \frac{\varepsilon}{4}.$$

Then, for any $f, \varphi \in S$

$$||f-\varphi||_{GM_{p\theta}^{w(\cdot)}} \leq \frac{3\varepsilon}{4} + \|w\|_{L_{\theta(0,\infty)}}\|A_\delta f - A_\delta\varphi\|_{L_p(B(0,R_2)\setminus B(0,R_1))}. \tag{20}$$

Hence, for any $\varphi_1, \varphi_2, \ldots, \varphi_m \in S$

$$\min_{j=1,2,\ldots,m}||f-\varphi_j||_{GM_{p\theta}^{w(\cdot)}} \leq \frac{3\varepsilon}{4} + \|w\|_{L_{\theta(0,\infty)}}\min_{j=1,2,\ldots,m}\left\|A_\delta f - A_\delta\varphi_j\right\|_{L_p(B(0,R_2)\setminus B(0,R_1))}. \tag{21}$$

Indeed, let j_0 be such that

$$||A_\delta f - A_\delta\phi_{j_0}||_{L_p(B(0,R_2)\setminus B(0,R_1))} = \min_{j=1,\ldots m}||A_\delta - A_\delta\phi_j||_{L_p(B(0,R_2)\setminus B(0,R_1))},$$

then

$$\min_{j=1,\ldots m}||f-\phi_j||_{L_p(B(0,R_2)\setminus B(0,R_1))} \leq ||f-\phi_{j_0}||_{L_p(B(0,R_2)\setminus B(0,R_1))}$$

and (20) with $\phi = \phi_{j_0}$ implies (21).

Finally, by the pre-compactness of the set $S_{\delta(\varepsilon)}$ in $L_p(B(0, R_2(\varepsilon)) \setminus B(0, R_1(\varepsilon)))$, for any $f \in S$ there exist $m(\varepsilon) \in \mathbb{N}$ and $f_{1,\varepsilon}, \ldots, f_{m(\varepsilon),\varepsilon} \in S$, such that

$$\min_{j=1,2,\ldots,m(\varepsilon)}\left\|A_{\delta(\varepsilon)}f - A_{\delta(\varepsilon)}f_{j,\varepsilon}\right\|_{L_p(B(0,R_2(\varepsilon))\setminus B(0,R_1(\varepsilon)))} \leq \frac{\varepsilon}{4}\left(\|w\|_{L_{\theta(0,\infty)}}\right)^{-1}.$$

Therefore, setting $\varphi_j = f_{j,\varepsilon}, j = 1, \ldots, m(\varepsilon)$, by inequality (21), for any $f \in S$, we obtain

$$\min_{j=1,2,\ldots,m(\varepsilon)}||f-f_{j(\varepsilon)}||_{GM_{p\theta}^{w(\cdot)}} \leq \varepsilon.$$

Thus, we have that $\varphi_j = f_{j,\varepsilon}, j = 1, \ldots, m(\varepsilon)$ is a finite ε-net in S with respect to the norm of $GM_{p\theta}^{w(\cdot)}$.

We conclude from this that the set S is totally bounded in $GM_{p\theta}^{w(\cdot)}$, or equivalently, the set S is pre-compact in $GM_{p\theta}^{w(\cdot)}$.
□

4. Pre-Compactness of Sets in the Global Morrey-Type Spaces in Terms of Uniform Equi-Continuity

In this section, we provide sufficient conditions for the pre-compactness of sets in the global Morrey-type spaces $GM_{p\theta}^{w(\cdot)}$ in terms of uniform equi-continuity.

The pre-compactness of sets in Morrey spaces $M_p^\lambda(\mathbb{R}^n)$ in terms of uniform equi-continuity was investigated in [6,11] and for generalized Morrey spaces $M_p^{w(\cdot)}(\mathbb{R}^n)$ it was investigated in [7,25].

Theorem 4. *Suppose that* $1 \le p \le \theta \le \infty$ *and* $w \in \Omega_{p\theta}$. *Suppose that a subset* S *of* $GM_{p\theta}^{w(\cdot)}$ *satisfies the following conditions:*

$$\sup_{f \in S} \|f\|_{GM_{p\theta}^{w(\cdot)}} < \infty, \tag{22}$$

$$\lim_{u \to 0} \sup_{f \in S} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} = 0, \tag{23}$$

$$\lim_{r \to \infty} \sup_{f \in S} \left\| f \chi_{^c B(0,r)} \right\|_{GM_{p\theta}^{w(\cdot)}} = 0. \tag{24}$$

Then S *is a pre-compact set in* $GM_{p\theta}^{w(\cdot)}(\mathbb{R})$.

To prove this theorem, we need the following statements.

Lemma 6. *Let* $1 \le p \le \theta \le \infty$, $w \in \Omega_{p\theta}$. *Then for all* $f \in GM_{p\theta}^{w(\cdot)}$ *and* $\delta > 0$ *we have*

$$\|A_\delta f - f\|_{GM_{p\theta}^{w(\cdot)}} \le \sup_{u \in B(0,\delta)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}}. \tag{25}$$

Proof. Let $z \in R^n$ and $\rho > 0$. Then by Hölder's inequality

$$\|A_\delta f - f\|_{L_p(B(z,\rho))} = \left(\int_{B(z,\rho)} \left| \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} f(y)dy - f(x) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{B(z,\rho)} \left| \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} (f(y) - f(x))dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\le \left(\int_{B(z,\rho)} \left(\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y) - f(x)|^p dy \right) dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{B(z,\rho)} \left(\frac{1}{|B(0,\delta)|} \int_{B(0,\delta)} |f(x+u) - f(x)|^p du \right) dx \right)^{\frac{1}{p}}$$

$$= \left(\frac{1}{|B(0,\delta)|} \int\limits_{B(0,\delta)} \left(\int\limits_{B(z,\rho)} |f(x+u) - f(x)|^p dx \right) du \right)^{\frac{1}{p}}.$$

Therefore,

$$\|A_\delta f - f\|_{GM_{p\theta}^{w(\cdot)}} = \sup_{z \in \mathbb{R}^n} \left\| w(\rho) \|A_\delta f - f\|_{L_p(B(z,\rho))} \right\|_{L_\theta(0,\infty)}$$

$$\leq \sup_{z \in \mathbb{R}^n} \left\| w(\rho) \left(\frac{1}{|B(0,\delta)|} \int\limits_{B(0,\delta)} \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)}$$

$$= \sup_{z \in \mathbb{R}^n} \left\| \frac{1}{|B(0,\delta)|} \int\limits_{B(0,\delta)} w(\rho)^p \|f(\cdot + u) - f(\cdot)\|^p_{L_p(B(z,\rho))} du \right\|^{\frac{1}{p}}_{L_\theta(0,\infty)}.$$

Since $\frac{\theta}{p} \geq 1$, by applying Minkowski's inequality for integrals, we obtain the following:

$$\|A_\delta f - f\|_{GM_{p\theta}^{w(\cdot)}} \leq \sup_{z \in \mathbb{R}^n} \left(\frac{1}{|B(0,\delta)|} \int\limits_{B(0,\delta)} \left(\int\limits_0^\infty w(\rho)^\theta \|f(\cdot + u) - f(\cdot)\|^\theta_{L_p(B(z,\rho))} d\rho \right)^{\frac{p}{\theta}} du \right)^{\frac{1}{p}}$$

$$\leq \left(\frac{1}{|B(0,\delta)|} \int\limits_{B(0,\delta)} \|f(\cdot + u) - f(\cdot)\|^p_{GM_{p\theta}^{w(\cdot)}} du \right)^{\frac{1}{p}}$$

$$\leq \sup_{u \in B(0,\delta)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}}.$$

□

Lemma 7. *Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then there exists $r_0 > 0$ such that for every $0 < r \leq r_0$ there exists $C_6 > 0$, depending only on r, n, p, θ, w, such that (below $C(\mathbb{R}^n)$ is the space of all bounded continuous functions on $\mathbb{R}^n$)*

(1) for every $f \in GM_{p\theta}^{w(\cdot)}$

$$\|A_r f\|_{C(\mathbb{R}^n)} \leq C_6 \|f\|_{GM_{p\theta}^{w(\cdot)}}, \tag{26}$$

(2) for every $f \in GM_{p\theta}^{w(\cdot)}$

$$\sup_{u \in B(0,r)} \|A_r f(\cdot + u) - A_r f(\cdot)\|_{C(\mathbb{R}^n)} \leq C_6 \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}}. \tag{27}$$

Proof. 1. Since a function $w \in \Omega_{p\theta}$ is not equivalent 0, there exists $r_0 > 0$, such that $\|w\|_{L_\theta(r_0,\infty)} > 0$. Let $0 < r \leq r_0$, $x \in R^n$. Then by Hölder's inequality

$$|A_r f(x)| \leq \frac{1}{|B(x,r)|^{\frac{1}{p}}} \|f\|_{L_p(B(x,r))}.$$

Hence,

$$\|w(\rho) A_r f(x)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left\| w(\rho) \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(r,\infty)}$$

$$\leq \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left\| w(\rho) \|f\|_{L_p(B(x,\rho))} \right\|_{L_\theta(r,\infty)},$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$, and

$$|A_r f(x)| \|w(\rho)\|_{L_\theta(r,\infty)} \leq \frac{1}{(v_n r^n)^{\frac{1}{p}}} \left\| w(\rho) \|f\|_{L_p(B(x,\rho))} \right\|_{L_\theta(0,\infty)},$$

That is why

$$\sup_{x\in\mathbb{R}^n} |A_r f(x)| \leq C_6 \sup_{x\in\mathbb{R}^n} \left\| w(\rho) \|f\|_{L_p(B(x,\rho))} \right\|_{L_\theta(0,\infty)} = C_6 \|f\|_{GM_{p\theta}^{w(\cdot)}}, \tag{28}$$

where $C_6 = \left(\|w\|_{L_\theta(r,\infty)} (v_n r^n)^{\frac{1}{p}} \right)^{-1}$.

2. Next, for every $x_1, x_2 \in B(0,r)$

$$|(A_r f)(x_1) - (A_r f)(x_2)| = \frac{1}{v_n r^n} \left| \int\limits_{B(x_1,r)} f(y)dy - \int\limits_{B(x_2,r)} f(y)dy \right| =$$

$$= (v_n r^n)^{-1} \left| \int\limits_{B(0,r)} f(z+x_1)dz - \int\limits_{B(0,r)} f(z+x_2)dz \right| \leq$$

$$\leq (v_n r^n)^{-1} \int\limits_{B(0,r)} |f(z+x_1) - f(z+x_2)|dz =$$

$$= (v_n r^n)^{-1} \int\limits_{B(x_2,r)} |f(s+x_1-x_2) - f(s)|ds \leq$$

$$\leq (v_n r^n)^{-\frac{1}{p}} \|f(\cdot + x_1 - x_2) - f(\cdot)\|_{L_p(B(x_2,r))}.$$

Therefore, similarly to the first step of the proof

$$\sup_{\substack{x_1,x_2\in R^n \\ |x_1-x_2|\leq r}} |(A_r f)(x_1) - (A_r f)(x_2)| \leq C_6 \sup_{\substack{x_1,x_2\in R^n \\ |x_1-x_2|\leq r}} \|f(\cdot + x_1 - x_2) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} =$$

$$= C_6 \sup_{u\in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}}.$$

Hence, we have (27). □

Lemma 8. *Let $1 \leq p, \theta \leq \infty$, $w \in \Omega_{p\theta}$. Then, there exists $C_7 > 0$, depending only on n, p, θ, w, such that for every, $r, R > 0$, and for all $f, g \in GM_{p\theta}^{w(\cdot)}$,*

$$\|A_r f - A_r g\|_{GM_{p\theta}^{w(\cdot)}} \leq C_7 \left(1 + R^{\frac{n}{p}}\right) \|A_r f - A_r g\|_{C(\overline{B(0,R)})} + \sup_{u\in B(0,R)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}}$$

$$+ \sup_{u\in B(0,R)} \|g(\cdot + u) - g(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} + \left\| f\chi_{{}^c B(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}} + \left\| g\chi_{{}^c B(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}}.$$

Proof. Indeed,

$$\|A_r f - A_r g\|_{GM^{w(\cdot)}_{p\theta}} \le \left\|(A_r f - A_r g)\chi_{B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}} + \left\|(A_r f - A_r g)\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}} = I_1 + I_2.$$

Next,

$$I_1 = \sup_{x\in R^n}\left\|w(\rho)\|A_r f - A_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,\infty)}$$
$$\le \sup_{x\in R^n}\left\|w(\rho)\|A_r f - A_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(0,1)}$$
$$+ \sup_{x\in R^n}\left\|w(\rho)\|A_r f - A_r g\|_{L_p(B(x,\rho)\cap B(0,R))}\right\|_{L_\theta(1,\infty)}$$
$$\le \|A_r f - A_r g\|_{C(\overline{B(0,R)})}\cdot$$
$$\cdot\left(\left\|w(\rho)(v_n\rho^n)^{\frac{1}{p}}\right\|_{L_\theta(0,1)} + \left\|w(\rho)(v_n R^n)^{\frac{1}{p}}\right\|_{L_\theta(1,\infty)}\right)$$
$$\le C_7\left(1 + R^{\frac{n}{p}}\right)\|A_r f - A_r g\|_{C(\overline{B(0,R)})},$$

where

$$C_7 = v_n^{\frac{1}{p}}\left(\left\|w(\rho)\rho^{\frac{n}{p}}\right\|_{L_\theta(0,1)} + \|w(\rho)\|_{L_\theta(1,\infty)}\right) < \infty,$$

since $w \in \Omega_{p\theta}$.

By Lemma 6

$$I_2 \le \|A_r f - f\|_{GM^{w(\cdot)}_{p\theta}} + \left\|(f-g)\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}} + \|A_r g - g\|_{GM^{w(\cdot)}_{p\theta}}$$
$$\le \sup_{u\in B(0,r)}\|f(\cdot+u) - f(\cdot)\|_{GM^{w(\cdot)}_{p\theta}} + \sup_{u\in B(0,r)}\|g(\cdot+u) - g(\cdot)\|_{GM^{w(\cdot)}_{p\theta}}$$
$$+ \left\|f\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}} + \left\|g\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}}.$$

□

Lemma 9. *Let* $1 \le p,\theta \le \infty$ *,* $w \in \Omega_{p\theta}$*. Then, for every* $r, R > 0$ *and for every* $f, g \in GM^{w(\cdot)}_{p\theta}$*,*

$$\|f - g\|_{GM^{w(\cdot)}_{p\theta}} \le C_7\left(1 + R^{\frac{n}{p}}\right)\|A_r f - A_r g\|_{C(\overline{B(0,R)})}$$
$$+2\sup_{u\in B(0,r)}\|f(\cdot+u) - f(\cdot)\|_{GM^{w(\cdot)}_{p\theta}} + 2\sup_{u\in B(0,r)}\|g(\cdot+u) - g(\cdot)\|_{GM^{w(\cdot)}_{p\theta}} \quad (29)$$
$$+ \left\|f\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}} + \left\|g\chi_{{}^c B(0,R)}\right\|_{GM^{w(\cdot)}_{p\theta}},$$

where $C_7 > 0$ *is the same as in Lemma 8.*

Proof. It suffices to notice that

$$\|f - g\|_{GM^{w(\cdot)}_{p\theta}} \le \|A_r f - f\|_{GM^{w(\cdot)}_{p\theta}} + \|A_r f - A_r g\|_{GM^{w(\cdot)}_{p\theta}} + \|A_r g - g\|_{GM^{w(\cdot)}_{p\theta}}$$

and apply Lemmas 6 and 8. □

Proof of Theorem 4. Let $S \subset GM^{w(\cdot)}_{p\theta}$ satisfy conditions (22)–(24).

Step 1. First, we show that the set $S_r = \{A_r f : f \in S\}$ is a pre-compact set in $C(\overline{B(0,R)})$.

Let $0 < r < r_0$, where r_0 is defined in Lemma 7, and let $R > 0$ be fixed. By using inequality (22) and inequality (26), we have

$$\sup_{f \in S} \|A_r f\|_{C(\overline{B(0,R)})} < \infty.$$

Using (27) and condition (23), we obtain

$$\lim_{u \to 0} \sup_{f \in S} \|A_r f(\cdot + u) - A_r f(\cdot)\|_{C(\overline{B(0,R)})} = 0.$$

Therefore, by the Ascoli–Arzela theorem, the set S_r is pre-compact in $C(\overline{B(0,R)})$, so the set S_r is totally bounded in $C(\overline{B(0,R)})$. Hence, for any $\varepsilon > 0$, there exists $f_1, \ldots, f_m \in S$ (depending on ε, r, and R), such that $\{A_r f_1, A_r f_2, \ldots, A_r f_m\}$ is a finite ε-net in S_r with respect to the norm of $C(\overline{B(0,R)})$. Therefore, for any $f \in S$, there exists $1 \leq j \leq m$, such that

$$\min_{j=1,\ldots,m} \|A_r f - A_r f_j\|_{C(\overline{B(0,R)})} < \varepsilon.$$

Step 2. Let us show that the set S is a pre-compact set in $GM_{p\theta}^{w(\cdot)}$.

Let $\{\varphi_1, \ldots, \varphi_m\}$ be an arbitrary finite subset of S. By inequality (29) for any $f \in S$ and any $j = 1, \ldots, m$, we have

$$\begin{aligned}
\|f - \varphi_j\|_{GM_{p\theta}^{w(\cdot)}} &\leq C_7(1 + R^{\frac{n}{p}}) \|A_r f - A_r \varphi_j\|_{C(\overline{B(0,R)})} \\
&+ 2 \sup_{u \in B(0,r)} \|f(\cdot + u) - f(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} + 2 \sup_{u \in B(0,r)} \|\varphi_j(\cdot + u) - \varphi_j(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} \\
&+ \left\| f \chi_{{}^cB(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}} + \left\| \varphi_j \chi_{{}^cB(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}} \\
&\leq C_7(1 + R^{\frac{n}{p}}) \|A_r f - A_r \varphi_j\|_{C(\overline{B(0,R)})} \\
&+ 4 \sup_{u \in B(0,r)} \sup_{g \in S} \|g(\cdot + u) - g(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} + 2 \sup_{g \in S} \left\| g \chi_{{}^cB(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\min_{j=1,\ldots,m} \|f - \varphi_j\|_{GM_{p\theta}^{w(\cdot)}} &\leq C_7(1 + R^{\frac{n}{p}}) \min_{j=1,\ldots,m} \|A_r f - A_r \varphi_j\|_{C(\overline{B(0,R)})} \\
&+ 4 \sup_{u \in B(0,r)} \sup_{g \in S} \|g(\cdot + u) - g(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} + 2 \sup_{g \in S} \left\| g \chi_{{}^cB(0,R)} \right\|_{GM_{p\theta}^{w(\cdot)}}.
\end{aligned} \tag{30}$$

Let $\varepsilon > 0$. First , by using condition (24), we choose $R(\varepsilon) > 0$, such that

$$\sup_{g \in S} \left\| g \chi_{{}^cB(0,R(\varepsilon))} \right\|_{GM_{p\theta}^{w(\cdot)}} < \frac{\varepsilon}{6}.$$

Next, by using condition (23), we choose $r(\varepsilon)$, such that

$$\sup_{u \in B(0,r(\varepsilon))} \sup_{g \in S} \|g(\cdot + u) - g(\cdot)\|_{GM_{p\theta}^{w(\cdot)}} < \frac{\varepsilon}{12}.$$

Since the set $S_{r(\varepsilon)}$ is pre-compact in $C(\overline{B(0,R(\varepsilon))})$, there exist $m(\varepsilon) \in \mathbb{N}$ and $f_{1,\varepsilon}, \ldots, f_{m(\varepsilon),\varepsilon} \in S$, such that for any $f \in S$

$$\min_{j=1,\ldots,m(\varepsilon)} \left\| A_{r(\varepsilon)} f - A_{r(\varepsilon)} f_{j,\varepsilon} \right\|_{C(\overline{B(0,R(\varepsilon))})} < \frac{\varepsilon}{3C_7(1 + R(\varepsilon)^{\frac{n}{p}})}.$$

Therefore, setting $\varphi_j = f_{j,\varepsilon}, j = 1, \ldots, m(\varepsilon)$, by inequality (30), for any $f \in S$, we obtain

$$\min_{j=1,\ldots,m(\varepsilon)} \|f - f_{j,\varepsilon}\|_{GM^{w(\cdot)}_{p\theta}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Then, we have that $\varphi_j = f_{j,\varepsilon}, j = 1, \ldots, m(\varepsilon)$ is a finite ε-net in S with respect to the norm of $GM^{w(\cdot)}_{p\theta}$.

So, the set S is pre-compact in $GM^{w(\cdot)}_{p\theta}$. □

5. Compactness of Commutators for the Riesz Potential

The main purpose of this section is to find sufficient conditions for the compactness of commutators for the Riesz Potential $[b, I_\alpha]$ on the global Morrey-type space $GM^{w(\cdot)}_{p\theta}(\mathbb{R}^n)$.

The next theorem contains sufficient conditions on w_1, w_2 ensuring the boundedness of I_α from $GM^{w_1(\cdot)}_{p_1\theta_1}$ to $GM^{w_2(\cdot)}_{p_2\theta_2}$ for some values of the parameters $\alpha, p_1, p_2, \theta_1, \theta_2$.

Theorem 5 (see [19]). *Let $1 < p_1 < p_2 < \infty$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $0 < \theta_1 \le \theta_2 < \infty$, $w_1 \in \Omega_{p_1\theta_1}, w_2 \in \Omega_{p_2\theta_2}$, then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p_2}} \right\|_{L_{\theta_2}(0,\infty)} \lesssim \|w_1(r)\|_{L_{\theta_1}(t,\infty)}, \tag{31}$$

for all $t > 0$ is sufficient for the boundedness of I_α from $GM^{w_1(\cdot)}_{p_1\theta_1}(\mathbb{R}^n)$ to $GM^{w_2(\cdot)}_{p_2\theta_2}(\mathbb{R}^n)$.

The following lemma gives the L_p-estimates for the commutators for the Riesz potential over balls.

Lemma 10 ([26]). *Let $1 < p_1 < \infty, 0 < \alpha < \frac{n}{p_1}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $b \in BMO(R^n)$.*

Then the inequality

$$\|[b, I_\alpha]f\|_{L_{p_2}(B(x_0,r))} \lesssim \|b\|_* r^{\frac{n}{p_2}} \int_{2r}^{\infty} \left(1 + ln\frac{t}{r}\right) t^{-\frac{n}{p_2}-1} \|f\|_{L_{p_1}(B(x_0,t))} dt \tag{32}$$

holds for any ball $B(x_0, r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

To formulate the following theorem on the boundedness of the Hardy operator in weighted Lebesgue spaces, we need the necessary notation.

Let u, w, and v be weight functions, that is locally integrable non-negative functions on $(0, \infty)$.

Denote by

$$Hg(t) := \int_0^t g(s)ds, \quad g \in \mathfrak{M}^+,$$

and

$$H^*g(t) := \int_t^\infty g(s)ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator and Copson operator, respectively.

Define

$$H_u(g)(t) := H(g \cdot u)(t) = \int_0^t g(s)u(s)ds, \quad g \in \mathfrak{M}^+,$$

$$H^*_u(g)(t) = H^*(g \cdot u)(t) = \int_t^\infty g(s) \cdot u(s)ds, \quad g \in \mathfrak{M}^+.$$

Also, define

$$U(t) := \int_0^t u(s)ds, \quad U_*(t) := \int_t^\infty u(s)ds, V(t) := \int_0^t v(s)ds, \quad V_*(t) := \int_t^\infty v(s)ds,$$

$$W(t) := \int_0^t w(s)ds, \quad W_*(t) := \int_t^\infty w(s)ds.$$

The following theorem gives a complete characterization of the weighted Hardy inequality on the cone of non-increasing functions.

Theorem 6 ([27]). *Let $1 < \theta_1 \le \theta_2 < \infty$. Let u, w, v be weight functions defined on $(0,\infty)$. Then inequality*

$$\|H_u^*(g)\|_{L_{\theta_2,w}(0;\infty)} \le C_8\|g\|_{L_{\theta_1,v}(0;\infty)}, g \in \mathfrak{M}^{\uparrow}$$

with the best constant C_8 holds if and only if the following conditions hold:

$$A_0^* := \sup_{t>0}\left(\int_t^\infty U_*^{\theta_2}(\tau)w(\tau)d\tau\right)^{\frac{1}{\theta_2}} V_*^{-\frac{1}{\theta_1}}(t) < \infty,$$

$$A_1^* := \sup_{t>0} W^{\frac{1}{\theta_2}}(t)\left(\int_t^\infty \left(\frac{U_*(\tau)}{V_*(\tau)}\right)^{\theta_1'} v(\tau)d\tau\right)^{\frac{1}{\theta_1'}} < \infty,$$

and in this case $C_8 \approx A_0^* + A_1^*$.

The following theorem provides sufficient conditions on w_1, w_2 ensuring the boundedness for the commutator $[b, I_\alpha]$ from $GM_{p_1\theta_1}^{w_1(\cdot)}$ to $GM_{p_2\theta_2}^{w_2(\cdot)}$ for some values of the parameters $\alpha, p_1, p_2, \theta_1, \theta_2$.

Theorem 7. *Let $1 < p_1 < p_2 < \infty$, $1 < \alpha < \frac{n}{p_1}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, $0 < \theta_1 \le \theta_2 < \infty$, $b \in BMO(\mathbb{R}^n)$, $w_1 \in \Omega_{p_1\theta_1}, w_2 \in \Omega_{p_2\theta_2}$, and w_1, w_2 satisfy the conditions*

$$A_2^* = \sup_{t>0}\left(\int_t^\infty \left[\left(\int_r^\infty (1+ln\frac{s}{r})s^{-\frac{n}{p_2}-1}ds\right)^{\theta_2} w_2(r)\right]dr\right)^{\frac{1}{\theta_2}} \cdot \left[\int_t^\infty w_1^{\theta_1}(s)ds\right]^{-\frac{1}{\theta_1}} < \infty, \quad (33)$$

$$A_3^* = \sup_{t>0}\left(\int_0^t w_2(s)\cdot s^{\frac{n}{p_2}}ds\right)^{\frac{1}{\theta_2}} \cdot \left(\int_t^\infty \left[\frac{\int_r^\infty (1+ln\frac{s}{r})s^{-\frac{n}{p_2}-1}ds}{\int_r^\infty w_1^{\theta_1}(s)ds}\right]^{\theta_1'} w_1^{\theta_1}(r)dr\right)^{\frac{1}{\theta_1'}} < \infty. \quad (34)$$

Then, the commutator $[b, I_\alpha]$ is bounded from $GM_{p_1\theta_1}^{w_1(\cdot)}(\mathbb{R}^n)$ to $GM_{p_2\theta_2}^{w_2(\cdot)}(\mathbb{R}^n)$. Moreover, there exists $C_9 > 0$, depending only on the numerical parameters and w_1, w_2, such that for all $f \in GM_{p_1\theta_1}^{w_1(\cdot)}$

$$\|[b, I_\alpha]f\|_{GM_{p_2\theta_2}^{w_2(\cdot)}} \le C_9\|b\|_*\|f\|_{GM_{p_1\theta_1}^{w_1(\cdot)}}.$$

Proof. By inequality (32) from Lemma 10, we have

$$\|[b, I_\alpha]f\|_{GM_{p_2\theta_2}^{w_2(\cdot)}} = \sup_{x\in\mathbb{R}^n}\left\|w_2(r)\|[b, I_\alpha]f\|_{L_{p_2}(B(x,r))}\right\|_{L_{\theta_2}(0,\infty)}$$

$$\lesssim \sup_{x\in\mathbb{R}^n}\|b\|_*\left\|w_2(r)r^{\frac{n}{p_2}}\int_r^\infty\left(1+ln\frac{t}{r}\right)t^{-\frac{n}{p_2}-1}\|f\|_{L_{p_1}(B(x,t))}dt\right\|_{L_{\theta_2}(0,\infty)}$$

$$\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n}\left(\int_0^\infty \left[w_2(r)r^{\frac{n}{p_2}}\int_r^\infty \left(1+ln\frac{t}{r}\right)t^{-\frac{n}{p_2}-1}\|f\|_{L_{p_1}(B(x,t))}dt\right]^{\theta_2}dr\right)^{\frac{1}{\theta_2}}.$$

Then, by Theorem 6, setting $w(r)=w_2(r)r^{\frac{n}{q}}$, $g(t)=\|f\|_{L_p(B(0,t))}$, $u(t)=(1+ln\frac{t}{r})t^{-\frac{n}{p_2}-1}$, $v(r)=w_1^{\theta_1}(r)$, under conditions (33) and (34), we have

$$\|[b,I_\alpha]f\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \le C_9\|b\|_* \sup_{x\in\mathbb{R}^n}\left(\int_0^\infty \left[w_1(r)\|f\|_{L_{p_1}(B(x,t))}dt\right]^{\theta_1}dr\right)^{\frac{1}{\theta_1}} = C_9\|b\|_*\|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}}.$$

□

Now, we present a theorem about the compactness of the operators $[b,I_\alpha]$ on global Morrey-type spaces from $GM^{w_1(\cdot)}_{p_1\theta_1}(\mathbb{R}^n)$ to $GM^{w_2(\cdot)}_{p_2\theta_2}(\mathbb{R}^n)$.

Theorem 8. *Let $1<p_1<p_2<\infty$, $0<\theta_1\le\theta_2<\infty$, $\frac{1}{p_2}=\frac{1}{p_1}-\frac{\alpha}{n}$, $w_1\in\Omega_{p_1\theta_1}$, $w_2\in\Omega_{p_2\theta_2}$, and the functions w_1,w_2 satisfy conditions (33) and (34). Moreover, let $b\in VMO(\mathbb{R}^n)$. Then, the commutator $[b,I_\alpha]$ is a compact operator from $GM^{w_1(\cdot)}_{p_1\theta_1}$ to $GM^{w_2(\cdot)}_{p_2\theta_2}$.*

To prove this theorem, we need the following auxiliary assertions.

Lemma 11 ([7])**.** *Let $n\in\mathbb{N}, 1<p_1<p_2<\infty$, $\frac{1}{p_2}=\frac{1}{p_1}-\frac{\alpha}{n}$. Then, there exists $C_{10}>0$, depending only on n,p_1,p_2, such that for any $\beta>0$, $\gamma\ge2\beta$, $t\in\mathbb{R}^n$, $r>0$, and for any $f\in L_p^{loc}(\mathbb{R}^n)$ satisfying the condition $supp f\subset\overline{B(0,\beta)}$*

$$\left\|(I_\alpha f)\chi_{^cB(0,\gamma)}\right\|_{L_{p_2}(B(t,r))} \le C_{10}\gamma^{\alpha-n}(min\{\gamma,r\})^{\frac{n}{p_2}}\beta^{n(1-\frac{1}{p_1})}\|f\|_{L_{p_1}(B(0,\beta))}. \tag{35}$$

Lemma 12 ([7])**.** *Let $n\in\mathbb{N}, 1<p_1<p_2<\infty$, $\frac{1}{p_2}=\frac{1}{p_1}-\frac{\alpha}{n}$. Then, there exists $C_{11}>0$, depending only on n,p_1,p_2, such that for any $\beta>0$, $\gamma\ge2\beta$, $t\in\mathbb{R}^n$, $r>0$, and for any $f\in L_p^{loc}(\mathbb{R}^n)$, for any $b\in L_\infty(\mathbb{R}^n)$ satisfying the condition $supp\, b\subset\overline{B(0,\beta)}$*

$$\left\|([b,I_\alpha]f)\chi_{^cB(0,\gamma)}\right\|_{L_{p_2}(B(t,r))} \le C_{11}\gamma^{\alpha-n}(min\{\gamma,r\})^{\frac{n}{p_2}}\beta^{n(1-\frac{1}{p_1})}\|b\|_{L_\infty(\mathbb{R}^n)}\|f\|_{L_{p_1}(B(0,\beta))}. \tag{36}$$

Proof Theorem 8. Let F be an arbitrary bounded subset of $GM^{w_1(\cdot)}_{p_1\theta_1}$. To prove Theorem 8, it suffices to show that conditions (22)–(24) of Theorem 4 hold for the set $S=\{[b,I_\alpha]: f\in F\}$, where $b\in VMO$.

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $VMO(\mathbb{R}^n)$, we only need to prove that the set $G=\{[b,I_\alpha]f : f\in F\}$, where $b\in C_0^\infty$, is pre-compact in $GM^{w_2(\cdot)}_{p_2\theta_2}$.

Suppose that $D>0$ and

$$\|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}} \le D, \text{for } f\in F.$$

By Theorem 7, we have

$$\sup_{f\in F}\|[b,I_\alpha]f\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \le C_9\cdot\|b\|_*\sup_{f\in F}\|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}} \le C_9\cdot D\|b\|_* <\infty.$$

This implies that condition (22) of Theorem 4 holds.

Now, let us prove that condition (24) of Theorem 4 holds for $[b,I_\alpha]$. Suppose that $supp\,\text{b}\subset\{x:|x|\le\beta\}$. By Lemma 12

$$\left\|([b,I_\alpha]f)\chi_{{}^c B(0,\gamma)}\right\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} = \sup_{x\in\mathbb{R}^n}\left\|w_2(r)\left\|([b,I_\alpha]f)\chi_{{}^c B(0,\gamma)}\right\|_{L_{p_2}(B(x,r))}\right\|_{L_{\theta_2}(0,\infty)}$$

$$\leq C_{11}\gamma^{\alpha-n}\beta^{n(1-\frac{1}{p_1})}\|b\|_{L_\infty(\mathbb{R}^n)}\|f\|_{L_{p_1}(B(0,\beta))}\left\|w_2(r)(min\{\gamma,r\})^{\frac{n}{p_2}}\right\|_{L_{\theta_2}(0,\infty)}.$$

For $r<\gamma$, we have $(min\{\gamma,r\})^{\frac{n}{p_2}}=r^{\frac{n}{p_2}}$. Using the condition $w_2\in\Omega_{p_2,\theta_2}$, we have

$$\|w_2(r)r^{\frac{n}{p_2}}\|_{L_{\theta_2}(0,t)}<\infty.$$

For $\gamma<r$, we have $(min\{\gamma,r\})^{\frac{n}{p_2}}=\gamma^{\frac{n}{p_2}}$. Using the condition $w_2\in\Omega_{p_2,\theta_2}$, we have

$$\|w_2(r)\gamma^{\frac{n}{p_2}}\|_{L_{\theta_2}(t,\infty)}=\gamma^{\frac{n}{p_2}}\|w_2(r)\|_{L_\theta(t,\infty)}<\infty.$$

Therefore, for $\gamma\geq 1$

$$\left\|w_2(r)(min\{\gamma,r\})^{\frac{n}{p_2}}\right\|_{L_{\theta_2}(0,\infty)}$$

$$\leq \theta_2^{(\frac{1}{\theta_2}-1)_+}\left(\left\|w_2(r)r^{\frac{n}{p_2}}\right\|_{L_{\theta_2}(0,t)}+\gamma^{n/p_2}\|w_2(r)\|_{L_{\theta_2}(t,\infty)}\right)\leq C_{13}\gamma^{n/p_2}<\infty,$$

where C_{13} is independent of $f\in F$ and γ.

Note that for any $f\in GM^{w_1(\cdot)}_{p_1\theta_1}$

$$\|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}}=\sup_{x\in\mathbb{R}^n}\left\|w_1(r)\|f\|_{L_{p_1}(B(x,r))}\right\|_{L_{\theta_1}(0,\infty)}$$

$$\geq\left\|w_1(r)\|f\|_{L_{p_1}(B(0,r))}\right\|_{L_{\theta_1}(\beta,\infty)}\geq\|w_1(r)\|_{L_{\theta_1}(\beta,\infty)}\|f\|_{L_{p_1}(B(0,\beta))}$$

By condition (33) $\|w_1(r)\|_{L_{\theta_1}(\beta,\infty)}>0$. Therefore for any $f\in F$

$$\|f\|_{L_{p_1}(B(0,\beta))}\leq\|w_1(r)\|^{-1}_{L_{\theta_1}(\beta,\infty)}\|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}}\leq\|w_1(r)\|^{-1}_{L_{\theta_1}(\beta,\infty)}D$$

So, for any $f\in F$ and $\gamma\geq 1$

$$\left\|([b,I_\alpha]f)\chi_{{}^c B(0,\gamma)}\right\|_{GM^{w_2(\cdot)}_{p_2\theta_2}}\leq C_{14}\gamma^{\alpha-n+\frac{n}{p_2}}D=C_{14}\gamma^{-n\left(1-\frac{1}{p_1}\right)}D,$$

where $C_{14}>0$ is independent of $f\in F$ and $\gamma\geq 1$.

Hence,

$$\lim_{\gamma\to\infty,}\sup_{f\in F}\left\|([b,I_\alpha]f)\chi_{{}^c B(0,\gamma)}\right\|_{GM^{w_2(\cdot)}_{p_2\theta_2}}=0.$$

This is the required condition (24).

Now we prove that condition (23) of Theorem 4 holds for the set $[b,I_\alpha](f)$, where $f\in F$. That is, we show that, for any $0<\varepsilon<\frac{1}{2}$ and if $|z|$ is sufficiently small depending only on ε, for every $f\in F$.

$$\|[(b,I_\alpha f)(\cdot+z)]-[b,I_\alpha]f(\cdot)\|_{GM^{w_2(\cdot)}_{p_2\theta_2}}\leq C_{15}D\varepsilon,$$

where $C_{15}>0$ is independent of f and ε.

Let ε be an arbitrary number, such that $0 < \varepsilon < \frac{1}{2}$. For $|z| \in \mathbb{R}^n$, we have that

$$[f, I_\alpha]f(x+z) - [b, I_\alpha]f(x) = \int\limits_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy - \int\limits_{\mathbb{R}^n} \frac{[b(x) - b(y)]f(y)}{|x-y|^{n-\alpha}}dy =$$

$$= \int\limits_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy - \int\limits_{\mathbb{R}^n} \frac{[b(x) + b(x+z) - b(x+z) - b(y)]f(y)}{|x-y|^{n-\alpha}}dy =$$

$$= \int\limits_{\mathbb{R}^n} \frac{[b(x+z) - b(y)]f(y)}{|x+z-y|^{n-\alpha}}dy + \int\limits_{\mathbb{R}^n} \frac{[b(y) - b(x+z)]f(y)}{|x-y|^{n-\alpha}}dy +$$

$$+ \int\limits_{\mathbb{R}^n} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}}dy =$$

$$= \int\limits_{\mathbb{R}^n} [b(y) - b(x+z)]\left(\frac{f(y)}{|x-y|^{n-\alpha}} - \frac{f(y)}{|x+z-y|^{n-\alpha}}\right)dy + \int\limits_{\mathbb{R}^n} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}}dy =$$

$$= \int\limits_{|x-y| > \frac{|z|}{\varepsilon}} \frac{[b(x+z) - b(x)]f(y)}{|x-y|^{n-\alpha}}dy$$

$$+ \int\limits_{|x-y| > \frac{|z|}{\varepsilon}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x+z-y|^{n-\alpha}}\right) \cdot [b(y) - b(x+z)]f(y)dy$$

$$+ \int\limits_{|x-y| \le \frac{|z|}{\varepsilon}} \frac{[b(y) - b(x)]f(y)}{|x-y|^{n-\alpha}}dy - \int\limits_{|x-y| \le \frac{|z|}{\varepsilon}} \frac{[b(y) - b(x+z)]f(y)}{|x+z-y|^{n-\alpha}}dy$$

$$= J_1 + J_2 + J_3 - J_4. \tag{37}$$

Since $b \in C_0^\infty(\mathbb{R}^n)$, we have

$$|b(x) - b(x+z)| \le |\nabla f(x)| \cdot |z| \le C|z|,$$

where $C = \max_{x \in \mathbb{R}^n} |(\nabla f)(x)|$.

Therefore,

$$|J_1| \le C_{16}|z| I_\alpha(|f|)(x).$$

By Theorem 5,

$$||J_1||_{GM^{w_2(\cdot)}_{p_2\theta_2}} \le C_{16}|z| \parallel I_\alpha(f) \parallel_{GM^{w_2(\cdot)}_{p_2\theta_2}} \le C_{16}|z| \parallel f \parallel_{GM^{w_1(\cdot)}_{p_1\theta_1}} \le C_{16}D|z|. \tag{38}$$

where $C_{16} > 0$ is independent of f.

For J_2, we have that

$$b(x+z) - b(y) \le 2 \parallel b \parallel_\infty \le C_{17}.$$

Therefore,

$$|J_2| \le C_{17}|z| \int\limits_{|x-y| > \frac{|z|}{\varepsilon}} \frac{|f(y)|}{|x-y|^n}dy \le C_{17} I_\alpha(|f|)(x).$$

Again, by Theorem 5, we obtain

$$\|J_2\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \le C_{17}\varepsilon \|I_\alpha(|f|)\|_{GM^{w_1(\cdot)}_{p_1\theta_1}} \le C_{17}\varepsilon \|f\|_{GM^{w_1(\cdot)}_{p_1\theta_1}} \le C_{17} \cdot D \cdot \varepsilon.$$

where $C_{17} > 0$ is independent of f and ε.

Regarding J_3. Since $b \in C_0^\infty$, we have $|b(x) - b(y)| \leq C|x - y|$.

$$|J_3| \leq C \int\limits_{|x-y| \leq \frac{|z|}{\varepsilon}} \frac{|f(y)|}{|x-y|^{n-\alpha-1}} dy \leq C\varepsilon^{-1}|z| \int\limits_{|x-y| \leq \frac{|z|}{\varepsilon}} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq C \cdot \varepsilon^{-1}|z| I_\alpha(|f|)(x).$$

Thus, we have

$$\| J_3 \|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \leq C \cdot \varepsilon^{-1}|z| \; \| I_\alpha(f) \|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \leq C \cdot \varepsilon^{-1}|z| \; \| f \|_{GM^{w_1(\cdot)}_{p_1\theta_1}} \leq \varepsilon^{-1} \cdot |z| \cdot D. \quad (39)$$

Similarly, using the estimate

$$|b(x+z) - b(y)| \leq C|x + z - y|,$$

we have

$$|J_4| \leq C \int\limits_{|x-y| \leq \frac{|z|}{\varepsilon}} |x + z - y|^{-n+1+\alpha} |f(y)| dy \leq C(\varepsilon^{-1}|z| + |z|) I_\alpha(|f|)(x+z).$$

Therefore,

$$\| J_4 \|_{GM^{w_2}_{p_2\theta_2}} \leq C \cdot (\varepsilon^{-1}|z| + |z|) \; \| f \|_{GM^{w_1}_{p_1\theta_1}} \leq C \cdot (\varepsilon^{-1}|z| + |z|) \cdot D. \quad (40)$$

Here, C does not depend on z and ε. Finally, from (37)–(40) by taking a $|z|$ to be sufficiently small, we have

$$\|[b, I_\alpha(f)(\cdot + z)] - [b, I_\alpha] f(\cdot)\|_{GM^{w_2(\cdot)}_{p_2\theta_2}}$$

$$\leq \|J_1\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} + \|J_2\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} + \|J_3\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} + \|J_4\|_{GM^{w_2(\cdot)}_{p_2\theta_2}} \leq C \cdot D \cdot \varepsilon.$$

This shows that the set $[b, I_\alpha](f), \; f \in F$ satisfies condition (23) of Theorem 4. Therefore, by Theorem 4, the set $[b, I_\alpha](f), \; f \in F$ is pre-compact in $GM^{w_2(\cdot)}_{p_2\theta_2}$, which completes the proof of Theorem 8. □

6. Conclusions

In this paper, we have established sufficient conditions for the compactness of sets in global Morrey-type spaces. Moreover, we have provided sufficient conditions for the compactness for the commutator $[b, I_\alpha]$ for the Riesz potential operator on global Morrey-type spaces $GM^{w(\cdot)}_{p\theta}(R^n)$. More specifically, we have proved that, if $b \in VMO(R^n)$, then $[b, I_\alpha]$ is a compact operator from $GM^{w_1(\cdot)}_{p_1\theta_1}$ to $GM^{w_2(\cdot)}_{p_2\theta_2}$.

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References

1. Morrey, C. On the solutions of quasi-linear elliptic partial differential equations. *Trans. Am. Math. Soc.* **1938**,*1*, 126–166. [CrossRef]
2. Coifman, R.R.; Rochberg, R.; Weiss, G. Factorization theorems for Hardy spaces in several variables. *Ann. Math.* **1976**, *103*, 611–635. [CrossRef]
3. Arai, R.; Nakai, E. An extension of the characterization of CMO and its application to compact commutators on Morrey spaces. *J. Math. Soc. Jpn.* **2020**, *2*, 507–539. [CrossRef]
4. Chen, J.; Hu, G. Compact commutators of rough singular integral operators. *Can. J. Math.* **2015**, *58*, 19–29. [CrossRef]
5. Iwaniec, T. L^p-theory of quasiregular mappings. In *Quasi Conformal Space Mappings*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 1992; Volume 1508, pp. 39–64.
6. Chen, Y.; Ding, Y.; Wang, X. Compactness of commutators of Riesz potential on Morrey space. *Potential Anal.* **2009**, *4*, 301–313. [CrossRef]
7. Bokayev, N.; Matin, D.; Akhazhanov, T.; Adilkhanov, A. Compactness of commutators for Riesz potential on generalized Morrey spaces. *Mathematics* **2024**, *12*, 304. [CrossRef]
8. Sawano, Y.; Shiai, S. Compact commutators on Morrey spaces with non-doubling measures. *Georgian Math. J.* **2008**, *2*, 353–376. [CrossRef]
9. Tao, J.; Yang, D.; Yuan, W.; Zhang, Y. Compactness characterizations of commutators on ball Banach function spaces. *Potential Anal.* **2023**, *58*, 645–679. [CrossRef]
10. Bandaliyev, R.A.; Górka, P.; Guliyev, V.S.; Sawano, Y. Relatively compact sets in variable exponent Morrey spaces on metric spaces. *Mediterr. J. Math.* **2021**, *6*, 1–23. [CrossRef]
11. Chen, Y.; Ding, Y.; Wang, X. Compactness of commutators for singular integrals on Morrey spaces. *Can. J. Math.* **2012**, *64*, 257–281. [CrossRef]
12. Wang, W.; Xu, J. Precompact sets, boundedness, and compactness of commutators for singular integrals in variable Morrey spaces. *J. Funct. Spaces* **2017**, *2017*, 3764142. [CrossRef]
13. Peetre, J. On the theory of $L^{p,\lambda}$ spaces. *J. Funct. Anal.* **1969**, *4*, 71–87. [CrossRef]
14. Adams, D.R. Lectures on L_p-potential theory. *Umea Univ.* **1981**, *2*, 1–74.
15. Mizuhara, T. Boundedness of some classical operators on generalized Morrey spaces. In *Proceedings of the Harmonic Analisis, ICM 90 Satellite Proceedings*; Springer: Tokyo, Japan, 1991; pp. 183–189.
16. Nakai, E. Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces. *Math. Nachr.* **1994**, *166* , 95–103. [CrossRef]
17. Guliyev, V.S. Generalized weighted Morrey spaces and higher order commutators of sublinear operators. *Eurasian Math. J.* **2012**, *3*, 33–61.
18. Burenkov, V.I. Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces II. *Eurasian Math. J.* **2013**, *4*, 21–45.
19. Burenkov, V.I.; Gogatishvili, A.; Guliyev, V.S.; Mustafaev, R. Boundedness of the Riesz potential in local Morrey-type spaces. *Potential Anal.* **2011**, *35*, 67–87. [CrossRef]
20. Ding, Y. A characterization of BMO via commutators for some operators. *Northeast. Math. J.* **1997**, *13*, 422–432.
21. Guliyev, V.S.; Bandaliyev, R.A. Embedding theorems between variable-exponent Morrey spaces. *Math. Notes* **2019**, *106*, 488–500.
22. Burenkov, V.I.; Guliyev, H.V. Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. *Stud. Math.* **2004**, *163*, 157–176. [CrossRef]
23. Yosida, K. *Functional Analysis*; Springer: Berlin, Germany, 1978.
24. Burenkov V.I.; Tararykova, T.V. An analog of Young's inequality for convolutions of functions for general Morrey-type spaces. *Proc. Steklov Inst. Math.* **2016**, *293*, 107–126. [CrossRef]
25. Bokayev, N.A.; Burenkov, V.I.; Matin, D.T. On pre-compactness of a set in general local and global Morrey-type spaces. *Eurasian Math. J.* **2017**, *8*, 109–115.
26. Guliyev, V.S.; Shukurov, P.S. On the boundedness of the fractional maximal operator, Riesz potential and their commutators in generalized Morrey spaces. *Oper. Theory Adv. Appl.* **2013**, *229*, 175–199.
27. Gogatishvili, A.; Mustafayev, R.C. Weighted iterated Hardy-type inequalities. *Math. Inequal. Appl.* **2017**, *20*, 683–728.