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A Formal Fuzzy Concept-Based Approach for Association Rule Discovery with Optimized Time and Storage

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Abstract: Association Rule Mining (ARM) relies on concept lattices as an effective knowledge representation structure. However, classical ARM methods face significant limitations, including the generation of misleading rules during data-to-formal-context mapping and poor handling of heterogeneous data types such as linguistic, continuous, and imprecise data. This study aims to address these limitations by introducing a novel fuzzy data structure called the "fuzzy iceberg lattice" and its corresponding construction algorithm. The primary objectives of this study are to enhance the efficiency of extracting and visualizing frequent fuzzy closed item sets and to optimize both execution time and storage requirements. The necessity of this research stems from the high computational cost and redundancy associated with traditional fuzzy approaches, which, while capable of managing quantitative and imprecise data, are often impractical for large-scale applications in real scenarios. The proposed approach incorporates a 'fuzzy min-max basis algorithm' to derive exact and approximate rule bases from the extracted fuzzy closed item sets, eliminating redundancy while preserving valuable insights. Experimental results on benchmark datasets demonstrate that the proposed fuzzy iceberg lattice outperforms traditional fuzzy concept lattices, achieving an average reduction of 74.75% in execution time and 70.53% in memory usage. This efficiency gain, coupled with the lattice's ability to handle crisp, quantitative, fuzzy, and heterogeneous data types, underscores its potential to advance ARM by yielding a manageable number of high-quality fuzzy concepts and rules.

Keywords: concept lattices; knowledge representation; association rule mining; formal fuzzy concepts; heterogeneous data; fuzzy iceberg lattice

MSC: 68T37; 68P01; 06B35; 68P20

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1. Introduction

Association rule mining (ARM) is a fundamental technique in data mining, instrumental in uncovering valuable patterns and relationships in large datasets to support decision-making through the discovery of relevant rules [1]. ARM identifies strong association rules (ARs) based on criteria such as support, confidence, and lift and has applications across multiple domains. In retail, ARM enhances market basket analysis and inventory management, while in healthcare, it aids in associating symptoms with diseases to improve diagnosis and treatment [2,3].

To address the limitations of traditional ARM approaches, Formal Concept Analysis (FCA) has been introduced as a mathematical framework for extracting closed itemsets

(CIs) via a structured lattice, enabling non-redundant association rule discovery without data loss [4,5]. However, FCA primarily handles crisp data, which restricts its application in contexts involving uncertain or quantitative data. To overcome this limitation, Fuzzy Formal Concept Analysis (FFCA) extends FCA by integrating fuzzy set theory, thus broadening its applicability in domains where data may not be purely binary [6]. Despite its utility, FFCA faces significant challenges, particularly due to the exponential growth of fuzzy concepts and the resulting rules as datasets scale in size and complexity. This growth introduces a #P-complete complexity in enumerating fuzzy concepts and leads to an exponential increase in association rule bases, making their generation and storage #P-hard [7].

Furthermore, as highlighted in recent ARM studies [8], mining high-dimensional data often results in an extensive set of rules, many of which may be redundant or irrelevant, complicating interpretation. The incorporation of fuzzy or uncertain data further adds complexity, requiring sophisticated methods for defining membership functions and interpreting results. These challenges underscore the need for ARM solutions that achieve a balance between interpretability and computational efficiency.

Numerous studies [9–12] have applied FFCA in ARM, yet they frequently encounter limitations in computational efficiency and memory consumption. Existing methods struggle with the exponential complexity of constructing a complete concept lattice and extracting association rules, especially as the data scale increases. For instance, while the approach in [11] utilized fuzzy-crisp concepts like our proposed approach, it relied on constructing a complete fuzzy lattice, which is computationally intensive. On the other hand, the approach in [10] utilized the crisp-fuzzy concepts that result in a more extensive number of fuzzy concepts that can result in very similar association rules. These limitations hinder the application of FFCA-based ARM methods in real-time and large-scale data processing, where timely and resource-efficient analysis is crucial. While recent advancements offer some improvements, there is still a gap in achieving a scalable, practical solution that maintains rule quality while mitigating computational overhead.

The increasing emphasis on handling large-scale data across diverse industries makes it essential to explore efficient ARM techniques, especially those that can accommodate complex data types. This research addresses this gap by introducing a novel fuzzy iceberg lattice, which is tailored to optimize ARM by focusing solely on frequent closed itemsets that exceed a predefined support threshold. Unlike traditional methods that build a complete concept lattice, this approach selectively retains only the essential upper portion, optimizing the process of mining strong association rules. Our method directly responds to recent calls for scalable approaches that can enhance ARM's applicability in modern data environments, making it distinct from previously published studies.

This study aims to address the following research questions:

- How does incorporating a user-centric approach, which considers subjective beliefs about data, impact the interpretability and effectiveness of the extracted rules?
- How to optimize the process of extracting association rules without losing critical information?
- What improvements can be achieved in computational efficiency and rule compactness when using the proposed fuzzy iceberg lattice over conventional methods?

These questions are addressed through a structured approach that includes the following: (1) mapping the dataset to a fuzzy representation using user-centric linguistic labels, enabling an interpretable transformation to a fuzzy variant without information loss; (2) constructing a fuzzy iceberg lattice focused on frequent closed itemsets, which leverages crisp intents and their corresponding fuzzy supports to filter out infrequent concepts early, thereby optimizing the extraction of association rules; and (3) generating concise fuzzy implication and association rules through a rule-based extraction algorithm that ensures no redundant rules are included.

This study contributes to the field in the following ways:

1. Formalizing the fundamental definitions of single-sided FFCA, in which a fuzzy concept includes a fuzzy extent and a crisp intent.
2. Presenting a method for mining concise fuzzy association rules from quantitative data, incorporating user-defined beliefs. The approach includes three main stages, data mapping, iceberg lattice construction, and association/implication rule generation.
3. Developing a novel algorithm for extracting a fuzzy-crisp iceberg lattice, where each node represents a crisp intent with fuzzy support values derived from the corresponding fuzzy extent.
4. Extending the classical algorithm from [13] by introducing fuzzification to extract a fuzzy association rule basis rather than binary rules.
5. Evaluating the proposed approach in comparison to existing methods [11,12] with respect to the reduction in the number of fuzzy concepts, rule compactness, and processing time.

The rest of this paper is organized as follows: Section 2 examines the fundamental concepts of FCA and FFCA. Section 3 reviews related research in mining association rules using FCA. Section 4 presents the proposed approach, including the algorithms for constructing the fuzzy iceberg lattice and extracting association rules. Experimental results on benchmark datasets are discussed in Section 5. Finally, Section 6 concludes the paper and outlines future research directions.

2. Classical and Fuzzy Formal Concept Analysis

Formal concept analysis (FCA) is essential for representing conceptual knowledge [14]. This section provides an overview of the classical and fuzzy FCA's fundamental concepts and definitions. For those interested in delving deeper into the topic, the works of [5,15] are some recommended sources.

Definition 1. *A formal context $\Bbbk$ can be defined as a triple (O, A, R) where O is a set of objects, A is a set of attributes, and $R \subseteq O \times A$ is a binary relation between objects and attributes. If $(O_1,\ A_1) \in R$, then object O_1 has attribute A_1.*

The formal context, a binary table with rows, columns, and incident relations, is the primary input for FCA algorithms that generate formal concepts and lattices.

Definition 2. *A formal concept extracted from a formal context $\Bbbk(O,\ A,\ R)$ is a pair $(X,\ Y)$ such that $X \subseteq\ O$ is the concept extent, $Y \subseteq\ A$ is the concept intent, and $X' = Y$ and $Y' = X$, given that X' and Y' are defined by Equations (1) and (2).*

$$X' = \{a \in A \mid (x, a) \in R\ \forall x \in X\} \tag{1}$$

Such that X' represents attributes' set shared by all objects in X

$$Y' = \{o \in O \mid (o, y) \in R\ \forall y \in Y\} \tag{2}$$

Such that Y' represents objects' set sharing all attributes in Y.

Definition 3. *A formal concept $(O_1,\ A_1)$ is a sub-concept of the concept $(O_2,\ A_2)$ if and only if its extent $O_1 \subseteq\ O_2$ and its intent $A_2 \subseteq\ A_1$. This sub-concept relationship is formally given as follows:*

$$(O_1, A_1) \leq (O_2, A_2) \Longleftrightarrow O_1 \subseteq O_2 \Longleftrightarrow A_2 \subseteq A_1 \tag{3}$$

Such that, $\leq$ demonstrates the sub-concept relationship operator.

The formal concept lattice $\underline{\beta}$ (O, A, R) is a lattice of all formal concepts, displaying sub-concept relationships through links. However, only frequent concepts are needed for strong association rules. On the contrary, an iceberg lattice displays only frequent concepts (frequent closed itemsets FCIs) in a sub-lattice satisfying a minimum support threshold [16]. Therefore, the iceberg lattice is suitable for extracting strong association rules.

FCA algorithms need a formal context in binary form. Multi-valued context (MVC) containing quantitative data then need to be converted into binary context using crisp scaling. This involves dividing each quantitative attribute into multiple non-overlapping intervals.

Example 1. *Table 1 shows a multi-valued context (MVC) with age and experience data for four instances (O0 to O3). Tables 2 and 3 demonstrate how this MVC is mapped to a binary context in Table 4. Age is divided into three intervals by the age scale context Table 2, and experience is divided into junior, middle, and senior categories (Table 3). The transformation into a binary context is achieved by evaluating each instance against the defined conditions of these scale contexts.*

Table 1. Example multi-valued context.

Object ID	Age	Experience Years
O0	10	0
O1	17	3
O2	45	9
O3	46	15

Table 2. Scale context for the age quantitative attribute.

Age (A)	Young	Youth	Old
$A \leq 16$	√		
$16 < A \leq 45$		√	
$A > 45$			√

Table 3. Scale context for the experience attribute.

Experience (E)	Junior	Middle	Senior
$E \leq 3$			
$3 < E \leq 5$		√	
$E > 5$			√

Table 4. Derived context after crisp scaling.

Object ID	Age A			Experience Years E		
	Young	Youth	Old	Junior	Middle	Senior
O0	1	0	0	1	0	0
O1	0	1	0	0	1	0
O2	0	1	0	0	0	1
O3	0	0	1	0	0	1

In this example, Table 1 shows that object O0 has an age of 10 years. According to the age scale in Table 2, the first condition states that any age equal or below 16 is classified as "young." Therefore, in the binary context shown in Table 4, O0 is assigned a "1" under the "young" attribute and "0" under other age categories ("youth" and "old"). Having this will finally result in a binary relation R between objects (O) and attributes (A), expressed using "0"s and "1"s to represent the presence or absence of each attribute, facilitating a concise relationship between attributes across instances.

Crisp scaling converts multi-valued contexts (MVC) into binary context for FCA algorithms, but it has drawbacks such as information loss and the sharp border problem [6]. As demonstrated in Example 1, a small age variation can alter object memberships. Furthermore, objects like O1 and O2 are inaccurately classified as identical (e.g., both classified as middle-aged with a membership of 1) despite actual differences. In contrast, Fuzzy FCA (FFCA) integrates fuzzy set theory to address these issues by fuzzifying quantitative attributes. Fuzzification considers an object's membership across a continuous range [0, 1], thereby providing a more nuanced representation than crisp scaling.

FFCA provides a robust framework for handling uncertainty and gradual membership in data. As shown in Figure 1, FFCA can be approached in the following two main ways:

1. *Full-sided FFCA*: This approach uses fuzzy concepts where both the extent (the set of objects) and the intent (the set of attributes) are fuzzy sets [17].
2. *Single-sided FFCA:* This category includes two variations as follows:
   - *Crisp-fuzzy FFCA*, where each fuzzy concept consists of a crisp extent (a precise set of objects) and a fuzzy intent (a set of attributes with varying degrees of membership) [18].
   - *Fuzzy-crisp FFCA*, in which the extent is a fuzzy set (with gradual membership degrees) while the intent is a crisp set (with exact membership) [16].

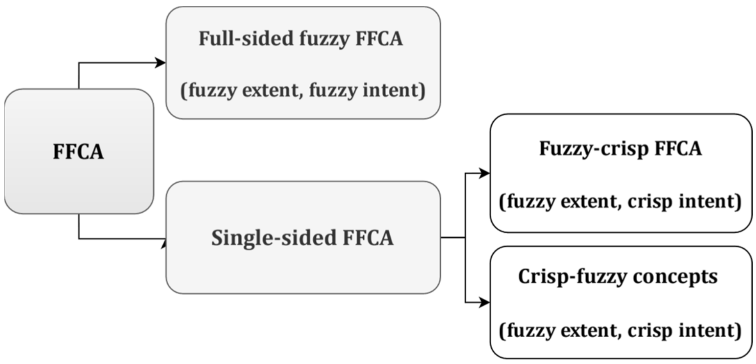

Figure 1. Different viewpoints of FFCA.

This study focuses on fuzzy-crisp FFCA, which is commonly used in association rule mining and ontology construction [10,19]. In this approach, fuzziness is applied to the extent, while the intent is treated as a crisp set, simplifying the representation of fuzzy concepts. This approach reduces complexity by eliminating fuzziness from one side of the concept, making it easier to analyze and interpret.

Definition 4. *A fuzzy context* $\hat{K}$ *is denoted as* $\left(O,\ A,\ \hat{R} : O \times A \to [0,1]\right)$ *such that O contains the objects set, A contains the attributes set, and the fuzzy incident relation* $\hat{R}$ *is between O and A. So, each object* $o \in O$ *has an attribute* $a \in A$ *to some extent degree* $\mu_{\hat{R}}(o, a)$.

FFCA effectively handles quantitative data by fuzzifying each attribute into linguistic labels using membership functions that map values to a range of [0, 1] [20]. For instance, Table 1 shows the "age" and "experience years" linguistic variables. Figure 2 illustrates how these variables are converted to fuzzy contexts, as seen in Table 5, retaining the membership degree. For instance, in Figure 2, the age linguistic variable is defined by two trapezoidal membership functions for young and old linguistic labels and one triangular membership function for the middle-age (youth) linguistic label. Triangular membership function $\mathrm{A(x)}$ is defined using Equation (4).

$$\mathrm{A(x)} = \begin{cases} 0 & \mathrm{x \le a \quad or \quad x \ge c} \\ \frac{\mathrm{x-a}}{\mathrm{b-a}} & \mathrm{a < x \le b} \\ \frac{\mathrm{c-x}}{\mathrm{c-b}} & \mathrm{b \le x < c} \end{cases} \tag{4}$$

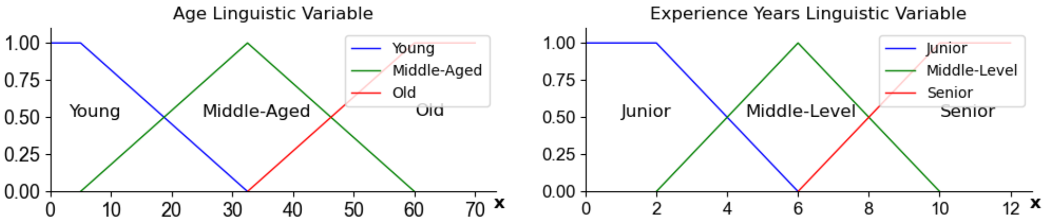

Figure 2. Definitions of age and experience linguistic variables. The colored lines depict different states within each linguistic variable: age (Young, Middle-Aged, Old) and experience (Junior, Middle-Level, Senior).

Table 5. Fuzzy context after fuzzifying MVC in Table 1 using membership functions depicted in Figure 2.

Object ID	Age A			Experience Years E		
	Young	**Youth**	**Old**	**Junior**	**Middle**	**Senior**
O0	0.82	0.18	0	1	0	0
O1	0.56	0.44	0	0.75	0.25	0
O2	0	0.55	0.45	0	0.25	0.75
O3	0	0.51	0.49	0	0	1

The triangular function A(x) is a linear function that defines a fuzzy set by the lower limit a, the middle value b, and the upper limit c, such that $a \leq\ b \leq\ c$.

In Figure 2, for example, the middle-aged linguistic label is a triangular membership function with $a = 5, b = 32.5$, and $c = 60$. Given age $x = 10$, $A(x)$ is evaluated by $\frac{10-5}{32.5-5} \simeq 0.18$.; therefore, the object $O0$ has a value of 0.18 in the youth label in Table 5.

The trapezoidal membership function is defined by Equation (5) and is characterized by four key points, $a,\ b,\ c$, and d. For example, the "young" trapezoidal function shown in Figure 2 is specified with parameters a = 0, b = 0, c = 5, and d = 32.5. Thus, for an object O0 with an age of 10, as shown in Table 1, the membership value for the "young" linguistic label is calculated as $T(10) = \frac{32.5-10}{32.5-5} \simeq 0.82$, which is presented in Table 5.

$$\mathrm{T(x)} = \begin{cases} 0 & \mathrm{x \leq a \quad or \quad x \geq d} \\ \frac{\mathrm{x-a}}{\mathrm{b-a}} & \mathrm{a < x < b} \\ 1 & \mathrm{b \leq x \leq c} \\ \frac{\mathrm{d-x}}{\mathrm{d-c}} & \mathrm{c < x < d} \end{cases} \tag{5}$$

According to Definition 4, the fuzzy context presented in Table 5 consists of a set of objects $O = \{O0,\ O1,\ O2,\ O3\}$, a set of attributes A = {Young, Youth, Old, Junior, MiddleExperience, Senior} , and the fuzzy relation $\hat{R}$ between objects in O and attributes in A. For instance, the relationship between O1 and young attribute is formally defined as $\mu_{\hat{R}}(O1, young)$, which quantifies the degree to which O1 possesses the attribute "young". In this case, $\mu_{\hat{R}}(O1, young)$ = 0.56, indicating that O1 is 56% "young." Similarly, O2 has a 55% membership in the "youth" category. This fuzzy representation provides a more refined and continuous characterization of attributes compared to traditional binary or crisp scaling, capturing subtle variations in attribute membership.

Definition 5. *Given a fuzzy context $\hat{K} = (O,\ A,\ \hat{R})$, a fuzzy concept $(\hat{X},\ Y)$ has a fuzzy extent $\hat{X} \subseteq\ O$ and a crisp intent $Y \subseteq\ A$, and $X' = Y$ and $Y' = \hat{X}$, where $\hat{X}'$ and Y' are formally given by Equations (6) and (7), respectively.*

$$\hat{X}' = \{y \in Y \mid \forall x \in \hat{X} : \mu_{\hat{R}}(\mathrm{x}, \mathrm{y}) \geq \mu_{\hat{X}}(x)\} \tag{6}$$

$\hat{X}'$ is a crisp set of attributes shared by all fuzzy objects in $\hat{X}$ and $\mu_{\hat{X}}(x)$ is the membership of object x in the fuzzy extent set $\hat{X}$.

$$Y' = \left\{ \frac{x}{\mu_{\hat{X}}(x)} \mid \mu_{\hat{X}}(x) = \min_{y \in Y} (\mu_{\hat{R}}(x, y)) \right\} \tag{7}$$

Y' is a fuzzy set of objects that share all attributes in Y, each with some extent degree.

For instance, consider Example 1 $\left(\left\{\frac{O2}{0.75}, \frac{O3}{1}\right\}, \{senior\}\right)$ as an example of a fuzzy-crisp concept according to Definition 5. In this concept, $\left\{\frac{O2}{0.75}, \frac{O3}{1}\right\}$ represents the fuzzy set defining the concept's extent, while $\{senior\}$ denotes the crisp set representing the concept's intent. This is a valid fuzzy-crisp concept because the following conditions are met: $\left\{\frac{O2}{0.75}, \frac{O3}{1}\right\}' = \{senior\}$ and $\{senior\}' = \left\{\frac{O2}{0.75}, \frac{O3}{1}\right\}'$, where ($'$) is the concept-forming operator known as the derivation operator. As clarified by Equation (6), when the derivation operator ($'$) is applied to a fuzzy extent such as $\left\{\frac{O2}{0.75}, \frac{O3}{1}\right\}$, it yields the set of attributes common to all objects within that fuzzy extent. On the other hand, when the derivation operator is applied to the crisp intent, e.g., {senior}, it returns a fuzzy set representing objects that share all attributes in the intent and their associated membership values.

Definition 6. *A fuzzy concept $(\hat{X}_1, Y_1)$ is a fuzzy sub-concept of $(\hat{X}_2, Y_2)$ if $\hat{X}_1 \subseteq \hat{X}_2$ and $Y_2 \subseteq Y_1$. This is denoted as $(\hat{X}_1, Y_1) \lesssim (\hat{X}_2, Y_2)$.*

Definition 7. *A fuzzy concept $(\hat{X}_1, Y_1)$ is a fuzzy direct predecessor of $(\hat{X}_2, Y_2)$ >if $(\hat{X}_1, Y_1) \lesssim (\hat{X}_2, Y_2)$, and there is no fuzzy concept $(\hat{X}_3, Y_3)$ such that $(\hat{X}_1, Y_1) \lesssim (\hat{X}_3, Y_3) \lesssim (\hat{X}_2, Y_2)$.*

A fuzzy lattice $\beta\ (O,\ A, \hat{R})$ visually organizes fuzzy concepts using the $\lesssim$ operator. Figure 3, derived from Table 5, illustrates this lattice, comprising 12 nodes representing fuzzy concepts by their fuzzy extent $\{object\_Id,\ \mu object\}$ and crisp intent. Each node condenses multiple attributes into a combined intent reflecting all superior concepts. For example, the "Middle_level" node combines attributes from predecessors like {Middle_Age, Middle_level}.

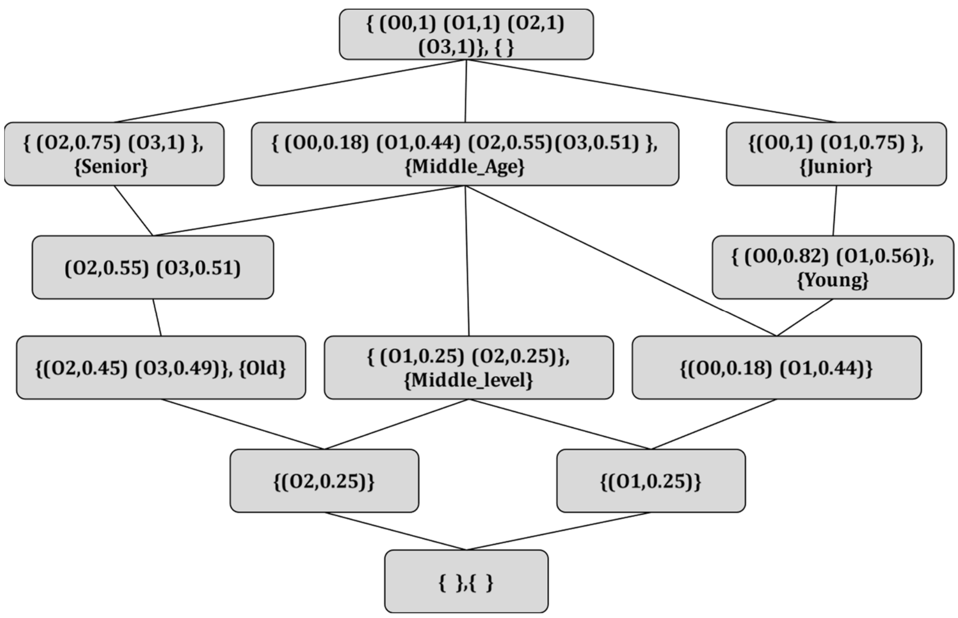

Figure 3. Fuzzy lattice derived from the fuzzy context in Table 5.

To construct the fuzzy concept lattice in Figure 3 from the fuzzy context in Table 5, we used the attribute-based algorithm provided in [20]. Starting with the top concept, which includes all objects and an empty intent, the algorithm derives each subsequent fuzzy concept by iteratively adding attributes (forming intents) and identifying the objects (fuzzy extents) that meet the membership criteria for those attributes, using the derivation operator provided by Equation (7). For instance, the concept with the intent {Senior} includes objects $O2$ and $O3$ in its fuzzy extent, with membership values 0.75 and 1, respectively, as both objects satisfy the "Senior" attribute to varying degrees. Similarly, the concept with the intent {Junior} has a fuzzy extent comprising $O0$ and $O1$, with membership values 1 and 0.75, respectively. For each concept, the fuzzy extent consists of objects that satisfy the attributes in the intent, while the crisp intent is formed by attributes shared by all objects in the extent. By organizing these concepts hierarchically based on subset relationships between extents and intents (Definition 7), we create a lattice structure, where each concept is connected to the others with which it shares direct subset relationships.

3. Literature Review

Crisp FCA-based approaches for association rule mining (ARM) excel in binary contexts but struggle with quantitative data, as shown in Example 1. In contrast, fuzzy FCA-based methods adeptly manage quantitative and imprecise data. Therefore, this section explores the application of fuzzy FCA in extracting association rules. Notable approaches include those detailed in [11,12,21,22].

The approach by Mguiris et al. [22] employs prime number encoding within a crisp-fuzzy FFCA framework for fuzzy association rule extraction. This method begins by encoding the fuzzy context using prime numbers to identify frequent fuzzy minimal generators (FFMGs) without computing closures directly. Subsequently, a lattice of FFMGs is constructed to facilitate the extraction of frequent closed itemsets (FCIs), which are then used to derive fuzzy implications and association rules. Despite its innovative approach to avoid closure computation, the method faces challenges such as computational complexity due to prime number encoding, potential scalability issues with large datasets, and interpretability concerns regarding the rules derived from the complex FFMG lattice.

Another crisp-fuzzy FFCA approach is discussed in [12], which extracts a set of representative association rule groups. These groups drastically reduced the amount of the extracted association rules (generic bases). An additional validation step checks how the rule promise impacts the rule conclusion using the structural equations model. Although this approach aims at reducing the number of rules, it still suffers from an extensive number of association rules compared to fuzzy-crisp approaches, like in [23]. Mao et al. [21] proposed a novel approach for constructing a crisp-fuzzy concept lattice by representing the fuzzy context as a weighted-complete graph. The authors demonstrate that using graph theory in the crisp-fuzzy concept lattice construction results in a more time-efficient algorithm.

Extracting fuzzy association rules using crisp-fuzzy FFCA is effective when the membership of each item is crucial but generates more FCIs than the corresponding fuzzy-crisp FFCA approaches. Furthermore, crisp-fuzzy FFCA yields very similar FCIs with a minor difference. Our proposed approach suggests a fuzzy-crisp iceberg lattice that reduces the number of generated FCIs by merging similar ones while preserving membership values. Each node in this lattice represents a crisp intent, and its fuzzy support preserves the membership values of the contributing objects.

Zou et al. [11] presented a fuzzy-crisp FFCA approach to build the entire fuzzy concept lattice incrementally by inserting one attribute at a time. This approach best suits fuzzy contexts prone to continuous attribute insertion. Nevertheless, it generates the entire formal fuzzy lattice, containing both frequent and infrequent concepts. Therefore, the approach of Zou et al. is computationally intensive and requires further storage. In addition, it does not generate implication rules with 100% confidence.

In contrast, our proposed approach utilizes the fuzzy-crisp paradigm to generate the fuzzy iceberg lattice, considering only the fuzzy FCIs. Therefore, the proposed approach is more efficient and requires less storage. Furthermore, the fuzzy membership of objects is aggregated in the fuzzy support. So, the suggested approach reduces the number of association rules without information loss and takes real-world quantitative and fuzzy data into account.

An iceberg lattice is a sub-concept lattice that highlights frequent closed itemsets (FCIs) above a support threshold, ignoring less frequent patterns. Therefore, it is regarded as the optimal method for extracting association rules due to its non-redundant and concise basis [24]. However, it still needs to be effectively applied to fuzzy-crisp FCA. Applying iceberg lattice to fuzzy FCA enables the effective handling of heterogeneous data (qualitative, quantitative, imprecise, and binary). While there is significant research on building binary iceberg lattices [24–27], only a few studies have utilized this approach for crisp-fuzzy FFCA, such as those of Mguiris et al. [16,22]. Crisp-fuzzy-based iceberg lattice suffers from the extraction of extensive frequent concepts, many of which are very similar in terms of the membership degrees of the concepts' intents. On the other hand, the proposed approach utilizes a fuzzy-crisp-based iceberg lattice that merges similar fuzzy concepts and combines membership degrees within the fuzzy support. Therefore, the proposed approach aims to decrease the number of the generated fuzzy rules and enhance performance and optimization in visualizing crisp intents with fuzzy support.

While notable advancements in FFCA have been achieved, several limitations persist. Prime number encoding-based methods [22] suffer from computational complexity and scalability issues in large datasets. Some crisp-fuzzy approaches [12] still generate an extensive number of rules, posing challenges in interpretability. Techniques involving graph theory [21] show improvements in time efficiency but may be too complex to implement on highly dimensional data. Incremental lattice construction approaches [11] generate comprehensive lattices but are resource-intensive, generating infrequent concepts and redundant rules. Furthermore, the extraction of numerous similar frequent concepts in crisp-fuzzy iceberg lattices [16,22] can complicate the analysis and reduce scalability. These limitations underscore the need for continued optimization, especially for approaches that handle large-scale or highly dimensional datasets with minimal redundancy and increased efficiency.

4. Proposed Approach

This section presents the proposed approach for extracting concise association rules from quantitative data. As depicted in Figure 4, the proposed approach consists of three stages, mapping quantitative attributes into a fuzzy representation, constructing a one-sided fuzzy iceberg lattice, and generating a condensed set of implication and association rules.

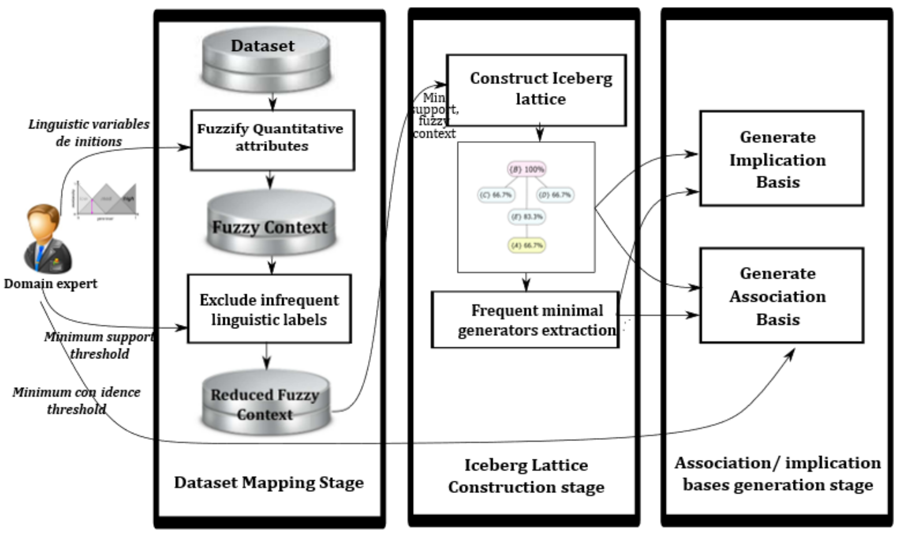

Figure 4. The entire architecture of the proposed approach for extracting association and implication bases from quantitative data.

4.1. Dataset Mapping Phase

The data mapping phase in our approach defines quantitative attributes using user-specified linguistic labels. Fuzzy membership functions [6,28], such as triangular and trapezoidal, are used to quantify degrees of membership in fuzzy sets. The data-sensitive method is applied for defining these functions [6].

To eliminate infrequent linguistic labels and save processing time, we evaluate the fuzzy support of each label using Equation (8). As a result, we obtain a reduced fuzzy context $K_r\left(A,\ O,\ \hat{R}\right)$ along with the fuzzy support for each linguistic label (i), which serves as the input for the second stage of our approach.

$$Fuzzy\ Support(i) = \frac{\sum_{x_j \in \hat{X}} \mu_i\left(x_j\right)}{|O|} \tag{8}$$

In this context, $|O|$ denotes the total number of objects in the dataset. The membership function $\mu_i\left(x_j\right) \in\ [0,\ 1]$ signifies the degree to which the object x_j belongs to the linguistic label i.

4.2. Fuzzy-Crisp Iceberg Lattice Phase

The proposed FuzzyIceberg algorithm (Algorithm 1) constructs the fuzzy iceberg lattice using the reduced fuzzy context $K_r\left(A, O, \hat{R}\right)$.

Algorithm 1: FuzzyIceberg$\left(\left(\hat{X}, Y\right), a\right)$

Input: (1) A reduced fuzzy context $K_r\left(A, O, \hat{R}\right)$, a minimum support $minSupport$, and the current attribute a
Output: Fuzzy closed itemsets in a fuzzy iceberg lattice
Begin
  For $i = a$ to $|\ A\ |$:
    If $i \notin\ Y$ **then**
      If $Y = O\prime$ **then**
        $F_{ext} \Leftarrow\ i\prime$
        $Fsupport \Leftarrow\ FuzzySupport(i)$
      Else
        $F_{ext} \Leftarrow\ tnorm\left(\hat{X},\ i\prime\right)$
        $Fsupport \Leftarrow ComputeFSupport(F_{ext})$
      End If
      If $Fsupport \geq\ minSupport$
        $Y_i \Leftarrow\ \{1, 2, \ldots.i\}$
        If $(Y_i \cap\ Y) =\ (Y_i \cap\ F'_{ext})$ ***then***
          $C_{new} \Leftarrow\ (F_{ext}\prime,\ Fsupport)$
          $Itemsets_f\ \Leftarrow\ Itemsets_f \cup C_{new}$
          $VisualizeNode\ (C_{new})$
          $C_{new}.Preds \Leftarrow\ DirectPreds(Y, C_{new})$
          If $F'_{ext} \neq\ A$ ***then***
            FuzzyIceberg $((F_{ext}, F_{ext}\prime), i+1)$
          End If
        End If
      End If
    End If
  End For
End Procedure

Algorithm 1 utilizes FuzzySupport (i) to represent the fuzzy support of each linguistic label (i). This algorithm is inspired by the principles of the CbO-type algorithms [29], focusing on extracting frequent and infrequent fuzzy concepts. In contrast, the FuzzyIceberg algorithm builds a fuzzy-crisp iceberg lattice where nodes correspond to frequent closed itemsets (FCIs) with crisp intents. The lattice's edges represent the sub-concept relationships between FCIs, ensuring each node's fuzzy support meets the minimum support.

The initial invocation of the FuzzyIceberg algorithm is FuzzyIceberg $((O, O'), 1)$, where the initial concept is $(O,\ O\prime)$, such that O comprises all objects with a membership degree of one. Equation (6) specifies $O\prime$ as the crisp set of attributes that are shared by all instances. In addition, a is initialized by 1, enabling the gradual processing of attributes numbered from 1 to the overall count of linguistic labels $|A|$. Line 12 is regarded as the canonical test for identifying novel fuzzy concepts, mathematically validated in CBO-based algorithms [23]. In addition, line 16 involves investigating the antecedents of the recently revealed concept based on Definition 7.

To further illustrate this, Figure 5 offers a visual representation of the iceberg lattice created from the fuzzy context in Table 5 with a 25% minimum support. It is evident that each node encompasses the frequent closed itemset (FCI) along with its corresponding fuzzy support. Once this lattice has been generated, the process of generating associations and implications commences.

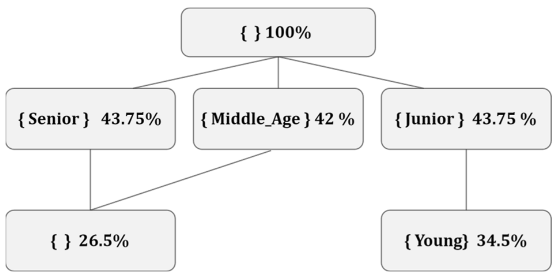

Figure 5. Fuzzy-based iceberg lattice generated from fuzzy context in Table 5 with 25% minimum support.

4.3. Association and Implication Bases Generation Phase

This stage aims to produce a condensed collection of significant association and implication rules that are not derivable from other rules. The implications generated are the association rules with 100% confidence, whereas the remaining rules with less than 100% confidence are referred to as approximate rules or associations [30]. To obtain both categories of rules, we have proposed Algorithm 2 ($FuzzyMinMax$ algorithm), which employs the crisp min-max basis approach [31] adapted for our fuzzy case.

Algorithm 2: $FuzzyMinMax$(FFCIs, MinGen, minConf)

For each $fci \in FFCIs$ **do**
  For each $M_g \in MinGen$ **do**
    If $fci \neq M_g$ **then**
      $ImpBasis = ImpBasis \cup (M_g \Rightarrow fci \setminus M_g),\ Supp = fci.Supp,\ Conf = 100\%$
    End If
  End For
  For each $Pred \in fci.Preds$ **do**
    For each $M_p \in Pred.MinGen$ **do**
      If $\frac{fci.Supp}{M_p.Supp} \geq minConf$ **then**
      $ApprBasis = ApprBasis \cup (M_p \longrightarrow fci \setminus M_p),\ Supp = fci.Supp,\ Conf = \frac{fci.Supp}{M_p.Supp}$
      End If
    End for
  End for
End For

The proposed $FuzzyMinMax$ algorithm process a comprehensive set of all the fuzzy frequent closed itemsets (FFCIs) that have been identified using the proposed $FuzzyIceberg$ algorithm during the iceberg lattice construction stage and the corresponding minimal generators M_g for each FCI. The initial phase of the $FuzzyMinMax$ algorithm (lines 1–6) involves the extraction of the implication basis $ImpBasis$ through an iterative process over the fuzzy frequent closed itemsets fci (line 2), during which it is ensured that fci is distinct from its minimal generator M_g (line 3). The resulting implication rule is then appended to the $ImpBasis$ on line 4.

The procedure of acquiring an approximate basis denoted as $ApprBasis$ necessitates the retrieval of the FFCIs' predecessors (line 7) and their corresponding minimal generators denoted by M_p (line 8). An illustration of this process can be observed in Table 6, which displays the non-redundant rule bases produced from the fuzzy iceberg lattice shown in Figure 4. Rule 1 serves as the implication basis with a confidence of 100%, with the remaining rules representing the approximate AR basis, whose confidence is below 100%.

Table 6. Non-redundant concise AR bases generated from fuzzy iceberg lattice in Figure 4.

Rule Number	Rule Bases	Support	Confidence
1	$Young \Rightarrow Junior$	34.50%	100%
2	$Junior \rightarrow Young$	34.50%	78.86%
3	$MiddleAge \rightarrow Senior$	26.50%	63.1%
4	$Senior \rightarrow MiddleAge$	26.50%	60.57%

5. Results and Discussion

All algorithms in the proposed approach are implemented in Java programming language and carried out on an Intel Core i5 2.30 GHz machine with 8 GB of RAM under Windows 10. We conducted experiments on crisp, quantitative, and fuzzy datasets to show the proposed approach's ability to generalize over different data types. The fuzzy set theory is an extension of classical set theory. Therefore, the proposed approach can handle crisp and fuzzy data with no fuzzification step. Nevertheless, the fuzzification step is only necessary to handle quantitative datasets.

The conducted experiments evaluate the proposed algorithms in comparison with recent related works, focusing on multiple aspects, including execution time, memory usage, the number of generated fuzzy concepts, and the quantity of extracted association rules. Table 7 provides a summary of the datasets used in the experiments. The benchmark datasets—Abalone, Iris, and Mushroom—were obtained from the UCI Repository. Additionally, we generated synthetic fuzzy datasets with a non-zero density percentage of 20%, possessing characteristics like those in [11].

Table 7. Datasets used for experiments.

Datasets	Data Type	∣O∣	∣A∣
Abalone	Quantitative/nominal	4177	19 after fuzzifying quantitative attributes with 2 linguistic labels.
Iris	Quantitative/nominal	150	15 after fuzzifying quantitative attributes with 3 linguistic labels.
Synthetic fuzzy (RandomFuzzy)	Fuzzy	20	200, 250, 300, 350, and 450
Mushroom	Nominal/crisp	8124	120

The fuzzy-mapped datasets (Abalone, Iris, and the synthetic fuzzy datasets) were used to assess the proposed iceberg lattice's performance in terms of processing time and storage efficiency relative to the fuzzy concept lattice. In contrast, the Mushroom dataset was used to compare the number of generated fuzzy concepts produced by the proposed approach against those from the Mguiris et al. approach, which employs the crisp-fuzzy paradigm.

The proposed approach utilizes the fuzzy-crisp FFCA paradigm, which combines similar fuzzy concepts while preserving their respective membership degrees. This merging process is crucial in significantly reducing the overall number of fuzzy concepts, yielding notable benefits in both time and storage requirements. To illustrate this advantageous characteristic, Figure 6 provides a visual representation of the fuzzy concept counts generated by the proposed approach in comparison to Mguiris et al. approach [12]. This experiment is conducted over the Mushroom dataset under various minimum support thresholds.

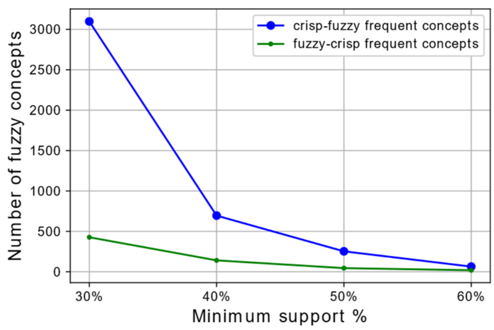

Figure 6. Number of fuzzy concepts generated by the proposed approach versus fuzzy concepts generated by [12,22] approaches over the Mushroom dataset.

The graph demonstrates that, when compared to the related approaches, the proposed approach generates an exceedingly lower count of fuzzy concepts. The resultant reduction in the number of fuzzy concepts enables a swifter generation of fuzzy association rules, thus enhancing the efficiency of the overall process.

Generating a vast number of fuzzy association rules poses a challenge regarding rule interpretation and practical usability. When faced with many rules, it becomes difficult for analysts and decision-makers to comprehend and extract meaningful insights from the rule set effectively. This problem is commonly referred to as the “rule explosion” or “rule clutter” issue [32,33].

We have experimented with the proposed approach versus the Mguiris approach [12] over the fuzzified abalone dataset. This experiment showed how the proposed approach generated a smaller number of association rules. For instance, with a minimum support of 80%, the proposed approach generated 82 association rules versus 903 rules generated by the Mguiris approach [12]. The significant reduction is evident because each item’s membership is not shown in the association rule (intent) but embedded in the fuzzy support. In addition, the proposed approach manipulates the quantitative attribute as a linguistic variable defined using a set of linguistic labels which leads to generating human-like rules and considering the actual membership of the object in the item via fuzzy support.

The work of Zou et al. in [11] represents a fuzzy-crisp FFCA approach for mining and updating ARs that incrementally builds the entire fuzzy concept lattice. Figure 7 shows the results of comparing the work in [11] with the proposed approach over synthetic fuzzy datasets, with a non-zero value percentage of 20%. The comparison result highlighted a significant reduction in the processing time achieved by the proposed approach. This reduction occurs due to the proposed fuzzy iceberg lattice rather than the entire fuzzy concept lattice. Additionally, Figures 8 and 9 depict how much less time (in *ms*) and memory consumption (in MB) it takes to make the fuzzy iceberg lattice than the entire fuzzy concept lattice over different types of datasets.

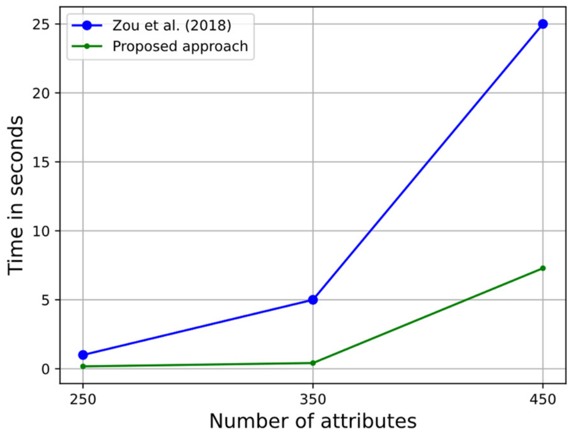

Figure 7. Processing time comparison between the proposed approach versus the Zou et al. (2018) [11] over the fuzzy synthetic datasets.

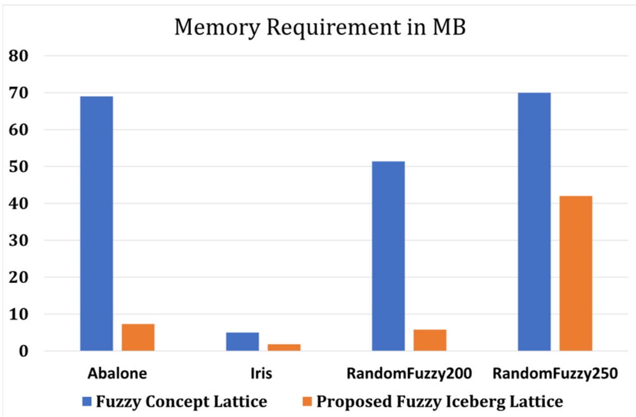

Figure 8. Memory consumption of constructing the entire fuzzy concept lattice vs. constructing the proposed fuzzy iceberg lattice.

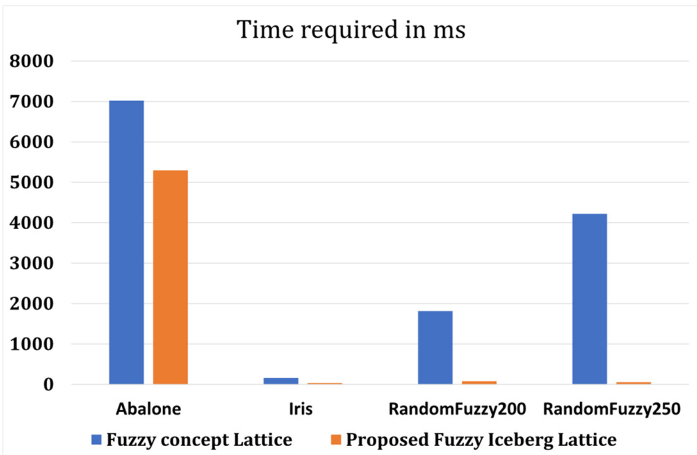

Figure 9. Time required to construct the entire fuzzy concept lattice vs. constructing the proposed fuzzy iceberg lattice.

Figure 10 illustrates a comparison between the total number of fuzzy concepts in the full fuzzy concept lattice and the reduced number of concepts in the fuzzy iceberg lattice (FFCI) for various datasets—Abalone, Iris, RandomFuzzy200, and RandomFuzzy250. The fuzzy iceberg lattice, which includes only the strong fuzzy concepts meeting a specified minimum support, has a notable reduction in the extracted concept count. This reduction, quantified by the reduction ratio, ranges from 45.18% for Abalone to over 90% for the Iris and synthetic datasets (RandomFuzzy200 and RandomFuzzy250). These results highlight the efficiency of the fuzzy iceberg lattice in simplifying the concept structure, significantly lowering the computational and memory requirements while retaining essential information for the association rule mining.

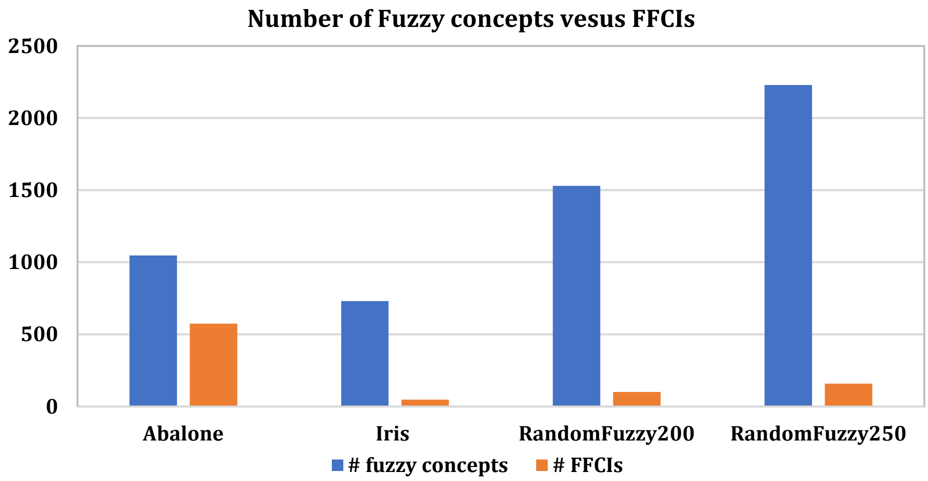

Figure 10. Comparison of concept counts in full fuzzy and iceberg lattices across various datasets.

6. Conclusions

This study addresses key limitations in traditional association rule mining (ARM) techniques, particularly the challenges of handling heterogeneous data and minimizing redundant rule generation. By introducing a novel fuzzy-crisp iceberg lattice structure based on Fuzzy Formal Concept Analysis (FFCA), the approach presented here successfully handles diverse data types, including binary, uncertain, and quantitative data. The fuzzy-crisp iceberg lattice algorithm effectively extracts frequent closed itemsets (FCIs) with their fuzzy supports while representing parent–child relationships among FCIs, providing a clear and efficient visualization of associations.

Our experimental results on benchmark datasets confirm that the proposed approach significantly reduces computational time and memory usage, achieving an average reduction of 74.75% in execution time and 70.53% in memory usage compared to traditional methods. The fuzzy min-max basis algorithm further refines the rule set by generating a concise, non-redundant collection of association and implication rules, which is crucial for improving both interpretability and practical application in ARM. These findings underscore the potential of the fuzzy-crisp iceberg lattice to enhance ARM by yielding a compact, high-quality set of fuzzy concepts and rules.

However, despite these advancements, the proposed approach has limitations. The current method does not support incremental updates, which restricts its performance when applied to dynamic datasets that evolve over time. Future research could explore the incremental construction of the fuzzy-crisp iceberg lattice, potentially improving processing efficiency and adaptability in real-time data environments. This enhancement would make the approach even more suitable for large-scale, continuously updated datasets, further advancing the practical applicability of ARM in diverse, data-rich domains.

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Data Availability Statement: The benchmark datasets used in the experiments are publicly available from the UCI Machine Learning Repository at https://archive.ics.uci.edu.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

Abbreviations

The following table provides a list of the abbreviations used throughout the paper as a quick reference.

Abbreviation	Full Term
FCA	Formal Concept Analysis
FFCA	Fuzzy Formal Concept Analysis
ARM	Association Rule Mining
ARs	Association Rules
CIs	Closed Itemsets
MVC	Multi-Valued Context
FCIs	Frequent Closed Itemsets
FFMGs	Frequent Fuzzy Minimal Generators
FFCIs	Fuzzy Frequent Closed Itemsets

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