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Supply Chain Analysis Based on Community Detection of Multi-Layer Weighted Networks

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Abstract: As the economic environment becomes more complex, improving supply chain resilience is critical for the effective operation and long-term sustainability of businesses. Real-world supply chains, which consist of various components such as goods, warehouses, and plants, often feature intricate network structures that pose challenges for resilience analysis. This paper addresses these challenges by proposing a framework for studying supply chains using multi-layer network community detection. The complex multi-mode supply chain network is transformed into single-mode, multi-layer weighted networks. A multi-layer weighted community detection method is proposed for identifying local clusters within these networks, revealing interconnected groups that highlight flexibility and redundancy in production capabilities across different plants and goods. An empirical study utilizing real data demonstrates that this clustering method effectively detects indirect capacity links between plants and goods. The insights derived from this method are useful for strategic capacity management, allowing businesses to respond more effectively to supply shortages and unexpected increases in demand.

Keywords: resilience analysis; supply chain; community detection

MSC: 90B06; 05C90; 05C22

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1. Introduction

As the socioeconomic environment becomes more complex, supply chains face a broader range of risks, such as natural disasters, political unrest, fuel shortages, pandemics, and terrorism [1–5]. A supply chain can be defined as a network of interconnected facilities that transport goods or services from producers to consumers [6]. These chains function as intricate systems involving plants, warehouses, and distribution channels, making efficient and sustainable management of these flows critical for businesses [7]. With rising economic instability, the likelihood of external shocks disrupting supply chains grows. Unexpected events, such as the COVID-19 pandemic, have revealed vulnerabilities in these networks, where disruptions at key nodes can significantly impair production and logistics, resulting in substantial economic losses, including declines in operating income and stock market performance [8]. For instance, the pandemic highlighted shortages of critical medical supplies, demonstrating how disruptions can spread throughout entire systems.

The importance of analyzing supply chain resilience is becoming more widely recognized. Resilience is defined as the ability to quickly restore supply chains to normal operations or increase efficiency following disruptions [9]. Ref. [10] emphasized that a company's ability to respond to disruptions more effectively than competitors can strengthen its market position, making resilience critical for both short-term survival and long-term success. Similarly, ref. [11] argued that resilience is an important strategy for supply chains operating in volatile environments. Ref. [12] encapsulated the essence of resilience from two different perspectives. The first is engineering resilience, which is measured by how quickly a system returns to equilibrium and how robust it is to disruptions. The second concept is social–ecological resilience, which is measured by the amount of disruption that a supply chain can endure before undergoing structural transformation. Ref. [13] viewed supply chain resilience through two perspectives: stability and adaptation. The stability-based view focuses on returning to normal states amid known risks, while the adaptation-based view emphasizes sustained value creation through flexibility and redundancy in the face of uncertainty. Scholars have also proposed extending risk management practices to include supply chains [14,15]. From a risk management perspective, resilience analysis entails identifying, evaluating, controlling, and monitoring risks that could destabilize supply chains.

However, supply chains are complex, dynamic networks of interconnected entities, resulting in significant distinctions between managing organizational risk and assessing supply chain resilience. Ref. [16] argued that resilience analysis should focus on the systemic properties of supply chains rather than isolated risk factors. Ref. [17] investigated the relationship between network complexity and supply chain resilience, specifically how disruptions affect them and how well they recover from such incidents. Their empirical studies revealed that network complexity is an important factor in determining supply chain recovery capabilities. Ref. [18] also emphasized the importance of the network perspective in analyzing supply chain resilience, arguing that tailored resilience strategies should be considered for different types of collaboration within supply networks. Inspired by these works, this paper investigates the network structure of supply chains and evaluates their characteristics using network analysis. By integrating quantitative data, such as production capacities, into the analysis, we aim to enhance visibility into supply chain flexibility and redundancy, resulting in greater resilience and the ability to withstand, adapt, or transform in response to disruptions.

In terms of network analysis, increasing redundancy is often employed to improve supply chain resilience. Companies can better address supply shortages and demand fluctuations by incorporating backup facilities or maintaining surplus inventory through strategic resource reallocation [19]. Ensuring spare capacity at critical points in the supply chain is critical for maintaining resilience and providing backup plans during disruptions [20]. Redundancy also increases operational flexibility, allowing for faster resource deployment and minimizing delays [21,22]. However, determining which segments of the supply chain should be duplicated and where capacity expansion is most effective remains a difficult task.

To address the complexity of supply chain networks, we represent a supply chain with diverse types of nodes as a multi-mode network. Here, "mode" refers to the category of nodes. For example, a supply chain that includes plants, goods, and warehouses can be modeled as a three-mode network, where each mode represents plants, goods, and warehouses. By modeling a real supply chain as a multi-mode network, we can treat it as a graph and apply network analysis techniques, such as community detection, which is particularly useful for identifying tightly connected groups within networks. Figure 1 illustrates the structure of a multi-mode network. This representation effectively captures the flow of production and distribution in the supply chain, with the nodes representing goods playing a central role. In the context of supply chain management, especially in relation to disruptions, we focus on the flexibility and redundancy in production capabilities for different goods. To explore the relationships among goods, we extract single-mode networks based on whether two goods share the same warehouse or can be produced by the same plant. As shown in the right panel of Figure 1, this results in multiple single-mode networks of goods, such as plant-based and warehouse-based networks. We then aggregate these single-mode networks into a multi-layer network, with a plant layer and a warehouse layer. This multi-layer network allows for a comprehensive analysis of the flexibility and redundancy among goods. Furthermore, we extend the analysis to weighted networks by incorporating quantitative information into the network structure, such as production capacity and resource availability. This integration enables a deeper understanding of the relationships between nodes, leading to more robust community detection results. By analyzing the distribution of nodes within these communities and identifying key bridging nodes, we can develop targeted redundancy strategies that not only enhance operational flexibility but also strengthen the overall resilience of the supply chain.

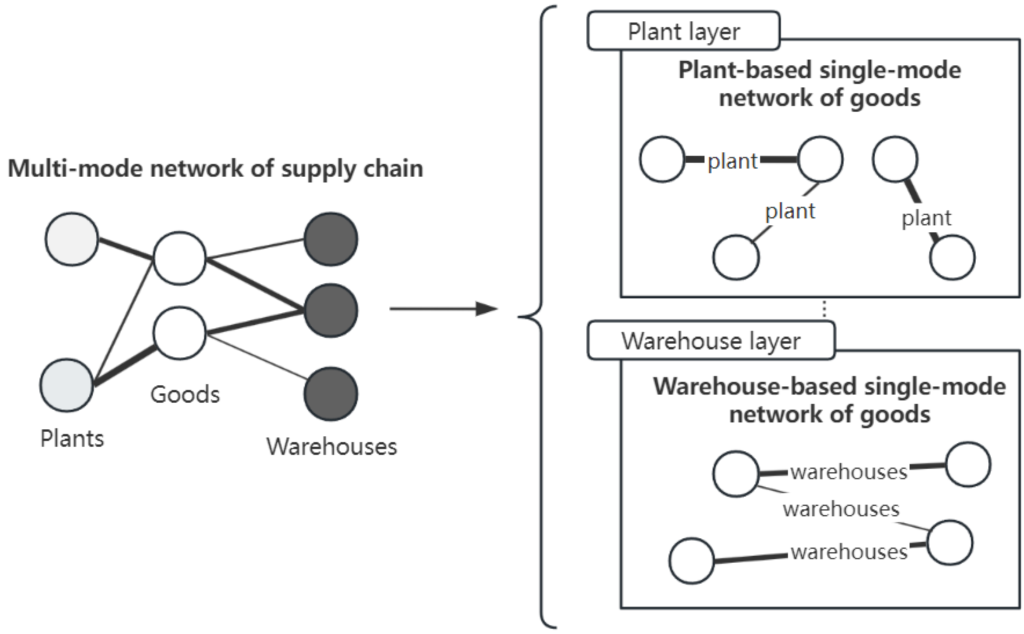

Figure 1. Multi-mode network of supply chain (**left**) and projected versions of single-mode networks (**right**).

In multi-layer networks, such as supply chains, community detection poses distinct challenges compared to single-layer networks [23]. The primary objective is to discover a unified community structure across multiple layers, revealing clusters of highly interconnected nodes. While methods originally developed for single-layer networks, such as modularity-based approaches, have been adapted to address the complexities of multi-layer networks, there is still a significant gap in the existing literature regarding the incorporation of quantitative information, such as production capacity and inventory levels, into these analyses. For instance, the modularity metric has been transformed into multi-layer modularity density, which aids in community detection across multiple layers [24]. Ref. [25] tried to combine connections within and between layers to find communities in multi-layer networks and showed that the aggregation of information from multiple layers contributes to the detection performance. Additionally, ensemble methods [26] combine independently detected communities from each layer to form a cohesive partition, and spectral clustering techniques have been improved to ensure consistent detection across layers [27]. Most existing research focuses on community detection using binary layers, such as binary adjacency matrices. However, integrating quantitative data, such as production capabilities, into these weighted multi-layer networks is critical for improving detection performance. By incorporating this information, researchers can achieve more accurate and robust community detection results.

In the face of increasing economic complexity and volatility, the resilience of supply chains has become a critical area of focus for both academic researchers and industry practitioners. Despite the growing body of literature on supply chain resilience, there remains a significant gap in understanding how to effectively leverage network structures to enhance resilience, particularly in the context of multi-layer weighted networks that capture the intricate relationships between various supply chain entities. In this study, we address this research gap by proposing a novel framework for supply chain analysis that utilizes community detection methods on multi-layer weighted networks. Our approach aims to identify local clusters within these networks, revealing interconnected groups that highlight flexibility and redundancy in production capabilities across different plants and goods. The supply chain is modeled as a weighted multi-mode network composed of various node types, allowing for a thorough examination of the underlying community structures. To achieve this, the multi-mode network of the supply chain is projected into several multi-layer networks that correspond to different modes, such as a multi-layer weighted network of production facilities. Using community detection techniques on these multi-layer weighted networks, critical local structures can be identified, improving supply chain planning and risk management. A thorough examination of multi-layer networks reveals global structures that may be overlooked when individual layers are examined separately. Identifying common patterns across layers significantly improves the accuracy of inference results. For local communities, the distribution and diversity of each node type are evaluated, with a focus on redundancy and flexibility across various supply chain segments. This framework is adaptable to a wide range of supply chain scenarios. Furthermore, by modeling the complex relationships between various supply chain entities using a multi-layer network structure, we use our proposed community detection methods to effectively integrate quantitative data, such as production capacity, with network characteristics. This comprehensive approach helps enterprises to recognize the importance of various node types in supply chain design and management, ultimately strengthening resilience and improving risk mitigation capabilities. The main contributions of this paper are threefold: (1) we model complex supply chains with a multi-modal network structure and thus propose the weighted multi-layer network community detection (WMLCD) method for analyzing the inherent structure of supply chains; (2) based on WMLCD, we present a resilience analysis framework that uncovers production capacity linkages among network nodes, providing valuable references for supply chain maintenance; and (3) we demonstrate the effectiveness of this method through both simulated and real-world supply chain data, illustrating how community detection results offer specific decision support for supply chain recovery. The insights from WMLCD help supply chain managers to make informed decisions. Tested with real-world data, WMLCD uncovers indirect connections that single-layer analyses may miss, enhancing strategies for capacity management, resource allocation, and overall resilience in global supply chains.

2. Methodology

2.1. Multi-Mode Weighted Network of Supply Chain

Suppose that a supply chain has N nodes, each indexed by $1 \leq i \leq N$. These nodes represent M different types, such as goods, plants, and warehouses. Consequently, the supply chain can be modeled as a multi-mode network with M modes, each corresponding to a specific type. For $1 \leq m \leq M$, let $\mathcal{S}_m$ represent the index set of nodes belonging to the m-th mode. Let $N_m = |\mathcal{S}_m|$ denote the cardinality of $\mathcal{S}_m$. Thus, we have $N = \sum_{m=1}^{M} N_m$. To describe the overall network structure, we define an $N \times N$ symmetric weighted adjacency matrix $A = (A_{ij})$, where A_{ij} represents the weight of the edge between nodes i and j. If there is no relationship between nodes i and j, then $A_{ij} = 0$. For nodes i and j with $A_{ij} > 0$, the weight A_{ij} quantifies the strength of their relationship. For instance, if nodes i and j refer to a good and a plant, respectively, the weight A_{ij} can be determined based on the production capacity of the plant to produce the good. If node j represents a warehouse, A_{ij} can be calculated using the percentage of storage allocated for good i in warehouse j. It is worth noting that there are no direct edges connecting nodes of the same mode; thus, we have $A_{ij} = 0$ for any i and j when there exists m such that $i, j \in \mathcal{S}_m$. Nevertheless, indirect relationships among nodes of the same mode can be inferred from the matrix A. For example, given two nodes representing goods i and j, if there is a plant node k such that

$A_{ik}A_{jk} > 0$, this indicates that these two goods can be produced by the same plant. In this way, the relationship between the two goods can be captured in terms of the substitutability of their production capacities.

As a representation of the supply chain, the weighted adjacency matrix A effectively captures the production and distribution dynamics of the supply chain. Typically, good nodes play pivotal roles in this network. The matrix A can be represented as a combination of block matrices as follows:

$$A = \begin{bmatrix} \mathbf{0} & A^{(12)} & \cdots & A^{(1M)} \\ A^{(12)} & \mathbf{0} & \cdots & A^{(2M)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(1M)} & A^{(2M)} & \cdots & \mathbf{0} \end{bmatrix},$$

where $\mathbf{0}$ denotes a zero matrix and $A^{(ml)}$ is an $N_m \times N_l$ matrix describing the relations among nodes from $\mathcal{S}_m$ and $\mathcal{S}_l$. For each mode m, the related network information is all collected in $M-1$ block matrices $\{A^{(1m)}, \cdots, A^{(mM)}\}$. It should be noted that the objective is to recover indirect relations among relationships between nodes of the same nodes. Therefore, we consider to derive single-mode networks for each mode m through projection on $\{A^{(1m)}, \cdots, A^{(mM)}\}$. For instance, a single-mode network for goods can be generated, with edges connecting any two goods that can be produced by the same plant or stored in the same warehouse.

2.2. Projected Weighted Multi-Layer Network

Direct relationships between nodes of the same type may not always exist. For example, two plants in the supply chain may not have a direct connection, such as through business cooperation. However, if these plants produce the same products, they may be implicitly related. In such cases, if one plant experiences operational difficulties, the other can continue production, preventing supply chain disruptions. To investigate these implicit relationships and increase supply chain redundancy, we project the original network into different configurations, such as single-mode networks of goods. This method enables us to identify community structures within the supply chain by analyzing projected single-mode networks.

For each mode m, given the block matrices $\{A^{(1m)}, \cdots, A^{(mM)}\}$, we define a projection operator $f(\cdot)$ to transform them into adjacency matrices of single-mode networks. Let $W^{(ml)} = f(A^{(ml)})$ denote the resulting matrix of the projection operator on $A^{(ml)}$. For $i, j \in \mathcal{S}_m$, we have

$$W_{ij}^{(ml)} = \sum_{k \in \mathcal{S}_l} \bar{A}_{ik}^{(ml)} \bar{A}_{jk}^{(ml)}, \tag{1}$$

where $\bar{A}_{ik}^{(ml)} = A_{ik}^{(ml)} / \sum_{i' \in \mathcal{S}_m} A_{i'k}^{(ml)}$ is a normalized version of $A_{ik}^{(ml)}$. Intuitively, the weight in (1) measures the similarity between nodes i and j according to the information from nodes of the mode l. For instance, if the mode l refers to plants, the similarity indicates how many production capacities are shared by the two nodes i and j. If the mode l refers to warehouses, $W_{ij}^{(ml)}$ will reflect the similarities in terms of storage capabilities. A higher value of $W_{ij}^{(ml)}$ indicates that i and j have more implicit relationships in the supply chain. Therefore, once the production or storage capacities of i are destructed, the manager of the supply chain can employ capacities related to j to help to restore the supply chain about i.

In this way, for mode m, a multi-later network with $M-1$ layers can be obtained, which is denoted as $\mathcal{W}^{(m)} = \{W^{(m1)}, \cdots, W^{(mM)}\}$, each of which is a similarity matrix generated using the projection operator $f(\cdot)$. Then, for each type of node, the objective is to perform community detection on the projected multi-layer network.

2.3. Community Detection of Projected Multi-Layer Network

Assume that, for each mode m, $1 \leq m \leq M$, there are K underlying communities. The objective of community detection is to divide N_m objects into K non-overlapping communities according to $\mathcal{W}^{(m)} = \{W^{(m1)}, \cdots, W^{(mM)}\}$. To ensure consistency in the underlying community structure across different layers, we make the following assumption.

Assumption 1. *(Structural Consistency) Layers $W^{(m1)}, \cdots, W^{(mM)}$ share the same community structure, i.e., K underlying communities. These communities are represented by K index sets $\Psi_1^{(m)}, \cdots, \Psi_K^{(m)}$. For $1 \leq k \leq K$, the set $\Psi_k^{(m)}$ contains indices of nodes belonging to the k-th community.*

To denote the latent community labels for nodes in $\mathcal{S}_m$, we define an $N_m \times K$ membership matrix as $Z^{(m)} = (z_{ik}^{(m)})$, where $z_{ik}^{(m)} = 1$ if $i \in \Psi_k^{(m)}$, and $z_{ik}^{(m)} = 0$ otherwise. Based on Assumption 1, in order to perform community detection, it suffices to estimate $Z^{(m)}$ via aggregating information from layers $W^{(m1)}, \cdots, W^{(mM)}$.

To model the latent community structure of a multi-layer network, we use a model setting similar to the stochastic block model (SBM), which has been widely used in theoretical community detection research. Specifically, the similarity between any two nodes is modeled via the following assumption.

Assumption 2. *(Node Similarity) Given the mode m and the layer l, for any two nodes i and j with $Z_{ik_1}^{(m)} = Z_{jk_2}^{(m)} = 1$, we assume that $W_{ij}^{(ml)}$, where the similarity between these two nodes in layer l can be denoted by*

$$W_{ij}^{(ml)} = b_{k_1k_2}^{(ml)} + \epsilon_{ij}^{(l)}, \tag{2}$$

where $b_{k_1k_2}^{(ml)}$ is the group similarity between the latent communities k_1 and k_2, and $\epsilon_{ij}^{(ml)}$ is a random noise with a mean of 0 and finite variance. For the mode m and the layer l, the true parameters of group similarities compose a $K \times K$ symmetric matrix

$$B^{(ml)} = \begin{bmatrix} b_{11}^{(ml)} & b_{12}^{(ml)} & \cdots & b_{1K}^{(ml)} \\ b_{12}^{(ml)} & b_{22}^{(ml)} & \cdots & b_{2K}^{(ml)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1K}^{(ml)} & b_{2K}^{(ml)} & \cdots & b_{KK}^{(ml)} \end{bmatrix}.$$

According to Assumption 2, the similarity between any two nodes is primarily determined by their corresponding latent communities. Specifically, for two latent communities k_1 and k_2, the expectations of similarities between any two nodes belonging to the k_1-th and k_2-th communities are identical. The matrix $B^{(ml)} = (b_{k_1k_2}^{(ml)})$ represents the true parameter matrix of similarity among the K communities in layer l. Moreover, we define an $N_m \times N_m$ matrix $\Sigma^{(ml)} = (\epsilon_{ij}^{(ml)})$ to collect all the noise terms. Then, using (2), the similarity matrix $W^{(ml)}$ can be decomposed as follows:

$$W^{(ml)} = Z^{(m)} B^{(ml)} \left\{ Z^{(m)} \right\}^\top * \Sigma^{(ml)}, \tag{3}$$

where $*$ denotes the Hadamard product. Based on (3), $Z^{(m)}$ can be estimated according to the observed similarity matrix $W^{(ml)}$. It is worth noting that all of the matrices $\{W^{(m1)}, \cdots, W^{(mM)}\}$ can contribute to the estimation of $Z^{(m)}$. Therefore, we aggregate information from $M - 1$ similarity matrices to obtain a comprehensive estimator. To better explain our framework, the assumptions related to the projected multi-layer network are shown in the upper panel of Figure 2. In addition, a toy example is provided in the lower panel of Figure 2 using the element $W_{ij}^{(ml)}$ for illustration. In practice, both matrices $Z^{(m)}$ and $B^{(ml)}$ are unobserved in practice and therefore need to be estimated.

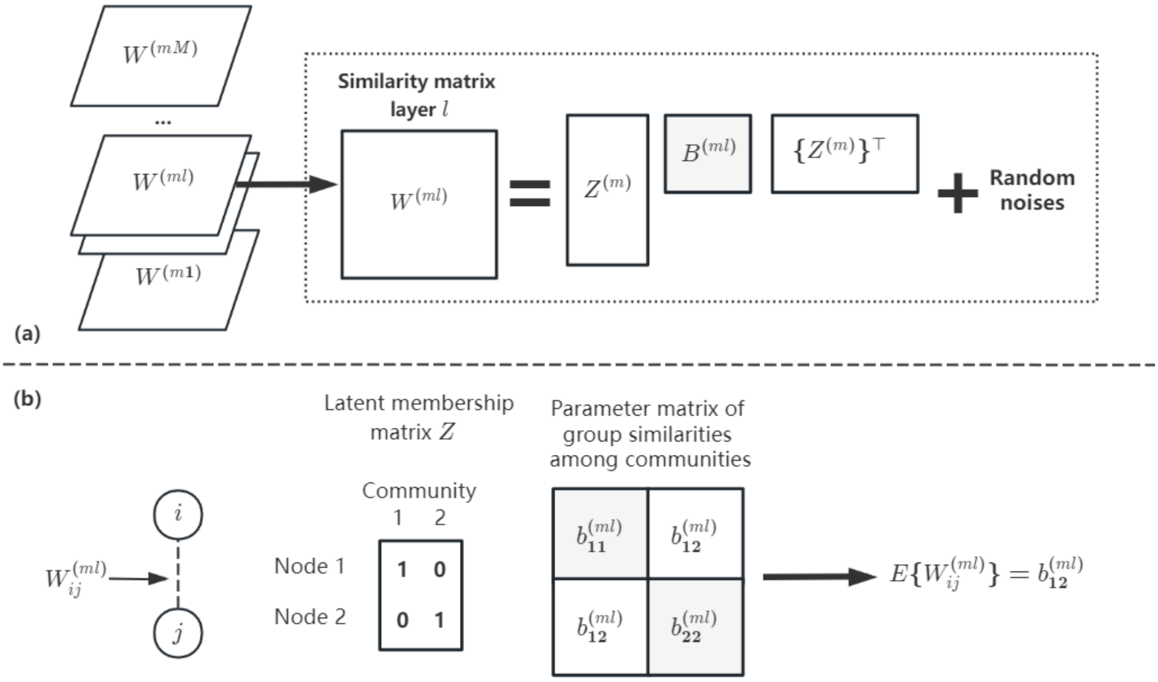

Figure 2. (**a**) The theoretical framework for the projected multi-layer network. (**b**) A toy example of the assumptions related to the element $W_{ij}^{(ml)}$.

In terms of community detection in multi-layer networks, our algorithmic principle is first defining an objective function that measures the quality of community partition and then optimizing this objective function. Ref. [27] showed that the least-squares estimator is effective in identifying cohesive community structures across multiple layers with robust performance. Inspired by [27], we estimate $Z^{(m)}$ with a least-squares estimator that minimizes the residual sum of squares as follows:

$$\widehat{Z}^{(m)} = \arg\min_{Z} \sum_{l=1, l\neq m}^{M} \sum_{i,j\in\mathcal{S}_m} \left\{ W_{ij}^{(ml)} - Z_{i\cdot}\widehat{B}^{(ml)} Z_{j\cdot}^{\top} \right\}^2, \tag{4}$$

where $Z_{i\cdot}$ represents the row vector corresponding to i within Z. The matrix $\widehat{B}^{(ml)}$ is consistent with $B^{(ml)}$, where

$$\begin{aligned} \widehat{B}_{k_1k_2}^{(ml)} &= \frac{\sum_{i,j\in\mathcal{C}_k(Z)} W_{ij}^{(ml)}}{|\mathcal{C}_k(Z)|\{|\mathcal{C}_k(Z)|-1\}} \quad \text{when } k_1 = k_2 = k, \\ \widehat{B}_{k_1k_2}^{(ml)} &= \frac{\sum_{i\in\mathcal{C}_{k_1}(Z), j\in\mathcal{C}_{k_2}(Z)} W_{ij}^{(ml)}}{|\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)|} \quad \text{when } k_1 \neq k_2, \end{aligned} \tag{5}$$

with $\mathcal{C}_k(Z) = \{i : Z_{ik} = 1\}$ and $|\cdot|$ denoting the number of elements within a set. The elements in (5) are used to estimate the parameter matrix $B^{(ml)}$ based on observations. Given a temporal membership matrix Z and $1 \le k_1 \ne k_2 \le K$, $\widehat{B}_{k_1k_2}^{(ml)}$ represents the sample mean of the similarity between communities k_1 and k_2. For $1 \le k \le K$, $\widehat{B}_{kk}^{(ml)}$ represents the sample mean of the similarity within the community k. If we have $Z_{ik_1} = Z_{jk_2} = 1$, the element $Z_{i\cdot}\widehat{B}^{(ml)}Z_{j\cdot}^{\top}$ in (4) is exactly the estimation of $b_{k_1k_2}^{(ml)}$ in (2), which is the group similarity between the latent communities k_1 and k_2.

By minimizing the least-squares loss function, the sample means of similarities tend to approximate the true parameters in $\{B^{(1m)}, \cdots, B^{(mM)}\}$. As a result, nodes belonging to the same underlying community are expected to be assigned to the same communities. To solve the minimization problem in (4), an approximately equivalent problem is considered, which is illustrated in Theorem 1.

Theorem 1. *Based on Assumption 1 and Assumption 2, for $1 \leq l \leq K$, assume that $|\mathcal{C}_k(Z)|(|\mathcal{C}_k(Z)| - 1) \approx |\mathcal{C}_k(Z)|^2$ for $1 \leq k \leq K$. Then, the minimization problem in (4) is equivalent to a maximization problem as follows:*

$$\widehat{Z}^{(m)} = \arg\max_{Z} \sum_{l=1, l\neq m}^{M} \sum_{1 \leq k_1, k_2 \leq K} |\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)| \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2. \tag{6}$$

Proof. For any node $i \in \mathcal{S}_m$, let $\psi_i(Z)$ denote the community label of i given a temporal membership matrix Z. We have $\psi_i(Z) = k$ if $Z_{ik} = 1$. Then, to prove this theorem, the objective function in (4) is decomposed into $M - 1$ components: $\{\sum_{i,j \in \mathcal{S}_m} \{W^{(ml)}_{ij} - \widehat{B}^{(ml)}_{\psi_i(Z)\psi_j(Z)}\}^2 : l = 1, \cdots, M, l \neq m\}$. For the l-th component, we have

$$\sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} - \widehat{B}^{(ml)}_{\psi_i(Z)\psi_j(Z)} \right\}^2 = \sum_{i,j \in \mathcal{S}_m} \left[\left\{ W^{(ml)}_{ij} \right\}^2 - 2 W^{(ml)}_{ij} \widehat{B}^{(ml)}_{\psi_i(Z)\psi_j(Z)} + \left\{ \widehat{B}^{(ml)}_{\psi_i(Z)\psi_j(Z)} \right\}^2 \right]$$

$$= \sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} \right\}^2 + \sum_{1 \leq k_1, k_2 \leq K} \sum_{\substack{i \in \mathcal{C}_{k_1}(Z) \\ j \in \mathcal{C}_{k_2}(Z)}} \left[-2 W^{(ml)}_{ij} \widehat{B}^{(ml)}_{k_1 k_2} + \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2 \right]$$

$$= \sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} \right\}^2 + \sum_{1 \leq k_1, k_2 \leq K} \sum_{\substack{i \in \mathcal{C}_{k_1}(Z) \\ j \in \mathcal{C}_{k_2}(Z)}} \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2 - 2 \sum_{1 \leq k_1, k_2 \leq K} \widehat{B}^{(ml)}_{k_1 k_2} \sum_{\substack{i \in \mathcal{C}_{k_1}(Z) \\ j \in \mathcal{C}_{k_2}(Z)}} W^{(ml)}_{ij}$$

$$= \sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} \right\}^2 - \sum_{1 \leq k_1 \neq k_2 \leq K} |\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)| \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2$$

$$- \sum_{1 \leq k \leq K} |\mathcal{C}_k(Z)|(|\mathcal{C}_k(Z)| - 1) \left\{ \widehat{B}^{(ml)}_{kk} \right\}^2.$$

To simplify the analysis, we adopt the approximation that $|\mathcal{C}_k(Z)|(|\mathcal{C}_k(Z)| - 1) \approx |\mathcal{C}_k(Z)|^2$ for $1 \leq k \leq K$. This approximation is simple to implement in practice, especially when the number of nodes is large enough. Then, we have

$$\sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} - \widehat{B}^{(ml)}_{\psi_i(Z)\psi_j(Z)} \right\}^2 \approx \sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} \right\}^2 - \sum_{1 \leq k_1, k_2 \leq K} |\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)| \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2.$$

Since this conclusion holds for $l = 1, \cdots, M$, we derive that the objective function in (4) can be approximated as follows:

$$\sum_{l=1, l\neq m}^{M} \sum_{i,j \in \mathcal{S}_m} \left\{ W^{(ml)}_{ij} \right\}^2 - \sum_{l=1, l\neq m}^{M} \sum_{1 \leq k_1, k_2 \leq K} |\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)| \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2. \tag{7}$$

It should be noted that maximizing the second term in (7) can achieve the same result as minimizing (7). Therefore, (4) is solved through the optimization of an approximately equivalent problem, i.e., the maximization of $\sum_{l=1, l\neq m}^{M} \sum_{1 \leq k_1, k_2 \leq K} |\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)| \left\{ \widehat{B}^{(ml)}_{k_1 k_2} \right\}^2$. □

Then, we investigate the calculation of $\widehat{Z}^{(m)}$ based on (6). Note that, given a temporal membership matrix Z, we have

$$\widehat{B}_{k_1k_2}^{(ml)} = \frac{\sum_{i\in\mathcal{C}_{k_1}(Z),j\in\mathcal{C}_{k_2}(Z)} W_{ij}^{(ml)}}{|\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)|} = \frac{Z_{\cdot k_1}^{\top} W^{(ml)} Z_{\cdot k_2}}{|\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)|}, \tag{8}$$

where $Z_{\cdot k}$ indicates the k-th column vector of Z. Based on (8), (6) can be rewritten as follows:

$$\begin{aligned}\widehat{Z}^{(m)} &= \arg\max_{Z} \sum_{l=1,l\neq m}^{M} \sum_{1\leq k_1,k_2\leq K} \frac{\left\{Z_{\cdot k_1}^{\top} W^{(ml)} Z_{\cdot k_2}\right\}^2}{|\mathcal{C}_{k_1}(Z)||\mathcal{C}_{k_2}(Z)|} \\ &= \arg\max_{Z} \sum_{l=1,l\neq m}^{M} \sum_{1\leq k_1,k_2\leq K} \left\{\widetilde{Z}_{\cdot k_1}^{\top} W^{(ml)} \widetilde{Z}_{\cdot k_2}\right\}^2 = \arg\max_{Z} \sum_{l=1,l\neq m}^{M} \left\|\widetilde{Z}^{\top} W^{(ml)} \widetilde{Z}\right\|_F^2\end{aligned} \tag{9}$$

where $\widetilde{Z}_{\cdot k} = Z_{\cdot k}/\sqrt{|\mathcal{C}_k(Z)|}$ and $\widetilde{Z} = [\widetilde{Z}_{\cdot 1}, \cdots, \widetilde{Z}_{\cdot K}]$ is an $N_m \times K$ matrix. According to (9), we further conclude that the solution of (6) can be obtained using a spectral clustering method, as shown in the following theorem.

Theorem 2. *The maximization problem in (6) is approximately equivalent to a spectral clustering problem as follows:*

$$\widehat{Z}^{(m)} = \arg\max_{Z:Z^{\top}Z=I_K} \mathrm{trace}\left(Z^{\top}\left[\sum_{l=1,l\neq m}^{M}\left\{W^{(ml)}\right\}^2\right]Z\right), \tag{10}$$

where I_K is a $K \times K$ identity matrix.

Proof. Note that, for $1 \leq k_1 \neq k_2 \leq K$, we have $\widetilde{Z}_{\cdot k_1}^{\top}\widetilde{Z}_{\cdot k_2} = 0$. For $1 \leq k \leq K$, $\widetilde{Z}_{\cdot k}^{\top}\widetilde{Z}_{\cdot k} = 0 = 1$. Therefore, $\{\widetilde{Z}_{\cdot 1}, \cdots, \widetilde{Z}_{\cdot K}\}$ is a group of orthogonal vectors and $\widetilde{Z}$ is an orthogonal matrix with $\widetilde{Z}^{\top}\widetilde{Z} = I_K$. For the orthogonal matrix $\widetilde{Z}$ and the symmetric matrices $\{W^{(m1)}, \cdots, W^{(mM)}\}$, we have

$$\begin{aligned}\sum_{l=1,l\neq m}^{M} \left\|\widetilde{Z}^{\top} W^{(ml)} \widetilde{Z}\right\|_F^2 &= \sum_{l=1,l\neq m}^{M} \mathrm{trace}\left(\widetilde{Z}^{\top} W^{(ml)} \widetilde{Z}\widetilde{Z}^{\top} W^{(ml)} \widetilde{Z}\right) \\ &\leq \mathrm{trace}\left(\widetilde{Z}^{\top}\left\{W^{(ml)}\right\}^2\widetilde{Z}\right) = \mathrm{trace}\left(\widetilde{Z}^{\top}\left[\sum_{l=1,l\neq m}^{M}\left\{W^{(ml)}\right\}^2\right]\widetilde{Z}\right)\end{aligned} \tag{11}$$

The inequality in (11) becomes an equality when $W^{(m1)}, \cdots, W^{(mM)}$ share the same community structure and $\widetilde{Z}$ consists of eigenvectors corresponding to the leading K eigenvalues of $\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2$. Therefore, the maximization of the objective function in (9) can be achieved by searching for an estimation $\widehat{Z}^{(m)}$ that maximizes $\mathrm{trace}(\widetilde{Z}^{\top}[\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2]\widetilde{Z})$.

Nevertheless, due to the constraint that all elements of $\widehat{Z}^{(m)}$ are binary, calculating the estimation may be costly in practice. To address this issue, we implement community detection with the restriction relaxed. We initially ignore the restriction on $\widehat{Z}^{(m)}$ and investigate $\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2$ through eigenvalue decomposition. The decomposition leads to $[\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2] = U^{\top}\Sigma U$, where $\Sigma = \mathrm{diag}(\lambda_1, \cdots, \lambda_{N_m})$ is a diagonal matrix with N_m eigenvalues and U is the corresponding $N_m \times N_m$ eigenvector matrix. Then, we take the K largest eigenvalues and corresponding eigenvectors. Let $\lambda_{(1)}, \cdots, \lambda_{(K)}$ denote the K largest eigenvalues. For $\lambda_{(k)}, 1 \leq k \leq K$, let $u_k \in \mathbf{R}^{N_m}$ denote the corresponding eigenvector. Using these eigenvectors, we create an $N_m \times K$ eigenvector matrix. Let U_K denote the eigenvector matrix with $U_K = [u_1^{\top}, \cdots, u_K^{\top}]$. It follows that the estimation $\widehat{Z}^{(m)}$ from (10) equals U_K. Hence, $\widehat{Z}^{(m)}$ can be viewed as a spectral embedding of $\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2$. Thus, we directly compute $\widehat{Z}^{(m)}$ using the spectral decomposition of $\sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2$.

In this way, we obtain an estimation, $\widehat{Z}^{(m)}$, which is a non-binary matrix. To implement community detection for N_m nodes, we can adopt classic clustering algorithms, such as the K-means algorithm, to the rows of $\widehat{Z}^{(m)}$. The resulting partition represents the final estimation for the membership vector, which directly indicates the estimated community labels for each node. $\square$

Based on Theorem 2, we can implement community detection within multi-layer networks using the classic spectral clustering algorithm. For the sake of brevity, we refer to this weighted multi-layer community detection approach as WMLCD. The computational process of WMLCD is illustrated as follows.

Step 1. Compute an aggregated matrix by summing the squares of all layers. This aggregation results in a matrix $\widetilde{W} = \sum_{l=1,l\neq m}^{M}\{W^{(ml)}\}^2$.

Step 2. Based on Theorem 2, compute the spectral embedding of the aggregated matrix $\widetilde{W}$ for estimating $Z^{(m)}$. We adopt the standard spectral clustering approach to achieve this goal. More specifically, the calculation of spectral embedding is implemented with 3 sub-steps.

Sub-step 2.1. Compile a Laplacian matrix $L = \widetilde{D}^{-1/2}\widetilde{W}\widetilde{D}^{-1/2}$, where $\widetilde{D}$ is a diagonal matrix with $\widetilde{D}_{ii} = \sum_{j=1}^{N_m}\widetilde{W}_{ij}$ for any $1 \leq i \leq N_m$.

Sub-step 2.2. Find the K largest eigenvalues of the Laplacian matrix L and stack the corresponding eigenvectors to form an $N_m \times K$ matrix $\widehat{V}$. Each row of $\widehat{V}$ refers to an individual node.

Sub-step 2.3. Apply the K-means method to cluster the N_m rows of $\widehat{V}$ into K clusters. The clustering result is a partition for N_m nodes. For the i-th row of $\widehat{V}$, let k_i denote the label outputted by the clustering.

Step 3. Output the final estimation, $\widehat{Z}^{(m)}$. For $1 \leq i \leq N_m$, we have $\widehat{Z}_{ik_i}^{(m)} = 1$ and $\widehat{Z}_{ik'}^{(m)} = 0$ for all $k' \neq k_i$.

The final estimate, $\widehat{Z}^{(m)}$, directly describes the detected community structure. We summarize the detailed steps of WMLCD in Algorithm 1. Unlike traditional community detection methods, which are typically designed for single-layer networks, WMLCD integrates information from multiple layers, making it more comprehensive. This method relies on spectral clustering of the aggregated network $\widetilde{W}$, allowing it to capture community structures across multiple layers rather than just one. As a result, the outcome is less influenced by noise or bias from any single layer. In addition, a common approach to handling multi-layer networks is to combine the matrices of multiple layers into an aggregated matrix, e.g., $\sum_{l=1,l\neq m}^{M} W^{(ml)}$, and then apply spectral clustering to this matrix. However, WMLCD offers advantages over this aggregation method. First, as mentioned above, the WMLCD approach is equivalent to optimizing the objective function in (4), providing statistical support for the least squares estimation. Second, the aggregated matrix $\widetilde{W}$ in WMLCD not only accounts for direct relationships between nodes but also incorporates their indirect similarities. Each element of $\widetilde{W}$, such as $\widetilde{W}_{ij}$, involves not just the relationship between nodes i and j but also their common neighbors. This enhances the identification of node structures. In the following simulation study, we further demonstrate through experimental results that our method improves community detection performance.

For each mode, we identify distinct clusters using WMLCD, each of which contains entities with strong implicit relationships. From a supply chain management perspective, entities within the same community exhibit overlapping production capacities, allowing for seamless capacity shifts between objects in the same community. To make the application of WMLCD more understandable, we present a toy example using community detection for goods, as shown in Figure 3. The detection results indicate that goods A and B belong to the same local community. They are grouped together because they can be produced by multiple plants and stored in multiple warehouses simultaneously. As a result, if the supply of good A is disrupted by external factors, the supply chain manager can quickly restore production capacity by reallocating resources from the plants and warehouses that produce good B. This identified community structure demonstrates potential flexibility and redundancy across goods, plants, and warehouses.

Algorithm 1 WMLCD algorithm

Input: K: number of communities; $\mathcal{W}^{(m)} = \{W^{(m1)}, \cdots, W^{(mM)}\}$: projected multi-layer network of mode m;
Compute an aggregated matrix $\widetilde{W} = \sum_{l=1, l\neq m}^{M} \{W^{(ml)}\}^2$;
Initialize an $N_m \times N_m$ zero matrix $\widetilde{D}$;
for $i \in \{1, \cdots, N_m\}$ **do**
  $\widetilde{D}_{ii} = \sum_{j=1}^{N_m} \widetilde{W}_{ij}$;
end for
Compute a Laplacian matrix $L = \widetilde{D}^{-1/2} \widetilde{W} \widetilde{D}^{-1/2}$;
Find the K largest eigenvalues of L and the corresponding eigenvectors. Stack the eigenvectors to form an $N_m \times K$ matrix $\widehat{V}$;
Apply the K-means method to cluster the N_m rows of $\widehat{V}$ into K clusters. Since each row of $\widehat{V}$ refers to an individual node, the clustering result leads to an partition for N_m nodes. For the i-th row of $\widehat{V}$, let k_i denote the label outputted by the clustering.
Initialize an $N_m \times K$ zero matrix $\widehat{Z}^{(m)}$;
for $i \in \{1, \cdots, N_m\}$ **do**
  Set $\widehat{Z}_{ik_i}^{(m)} = 1$;
end for
Output: An estimation of the membership matrix $\widehat{Z}^{(m)}$.

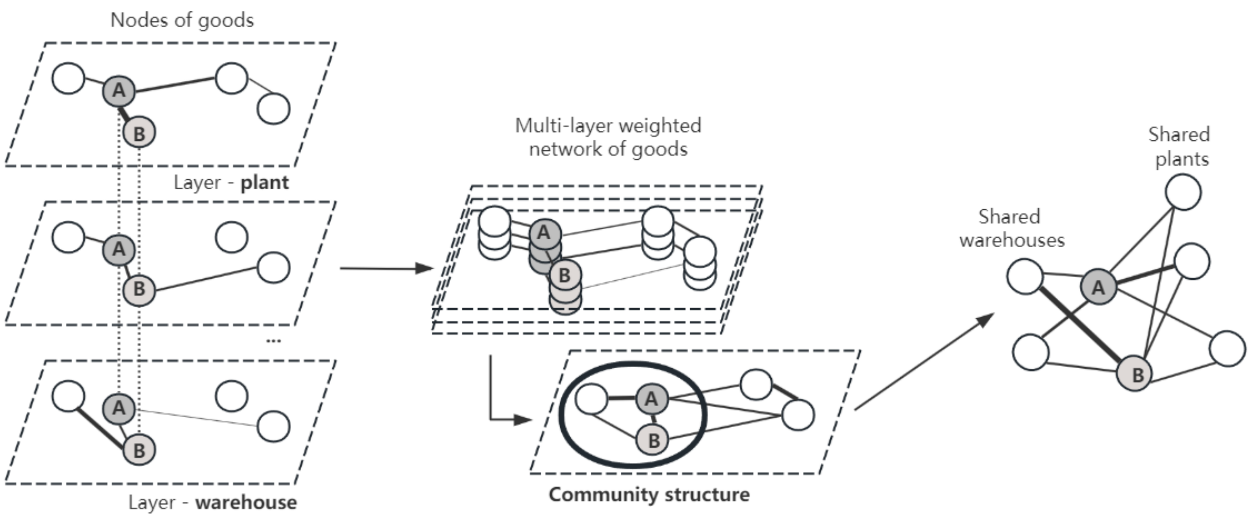

Figure 3. Relationships in production capacity among objects within the same community.

3. Simulation Study

To evaluate the performance of the multi-layer community detection approach for supply chain networks, we conduct a simulation study utilizing simulated multi-layer supply chain networks. We model a supply chain encompassing three distinct types of nodes: goods, plants, and warehouses. We then focus on the community detection within the goods. The entire supply chain is mapped onto two single-mode weighted networks, denoted as $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$. These networks correspond to the similarity matrices that are derived from plant-based and warehouse-based relationships, respectively. The process of generating the simulated networks $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$ involves the following steps:

Step 1. Specify the number of goods N_m and the number of communities K. Then, randomly generate an $N_m \times K$ membership matrix Z. For each good $1 \leq i \leq N_m$, we randomly pick k from $\{1, \cdots, L\}$ as its latent community label. We set $Z_{ik} = 1$ and $Z_{ik'} = 1$ for $k' \neq k$.
Step 2. Specify two parameter matrices of similarities $B^{(\mathrm{P})}$ and $B^{(\mathrm{W})}$, which correspond to $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$, respectively. To simplify the simulation setting, we use P_{inner} and P_{outer} to denote the strength of the similarity between two nodes within the same community and that between nodes from different communities. Then, we set $B^{(\mathrm{P})}$ and $B^{(\mathrm{W})}$ as

$$B^{(\mathrm{P})} = B^{(\mathrm{W})} = \begin{bmatrix} P_{\mathrm{inner}} & P_{\mathrm{outer}} & P_{\mathrm{outer}} \\ P_{\mathrm{outer}} & P_{\mathrm{inner}} & P_{\mathrm{outer}} \\ P_{\mathrm{outer}} & P_{\mathrm{outer}} & P_{\mathrm{inner}} \end{bmatrix}.$$

Step 3. Generate weighted matrices $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$ as $W^{(\mathrm{P})} = ZB^{(\mathrm{P})}Z^\top * \Sigma^{(\mathrm{P})}$ and $W^{(\mathrm{W})} = ZB^{(\mathrm{W})}Z^\top * \Sigma^{(\mathrm{W})}$, respectively. Here, $\Sigma^{(\mathrm{P})}$ and $\Sigma^{(\mathrm{W})}$ are both $N_m \times N_m$ matrices of noise terms. The elements of $\Sigma^{(\mathrm{P})}$ and $\Sigma^{(\mathrm{W})}$ are independently picked from $\mathrm{N}(0, 0.01)$.

Given simulated weighted matrices $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$, we then apply the WMLCD method to implement community detection for goods. To better simulate the structural characteristics where nodes within the same community are closely connected and nodes in different clusters are sparsely connected, we adopt a setting with $P_{\mathrm{inner}} = 0.5$ and $P_{\mathrm{outer}} = 0.05$. In this setting, the connection strength of an edge within a community is set to be ten times that between different communities. Specifically, we set $P_{\mathrm{inner}} = 0.5$ and $P_{\mathrm{outer}} = 0.05$. To compare the performance of WMLCD with traditional methods, we consider the following baseline methods.

- Single-layer community detection (SLCD): Conduct community detection with the standard spectral clustering algorithm using only one layer selected from the project networks.
- Community detection on aggregated network (AN): Aggregate multiple layers by directly summing the corresponding adjacency matrices. Then, conduct community detection with the standard spectral clustering algorithm using the aggregated matrix.

Regarding the evaluation criteria, we employ metrics that have been commonly used for evaluating clustering performance, including normalized mutual information (NMI), adjusted Rand index (ARI), and accuracy based on the Hungarian algorithm (ACC). To provide a robust measurement of the clustering performance, we report the average results obtained from 200 random replications.

The simulation results are shown in Figure 4. We label the single-layer community detection based on the plant layer and that based on the warehouse layer as SLCD-1 and SLCD-2, respectively. In Figure 4, we explore the community detection performance with varying network sizes, which are determined by the number of nodes. Based on Figure 4, we can draw the following conclusions. Firstly, WMLCD consistently performs better than other methods across all network sizes, especially in larger networks. Secondly, the results indicate that the performance of WMLCD improves as the network size grows, with higher NMI, ARI, and ACC scores. This suggests that WMLCD is a promising solution for capturing complex interactions within large networks. Thirdly, SLCD-1 and SLCD-2 perform worse compared to WMLCD and AN, showing that relying on only one layer of the networks will heavily limit the community detection performance. In summary, the simulation results show that the WMLCD method can significantly enhance the ability to identify closely connected areas within supply chain networks and thus contribute to downstream analyses of supply chains.

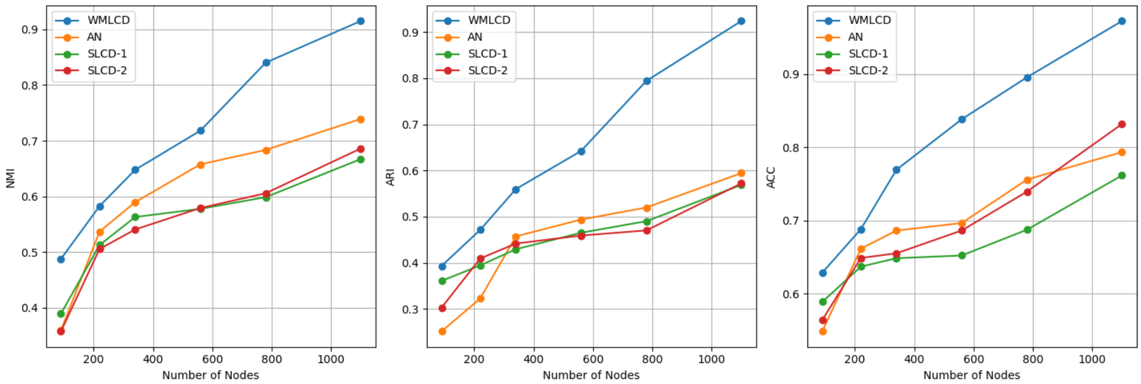

Figure 4. Performances of different community detection methods on simulated networks of goods.

4. Empirical Study

In this section, we demonstrate the effectiveness of the proposed method through an empirical study on a real supply chain dataset. This dataset originates from the central database system of one of Bangladesh's largest fast-moving consumer goods companies [28]. Covering the period from January 1, 2023 to August 9, 2023, the dataset includes 221 daily records related to the production and storage of goods. It contains no missing values or outliers and anonymizes trade or company names to protect data privacy. All entities within the supply chain are assigned anonymized identifiers. The dataset includes three types of nodes: goods, plants, and warehouses. The supply chain consists of 13 warehouses, 25 plants, and 41 goods. Production and storage data are presented in a time-series format, detailing plants' daily output and each warehouses' daily storage related to the goods. Figure 5 depicts a multi-mode network, demonstrating the supply chain's resilience through network density. Nodes represent various commodities, plants, and warehouses, and are color-coded to differentiate between node types. Edges represent the connections between these nodes, such as plants producing goods or goods being stored in a warehouse.

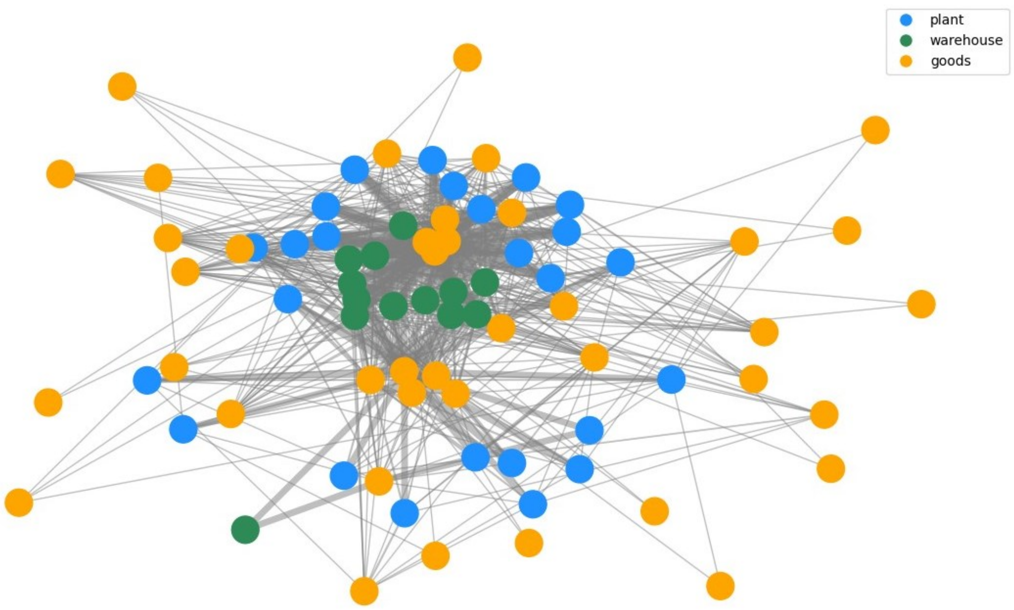

Figure 5. Weighted multi-mode supply chain network.

The nodes of goods are pivotal nodes in the multi-mode network shown in Figure 5. For each good in the records, we calculate the daily production volume, daily storage capacity, and numbers of associated factories and warehouses. Both production and storage capacities are measured in units of goods. Table 1 presents basic statistics about all products. To study the community structure among goods, we construct weighted multi-mode networks with goods as nodes based on these records. Specifically, we generate two matrices, $A^{(\mathrm{P})}$ and $A^{(\mathrm{W})}$, representing the plant–good and warehouse–good relationships, respectively. We define $A^{(\mathrm{P})} = (A^{(\mathrm{P})}ik)$ as the plant–good relationship matrix, where each element $A^{(\mathrm{P})}ik$ indicates the average daily production volume of good i at plant k. Similarly, $A^{(\mathrm{W})} = (A^{(\mathrm{W})}_{ik})$ represents the warehouse–good relationship matrix, where $A^{(\mathrm{W})}_{ik}$ denotes the average daily storage capacity for good i at warehouse k. In Figure 5, each line represents an edge between two nodes, and a thicker line indicates a higher weight for the edge. Using production volume as the weight of the matrix for spectral clustering of the supply chain network can more accurately reflect plant production capacity, making the clustering results more aligned with actual production conditions and improving the clustering's interpretability. For instance, nodes with high production volumes may indicate a more important role in the supply chain. By considering production volume, resources can be allocated more rationally, and production plans can be optimized to respond to changes in market demand. Finally, production volume weights can help to identify key production nodes that are sensitive to market changes, allowing for rapid adjustments in production strategies during market fluctuations.

Table 1. Basic statistics of all goods.

	Min	25% Quantile	Median	Mean	75 % Quantile	Max
Production [1]	0	0.066	2.446	12.916	9.717	123.456
Storage	0	0	2.193	12.795	9.655	122.903
#plants	1	1	7	6.805	12	13
#warehouses	0	1	7	6.685	12	13

[1] The statistics "Production", "Storage", "#plants", and "#warehouses" denote average daily production, average daily storage capability, number of associated plants, and number of associated warehouses, respectively.

By analyzing this dataset, we can gain a deeper understanding of the company's production and warehousing networks, laying the groundwork for future supply chain resilience analyses. In the following sections, we will investigate the independent predictive model networks for goods and plants in order to identify redundancies and flexibilities in the supply chain.

Projected Network of Goods

To demonstrate the use of the proposed community detection framework, we conduct a resilience analysis on the projected network of goods. Assessing the network's resilience to risks and recovery capabilities allows us to identify potential vulnerabilities and recommend targeted improvement strategies. Goods are at the core of the supply chain, serving as the primary objects of consumer demand. They are critical in meeting market demands and easing the transition from production to consumption. Precise inventory management reduces costs, prevents shortages or overstocking, and improves supply chain responsiveness and flexibility. As described in Section 2, indirect relations create single-mode goods networks with two layers: the plant layer and the warehouse layer.

According to the methodology in Section 2, we project the original three-mode network with plants, goods, and warehouses into two single-mode networks of goods. Based on $A^{(\mathrm{P})}$ and $A^{(\mathrm{W})}$, i.e., the matrices of plant–good relationships and warehouse–good relationships, we generate plant layer $W^{(\mathrm{P})}$ and $W^{(\mathrm{W})}$ according to (1). Specifically, for any two goods i and j,

$$W_{ij}^{(\mathrm{P})} = \sum_{k=1}^{N_{\mathrm{P}}} \frac{A_{ik}^{(\mathrm{P})} A_{jk}^{(\mathrm{P})}}{\left\{\sum_{i'=1}^{N_{\mathrm{G}}} A_{i'k}^{(\mathrm{P})}\right\}^2}, \quad W_{ij}^{(\mathrm{W})} = \sum_{k=1}^{N_{\mathrm{W}}} \frac{A_{ik}^{(\mathrm{W})} A_{jk}^{(\mathrm{W})}}{\left\{\sum_{i'=1}^{N_{\mathrm{G}}} A_{i'k}^{(\mathrm{W})}\right\}^2},$$

where $N_{\mathrm{P}}, N_{\mathrm{W}}, N_{\mathrm{G}}$ are the numbers of plants, warehouses, and goods, respectively. Then, we apply the WMLCD method to implement community detection on $\{W^{(\mathrm{P})}, W^{(\mathrm{W})}\}$. For the plant layer, there are 360 edges. The network density of this layer is 0.462, indicating a moderate level of connectivity and numerous connections between nodes. The warehouse layer has 665 edges and a network density of 0.853. The results imply that various goods may share resources, processes, or production lines during the production process.

The detection of communities within the multi-layer network of goods, as shown in Figure 6, has produced results that can be visually articulated by reordering goods in the adjacency matrix. This graphical representation clearly distinguishes between two distinct communities of goods, indicating the presence of separate clusters within the network.

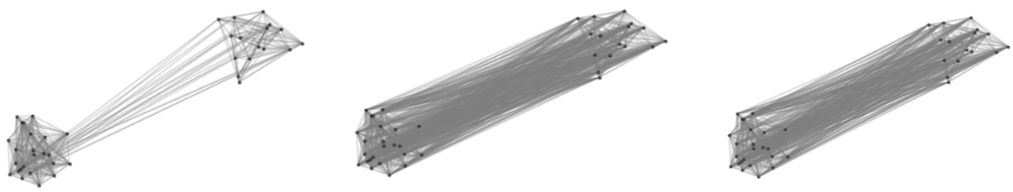

Figure 6. Adjacency matrices of goods in the plant layer and warehouse layer.

Clustering analysis provides valuable insights into the underlying connections between goods in the supply chain network. Goods from the same community typically share significant similarities in the plants and warehouses that they interact with. The clustering results provide valuable insights into the hidden relationships between goods in the supply chain network. Goods clustered into the same community often share significant overlaps in the plants and warehouses that they utilize. This finding has substantial implications for supply chain management, as identifying goods communities can enhance risk management and improve supply chain flexibility. By identifying these communities, businesses can develop backup plans for critical goods to prevent disruptions at specific nodes, such as plants or warehouses, from affecting the entire supply chain. For example, if goods A and B belong to the same community and the facility for good A becomes inoperable, the facilities associated with good B can be used to identify an alternative source. Such strategic coordination improves capacity management and strengthens the supply chain's flexibility and responsiveness to unexpected challenges.

Beyond increasing supply chain resilience, community detection of goods can significantly improve inventory management by reducing redundant stock and holding costs. Additionally, it can lead to cost savings by simplifying transportation and distribution procedures. By grouping and shipping goods that are often together, we can lower shipping costs and speed up delivery times. Better coordination of plant and warehouse operations leads to overall operational improvements. By reducing excess inventory, we can save money on storage. Furthermore, by improving transportation efficiency, we can reduce shipping costs and deliver goods to customers more quickly.

During the COVID-19 crisis, numerous enterprises experienced significant disruptions in their supply chains. Those who had a thorough understanding of the shared goods across multiple plants and warehouses were better able to quickly adapt to the situation by reallocating stock and resources to meet changing demands. For instance, an enterprise that is aware of the presence of critical medical supplies at both production facilities and distribution hubs may be able to maintain a consistent flow of supplies to healthcare institutions even if some facilities are affected by lockdown measures or experience supply chain delays.

Figure 7 shows the network structures of the two communities based on the clustering results. The edges represent relationships between plants, such as shared plant or storage locations, raw material requirements, product interdependencies, and other influencing factors.

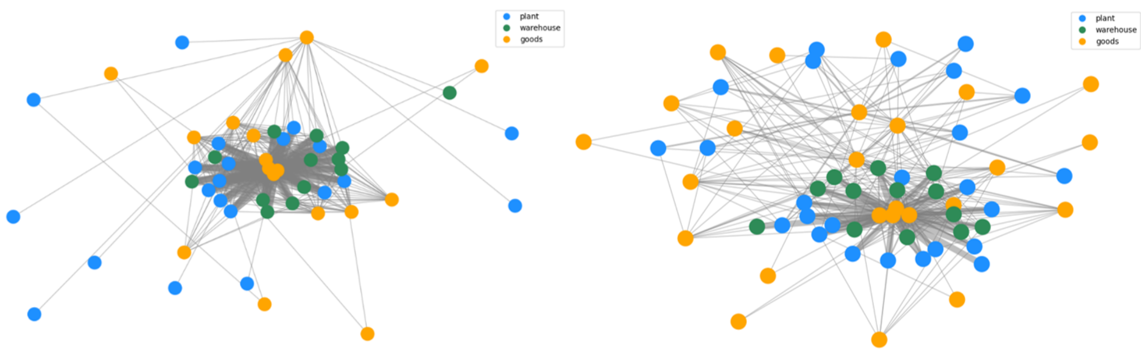

Figure 7. Cluster 1 (**left**) and Cluster 2 (**right**) within the weighted projected network of goods.

After clustering, as shown in Figure 7, Cluster 1 includes 25 goods along with their associated plants and warehouses, while Cluster 2 contains 16 goods and exhibits a more evenly distributed network compared to Cluster 1. Structurally, Cluster 2 shows a higher degree of interconnectivity, with individual goods produced by a larger number of plants and stored in more warehouses. Examining production data, the thickness of the connecting lines indicates that production and storage in Cluster 2 are more balanced than in Cluster 1. In Cluster 2, the production of any single item is less likely to be overly concentrated, reducing the risk that a single product is exclusively produced by one factory. From a stability perspective, production data reveal that Cluster 2 is a more stable group, with lower variance and a smaller coefficient of variation, indicating less fluctuation in production. By contrast, Cluster 1 shows less stability, with production volumes that fluctuate more over time. This greater volatility in Cluster 1 may reflect higher levels of uncertainty, influenced by factors such as environmental shifts or management differences, which can lead to significant swings in production. These fluctuations introduce challenges in supply chain management, requiring closer monitoring and response. Conversely, the stability of Cluster 2 helps to buffer the supply chain against disruptions, mitigating volatility and enhancing the overall resilience of production and development. This stability complements the efficiency gains in industrial development by strengthening both production and operational resilience. Further discussion on the flexibility and stability of these clusters is provided in the following sections.

(1) Flexibility Analysis of goods

Flexibility in the supply chain is defined as the ability to respond quickly to changes in market demand or other unforeseen circumstances, demonstrating the system's agility and adaptability in a volatile environment.

Producing a product in multiple plants allows for the quick reassignment of production tasks to other facilities if problems arise, such as machinery breakdowns or raw material shortages. This approach ensures consistent production and stability. Consequently, the greater the number of plants and warehouses linked to a product, the more swiftly production schedules, supply chain tactics, and logistics can be modified to accommodate changes in market demand. For instance, in the event of a sudden increase in order volume, plants can allocate resources quickly, reduce the time and costs associated with production changes, and accelerate the rate of production response.

Goods nodes located further from the network center are associated with fewer plants and warehouses, implying less flexibility. As shown in Figure 8, Cluster 2 has an average association of 14 plants and warehouses per good, with a median of more than 15, indicating strong production and storage flexibility. The extensive network of production plants and storage facilities enables broader distribution, allowing for quick adjustments to production and distribution strategies during crises such as natural disasters, equipment malfunctions, or logistical disruptions. This ensures the supply chain's continuity. Furthermore, the variety of manufacturing and warehousing sites enables goods in this category to respond quickly to market demand fluctuations and shifts. Goods in Cluster 1, which are associated with fewer plants and warehouses, have lower supply chain resilience. The clustering of production and storage facilities makes the supply chain more susceptible to disruptions during emergencies, resulting in longer recovery times. At the same time, Cluster 2 has a lower standard deviation than Cluster 1, indicating a more concentrated distribution of values.

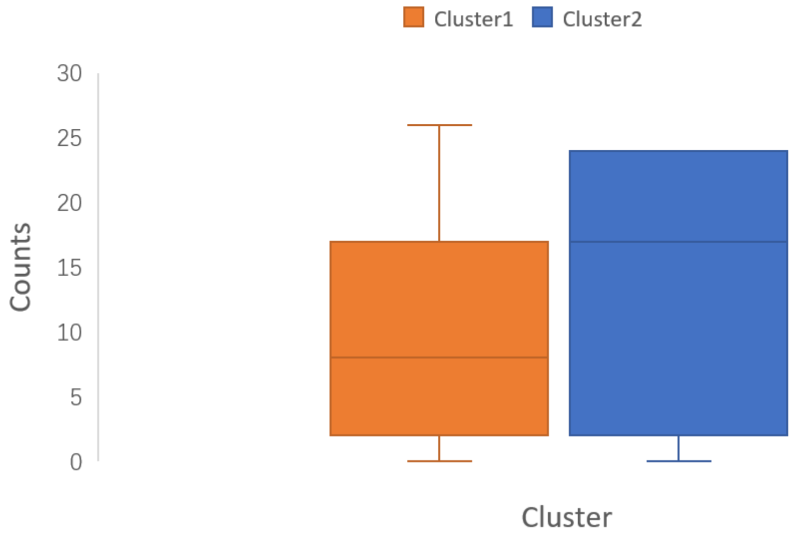

Figure 8. The total number of plants and warehouses associated with goods for different clusters.

Meanwhile, it should be noted that some plants in Cluster 1 produce only one product. This singular production model limits the supply chain's flexibility. When faced with market changes or emergencies, these plants struggle to quickly adjust their production schedules, resulting in a significant increase in the time required to respond to critical situations. Moreover, centralized production may cause critical links in the supply chain to become overly reliant on certain plants. If any of these plants fail, the entire supply chain's stability and responsiveness will suffer significantly. In contrast, Cluster 2 has a more diverse supply chain structure. This cluster has tighter and more diverse connections between different types of nodes, such as plants and warehouses. This means that the same product can be manufactured by several plants and stored in a variety of warehouses. This diversified approach to production and storage significantly improves the supply chain's flexibility and robustness. To illustrate the differences in flexibility more intuitively, we use two local network examples shown in Figure 9. Node MAHS025K is only connected to 8 nodes, whereas Node SOS001L12P is connected to a total of 24 nodes, including 12 plants and 12 warehouses, far outnumbering Node MAHS025K. From the standpoint of supply chain resilience assessment, Node SOS001L12P is relatively more fragile and may necessitate additional management measures to maintain corresponding production capacity.

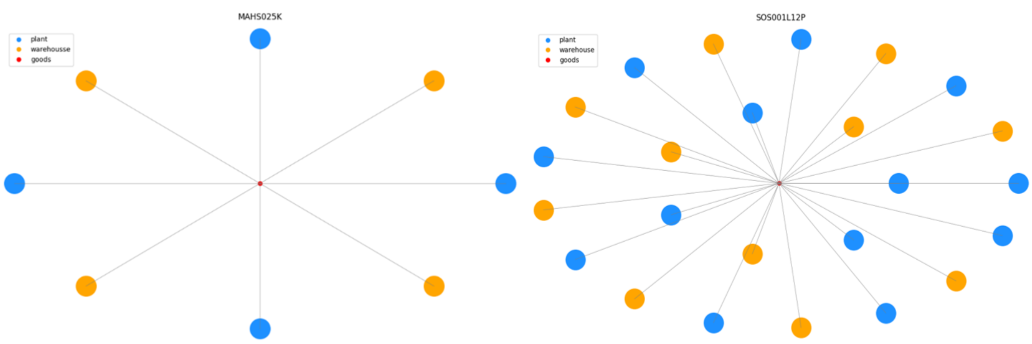

Figure 9. Local networks of Node MAHS025K (**left**) and Node SOS001L12P (**right**).

Furthermore, the community detection results provide detailed decision-making support for managing production capacities. Because the community detection results reflect the implicit connections between two goods in terms of production capacity, if one good's production capacity is damaged, it can be restored by using the production capacity of the other. Figure 10 shows an example based on two good nodes, MAR02K12P and MAR01K24P. They share three plants and three warehouses in their supply chain. The figure clearly shows that the good MAR01K24P, represented by the node on the right, can be produced at more plants and stored in more warehouses. In contrast, the good MAR02K12, represented by the node on the left, has fewer neighboring nodes, indicating a sparser production network. Within such a network structure, if the production of MAR02K12 suffers from plant closures, equipment failures, or similar events, a possible solution is for those plants shared by the two goods to adjust their production plans in a timely manner. Nodes with denser connections can support those with sparser connections, allowing for a more flexible response to emergencies and increasing supply chain resilience. The ability to support and adjust to each other is critical throughout the supply chain network. It not only helps to reduce the risk of production interruptions and inventory shortages but it also ensures that the supply chain is continuous and stable. To achieve more sustainable development, managers should consider improving emergency response plans, strengthening supply chain monitoring, and optimizing production plans to continuously improve the supply chain's resilience.

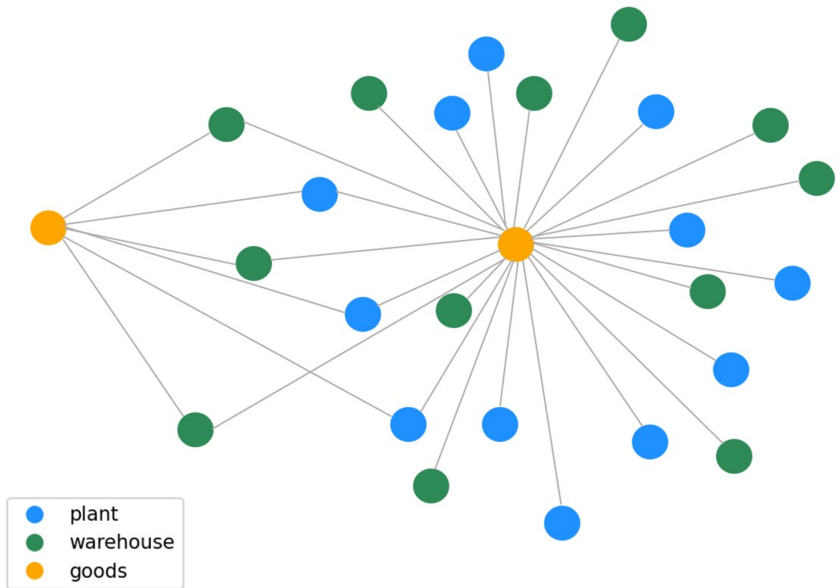

Figure 10. Local networks of Node MAR02K12P (**left**) and MAR01K24P (**right**).

(2) Stability Analysis of goods

Stability is a key component of supply chain management, indicating that the supply chain can keep its operations running and its services consistent in the face of market demand fluctuations, disruptions in raw material supply, natural disasters, political unrest, or other unpredictable events. A stable supply chain ensures that the production flow, information flow, and cash flow run smoothly, with fewer interruptions and delays, protecting the company's delivery capabilities and customer satisfaction. Moreover, stability reduces risks, improves the overall efficiency and cost-effectiveness of the supply chain, and allows businesses to maintain a competitive advantage in the market.

By analyzing plant production output, we can create a line chart displaying the average output of plants in the two clusters. The chart demonstrates that the orange line representing Cluster 2 fluctuates less than the blue line representing Cluster 1, indicating that Cluster 2's overall production is more stable. Furthermore, we calculated the coefficient of variation for the average output of the two clusters and found it to be 0.584 for Cluster 1 and 0.554 for Cluster 2. The coefficient of variation is a statistical measure of data dispersion that is calculated as the standard deviation divided by the mean. The smaller the coefficient of variation, the more concentrated and stable the data distribution. Thus, this calculation also supports the conclusion that Cluster 2 is superior in terms of stability. By analyzing plant production output, we can create a line chart displaying the average output of plants in the two clusters. The chart indicates that the orange line representing Cluster 2 fluctuates less than the blue line representing Cluster 1, indicating that Cluster 2's overall production is more stable.

5. Practical Implications for Supply Chain Resilience

This paper proposes the WMLCD method for community detection within complex supply chains, uncovering the structural information embedded in supply chain networks. We further discuss how community detection results based on WMLCD can support decision making in real-world supply chain management, with a particular focus on enhancing supply chain resilience. The detection results of WMLCD offer significant potential for enhancing supply chain resilience by providing valuable insights for supply chain network management and optimization. First, applying community detection techniques can optimize supply chain structures and increase the flexibility of production and logistics. For instance, by identifying clusters within the supply chain network, organizations can uncover groups with synergistic strengths. These clusters can facilitate rapid adjustments in production and mutual support among members in response to supply chain disruptions, such as those caused by natural disasters or sudden shifts in demand. For example, in the food and beverage supply chain, community detection could reveal clusters of regional producers and distributors that, in times of disruption, could quickly shift supply routes to maintain availability in affected areas.

Furthermore, detection results of WMLCD can guide the standardization of product designs and processes, making it easier for companies to switch between suppliers or shift production from one factory to another. This ability to pivot across suppliers or production sites reduces risks associated with sudden supplier changes, particularly for industries with complex manufacturing processes like electronics or automotive parts. By creating interchangeable components and establishing flexible production protocols, enterprises can reduce downtime and maintain continuity in the face of disruptions, thereby bolstering resilience.

Lastly, WMLCD can aid in the dynamic monitoring of supply chain data, helping organizations to anticipate risks and implement proactive interventions. By regularly analyzing supply chain networks, companies can identify potential bottlenecks or risk points within the chain—such as suppliers in vulnerable locations or routes susceptible to delays. In the pharmaceutical supply chain, for instance, where delays in critical component deliveries can have severe consequences, community detection can help to develop alternative routes or strategies to pre-emptively address these risks, ensuring uninterrupted access to essential products. Overall, integrating community detection into supply chain management fosters an adaptive, resilient network capable of withstanding various external shocks.

6. Conclusions

Analyzing supply chain resilience is critical for efficient production and long-term growth. In today's rapidly changing and uncertain economic landscape, a practical approach to evaluating supply chains is to assess resilience through flexibility and redundancy. This paper presents a robust framework for analyzing supply chains through a multi-layer network community detection approach, offering an innovative way to uncover production capacity linkages between products within complex supply networks. By converting the complex, multi-modal supply chains into a series of single-mode multi-layer weighted networks, we propose the WMLCD method to comprehensively identify interconnected units within the network. This approach highlights clusters of closely linked components, facilitating a deeper understanding of inter-product dependencies and collaborative potential.

The application of WMLCD reveals critical relationships between network entities based on production capacities, providing a foundation for informed decision making in supply chain management. Such insights are particularly valuable in scenarios where disruptions occur. For example, if the production or supply of a particular item is compromised, our results can pinpoint specific factories, warehouses, or other nodes that should adjust their capacity to help restore the disrupted flow. In this way, the community detection results support proactive interventions, minimizing the negative impacts of supply chain interruptions. Utilizing a real-world dataset, our study shows how multi-layer network community detection effectively brings to light indirect connections between production plants and storage facilities. These findings extend beyond immediate production relationships, capturing subtler yet vital links that traditional single-layer analyses might overlook. Ultimately, the community-detection-based insights enhance capacity management and strategic resource allocation, empowering businesses to respond agilely to both unexpected demand surges and supply disruptions, and to build resilience into their supply chain strategies. This framework thus provides crucial support for sustaining operational continuity and mitigating risks within increasingly interconnected global supply networks.

In addition, we discuss the limitations of WMLCD and potential directions for future research. This paper does not address large-scale supply chain networks, which involve a high number of nodes. It is important to note that WMLCD requires projecting network matrices and performing eigenvalue decomposition for community detection, which can be computationally demanding when applied to large networks. As a result, WMLCD may be less efficient compared to multi-layer community detection methods that are based on local search [29,30]. For future work, we plan to enhance WMLCD's performance on large-scale datasets by exploring subsampling methods and distributed computing solutions. We also aim to test the method on additional supply chain datasets to verify its effectiveness across different contexts. Furthermore, we intend to integrate multi-layer network clustering with capacity forecasting to provide more comprehensive insights into supply chain resilience analysis.

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