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Existence, Uniqueness and Asymptotic Behavior of Solutions for Semilinear Elliptic Equations

Lin-Lin Wang, Jing-Jing Liu and Yong-Hong Fan *

School of Mathematics and Statistics Sciences, Ludong University, Yantai 264025, China;
llwang@ldu.edu.cn (L.-L.W.); l17860368796@163.com (J.-J.L.)
* Correspondence: yhfan@ldu.edu.cn

Abstract: A class of semilinear elliptic differential equations was investigated in this study. By constructing the inverse function, using the method of upper and lower solutions and the principle of comparison, the existence of the maximum positive solution and the minimum positive solution was explored. Furthermore, the uniqueness of the positive solution and its asymptotic estimation at the origin were evaluated. The results show that the asymptotic estimation is similar to that of the corresponding boundary blowup problems. Compared with the conclusions of Wei's work in 2017, the asymptotic behavior of the solution only depends on the asymptotic behavior of $b(x)$ at the origin and the asymptotic behavior of g at infinity.

Keywords: semilinear elliptic differential equation; upper and lower solution; blowup problem; uniqueness; asymptotic behavior

MSC: 35B09; 35J20; 35J60; 35J70

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1. Introduction

In the theoretical and applied research of differential equations, the boundary blowup problem, or large solution problem, is a hot topic. The problem originated from the research of Riemannian surface theory and automatic-function theory with negative curvature constant. Later, it was found that the boundary blowup problem appeared in many fields, such as stable constrained stochastic control, superdiffusion process in stochastic process, differential geometry, concentrated wealth effect in ecology, electric potential in hot hollow metal shells, high-speed diffusion in a chemical reaction, etc. These discoveries have prompted many mathematicians to turn their attention to the research of boundary blowup, leading to the continuous development and fine-tuning of the boundary blowup theory.

From the point of view of the research content, the boundary blowup problem mainly focuses on the existence, asymptotic property, and uniqueness of the solution, as well as the asymptotic expansion of the solution at the boundary. The research can be traced back to 1916 by L. Bieberbach [1]. He first studied the following semilinear elliptic equation with an exponential source, which is expressed as follows:

$$\begin{cases} \Delta u = e^u, \ x \in \Omega, \\ u = \infty, \ x \in \partial\Omega. \end{cases}$$

Regarding the study of boundary blowup problems, a landmark result should be attributed to J.B. Keller [2] and R. Osserman [3]. They revealed that a sufficient and necessary condition for the existence of a solution of the equation

$$\begin{cases} \Delta u = f(u), u > 0, \ x \in \Omega, \\ u = \infty, \ x \in \partial\Omega. \end{cases}$$

is that the inequality $\frac{1}{\sqrt{2}}\int_u^\infty\left[\int_{s_0}^t f(s)ds\right]^{-1/2}dt < +\infty$ is true. This condition came to be known as the Keller–Osserman condition. It played a very important role in subsequent research on the boundary blowup problem. For research on boundary blowup problems, we may refer to [4–7], among others.

Later, research mainly focused on the following semilinear elliptic differential equation:

$$-\Delta u = a(x)u - b(x)g(u),\ x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a domain with C^2 boundary $\partial\Omega$; $a(x)$ is a continuous function in $\Omega_0 \subset \Omega$; $b(x)$ is a non-negative continuous function in $\overline{\Omega}$; and $g : \mathbb{R}^1 \to \mathbb{R}^1$ is also assumed to be a non-negative continuous function.

The classical Logistic equation

$$\begin{cases} -\Delta u = \lambda u - b(x)u^p,\ x \in \Omega, \\ u = 0,\ x \in \partial\Omega, \end{cases} \tag{2}$$

is a special case of (1); here, $\lambda \in \mathbb{R}^1$ can be seen as a parameter, and $p > 1$ is a constant. In [8], Professor Du studied (2) thoroughly and obtained the following theorem:

Theorem 1 (Theorem 5.1 [8])**.** *Let λ_1 denote the first eigenvalue of the operator $-\Delta$ with zero Dirichlet boundary condition; then, (2) has no positive solution when $\lambda \leq \lambda_1$ and it has a unique solution when $\lambda > \lambda_1$.*

Many authors focus their study on (1), and $a(x)$ may be allowed to be unbounded in Ω, a classical function is $\frac{1}{|x|^2}$ (usually under this situation, $0 \in \Omega$). As we all know, in this case, this term is usually called the Hardy potential or inverse square potential. For the study of semilinear elliptic differential equations with the Hardy potential, one can refer to [9–16].

For the study of elliptic differential equations with singular coefficients, the method of upper and lower solutions is usually used. Many scholars choose the corresponding solution of the boundary blasting problem as the upper solution; then, by constructing the appropriate lower solution, they investigate the existence of the solution with zero Dirichlet boundary conditions. In the literature, the asymptotic behavior at the singularity has been studied, but the relationship between the asymptotic behavior and the corresponding boundary blowup problem has not been specified. In this article, we hope to make some breakthroughs in this area. In this research, for elliptic equations with singular coefficients, we determine the relationship between the blowup solutions with zero Dirichlet boundary conditions at singular points and the blowup solutions at the boundary for the corresponding boundary blowup problem. Therefore, in the future, when we study the blowup behavior of solutions at singular points for elliptic equations with singular coefficients, we may first study the asymptotic behavior of solutions at the boundary for the corresponding boundary blowup problem and then study the relationship between them. We believe that many interesting results could be obtained from this kind of research.

Recently, Cîrstea studied the following equation in [12]:

$$-\Delta u = \lambda\frac{u}{|x|^2} - b(x)h(u),\ x \in \Omega\backslash\{0\}, \tag{3}$$

where λ is a parameter, and $-\infty < \lambda \leq \frac{(N-2)^2}{4}$ and $0 \in \Omega$. Here, $b(x)$ is a positive continuous function in $\overline{\Omega}\backslash\{0\}$, which behaves near the origin as a regularly varying function at zero with index θ greater than -2. The nonlinearity h is assumed to be continuous on $\mathbb{R}^1$ and positive on $(0,\infty)$ with $h(0) = 0$ such that $h(t)/t$ is bounded for small $t > 0$. The author completely classified the behavior near zero of all positive solutions for (3) when h is regularly varying at ∞ with index q greater than 1. In particular, as an application of his main result, he chose $h(t) = t^q(logt)^{\alpha_1}$ as $t \to \infty$ and $b(x) = |x|^\theta(-log|x|)^{\alpha_2}$ as $|x| \to 0$, where α_1 and α_2 are any real numbers. We can easily see that $h(t) = t^p e^{\alpha t} (p > 1$ and $\alpha > 0)$ does not satisfy the above conditions.

Recently, Wei and Du [17] considered

$$-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\sigma u^p,\ x \in \Omega \backslash \{0\}, \tag{4}$$

under the conditions $\lambda > \frac{(N-2)^2}{4}$, $p > 1$, $\sigma > -2$ and obtained the following theorem:

Theorem 2. *Suppose that $u(x)$ is an arbitrary positive solution of (4) with zero Dirichlet boundary conditions; then,*

$$\lim_{|x|\to 0} |x|^{\frac{\theta+2}{p-1}} u(x) = l^{\frac{1}{p-1}},$$

where

$$l = \lambda + \frac{\theta+2}{p-1}\left(\frac{\theta+2}{p-1} + 2 - N\right).$$

Remark 1. *From the proof of Theorem 2 in [17], we can easily see that the asymptotic behavior of the solution for (4) with zero Dirichlet boundary conditions at the origin depends not only on λ and N but also on the asymptotic behavior of $b(x)$ at the origin and the asymptotic behavior of g at infinity.*

In this paper, we consider

$$-\Delta u = \lambda \frac{u}{|x|^2} - b(x) u^p e^{\alpha u},\ x \in \Omega \backslash \{0\}, \tag{5}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 3)$ is a bounded smooth region, and $0 \in \Omega$, $\lambda \in \mathbb{R}^1$, $b(x)$ is a positive continuous function in $\overline{\Omega} \backslash \{0\}$; $p > 1$ and $\alpha > 0$ are two constants. We want to know whether the conclusions similar to those in Theorem 1 and Theorem 2 still hold true.

2. Preliminaries

For convenience, we briefly explain the notations and some lemmas, which will be used hereafter.

As usual, suppose $p \geq 1$, k is a non-negative integer, $C^m(\Omega)$ denotes the function space such that U and $D^\alpha U(|\alpha| \leq m)$ are all continuous in Ω, $C^\infty(\Omega) = \cap_{m=0}^\infty C^m(\Omega)$, $C_0^\infty(\Omega)$ denotes all spaces of $C^\infty(\Omega)$ that have compact support sets in Ω, $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. $C^{0,\mu}(\overline{\Omega})$ denotes the space of all μ-Holder continuous functions in $\overline{\Omega}$. $W_0^{1,p}(\Omega)$ is the normal Sobolev space, and $W_0^{1,2}(\Omega) = H_0^1(\Omega)$. The norm in $L^q(\Omega)$ is defined as follows:

$$\|u\|_q = \left(\int_\Omega |u|^q\right)^{1/q}.$$

Lemma 1 (Hardy inequality [18])**.** *Suppose $1 < p < N$, $u \in W_0^{1,p}(\Omega)$. We have*

$$\int_\Omega \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{N-p}\right)^p \int_\Omega |\nabla u|^p dx.$$

In particular, when $p = 2$, the inequality is

$$\int_\Omega \frac{|u|^2}{|x|^2} dx \leq \left(\frac{2}{N-2}\right)^2 \int_\Omega |\nabla u|^2 dx.$$

Denote H as the Hardy constant; that is, H is the best constant to ensure the following formula holds true:

$$\int_\Omega \frac{u^2}{|x|^2}dx \le \frac{1}{H}\int_\Omega |\nabla u|^2 dx,\ \forall u \in H_0^1(\Omega).$$

From Lemma 1, the best Hardy constant is $H = \frac{(N-2)^2}{4}$, and as we all know, $H = \frac{(N-2)^2}{4}$ could not be obtained in $H_0^1(\Omega)$, while it can be expressed as follows:

$$H = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2}dx}.$$

Let $\omega \subset \mathbb{R}^N (N \geqslant 3)$ be a bounded and smooth domain and denote $\lambda_1[c(x), b(x), \omega]$ (for short, $\lambda_1[c, b, \omega]$) as the first eigenvalue for the following boundary value problem:

$$\begin{cases} -\Delta u + c(x)u = \mu b(x)u,\ x \in \omega, \\ u = 0,\ x \in \partial\omega, \end{cases}$$

where $c(x)$ is a continuous function on $\overline{\omega}$, and $c(x) > 0$, $b(x)$ is a non-negative continuous function on $\overline{\omega}$. We denote

$$\lambda_1[b,\omega] = \lambda_1[0,b,\omega],\ \lambda_1(\omega) = \lambda_1[0,1,\omega],\ \omega^\delta = \{x | x \in \omega \text{ and } |x| > \delta \text{ for } \delta > 0\}. \quad (6)$$

From the work of Cheng [19] and Wei [20], we have the following:

Lemma 2 (Proposition 1 [20])**.** *Let $\lambda_1[c, b, \omega]$ be defined as above; then,*

(i) If $b_1(x) \le b_2(x)$ in ω; then, $\lambda_1[c, b_2, \omega] \le \lambda_1[c, b_1, \omega]$ and the equality holds if and only if $b_1(x) \equiv b_2(x)$;

(ii) If $c_1(x) \le c_2(x)$ in ω; then, $\lambda_1[c_1, b, \omega] \le \lambda_1[c_2, b, \omega]$ and the equality holds if and only if $c_1(x) \equiv c_2(x)$;

(iii) If $0 < \delta_1 < \delta_2$ in ω; then, $\lambda_1[c, b, \omega^{\delta_1}] < \lambda_1[c, b, \omega^{\delta_2}]$ and $\lambda_1[c, b, \omega^\delta] \to \lambda_1[c, b, \omega]$ as $\delta \to 0^+$.

Lemma 3 (Proposition 2 [20])**.** *If $\varepsilon \to 0$, then $\lambda_1[\frac{1}{|x|^2+\varepsilon}, \omega] \to H$.*

Lemma 4 ([19])**.** *Let ω^δ be defined as in (6); we have*

$$\lim_{\delta\to 0^+} \lambda_1[|x|^{-2}, \omega^\delta] = \frac{(N-2)^2}{4}.$$

From Lemma 2, we know that $\lambda_1[|x|^{-2}, \omega^\delta]$ increases as δ increases.

The following four lemmas and their proofs can be found in [8]:

Lemma 5 (Lemma 5.6 [8])**.** *Suppose ω is a bounded domain in $\mathbb{R}^N (N \geqslant 3)$, $\alpha(x) \in C(\omega)$ and $||\alpha(x)||_\infty < +\infty$, $\beta(x) \in C(\omega)$, and $\beta(x)$ is non-negative and not identically zero. Let $u_1, u_2 \in C^1(\omega)$ be positive in ω and satisfy in the weak sense*

$$\Delta u_1 + \alpha(x)u_1 - \beta(x)g_1(u_1) \le 0 \le \Delta u_2 + \alpha(x)u_2 - \beta(x)g_1(u_2),\ x \in \omega,$$

and

$$\lim_{x\to\partial\omega} \sup(u_2 - u_1) \le 0,$$

where $g_1(u)$ is continuous and $\frac{g_1(u)}{u}$ is strictly increasing for u in the range $\min\{u_1, u_2\} < u < \max\{u_1, u_2\}$. Then,

$$u_2 < u_1.$$

Lemma 6 (Theorem 6.6 [8]). *Suppose $f(u)$ is continuous on $\mathbb{R}^1$; there exists $s_0 > 0$, $h(u)$ is a non-decreasing continuous function for $u \geq s_0$ such that*

$$f(u) \geq h(u) > 0$$

and

$$\int_{s_0}^{\infty} \left[\int_{s_0}^{t} h(s) ds \right]^{-1/2} dt < \infty.$$

If Ω is a bounded Lipschitz region and there exists a $v_* \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$\Delta v_* \geq f(v_*),\ x \in \Omega,$$

then the equation

$$\begin{cases} \Delta u = f(u),\ x \in \Omega, \\ u = \infty,\ x \in \partial\Omega, \end{cases} \tag{7}$$

has at least a solution $u \in C^1(\Omega)$ such that $u \geq v_*$ in Ω, and among all such solutions, there are the largest positive solution u^* and the smallest positive solution u_*.

Lemma 7 (Theorem 6.8 [8]). *Suppose Ω and $f(u)$ satisfy the conditions in Lemma 6, then (7) has a solution u, and if $\partial\Omega \in C^2$; then,*

$$\lim_{d(x) \to 0} \frac{\Psi_f[u(x)]}{d(x)} = 1,$$

where

$$d(x) = d(x, \partial\Omega)$$

and

$$\Psi_f[u] = \frac{1}{\sqrt{2}} \int_{u}^{\infty} \left[\int_{s_0}^{t} f(s) ds \right]^{-1/2} dt.$$

Lemma 8 (Theorem 6.10 [8]). *Suppose Ω and $f(u)$ satisfy the conditions in Lemma 6; furthermore,*

$$\lim_{t \to +\infty} \frac{\Psi_f(\beta t)}{\Psi_f(t)} > 1,\ \forall \beta \in (0, 1).$$

$f(0) = 0$, $\frac{f(t)}{t}$ does not decrease on $t > 0$; then, Equation (7) has a unique non-negative solution.

3. Main Results

Now, we consider Equation (5) with zero Dirichlet boundary conditions as follows:

$$\begin{cases} -\Delta u = \frac{\lambda u}{|x|^2} - b(x) u^p e^{\alpha u},\ x \in \Omega \backslash \{0\}, \\ u = 0,\ x \in \partial\Omega. \end{cases} \tag{8}$$

First, we have

Theorem 3. *Let H be defined as in Section 2; then, (8) has no positive solution when $\lambda \leq H$, and it has at least a minimal positive solution and a maximal positive solution when $\lambda > H$.*

Proof. Assume that $u \in C^1(\Omega)$ is a positive solution of (8); from the first equation of (8), we have

$$\int_{\Omega} -u \Delta u dx = \int_{\Omega} \frac{\lambda u^2}{|x|^2} dx - \int_{\Omega} b(x) u^{p+1} e^{\alpha u} dx.$$

Using the integration by parts, we deduce

$$\int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega \frac{u^2}{|x|^2} dx - \int_\Omega b(x) u^{p+1} e^{\alpha u} dx,$$

and therefore,

$$H \le \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \frac{u^2}{|x|^2} dx} = \lambda - \frac{\int_\Omega b(x) u^{p+1} e^{\alpha u} dx}{\int_\Omega \frac{u^2}{|x|^2} dx} < \lambda.$$

From the above inequality, we know if $\lambda \le H$, then (8) has no positive solution. This completes the proof of the first part of Theorem 3.

Now, we suppose $\lambda > H$. By Lemma 4, $\exists \delta_0 > 0$ such that for any $\delta \le \delta_0$, $\lambda_1\left[|x|^{-2}, \Omega^\delta\right] < \lambda$. Let $\phi_\delta > 0$ be the solution of

$$\begin{cases} -\Delta u = \lambda_1\left[|x|^{-2}, \Omega^\delta\right] \frac{u}{|x|^2},\ x \in \Omega^\delta, \\ u = 0,\ x \in \partial\Omega^\delta. \end{cases} \tag{9}$$

Then, we can easily obtain that for any $\varepsilon > 0$ sufficiently small,

$$\begin{cases} -\Delta(\varepsilon\phi_\delta) = \varepsilon\lambda_1\left[|x|^{-2}, \Omega^\delta\right] \frac{\phi_\delta}{|x|^2} < \varepsilon\lambda \frac{\phi_\delta}{|x|^2} - \varepsilon^p b(x) \phi_\delta^p e^{\alpha\varepsilon\phi_\delta},\ x \in \Omega^\delta, \\ \varepsilon\phi_\delta = 0,\ x \in \partial\Omega^\delta, \end{cases}$$

which means that $\varepsilon\phi_\delta$ is a lower solution of the following equation:

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} - b(x) u^p e^{\alpha u},\ x \in \Omega^\delta, \\ u = 0,\ x \in \partial\Omega^\delta. \end{cases} \tag{10}$$

On the other hand, consider the following equation:

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} - b(x) u^p e^{\alpha u},\ x \in \Omega^\delta, \\ u = \infty,\ x \in \partial\Omega^\delta. \end{cases} \tag{11}$$

Notice that

$$-\Delta u = \lambda \frac{u}{|x|^2} - b(x) u^p e^{\alpha u} \ge \lambda \frac{u}{M^2} - b_u u^p e^{\alpha u},$$

where $|x| \le M$ for any $x \in \overline{\Omega}$ and $b_u = \max_{x \in \overline{\Omega_\delta}} b(x)$. Let

$$f(u) = b_u u^p e^{\alpha u} - \lambda \frac{u}{M^2}.$$

Then, by Lemma 6 and Lemma 8, we know that the equation

$$\begin{cases} -\Delta u = \lambda \frac{u}{M^2} - b_u u^p e^{\alpha u},\ x \in \Omega^\delta, \\ u = \infty,\ x \in \partial\Omega^\delta, \end{cases}$$

has a unique non-negative solution $u_{\delta,\infty}(x) \in C^1(\Omega^\delta)$, which means that $u_{\delta,\infty}(x)$ is an upper solution of (10). Then based on the upper and lower theorem, Equation (10) has at least a maximal positive solution and a minimal positive solution. By Lemma 5, we know that all solutions of (10), if exist, must be unique. Denote it as $u_\delta(x)$; then, $0 < \varepsilon\phi_\delta \le u_\delta(x) \le u_{\delta,\infty}(x)$. From Lemma 2, we know that $u_\delta(x)$ decreases as δ decreases, so $\underline{u}(x) := \lim_{\delta\to 0^+} u_\delta(x)$ is well defined in $\overline{\Omega}\backslash\{0\}$. By the regularity of elliptic equations, $\underline{u}(x)$ is a solution of (8); now, we prove that $\underline{u}(x)$ is a minimal positive solution of (8). In fact, for any positive solution u of (8), u is an upper solution of (10); then, Lemma 5 implies that $u \ge u_\delta(x)$, and thus, $u \ge \underline{u}(x)$.

On the other hand, we consider the following equation:

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} - b(x)u^p e^{\alpha u}, \ x \in \Omega^\delta, \\ u = 0, \ x \in \partial\Omega, \\ u = n, \ |x| = \delta. \end{cases} \tag{12}$$

Notice that $u_\delta(x)$ is a lower solution of (12) and $u_{\delta,\infty}(x)$ is an upper solution of (12); then, the upper and lower theorem and Lemma 5 imply that (12) has a unique positive solution $u_{\delta,n}(x)$ such that

$$u_\delta(x) \le u_{\delta,n}(x) \le u_{\delta,\infty}(x). \tag{13}$$

It is easy to see that $\{u_{\delta,n}(x)\}$ is increasing in n, so $\lim_{n\to\infty} u_{\delta,n}(x)$ exists; denote it as $U_\delta(x)$, and $U_\delta(x)$ is a solution of (12) with $n = \infty$. Similarly, $\lim_{\delta\to 0^+} U_\delta(x)$ exists; denote it as $\overline{u}(x)$. Then, it is a positive solution of (8) and it is also the maximal positive solution of (8). This completes the proof. □

The lemma below will be useful for our study. Since the proof is relatively elementary, we omit it.

Lemma 9. *Let Ψ_g be defined as in Lemma 7 with f replaced by g; then, we have*
(i) $\Phi'_g(t) = -\sqrt{2G(\Phi_g(t))}$ and $\Phi''_g(t) = g(\Phi_g(t))$, where $G(t) = \int_{s_0}^t g(s)ds$;
(ii) $\Phi_g(0) = +\infty$ and $\Phi_g(t) > 0$ for $t > 0$.

Theorem 4. *Suppose that there exists a constant $\beta \ge 0$ and $b(x)$ satisfies*

$$\lim_{|x|\to 0^+} \frac{b(x)}{|x|^\beta} = c, \tag{14}$$

if $\lambda > H$, then (8) has a unique positive solution u such that

$$\lim_{|x|\to 0^+} \frac{u(x)}{\Phi_g(\xi|x|^r)} = 1, \tag{15}$$

where Φ_g is the inverse of Ψ_g and $g(t) = t^p e^{\alpha t}$, $\xi = \sqrt{\frac{c}{r}}$, $r = \frac{\beta+2}{2}$.

Proof. Notice that

$$\lim_{u\to+\infty} \frac{G(u)}{g(u)} = \frac{1}{\alpha}, \tag{16}$$

and thus,

$$\begin{aligned} \lim_{t\to 0^+} t\sqrt{G(\Phi_g(t))} &= \lim_{t\to 0^+} \frac{\int_{\Phi_g(t)}^{+\infty} \frac{ds}{\sqrt{2G(s)}}}{\left[G(\Phi_g(t))\right]^{-1/2}} \\ &= \lim_{t\to 0^+} \frac{-\left[2G(\Phi_g(t))\right]^{-1/2}}{-1/2\left[G(\Phi_g(t))\right]^{-3/2} g(\Phi_g(t))} \\ &= \frac{\sqrt{2}}{\alpha}, \end{aligned} \tag{17}$$

by Lemma 9(i), (16) and (17), we have

$$\lim_{t\to 0^+} \frac{t\Phi''_g(t)}{\Phi'_g(t)} = \lim_{t\to 0^+} \frac{tg(\Phi_g(t))}{-\sqrt{2G(\Phi_g(t))}} = -1.$$

From (17), we have for any $\alpha > 0$,

$$\lim_{t\to 0^+} \Phi_g(t)t^\alpha = 0,$$

and thus,

$$\lim_{t\to 0^+} \frac{\Phi_g(t)t^{2/r}}{t\Phi_g'(t)} = 0 \text{ and } \lim_{t\to 0^+} \frac{\Phi_g(t)}{t\Phi_g'(t)} = -\infty.$$

Also notice that (14) holds, and we can easily infer that for any $0 < \varepsilon < \min\{\frac{r}{1+r^2}, c\}$, there exists a positive constant δ such that

$$(c-\varepsilon)|x|^\beta \le b(x) \le (c+\varepsilon)|x|^\beta, \text{ for } 0 < |x| < 2\delta, \tag{18}$$

and

$$-1-\varepsilon < \frac{t\Phi_g''(t)}{\Phi_g'(t)} < -1+\varepsilon, \tag{19}$$

$$-\varepsilon < r(N-1)t + \frac{\lambda\Phi_g(t)t^{2/r}}{t\Phi_g'(t)} < \varepsilon, \tag{20}$$

$$\frac{\lambda\Phi_g(t)}{t\Phi_g'(t)} + 2r(N-1) \le 0, \tag{21}$$

for any $0 < t < 2\delta$ and $\lambda > H$.

By Lemma 4 and $\lambda > H$, we know there exists a positive constant $\sigma_1 < 2\delta$ such that

$$H < \lambda_1\left[|x|^{-2}, \Omega_{\sigma_1}^{2\delta}\right] < \lambda,$$

where

$$\Omega_{\sigma_1}^{2\delta} := \{x \in \Omega | \sigma_1 < |x| < 2\delta\}.$$

Let $\underline{\xi} = \sqrt{\frac{(c+\varepsilon)(1+\varepsilon)}{r-2\varepsilon-\varepsilon r^2}}$, $\overline{\xi} = \sqrt{\frac{(c-\varepsilon)(1-\varepsilon)}{r+\varepsilon r^2+2\varepsilon}}$, for any $\sigma \in (0, \sigma_1)$, $\sigma < |x| < 2\delta$; define

$$\overline{v}_\sigma = \Phi_g\big(\overline{\xi}(|x|-\sigma)^r\big),$$

and for any $\sigma \in (0, \sigma_1)$, $2\sigma < |x| + \sigma < 2\delta$, define

$$\underline{v}_\sigma = \Phi_g(\underline{\xi}(|x|+\sigma)^r).$$

Let $t = \overline{\xi}(|x|-\sigma)^r$; then,

$$\overline{v}_\sigma = \Phi_g\big(\overline{\xi}(|x|-\sigma)^r\big) = \Phi_g(t).$$

Notice that $\Phi_g'(t) = -\sqrt{2G(\Phi_g(t))}$, so

$$\frac{\partial \overline{v}_\sigma}{\partial x_i} = \Phi_g'(t)\overline{\xi} r(|x|-\sigma)^{r-1}\frac{x_i}{|x|},$$

$$\begin{aligned}\frac{\partial^2 \overline{v}_\sigma}{\partial x_i^2} &= \Phi_g''(t)\overline{\xi}^2 r^2(|x|-\sigma)^{2r-2}\frac{x_i^2}{|x|^2} + \Phi_g'(t)\overline{\xi} r(r-1)(|x|-\sigma)^{r-2}\frac{x_i^2}{|x|^2} \\ &\quad + \Phi_g'(t)\overline{\xi} r(|x|-\sigma)^{r-1}\frac{1}{|x|} - \Phi_g'(t)\overline{\xi} r(|x|-\sigma)^{r-1}\frac{x_i^2}{|x|^3};\end{aligned}$$

thus,

$$\begin{aligned}\Delta\overline{v}_\sigma &= \Phi_g''(t)\overline{\xi}r^2(|x|-\sigma)^{2r-2}+\Phi_g'(t)\overline{\xi}r(r-1)(|x|-\sigma)^{r-2}\\ &\quad+\Phi_g'(t)\overline{\xi}r(|x|-\sigma)^{r-1}\frac{N-1}{|x|},\end{aligned}$$

and therefore, for any $\sigma<|x|<2\delta$, we have

$$\begin{aligned}&-\Delta\overline{v}_\sigma-\frac{\lambda}{|x|^2}\overline{v}_\sigma+b(x)g(\overline{v}_\sigma)\\ =\;&-\Phi_g''(t)r^2\overline{\xi}(|x|-\sigma)^{2r-2}-\Phi_g'(t)r(r-1)\overline{\xi}(|x|-\sigma)^{r-2}\\ &-\Phi_g'(t)r\overline{\xi}(|x|-\sigma)^{r-1}\frac{N-1}{|x|}-\frac{\lambda}{|x|^2}\Phi_g(t)+b(x)g(\Phi_g(t))\\ \geq\;&-\Phi_g'(t)\overline{\xi}(|x|-\sigma)^{r-2}\Bigg[\frac{r^2t\Phi_g''(t)}{\Phi_g'(t)}+r(r-1)+r(N-1)\frac{|x|-\sigma}{|x|}\\ &+\frac{\lambda\Phi_g(t)(|x|-\sigma)^2}{t\Phi_g'(t)|x|^2}-\frac{(c-\varepsilon)\Phi_g''(t)(|x|-\sigma)^2|x|^\beta}{t\Phi_g'(t)}\Bigg].\end{aligned}\tag{22}$$

From $\beta\geq0$, we have

$$-\frac{(c-\varepsilon)\Phi_g''(t)(|x|-\sigma)^2|x|^\beta}{t\Phi_g'(t)}\geq-\frac{(c-\varepsilon)t\Phi_g''(t)}{\overline{\xi}^2\Phi_g'(t)}.$$

By (19) and (20), we can obtain

$$\begin{aligned}&-\Phi_g'(t)\overline{\xi}(|x|-\sigma)^{r-2}\Bigg[\frac{r^2t\Phi_g''(t)}{\Phi_g'(t)}+r(r-1)+r(N-1)\frac{|x|-\sigma}{|x|}\\ &+\frac{\lambda\Phi_g(t)(|x|-\sigma)^2}{t\Phi_g'(t)|x|^2}-\frac{(c-\varepsilon)t\Phi_g''(t)}{\overline{\xi}^2\Phi_g'(t)}\Bigg]\\ \geq\;&-\varepsilon\Phi_g'(t)\overline{\xi}(|x|-\sigma)^{r-2}\\ \geq\;&0.\end{aligned}$$

Similarly, notice that $2\sigma<|x|+\sigma<2\delta$ implies $b(x)<(c+\varepsilon)|x|^\beta$ and $\sigma<|x|<2\delta$; thus, by (19)–(21), we have

$$\begin{aligned}&-\Delta\underline{v}_\sigma-\tfrac{\lambda}{|x|^2}\underline{v}_\sigma+b(x)g(\underline{v}_\sigma)\\ &\leq-\Phi_g'(s)\underline{\xi}(|x|+\sigma)^{r-2}\Big[\tfrac{r^2s\Phi_g''(s)}{\Phi_g'(s)}+r(r-1)+r(N-1)\tfrac{|x|+\sigma}{|x|}\\ &+\tfrac{\lambda\Phi_g(s)(|x|+\sigma)^2}{s\Phi_g'(s)|x|^2}-\tfrac{(c+\varepsilon)\Phi_g''(s)(|x|+\sigma)^2|x|^\beta}{s\Phi_g'(s)}\Big]\\ &\leq-\Phi_g'(s)\underline{\xi}(|x|+\sigma)^{r-2}\Big[r^2(-1+\varepsilon)+r(r-1)+2r(N-1)+\tfrac{\lambda\Phi_g(s)}{s\Phi_g'(s)}+\tfrac{(c+\varepsilon)(1+\varepsilon)}{\underline{\xi}^2}\Big]\\ &\leq\varepsilon\Phi_g'(s)\underline{\xi}(|x|+\sigma)^{r-2}\leq0,\end{aligned}$$

where $s=\underline{\xi}(|x|+\sigma)^r$.

Let u be any positive solution of (8) and consider the equation

$$\begin{cases}-\Delta w=\frac{\lambda}{|x|^2}w-b(x)g(w),\ x\in\Omega_\sigma^{2\delta},\\ w=\infty,\ |x|=2\delta,\\ w=0,\ |x|=\sigma,\end{cases}$$

using similar methods as that in Theorem 3 and for any $\sigma \in (0, \sigma_1)$, $H < \lambda_1\left[|x|^{-2}, \Omega_{\sigma}^{2\delta}\right] < \lambda_1\left[|x|^{-2}, \Omega_{\sigma_1}^{2\delta}\right] < \lambda$, it is easy to see that the above problem has a minimum positive solution w. Let $v = \overline{v}_\sigma + w$; then,

$$-\Delta v \geq \frac{\lambda}{|x|^2} v - b(x)g(v),\ x \in \Omega_{\sigma}^{2\delta},$$

here, we use the fact that

$$g(\overline{v}_\sigma + w) \leq g(\overline{v}_\sigma) + g(w).$$

In addition,

$$\begin{cases} v|_{|x|=2\delta} = \infty > u|_{|x|=2\delta}, \\ v|_{|x|=\sigma} = \infty > u|_{|x|=\sigma}, \end{cases}$$

by Lemma 5, we have

$$w + \overline{v}_\sigma \geq u,\ x \in \Omega_{\sigma}^{2\delta}.$$

Similarly,

$$u + w \geq \underline{v}_\sigma,\ x \in \Omega_{\sigma}^{2\delta},$$

let $\sigma \to 0$; then, we obtain

$$\Phi_g\big(\overline{\xi}(|x|)^r\big) + 2w \geq u + w \geq \Phi_g\big(\underline{\xi}(|x|)^r\big),\ 0 < |x| < 2\delta,$$

and thus,

$$\lim_{|x|\to 0^+} \frac{\Phi_g\big(\overline{\xi}(|x|)^r\big)}{\Phi_g\big(\underline{\xi}(|x|)^r\big)} \geq \lim_{|x|\to 0^+} \frac{u}{\Phi_g\big(\underline{\xi}(|x|)^r\big)} \geq 1,$$

on the other hand,

$$\lim_{\varepsilon\to 0^+} \underline{\xi} = \lim_{\varepsilon\to 0^+} \overline{\xi} = \xi;$$

therefore,

$$\lim_{|x|\to 0^+} \frac{u}{\Phi_g\big(\xi(|x|)^r\big)} = 1. \tag{23}$$

Suppose u_1 and u_2 are two arbitrary positive solutions of (8); then, (23) implies that

$$\lim_{|x|\to 0^+} \frac{u_1(x)}{u_2(x)} = 1,$$

which means that for any $0 < \varepsilon < 1$, there exists $\delta > 0$ small enough such that when $|x| < \delta$,

$$(1-\varepsilon)u_2(x) < u_1(x) < (1+\varepsilon)u_2(x), \tag{24}$$

consider the following equation:

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} - b(x)u^p e^{\alpha u},\ x \in \Omega^{\delta/2}, \\ u = 0,\ x \in \partial\Omega, \\ u = u_1(x),\ |x| = \frac{\delta}{2}. \end{cases} \tag{25}$$

Notice that

$$\begin{aligned} -\Delta[(1-\varepsilon)u_2(x)] &= \lambda \frac{(1-\varepsilon)u_2(x)}{|x|^2} - b(x)(1-\varepsilon)[u_2(x)]^p e^{\alpha u_2(x)} \\ &\leq \lambda \frac{(1-\varepsilon)u_2(x)}{|x|^2} - b(x)(1-\varepsilon)^p[u_2(x)]^p e^{\alpha(1-\varepsilon)u_2(x)},\ x \in \Omega^{\delta/2}, \end{aligned}$$

by (24) and notice that $u_2(x)$ is a positive solution of (8); we have

$$\begin{aligned}(1-\varepsilon)u_2(x) &= 0 \le u_1(x),\ x\in\partial\Omega,\\ (1-\varepsilon)u_2(x) &\le u_1(x),\ |x| = \frac{\delta}{2};\end{aligned}$$

then, Lemma 5 implies that

$$(1-\varepsilon)u_2(x) \le u_1(x),\ \text{for any } x\in\Omega^{\delta/2}.$$

Similarly, we can obtain

$$u_1(x) \le (1+\varepsilon)u_2(x),\ \text{for any } x\in\Omega^{\delta/2}.$$

Let $\varepsilon\to 0^+$; we have

$$u_2(x) \equiv u_1(x),\ \text{for any } x\in\Omega\backslash\{0\}.$$

This completes the proof. □

4. Numerical Example

In this section, we give some numerical examples to verify Theorem 4. For simplicity, we suppose that $N = 3$, $\Omega = B(0;1)$ and $b(x) = c|x|^\beta$, which shows that (14) holds true. According to [21], we know that all the solutions of (8) are radially symmetric. Let $u(x) = u(\rho)$, $\rho = |x|$; then, (8) can be transformed to the following second-order ordinary differential equations with singular coefficients:

$$\begin{cases} -u''(\rho) - \frac{(N-1)}{\rho}u'(\rho) = \frac{\lambda u(\rho)}{\rho^2} - c\rho^\beta u^p(\rho)e^{\alpha u(\rho)},\ 0<\rho<1,\\ u(\rho) > 0, 0<\rho<1,\\ u(1) = 0.\end{cases} \tag{26}$$

In order to use the numerical computation method for the ordinary differential equation, we assume that $\rho = 1-t$ and set $u(\rho) = y_1(t)$, $u'(\rho) = -y_1'(t) = -y_2(t)$; then, (26) can be transformed into

$$\begin{cases} y_1'(t) = y_2(t)\\ y_2'(t) = \frac{N-1}{1-t}y_2(t) - \frac{\lambda y_1(t)}{(1-t)^2} + c(1-t)^\beta y_1^p(t)e^{\alpha y_1(t)},\ 0<t<1,\\ y_1(0) = 0.\end{cases} \tag{27}$$

Example 1. *Assume that $\beta = 1$, $p = 2$, $\alpha = 1$, $c = 4$ and $\lambda = 1$. It is easy to verify that all conditions in Theorem 4 hold. Then, by Theorem 4, we have*

$$\lim_{t\to 1^-}\frac{y_1(t)}{\Phi_g(\xi(1-t)^r)} = 1,$$

where $\xi = \sqrt{\frac{8}{3}}$, $r = \frac{3}{2}$.

Because the function Φ_g can not be easily obtained from the inverse of Ψ_g, we consider the numerical solution of the function Φ_g; notice that the function Φ_g satisfies

$$\begin{cases} \Phi_g'(t) = -\sqrt{2G(\Phi_g(t))},\\ \Phi_g(0) = +\infty.\end{cases}$$

Let $\Phi_g(\xi(1-t)^r) = z(t)$ and $s = \xi(1-t)^r$; then, we have

$$z'(t) = \xi r(1-t)^{r-1}\sqrt{2G(z(t))},\ z(1) = +\infty, \tag{28}$$

where

$$G(u) = e^u(u^2 - 2u + 2) - 2.$$

The graph of numerical boundary blowup solutions for Equations (27) and (28) is presented in Figure 1.

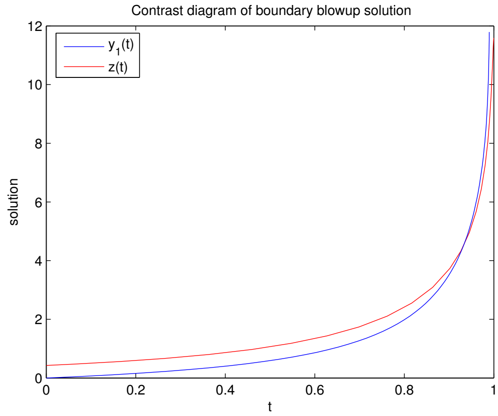

Figure 1. Contrast diagram of boundary blowup solution, $y_1(t) = u(\rho), z(t) = \Phi_g(\xi\rho^r), \rho = |x|$, $t = 1 - \rho$. The initial condition for Equation (27) is $[0\ 0.6175]$, and the initial condition for Equation (28) is 0.4282.

5. Conclusions

In this paper, a class of semilinear elliptic differential equations was investigated. By constructing the inverse function and using the method of upper and lower solutions and the principle of comparison, the existence of maximum positive solution and minimum positive solution was explored. Furthermore, the uniqueness of the positive solution and its asymptotic estimation at the origin were investigated. The results show that the asymptotic estimation is similar to that of the corresponding boundary blowup problems. It is worth mentioning that the above methods can also be used to deal with the case of $\alpha = 0$.

From Theorem 4 and its proof, we can easily see that the asymptotic behavior of the solution for (8) at the origin does not depend on λ and N; it only depends on the asymptotic behavior of $b(x)$ at the origin and the asymptotic behavior of g at infinity. Comparing Theorem 4 with Theorem 2, we see that the asymptotic behavior of the solution at the origin when $\alpha = 0$ is fundamentally different from that when $\alpha \neq 0$. From this point of view, we believe that $\alpha = 0$ is a branch point. On the other hand, we only consider the case of $\beta \geq 0$; compared with the work of Wei [17] and Du [8], we have

Conjecture. The conclusion of Theorem 4 is also true when $-2 < \beta < 0$.

In fact, the following numerical examples (see Figure 2) also support this conjecture:

Example 2. *Consider (27) and (28) again. Assume that $\beta = -1$, $p = 2$, $\alpha = 1$, $c = 4$ and $\lambda = 1$. Then, $\xi = 2\sqrt{2}$, $r = \frac{1}{2}$.*

Now, we discuss how the solution of Equation (8) changes at the origin as the parameters $\beta$and c change. We only consider the single-parameter variation case. By Theorem 4, we have

$$\frac{\partial u}{\partial c} \approx \Phi_g'(s)\frac{|x|^r}{2\sqrt{cr}},\ \frac{\partial u}{\partial \beta} \approx \Phi_g'(s)\left[-\frac{\sqrt{c}}{4r^{3/2}} + \ln|x|\right]|x|^r,\ 0 < |x| << 1,$$

notice that $\Phi_g'(s) < 0$, for $s > 0$; then,

$$\frac{\partial u}{\partial c} < 0,\ \frac{\partial u}{\partial \beta} > 0,\ 0 < |x| << 1,$$

which shows that c can reduce the blowup rate of the solution, while β, on the contrary, can accelerate the blowup of the solution. The numerical results also support these results (see Figures 3 and 4).

Let $p = 2$, $\alpha = 1$, $\lambda = 1$. When c changes, we assume $\beta = 1$; when β changes, we assume $c = 4$.

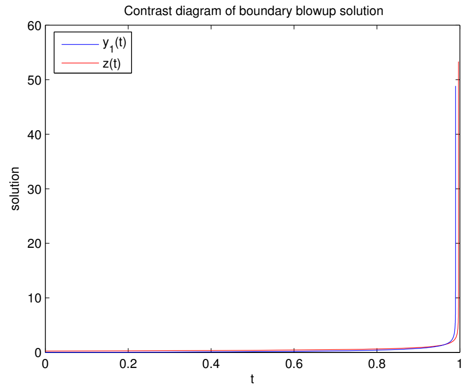

Figure 2. Contrast diagram of boundary blowup solution when $\beta = -1$, $y_1(t) = u(\rho), z(t) = \Phi_g(\xi\rho^r), \rho = |x|, t = 1 - \rho$. The initial condition for Equation (27) is $[0\ 0.14]$, and the initial condition for Equation (28) is 0.25.

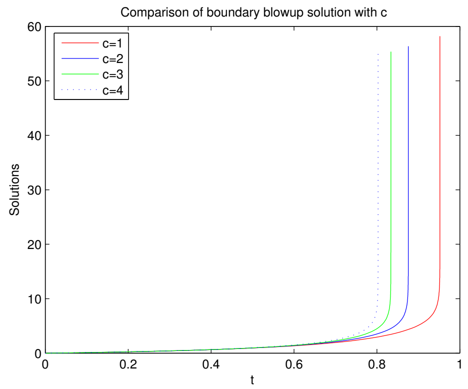

Figure 3. Boundary blowup solution when $c = 1, 2, 3, 4$, respectively. The initial condition for Equation (27) is $[0\ 1]$.

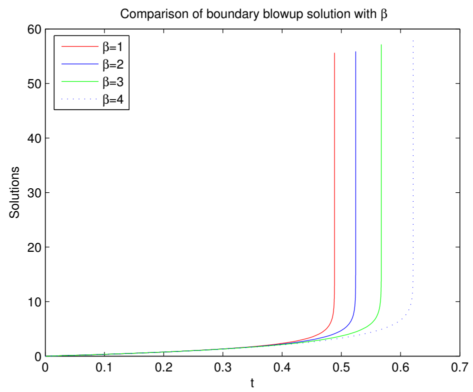

Figure 4. Boundary blowup solution when $\beta = 1, 2, 3, 4$, respectively. The initial condition for Equation (27) is $[0\ 3]$.

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References

1. Bieberbach, L. $\delta u = e^u$ und die automorphen Funktionen. *Math. Ann.* **1916**, *77*, 173–212. [CrossRef]
2. Keller, J.B. On solution of $\delta u = f(u)$. *Commun. Pure Appl. Math.* **1957**, *10*, 503–510. [CrossRef]
3. Osserman, R. On the inequality $\delta u \geq f(u)$. *Pac. J. Math.* **1957**, *7*, 1641–1647. [CrossRef]
4. Lair, A.V.; Mohammed, A. Necessary and sufficient conditions for the existence of large solutions to semilinear elliptic equations with gradient terms. *J. Differ. Equ.* **2023**, *374*, 593–631. [CrossRef]
5. Wei, Y.F.; Chen, C.S.; Yang, H.W.; Xiu, Z.H. Existence and nonexistence of entire large solutions to a class of generalized quasilinear Schrödinger equations. *Appl. Math. Lett.* **2022**, *133*, 108296. [CrossRef]
6. Esposito, F.; Trombetta, A. Qualitative properties of blow-up solutions to some semilinear elliptic systems in non-convex domain. *Nonlinear Anal. Real World Appl.* **2023**, *71*, 103804. [CrossRef]
7. Ji, J.W.; Deng, H.Y.; Jiang, F.D. On the boundary blow-up problem for real $n-1$ Monge–Ampère equation. *Nonlinear Anal.* **2025**, *250*, 113669. [CrossRef]
8. Du, Y.H. *Order Structure and Topological Methods in Nonlinear Partial Differential Equations, Maximum Principles and Applications*; World Scientifi: Beijing, China, 2006; Volume 1.
9. Brizes, H.; Nirenberg, L. Positive solutions of nonlinear elliptic equations involving critical sobolev exponents. *Commun. Pure Appl. Math.* **1983**, *36*, 437–477. [CrossRef]
10. Jannelli, E. The role played by space dimension in elliptic critical problems. *J. Differ. Equ.* **1999**, *156*, 407–426. [CrossRef]
11. Lopezgomez, J. Large solutions, metasolutions, and asymptotic behaviour of the regular positive solutions of sublinear parabolic problems. *Electron. J. Differ. Equ. Conf.* **2000**, *5*, 135–171.
12. Cîrstea, F.C.; Fărcăşeanu, M. Sharp existence and classification results for nonlinear elliptic equations in $\mathbb{R}^n \setminus \{0\}$ with hardy potential. *J. Differ. Equ.* **2021**, *292*, 461–500. [CrossRef]
13. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order*; Springer: New York, NY, USA, 1983.
14. Chirillo, G.; Montoro, L.; Muglia, L.; Sciunzi, B. Existence and regularity for a general class of quasilinear elliptic problems involving the hardy potential. *J. Differ. Equ.* **2023**, *349*, 1–52. [CrossRef]
15. Cao, L. Existence of stable standing waves for the nonlinear schrödinger equation with the hardy potential. *Discret. Contin. Syst.-B* **2023**, *28*, 1342–1366. [CrossRef]
16. Cîrstea, F.C. A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials. *Mem. Am. Math. Soc.* **2014**, *227*, 1–97.
17. Wei, L.; Du, Y.H. Exact singular behavior of positive solutions to nonlinear elliptic equations with a Hardy potential. *J. Differ. Equ.* **2017**, *262*, 3864–3886. [CrossRef]
18. Ghoussoub, N.; Yuan, C. Multiple solutions for quasilinear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Am. Math. Soc.* **2000**, *352*, 5703–5743. [CrossRef]
19. Cheng, X.Y.; Feng, Z.S.; Wei, L. Positive solutions for a class of elliptic equations. *J. Differ. Equ.* **2021**, *275*, 1–26. [CrossRef]
20. Wei, L.; Feng, Z.S. Isolated singularity for semilinear elliptic equations. *Discret. Contin. Dyn. Syst.* **2015**, *35*, 3239–3252. [CrossRef]
21. Gidas, B.; Ni, W.M.; Nirenberg, L. Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **1979**, *68*, 209–243. [CrossRef]