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Cyclic Structure, Vertex Degree and Number of Linear Vertices in Minimal Strong Digraphs

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Abstract: Minimal Strong Digraphs (MSDs) can be regarded as a generalization of the concept of tree to directed graphs. Their cyclic structure and some spectral properties have been studied in several articles. In this work, we further study some properties of MSDs that have to do with bounding the length of the longest cycle (regarding the number of linear vertices, or the maximal in- or outdegree of vertices); studying whatever consequences from the spectral point of view; and giving some insight about the circumstances in which an efficient algorithm to find the longest cycle contained in an MSD can be formulated. Among other properties, we show that the number of linear vertices contained in an MSD is greater than or equal to the maximal (respectively minimal) in- or outdegree of any vertex of the MSD and that the maximal length of a cycle contained in an MSD is lesser than or equal to $2n - m$ where n, m are the order and the size of the MSD, respectively; we find a bound for the coefficients of the characteristic polynomial of an MSD, and finally, we prove that computing the longest cycle contained in an MSD is an NP-hard problem.

Keywords: minimal strong digraphs; maximum length directed cycles; linear vertex; external chain; characteristic polynomial; NP-hard problem

MSC: 68R10

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1. Introduction

A Minimal Strong Digraph (MSD) is a strong digraph in which the deletion of any arc yields a non-strongly connected digraph. In [1,2] a compilation of the properties properties of MDSs can be found. Additionally, in [2] a comparative analysis between MSDs and non-directed trees, where a series of the analog properties of both types of graphs, is presented. In this sense, MSDs gain interest as a counterpart of trees in the context of directed graphs.

There are several other reasons to justify the interest in studying MSDs. One of them is the relationship between MSDs and nearly reducible $(0, 1)$-matrices (via the adjacency matrix; see, for instance, [3,4]) and the non-negative inverse eigenvalue problem (see [5]): given real numbers $k_1, k_2, \ldots, k_n$, find the necessary and sufficient conditions for the existence of a non-negative matrix A of order n with characteristic polynomial $x^n + k_1 x^{n-1} + k_2 x^{n-2} + \cdots + k_n$. The coefficients of the characteristic polynomial are closely related to the cycle structure of the weighted digraph with adjacency matrix A by means of the theorem of the coefficients [6], and the irreducible matricial realizations of the polynomial (which are identified with strongly connected digraphs [3]) can easily be reduced to the class of Minimal Strong Digraphs. Hence, a better understanding of the cyclic structure of MSDs could lead to results on spectral theory.

Another goal for our work is trying to take advantage of the fact that minimality among SDs is a very restrictive condition. For instance, it is well known that the size of an MSD of order n is bounded by $2(n-1)$. We think that the fact that the class of MSDs is comparatively small, together with the properties obtained in [2], pointing out relationships between the size of the longest cycle in an MSD and the number of linear vertices, could lead to finding an algorithm of polynomial complexity to find the longest cycle in an MSD. Note that finding the longest cycle on a SD is an NP-hard problem.

Our work plan is, thus, to further study the properties of MSDs that could give a better understanding of their cyclic structure, especially those having to do with bounding the length of the longest cycle (regarding the number of linear vertices, or the maximal in- or outdegree of vertices); studying whatever consequences from the spectral point of view; and finally trying to devise an efficient algorithm to find the longest cycle in an MSD. The first steps are accomplished, but we have to accept that the restrictions we obtain to bound the length of cycles in an MSD are not enough to simplify the search of the longest cycle. In fact, we prove that finding the longest cycle in an MSD is NP-hard. Nevertheless, we think that the new properties of MSDs that we are able to prove are interesting in and of themselves, insofar as they progress the way of understanding the cyclic structure of MSDs, and hence they can lead to advances in spectral theory.

The outline of the article is as follows: In Section 2, we introduce some notations and review several results on MSDs. In Section 3, we study the relationship between the length of the longest cycle, the number of linear vertices, and the maximal in- or outdegree of vertices. We also state some MSD properties, regarding chains and its contraction, that arise from the ear decomposition. In Section 4, we state a bound for the coefficients of the characteristic polynomial of an MSD, extending the results of [2]. In Section 5, we prove that the problem of finding the longest cycle in an MSD is NP-hard. Finally, we draw some conclusions.

2. Notation and Basic Properties

In this paper, we use some concepts and basic results about graphs that are described below, in order to fix the notation [1,2,7–13].

Let $D = (V, A)$ be a digraph. If $(u, v) \in A$ is an arc of D, we say that u is the tail (or initial vertex) and v the head (or final vertex) of the arc, and we denote the arc by uv. We shall consider only directed paths and directed cycles. We shall denote by $n = |V|$ and by $m = |A|$ the order and the size of D, respectively.

In a strongly connected digraph, the indegree $d^-(v)$ and the outdegree $d^+(v)$ of every vertex v are greater than or equal to 1. We shall say that v is a linear vertex if it satisfies $d^+(v) = d^-(v) = 1$.

An arc uv in a digraph D is transitive if there exists another uv-path disjoint to the arc uv. A digraph is called a minimal digraph if it has no transitive arcs.

The contraction of a subdigraph consists in the reduction in the subdigraph to a unique vertex $\bar{v}$. Note that the contraction of a cycle of length q in an SD yields another SD. In such a process, $q-1$ vertices and q arcs are eliminated. Given a cycle C_q, let $\bar{v}$ be the vertex corresponding to C_q after contraction. We shall denote by $d^-(C_q) = d^-(\bar{v})$ (respectively $d^+(C_q) = d^+(\bar{v})$). Note that $d^+(C_q) = \sum_{v \in C_q}(d^+(v) - 1)$ (and the same with $d^-(v)$).

Some basic properties concerning MSDs can be found in [1,2,8,14,15].

We summarize some of them: The size of an MSD digraph D of order $n \geq 2$ verifies $n \leq m \leq 2(n-1)$ [1]. The contraction of a cycle in an MSD preserves the minimality, that is, it produces another MSD; hence, if we contract a strongly connected subdigraph in a minimal digraph, the resulting digraph is also minimal, and each MSD of order $n \geq 2$ has at least two linear vertices.

If C_q is a cycle contained in an MSD D, then the number of linear vertices of D is greater than or equal to $\left\lfloor \frac{q+1}{2} \right\rfloor$. An MSD factors into a rooted spanning tree and a forest of reversed rooted trees (Theorem 20 [2]). Finally, we will use the next result.

Lemma 1 ([2])**.** *If an MSD contains a cycle C_2, then the vertices on the cycle are linear vertices or cut points.*

3. Lower Bounds of the Number of Linear Vertices of an MSD

Let D be an MSD and C_q a cycle contained in D.

In this section, we show some results obtained through the analysis of the degree of the vertices, especially those with a high degree.

Proposition 1. *Let $D = (V, A)$ be an MSD, λ the number of linear vertices of D, and $v \in V$ a vertex such that v is contained in each cycle of D. Then, $\lambda \geq max(d^-(v), d^+(v))$.*

Proof. If D is a cycle, then $d^-(v) = d^+(v) = 1$; therefore, $\lambda \geq 2 > max(d^-(v), d^+(v)) = 1$, and the proof is completed.

Otherwise, let $C_q = v, u_1, \dots, u_{q-1}, v$ be a cycle contained in D. By definition of MSD, each arc of D is contained in at least one directed cycle of D, or else D would not be strongly connected. Since v is contained in each cycle of D, then each arc wu_i such that $w \notin C_q$ is contained in a cycle $v, \dots, w, u_i, \dots, v$ for $1 \leq i \leq q-1$. In a similar way, each arc u_iw such that $w \notin C_q$ is contained in a cycle $v, \dots, u_i, w, \dots, v$ for $1 \leq i \leq q-1$.

We shall prove that in C_q, there must exist at least one linear vertex. Let us, in fact, suppose, by contradiction, that $u_i \in C_q$ is not a linear vertex for $1 \leq i \leq q-1$. Hence, $d^-(u_1) = 1$ or else the arc vu_1 would be transitive in D. In fact, if $d^-(u_1) > 1$, since v is contained in each cycle, v is a vertex reached by walking in reverse direction from u_1 using an arc u'_1u_1 different from vu_1 (such an arc exists because of $d^-(u_1) > 1$), and then a vu_1-path (not containing the arc vu_1) can be obtained by concatenation of a vu'_1-path with the arc u'_1u_1.

Then, $d^+(u_1) > 1$ since $d^-(u_1) = 1$ and we are assuming that u_1 is not linear. Let $u''_1 \neq u_2$ be the vertex defined by the corresponding arc $u_1u''_1 \in D$.

Now the following result will be proved for all u_i, $2 \leq i \leq q-1$: $d^-(u_i) = 1$ and there is an arc $u_iu''_i$ with $u''_i \neq u_{i+1}$. To show this, the following reasoning is applied iteratively for each vertex, starting from u_2. First, we remark that $d^-(u_i) = 1$. Otherwise, the arc $u_{i-1}u_i$ would be transitive in D because an $u_{i-1}u_i$-path would exist, not containing the arc $u_{i-1}u_i$. In fact, since v is contained in each cycle, v is a vertex reached walking in reverse direction from u_i starting with an arc u'_iu_i different from $u_{i-1}u_i$ (such an arc exists since $d^-(u_i) > 1$). Also, v is a vertex reached walking from u_{i-1} starting with the arc $u_{i-1}u''_{i-1}$. Then, a $u_{i-1}u_i$-path would be obtained by concatenation of the arc $u_{i-1}u''_{i-1}$ with the $u''_{i-1}v$-path, the vu'_i-path, and the arc u'_iu_i.

$d^+(u_i) > 1$ also holds because $d^-(u_i) = 1$ and, by hypothesis, u_i is not a linear vertex. Let $u''_i \neq u_{i+1}$ be the vertex defined by the arc $u_iu''_i$ belonging to D.

Finally, let us show that the arc $u_{q-1}v$ is transitive. In fact, since v is contained in each cycle, v is a vertex reached walking from u_{q-1}, starting with the arc $u_{q-1}u''_{q-1}$. The $u_{q-1}v$-path obtained by concatenation of the arc $u_{q-1}u''_{q-1}$ with the $u''_{q-1}v$-path proves that $u_{q-1}v$ is transitive. This fact contradicts the minimality of D.

We have still to prove that the linear vertices reached for each outgoing (respectively, incoming) arc from (respectively, to) v are all different. Let vu_1 and vu'_1 be two arcs in D. From vu_1, as we have seen, we can construct a path $v, u_1, \dots, u_k$ such that $d^-(u_i) = 1$ for $1 \leq i \leq k$ and $d^+(u_i) > 1$ for $1 \leq i \leq k-1$, and u_k is linear (note that k can be 1, but it must exist) as we have proved previously. Now, in a similar way, we construct a path $v, u'_1, \dots, u'_l$ such that $d^-(u'_j) = 1$ for $1 \leq i \leq l$ and $d^+(u'_j) > 1$ for $1 \leq i \leq l-1$ and u'_l is linear.

The paths $v, u_1, \dots, u_k$ and $v, u'_1, \dots, u'_l$ cannot rejoin after they leave v since all the indegrees of their vertices are 1. Hence, $u_k \neq u'_l$. The proof is completed. □

Proposition 2. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, $v \in V$ a vertex of D, and λ the number of linear vertices contained in D. Then, $\lambda \geq max(d^-(v), d^+(v))$.*

Proof. If $D = C_n$ then $d^-(v) = d^+(v) = 1$, therefore $\lambda = q \geq 2$ and the proof is completed.

Otherwise, we obtain an MSD D' from D by the contraction of all cycles that do not contain the vertex v. Note that v is a vertex contained in each cycle of D'. Then, by Proposition 1 $\lambda_{D'} \geq max(d^-(v), d^+(v))$ where $\lambda_{D'}$ is the number of linear vertices of D'. Note also that v preserves in D' all its incident arcs. Next we expand the cycles contracted previously. In this process, the linear vertices are maintained. Indeed, if we expand a linear vertex corresponding to a cycle of length greater than two, this fact is obvious. And, if we expand one corresponding to a cycle of length two, the result follows from Lemma 1, since for cycles of length two, the contracted vertex in D' will contain at least one existing linear vertex in D; hence, the number of linear vertices in D is not less than the number of linear vertices contained in D'. The proof is completed. □

Corollary 1. *Let $D = (V, A)$ be an MSD, C_q a cycle contained in D, and μ the number of linear vertices contained in D but not contained in C_q. Then, $\mu \geq max(d^-(C_q), d^+(C_q))$.*

Proof. If $D = C_q$, then $\mu = d^-(C_q) = d^+(C_q) = 0$, and the proof is completed.

Otherwise, we obtain an MSD D' from D by contracting C_q in a unique vertex v'. Note that the number of linear vertices of D' is precisely μ. The application of Proposition 2 then implies that $\mu \geq max(d^-(v'), d^+(v')) = max(d^-(C_q), d^+(C_q))$ and we are finished. □

As we mentioned in Section 2, if there is a cycle $C_q \in D$, the number of linear vertices of D is greater than or equal to $\left\lfloor \frac{q+1}{2} \right\rfloor$; see [8]. We ratify this result with a new, shorter proof, by using the previous properties.

Corollary 2. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, C_q a cycle contained in D, and λ the number of linear vertices contained in D. Then, $\lambda \geq \left\lfloor \frac{q+1}{2} \right\rfloor$.*

Proof. Let ν be the number of linear vertices contained in C_q, and μ the rest of linear vertices of D. Then, $\lambda = \mu + \nu$, and we know by Corollary 1 that $\mu \geq max(d^+(C_q), d^-(C_q))$. Since $d^+(C_q) + d^-(C_q) \geq q - \nu$, we have that

$$\mu \geq max(d^+(C_q), d^-(C_q)) \geq \left\lceil \frac{q-\nu}{2} \right\rceil, \tag{1}$$

and then

$$\lambda = \mu + \nu \geq \left\lceil \frac{q-\nu}{2} \right\rceil + \nu = \left\lceil \frac{q+\nu}{2} \right\rceil \geq \left\lceil \frac{q}{2} \right\rceil = \left\lfloor \frac{q+1}{2} \right\rfloor. \tag{2}$$

The proof is completed. □

As a consequence of Corollary 2, we obtain an upper bound for the maximum length of a cycle contained in an MSD.

Corollary 3. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, C_l a cycle with maximal length l contained in D, and λ the number of linear vertices contained in D. Then, $l \leq 2\lambda$.*

Proof. By Corollary 2, we know that

$$\lambda \geq \left\lfloor \frac{l+1}{2} \right\rfloor, \tag{3}$$

then

$$l \leq 2\lambda. \tag{4}$$

The proof is completed. □

Since every vertex contained in an MSD must be contained in at least one directed cycle, we can obtain two different bounds for the number of linear vertices, one from the vertex degree and one from the cycle length. The next result somehow combines the two aforementioned bounds.

Corollary 4. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, C_q a directed cycle of length q contained in D, $u \in C_q$ a vertex of D, $d(u) = d^+(u) + d^-(u)$, and λ the number of linear vertices contained in D. Then,*

$$\lambda \geq \left\lfloor \frac{q + d(u)}{2} \right\rfloor - 1. \tag{5}$$

Proof. As we did in the proof of Corollary 2, we call ν the number of linear vertices contained in C_q and μ the rest of linear vertices of D. The value of ν tends to be smaller, as there are more paths between the vertices contained in the cycle C_q. Then, for any vertex u contained in the cycle C_q, we obtain the following inequality:

$$\begin{aligned} &\nu + (d^+(C_q) - (d^+(u) - 1)) + (d^-(C_q) - (d^-(u) - 1)) + 1 \geq q \\ &\qquad \Rightarrow d^+(C_q) + d^-(C_q) \geq q - \nu + d(u) - 3. \end{aligned} \tag{6}$$

Combining it with Corollary 1 ($\mu \geq max(d^+(C_q), d^-(C_q))$), we obtain

$$\mu \geq \max(d^+(C_q), d^-(C_q)) \geq \left\lceil \frac{q - \nu + d(u) - 3}{2} \right\rceil \tag{7}$$

and finally

$$\begin{aligned} \lambda = \mu + \nu &\geq \left\lceil \frac{q - \nu + d(u) - 3}{2} \right\rceil + \nu = \left\lceil \frac{q + \nu + d(u) - 1}{2} \right\rceil - 1 \\ &\geq \left\lceil \frac{q + d(u) - 1}{2} \right\rceil - 1 = \left\lfloor \frac{q + d(u)}{2} \right\rfloor - 1. \end{aligned} \tag{8}$$

The proof is completed. □

Corollary 4 can be useful when a vertex u with a high degree is contained in the cycle C_q (see examples in Figures 1 and 2). However, if the vertex u is not contained in the cycle, the number of linear vertices contained in the MSD could be much higher than the number of linear vertices obtained with this bound. For instance, in the examples in Figures 3 and 4), if $q = 10$ and the vertices in the cycle have degree 2 or less, the bound given by Corollary 4 would be 5, but the number of linear vertices would be at least 14; an analogous example is showed in Figure 5.

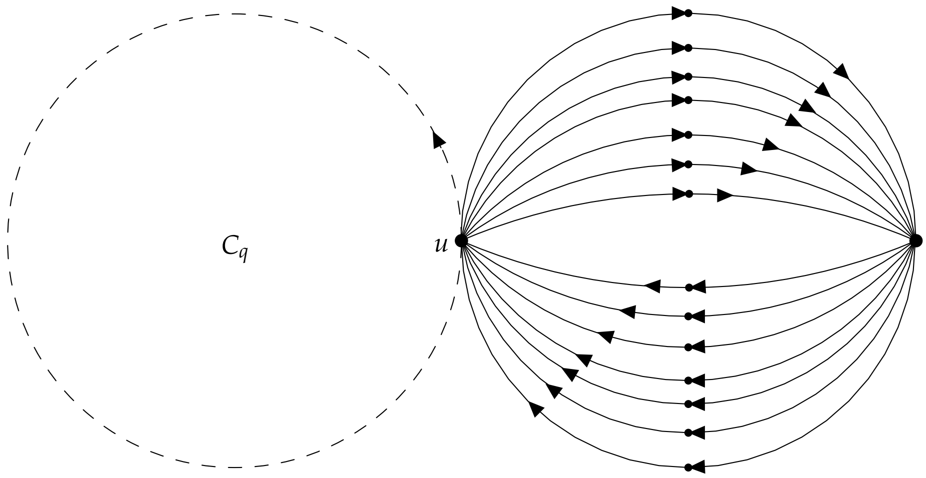

Figure 1. Example 1 for Corollary 4 where C_q contains a vertex u with high in- and out-degree.

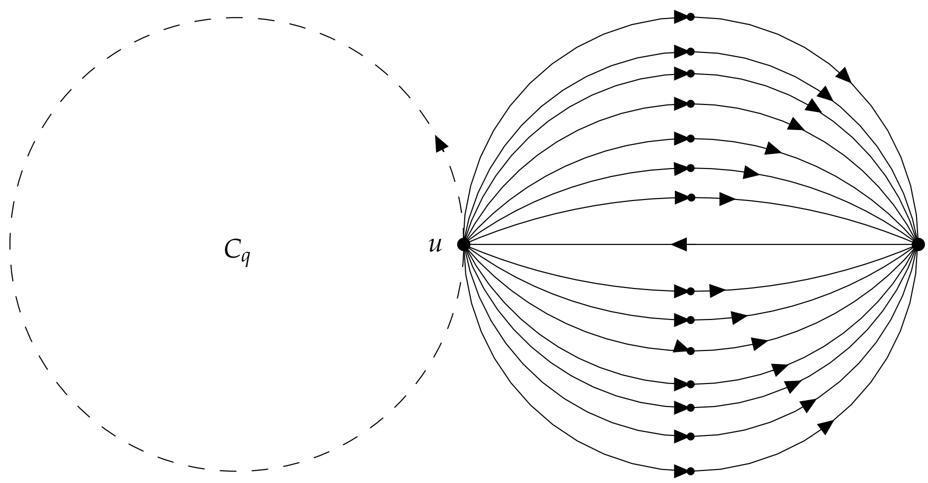

Figure 2. Example 2 for Corollary 4 where C_q contains a vertex u with high out-degree.

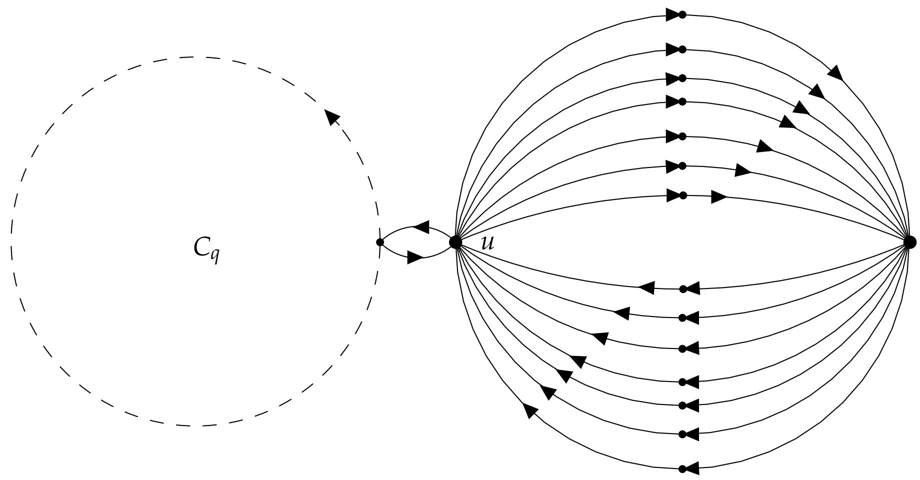

Figure 3. Example 1 of an MSD, in which there is a vertex with high degree (input and output) and is not contained in the cycle C_q.

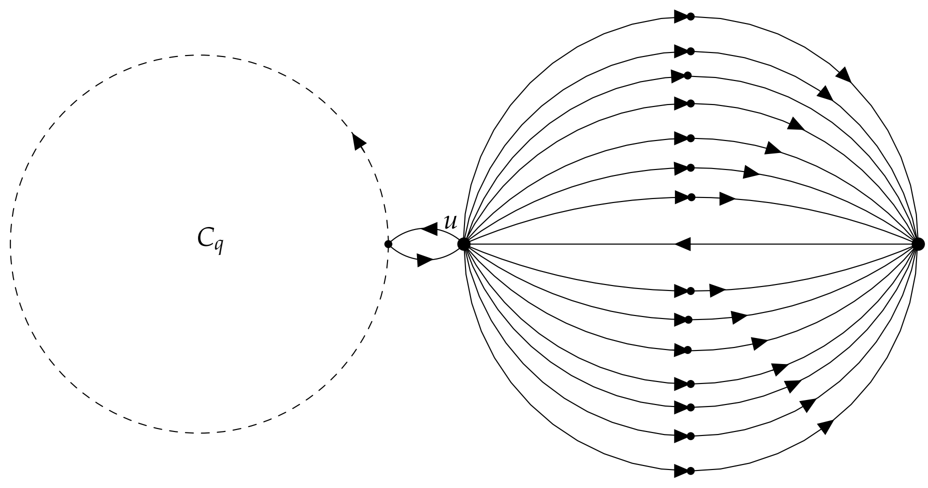

Figure 4. Example 2 of an MSD, in which there is a vertex with high output degree and is not contained in the cycle C_q.

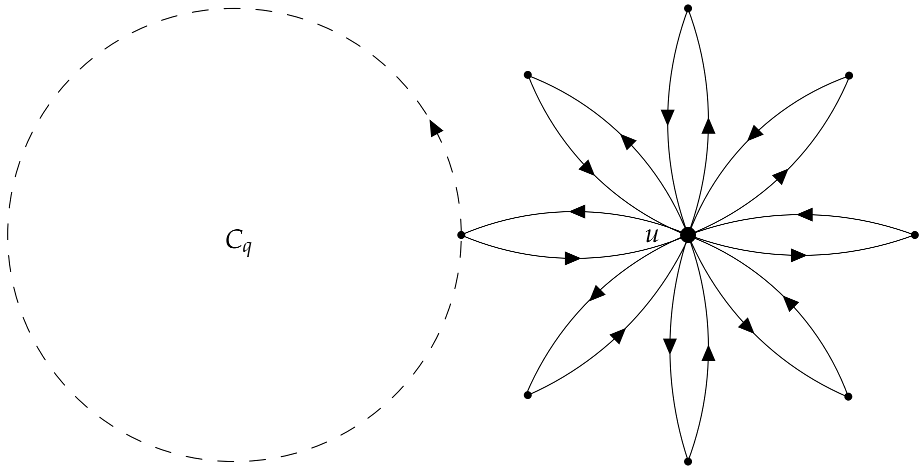

Figure 5. Example 3 of an MSD, in which there is a vertex with high degree (input and output) and is not contained in the cycle C_q.

Proposition 3. *Let $D = (V, A)$ be an MSD, and let C_q be a cycle of length q contained in D. Then, $q \leq 2n - m$.*

Proof. We obtain an MSD D' by contraction of C_q in a unique vertex v', and then $n' = n - q + 1$ and $m' = m - q$. Hence, since

$$m' \leq 2(n' - 1), \tag{9}$$

we obtain

$$m - q \leq 2(n - q), \tag{10}$$

and finally

$$q \leq 2n - m. \tag{11}$$

The proof is completed. □

Other Properties of MSDs

In [2,14], some results about ear decomposition are proved. We use these previous results to show the next properties of MSDs.

Definition 1. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, and let $v_1, \ldots, v_l$ be a path contained in D. We say that the v_1v_l-path is a* chain *with length l if $d^-(v_i) = d^+(v_i) = 1$ for all $1 \leq i \leq l$.*

Note that an isolated linear vertex is a chain of length 1.

Definition 2. *Let $D = (V, A)$ be an MSD of order $n \geq 2$, let $v_1, \ldots, v_l$ be a chain contained in D, and let D' be the digraph obtained from D by the elimination of the v_1v_l-path. We say that the v_1v_l-path is an* external chain *with length l if D' preserves the strong connection.*

Proposition 4. *Let $D = (V, A)$ be an MSD of order $n \geq 2$ and C_q a cycle contained in D such that $D \neq C_q$. Then, in D, there exists at least one external chain.*

Proof. We use the ear decomposition shown in Theorem 20 in [2] in a similar way to how it was used in the proof of the property that affirms that an MSD factors into a rooted spanning tree and a forest of reversed rooted trees.

Let us consider an ear decomposition of D, $\mathcal{E} = P_0, \ldots, P_k$. Since D is an MSD, each ear P_j ($0 \le j \le k$) contains at least one new vertex and two new arcs, with respect to $\bigcup_{i=0}^{j-1} V_i$ and $\bigcup_{i=0}^{j-1} A_i$, respectively.

Then, it is clear that the last ear $P_k = v_0^k \ldots v_{s_k}^k$ completes the construction of D, and $Q_k = v_1^k \ldots v_{s_k-1}^k$ is a chain of linear vertices, whose first and last vertex are joined to vertices of a minimal and strongly connected digraph D'. Hence, $D' = D - Q_k$ is an MSD, and therefore Q_k is an external chain of length $l = s_k - 1 \ge 1$. Trivially, we can say that if $D = C_q = P_0$, then there is no external chain contained in D. The proof is completed. □

Note that D' is an MSD with $n - l$ vertices and $m - l - 1$ arcs. Note also that if $P_0 = C_q$, with C_q as a maximal length cycle contained in D, and there exists any external chain with length $l \ge 1$ not contained in C_q, then $q \le n - l$.

Proposition 5. *Let $D = (V, A)$ be an MSD, and let $v_1 v_l$-path be a chain contained in D with length $l < n$. Then, the contraction of all vertices of the $v_1 v_l$-path in a unique vertex preserves the minimality, that is, it produces another MSD D' with $n - l + 1$ vertices and $m - l + 1$ arcs.*

Proof. Let D' be the digraph obtained by the contraction of all vertices of the $v_1 v_l$-path in a unique vertex v'. Let n' be the number of vertices and m' the number of arcs of D'. In D', all vertices of the $v_1 v_l$-path are suppressed, but it contains the vertex $v' \notin D$, and then $n' = n - l + 1$. Since $d^-(v_i) = d^+(v_i) = 1$ for all $1 \le i \le l$, we have $m' = m - l + 1$. Now, let us assume that there are transitive arcs in D'. If we expand v', these transitive arcs would also exist in D, contradicting the minimality of D. Hence, D' is minimal. Since $n > l$, then a vertex $w \notin v_1 v_l$-path, exists also in D', and D' contains a wv'-path and a $v'w$-path. Therefore, D' is strongly connected. The proof is completed. □

Proposition 6. *Let $D = (V, A)$ be an MSD such that D is not a cycle. Then, there is not a cycle in D that contains all linear vertices of D.*

Proof. Let us suppose that C_q contains all linear vertices of D. We can obtain an MSD D' by contraction of C_q in a unique vertex v'. We know that D' must contain at least two linear vertices, and at least one of them is different from v'. Then, it is clear that there exists at least one linear vertex that is contained in D but is not contained in C_q. The proof is completed. □

Let D be an MSD such that D is not a cycle, and λ be the number of linear vertices contained in D. From the proposition above, it is trivial to see that a cycle C_q contained in D will contain at most $\lambda - 1$ linear vertices of D.

4. Upper Bounds for the Coefficients of the Characteristic Polynomial of MSDs

In [2], some results about bounds of the coefficients of the characteristic polynomial of an MSD are proved. In particular, it is shown that the independent term must be 1, 0, or -1. We follow the lines of that proof to generalize that bound.

Proposition 7. *Let $D = (V, A)$ be an MSD, and let $x^n + k_1 x^{n-1} + \cdots + k_i x^{n-i} + \cdots + k_{n-1} x + k_n$ be the characteristic polynomial of the adjacency matrix of D. Then,*

$$|k_i| \le \binom{n}{i} \tag{12}$$

Proof. We claim that any subset of i vertices can be covered by disjoint cycles in at most one manner. In fact, take any subset $A \subset V$, with $|A| = i$, and consider the subdigraph D' to be generated by that A. Now, D' is a subdigraph of an MSD, so it has no transitive arcs. If it is not strongly connected, we can add arcs, one by one, until we obtain a strongly connected digraph D'' that would be minimal. Therefore, D'' would be an MSD, and the aforementioned result of [2] implies that there is at most one covering of the vertices of D'' (that is, of A) by disjoint cycles.

The coefficients theorem for digraphs allows us to finish the proof. □

5. MSD Properties Associated to Results of Algorithms Complexity

It was well known that minimality is a very strict condition in the family of strong digraphs implying, for instance, the size limitation $n \leq m \leq 2(n-1)$. As we have seen in previous sections, MSDs also exhibit strong constraints on the number of linear vertices and maximum in- and outdegrees of vertices, regarding the length of the longest directed cycle. Unfortunately, these constraints are not enough to construct an efficient algorithm finding the longest cycle in an MSD.

A proof that an MSD can be converted into a directed cycle by successively eliminating external chains is given in [16]. However, this process does not guarantee that the resulting directed cycle will have a maximum length. Figure 6 shows an MSD where the longest directed cycle is given by $u_1, u_2, u_3, u_4, u_5, u_1$, but this cycle will be obtained only in the case that the external chains eliminated are those formed by the u_6-path and u_7-path. Nevertheless, there is no an efficient algorithm that can determine the deletion of these chains and the non-deletion of the external chain formed by the vertex u_5-path because if this chain is deleted, then the longest cycle of the MSD will also have been eliminated.

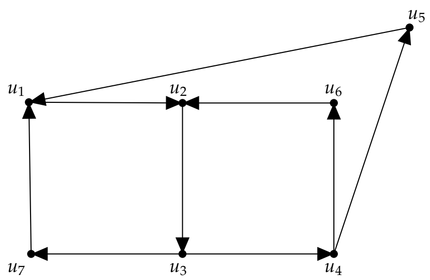

Figure 6. Example of an MSD that contains three external chains.

Theorem 1. *Computing a cycle with maximal length in an MSD is an NP-hard problem.*

Proof. We can reduce the problem of computing a cycle with maximal length in a strongly connected digraph to the problem of computing a cycle with maximal length in an MSD.

Let $D' = (V', A')$ be a strong digraph. We can build an MSD $D = (V, A)$ from D' as follows. For each arc $v'_i v'_j \in A'$, we add an intermediate vertex v_{ij}. We thus obtain

$$V = V' \cup \{v_{ij} \mid v'_i v'_j \in A'\} \tag{13}$$

$$A = \{v'_i v_{ij} \mid v'_i v'_j \in A'\} \cup \{v_{ij} v'_j \mid v'_i v'_j \in A'\} \tag{14}$$

Note that the strong connection of D' implies that D is trivially strongly connected. Note also that no arc of D can be transitive since every arc has a linear vertex v_{ij} as start- or endpoint. Hence, D is in fact an MSD.

Now, we remark that there is a one-to-one correspondence between cycles in D and cycles in D': for every cycle C'_q in D', a cycle C_{2q} arises in D, and all the cycles in D are generated in this way.

We conclude that any algorithm allowing us to compute the longest cycle of an MSD would then be able to compute the longest cycle of any SD, too. Since the problem of computing the longest cycle in a strongly connected digraph is NP-hard [14], then the theorem is proved. □

Theorem 2. *Let $D = (V, A)$ be an MSD. Finding a cycle contained in D with length $2n - m$ is an NP-complete problem.*

Proof. We can reduce the problem of determining if a digraph is Hamiltonian to the problem of determining if an MSD has a cycle of length $2n - m$.

Let $D' = (V', A')$ be a digraph. If D' is strongly connected, the same procedure used in the previous proof yields an MSD $D = (V, A)$ (if D' is not strongly connected, then it cannot be Hamiltonian). The order of D verifies $n = n' + m'$, and the size holds $m = 2m'$. Hence, finding a cycle in D with length $2n - m = 2(n' + m') - 2m' = 2n'$ would imply finding a n'-cycle in D', that is, determining if D is Hamiltonian. Since determining whether a digraph is Hamiltonian is an NP-complete problem, the theorem is proved. □

6. Conclusions

In this work, we have found some new properties regarding MSDs. The first set of properties has to do with the number of linear vertices in an MSD. We have seen that the existence of a vertex with a high in- or outdegree implies a high number of linear vertices. Furthermore, we have used this fact to give a simpler proof of the lower bound of linear vertices that we obtained in [8], where the existence of a q cycle implies at least $\lfloor (q+1)/2 \rfloor$ linear vertices. We have also proved that chains of consecutive linear vertices in an MSD can be contracted without loss of minimality. We feel that further research along these lines could give, from one side, a result linking maximal cycle lengths, maximal in- or outdegrees, and improved estimations of the number of linear vertices, as well as a better understanding of the cycle properties that can lead to spectral properties, such as the characterization of polynomials that can be realized as characteristic polynomials of MSDs. In this regard, we have proved a bound for the coefficients of such polynomials, advancing along the lines given in [2].

Since the number of linear vertices in an MSD is easily computed, we wanted to explore the possibility that the maximal length of a cycle could be bounded so as to allow to construct a polynomial complexity algorithm to find the longest cycle. Unfortunately, that is not the case, and we have proved that the search of a maximal length cycle in an MSD is NP-hard. Still, it can be interesting to look for a subset of MSDs for which the search for maximal length cycles can be performed efficiently. This kind of result could arise, also, by further study of the properties that we pointed out in the paragraph above.

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References

1. García-López, J.; Marijuán, C. Minimal strong digraphs. *Discrete Math.* **2012**, *312*, 737–744. [CrossRef]
2. García-López, J.; Marijuán, C.; Pozo-Coronado, L.M. Structural properties of minimal strong digraphs versus trees. *Linear Algebra Appl.* **2018**, *540*, 203–220. [CrossRef]
3. Brualdi, R.A.; Ryser, H.J. *Combinatorial Matrix Theory*; Cambridge University Press: New York, NY, USA, 1992.
4. Hedrick, M.; Sinkhorn, R. A special class of irreducible matrices—The nearly reducible matrices. *J. Algebra* **1970**, *16*, 143–150. [CrossRef]
5. Torre-Mayo, J.; Abril-Raymundo, M.R.; Alarcia-Estévez, E.; Marijuán, C.; Pisonero, M. The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial, EBL digraphs. *Linear Algebra Appl.* **2007**, *426*, 729–773. [CrossRef]
6. Cvetkovic, D.; Doob, M.; Sachs, H. *Spectra of Graphs: Theory and Application*; Academic Press: Cambridge, MA, USA, 1980.
7. Verstraete, J. *Introduction to Graph Theory*; Department of Mathematics of University of California at San Diego: San Diego, CA, USA, 2020.
8. Arcos-Argudo, M.; García-López, J.; Pozo-Coronado, L.M. Structure of cycles in Minimal Strong Digraphs. *Discret. Appl. Math.* **2019**, *263*, 35–41. [CrossRef]
9. Zverovich, V. *Modern Applications of Graph Theory*; Oxford University Press: New York, NY, USA, 2021.

10. Diestel, R. *Graph Theory*; Springer: Cham, Switzerland, 2024.
11. O'Regan, G. M*athematical Foundations of Software Engineering—Chapter 7: Graph Theory*; Springer: Cham, Switzerland, 2023.
12. Kumar, Y.S. *Advanced Graph Theory*; Springer: Cham, Switzerland, 2023.
13. Lecca, P.; Carpentieri, B. *Introduction to Mathematics for Computational Biology*; Springer: Cham, Switzerland, 2023.
14. Bang-Jensen, J.; Gutin, G. *Digraphs: Theory, Algorithms and Applications*, 2nd. ed.; Springer: London, UK, 2009.
15. Berge, C. *Graphes*; North-Holland: Amsterdam, The Netherlands, 1991.
16. Chen, Z.B.; Zhang, F.J. Bounds of the longest directed cycle length for minimal strong digraphs. *Discret. Math.* **1988**, *68*, 9–13. [CrossRef]