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Strong Convergence of Truncated EM Method for Stochastic Volterra Integral Differential Equations with Hölder Diffusion Coefficients

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Abstract: The strong convergence of numerical solutions is studied in this paper for stochastic Volterra integral differential equations (SVIDEs) with a Hölder diffusion coefficient using the truncated Euler–Maruyama method. Firstly, the numerical solutions of SVIDEs are obtained based on the Euler–Maruyama method. Then, the pth moment boundedness and strong convergence of truncated the Euler–Maruyama numerical solutions are proven under the local Lipschitz condition and the Khasminskii-type condition. Finally, the convergence rate of the truncated Euler–Maruyama method of the numerical solutions is also discussed under some suitable assumptions.

Keywords: stochastic Volterra integral differential equations; Khasminskii-type condition; strong convergence; local Lipschitz condition; truncated Euler–Maruyama method

MSC: 65P40; 37H30

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1. Introduction

There are many fields in which Volterra integral differential equations (VIDEs) are used, including control theory, economics, engineering, physical chemistry, and their theoretical and numerical analysis research, which have also received widespread attention from researchers; see [1–6] and the references therein. However, integral equations are affected by noise and uncertain factors in practical applications. Therefore, stochastic Volterra integral differential equations (SVIDEs) have been applied to describe the phenomena of these uncertain factors, which actuates that more and more researchers are paying attention to the study of SVIDEs [7]. For example, Zhang [8] noted Euler schemes and large deviations for stochastic Volterra equations with singular kernels, Amir Haghighi [9] noted the convergence of a partially truncated Euler–Maruyama method for SDEs with superlinear piecewise continuous drift and Hölder diffusion coefficients, and Mao [10] studied the stability of the following stochastic Volterra integral differential equations:

$$\mathrm{d}z(t) = F(z(t),t) + G\left(\int_0^t g(t,s)z(s)\mathrm{d}s, t\right)\mathrm{d}w(t),$$

where w is a Brownian motion. Mao and Riedle [11] examined the mean square stability of nonlinear SVIDEs as follows:

$$\mathrm{d}z(t) = \left[F(z(t),t) + G\left(\int_0^t k_1(t,s)z(s)\mathrm{d}s, t\right)\right]\mathrm{d}t + H\left(\int_0^t k_2(t,s)z(s)\mathrm{d}s, t\right)\mathrm{d}w(t),$$

where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The kernel functions $k_1 : \mathbb{D} \to \mathbb{R}$ and $k_2 : \mathbb{D} \to \mathbb{R}$ belong to $C^1(\mathbb{D})$, in which $\mathbb{D} := \{(t,s) : 0 \le s \le t \le T\}$. When $i = 1,2$, we have

$$\|k_i\|_\infty = \max_{(t,s)\in\mathbb{D}} |k_i(t,s)|.$$

Due to the inability to obtain the exact solutions for most nonlinear SVIDEs, the numerical solutions are often solved to approximate the exact solutions of the equations. Mao [12] constructed a convergent specific numerical method to study the stochastic differential equations, i.e., the truncated Euler–Maruyama (EM) method, in which the equations satisfy local Lipschitz- and Khasminskii-type conditions. Due to the low computational cost and acceptable convergence order, the truncation method has received increasing attention. Thus, Zhang [13] proposed a truncation EM method under non-global Lipschitz conditions for SVIDEs and considered its moment's boundedness and L^q-convergence. Compared with the implicit EM method, the explicit EM method is more attractive to researchers because of its simple algebraic structure, low computational cost, and ideal convergence order. Meanwhile, Mao [12] noted that truncated EM methods are strongly convergent if the coefficients of stochastic differential equations meet local Lipschitz conditions and Khasminskii-type conditions. Reference [13] pointed out that for generalized stochastic differential Equation (1), the classical Euler numerical methods are divergent in the sense of moment, while Wei et al. [14] showed that the shortened EM techniques are strongly convergent for generalized SVIDEs, and David [15] noted that for stochastic Volterra equations with Hölder diffusion coefffcients, it was found that the diffusion coefficients of many important SVIDs satisfy the Hölder continuity condition. Therefore, in this paper, SVIDEs with Hölder diffusion coefficients will be studied.

$$\mathrm{d}z(t) = F\Big(z(t), \int_0^t k_1(t,s)z(s)\mathrm{d}s\Big)\mathrm{d}t + G(z(t))\mathrm{d}B(t). \tag{1}$$

Notation: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ denotes the complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ that meets the usual conditions, i.e., $\mathcal{F}_t$ is right continuous and $\mathcal{F}_0$ contains all P-null sets. $\mathbb{E}$ denotes the mathematical expectation associated with the probability $\mathbb{P}$. $B(t)$ is a standard Brown motion defined on the above probability space. $C^2(\mathbb{R}^2 \times \mathbb{R}_+; \mathbb{R})$ represents the set of function $V(z,t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ that has first-order and second-order continuous derivatives with respect to t and x, respectively. $L^p(\mathbb{R}_+; \mathbb{R}^d)$ consists of all measurable, $\mathcal{F}_t$-compatible stochastic processes $\psi(t,\omega)$ that satisfy $\int_0^T |\psi(t)|^p \mathrm{d}t < \infty$ $a.s.$ for all $T > 0$. It is said that $\tau(\omega)$ is a stopping time of $\mathcal{F}_t$ if a stochastic process $\tau(\omega)$ taking a value on $[0, +\infty])$ satisfies $\{\omega : \tau(\omega) < t\}$.

2. Preliminaries

Integrating differential Equation (1) with respect to time t, where t belongs to [0, T] and $z(0) = z_0$, we can refer to reference [16] and obtain

$$z(t) = z_0 + \int_0^t F\Big(z(s), \int_0^s k_1(t,s)z(s)\mathrm{d}s\Big)\mathrm{d}s + \int_0^t G(z(s))\mathrm{d}B(s), \tag{2}$$

In the following, we present some assumptions for the drift coefficient $F(z,u)$ and diffusion coefficient $G(u)$.

The drift coefficient $F(z,u)$ is a Borel-measurable function on the interval $[0,T]$ that satisfies the following conditions:

A.1. The drift coefficient $F(z,u)$ meets the local Lipschitz condition, which implies that for every $R > 0$, there exists a positive constant C_R such that for every $z, \bar{z}, u, \bar{u} \in \mathbb{R}$ and $|z| \vee |\bar{z}| \vee |u| \vee |\bar{u}| \le R$, we have

$$|F(z,u) - F(\bar{z},\bar{u})| \le C_R(|z-\bar{z}| + |u-\bar{u}|).$$

A.2. The drift coefficient $F(z,u)$ meets the one-sided Lipschitz condition with respect to z, i.e., there is a positive constant L_1 such that

$$(z-\bar{z})(F(z,u)-F(\bar{z},u)) \leq L_1|z-\bar{z}|^2, \quad z,\bar{z},u \in \mathbb{R}.$$

A.3. The drift coefficient $F(z,u)$ meets the Khasminskii-type requirement, i.e., there exists a positive constant $L_1 > 0$ such that

$$z^T F(z,u) \leq L_1(1+|z|^2+|u|^2), \quad z,u \in \mathbb{R}.$$

A.4. There exist constants $L_1 > 0$ and $u > 0$ such that

$$|F(z,u)-F(\bar{z},\bar{u})| \leq L_1(1+|z|^r+|\bar{z}|^r+|u|^r+|\bar{u}|^r)(|z-\bar{z}|+|u-\bar{u}|), \quad \forall z,\bar{z},u,\bar{u} \in \mathbb{R}.$$

Remark 1. *It is concluded from A.4 that*

$$|F(z,u)-F(z,\bar{u})| \leq L_1(1+|z|^r+|u|^r+|\bar{u}|^r)|u-\bar{u}|, \quad \forall z,u,\bar{u} \in \mathbb{R}.$$

A.5. Diffusion coefficient $G(u)$ is Borel sigma-algebra on the interval $[0,T]$ and satisfies the Hölder continuity requirement, i.e., there are constants $0 < \alpha < \frac{1}{2}$ and $L_2 > 0$ such that

$$|G(z)-G(u)| \leq L_2|z-u|^{\frac{1}{2}+\alpha}, \quad z,u \in \mathbb{R}.$$

Under assumptions A.1–A.4, it is easy to obtain the well-posed solution in the process of reference [17]'s similarity proof.

Lemma 1. *SVIDE (1) satisfies the Khasmiskii-type condition for the drift coefficient $F(z,u)$ and diffusion coefficient $G(z)$, under assumption A.3 and assumption A.5, i.e., there exist $q > 2$ and $L_1 = L_1(q) > 0$ such that*

$$z^T F(z,u) + \frac{q-1}{2}|G(z)|^2 \leq L_1(1+|z|^2+|u|^2).$$

Proof. By assumption A.5, we obtain that

$$|G(z)-G(u)| \leq |G(z)-G(0)| \leq L_2|z|^{\frac{1}{2}+\alpha}, \quad z,u \in \mathbb{R},$$

which means that

$$|g(z)|^2 \leq (L_2|z|^{\frac{1}{2}+\alpha}+|G(0)|)^2 \leq 2({L_2}^2|z|^{\frac{1}{2}+\alpha}+|G(0)|^2) \leq L_3(1+|z|^2).$$

According to hypothesis A.3, we have

$$\begin{aligned} z^T F(z,u) + \frac{q-1}{2}|G(z)|^2 &\leq L_1\left(1+|z|^2+|u|^2\right) + \frac{L_3(q-1)}{2}\left(1+|z|^2\right) \\ &\leq L_1\left(1+|z|^2+|u|^2\right). \end{aligned}$$

The proof is complete. □

3. Numerical Analysis

In this section, the truncated Euler–Maruyama method and its boundedness in the sense of moment will be analyzed.

3.1. Truncated Euler–Maruyama Method

Because of the classical Euler method's divergence with superlinear growth coefficients (see [18]), the EM technique is utilized to calculate the numerical solutions of SVIDEs. More specifically, the EM method does not directly find the analytical solution $z(t)$ but finds a series of points $z_n(n = 1, 2, 3, \cdots)$ to approximate the analytical solution on the interval where the solution exists. Here, to obtain the truncated EM solution, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly monotonically increasing function with $\phi(r) \to \infty$ $(r \to \infty)$ and

$$\sup_{|z|\vee|u|\leq r} |F(z,u)| \leq \phi(r), \quad \forall r \geq 1.$$

Let ϕ^{-1} be the inverse function of ϕ. Then, $\phi^{-1} : [\phi(0), \infty] \to \mathbb{R}_+$ is a continuous and strictly increasing function. In addition, for given $\Delta^* \in (0, 1]$, let $\psi : (0, \Delta^*] \to (0, \infty)$ be a strictly increasing function with

$$\psi(\Delta^*) \geq \phi(2), \quad \lim_{\Delta\to 0} \psi(\Delta) = 0 \quad \Delta^*\psi(\Delta) \leq 1, \Delta \in (0, \Delta^*).$$

For given step size $\Delta \in (0, \Delta^*)$, the discretization scheme of the equation is as follows:

$$Z_{k+1} = Z_k + F_\Delta(Z_k)\Delta z + G_\Delta(Z_k)\Delta B_k, \quad Z_0 = z_0,$$

where

$$F_\Delta(z,u) = F\left(|z| \wedge \phi^{-1}(\psi(\Delta))\frac{z}{|z|}, u \wedge \phi^{-1}(\psi(\Delta))\frac{u}{|u|}\right), \quad z, u \in \mathbb{R}. \tag{3}$$

When $z = 0$, let $\frac{z}{|z|} = 0$. Then,

$$|F_\Delta(z,u)| \leq \phi(\phi^{-1}\psi(\Delta)) = \psi(\Delta), \quad \forall z, u \in \mathbb{R} \tag{4}$$

It is evident that the truncation functions $F(z,u)$ are bounded, whether $F_\Delta(z,u)$ is bounded or not. Furthermore, it has been demonstrated that these shortened functions maintain the Khasminskii-type condition.

Remark 2. *The truncation technique employed here guarantees that the moments of the numerical solution are bounded. Since the diffusion coefficient $G(\cdot)$ satisfies the linear growth condition, it is unnecessary to truncate $G(\cdot)$.*

From references [12,13], we can easily obtain the following result.

Lemma 2. *Under the conditions of Lemma 1, for every $\Delta \in (0, \Delta^*]$, we have*

$$z^T F(z,u) + \frac{q-1}{2}|G(z)|^2 \leq L_1(1 + |z|^2 + |u|^2), \quad z, u \in \mathbb{R}. \tag{5}$$

Proof. Let $z_\triangle(0) = z_0$ be the initial value. Then, the truncated EM numerical scheme of (1) is

$$\begin{aligned} Z_\Delta(t_{k+1}) &= Z_\Delta(t_k) + F_\Delta\left(Z_\Delta(t_k), \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} k_1(t_k,s)Z_\Delta(t_i)\mathrm{d}s\right)\Delta + G(Z_\Delta(t_k))\Delta B_k \\ &= Z_\Delta(t_k) + F_\Delta(Z_\Delta(t_k), U_\Delta(t_k))\Delta + G(Z_\Delta(t_k))\Delta B_k, \end{aligned} \tag{6}$$

from which we know that $Z_\Delta(t_k) \approx z(t_k)$ when $t_k = k\Delta$, where $\Delta = \frac{T}{M}$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and

$$U_\Delta(t_k) = \sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}} k_1(t_k,s)Z_\Delta(t_i)\mathrm{d}s,$$

with $k = 0, 1, \cdots, M-1$, and $M \in \mathbb{N}$. Further, according to $\{z_k\}_{k\geq 0}$, we introduce the following steps

$$\bar{z}_\Delta(t) = \sum_{k=0}^{\infty} Z_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad t \geq 0.$$

Then, the continuous truncated EM solution can be defined as follows:

$$z_\Delta(t) = z_0 + \int_0^t F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s + \int_0^t G(\bar{u}_\Delta(s))\mathrm{d}B(s), \tag{7}$$

where

$$\bar{u}_\Delta(s) = \int_0^s k(s,v)\bar{z}_\Delta(v)\mathrm{d}v, \quad t \geq 0. \tag{8}$$

□

Remark 3. *Specifically, under the continuous time and continuous sample conditions defined above, for all $k = 1, 2, \cdots, M-1$, truncating the EM solution satisfies*

$$z_\Delta(t_k) = \bar{z}_\Delta(t_k) = z_\Delta(t_k),$$

where $z_\Delta(t)$ is an Itô process with Itô differentiation:

$$\mathrm{d}z_\Delta(t) = F\left(\bar{z}_\Delta(t), \int_0^t k_1(t,s)\bar{z}_\Delta(s)\mathrm{d}s\right)\mathrm{d}t + G(\bar{z}_\Delta(t))\mathrm{d}B(t).$$

3.2. Moment Boundedness of Numerical Solutions for Truncated EM Method

This subsection demonstrates the boundedness of truncated EM solutions in the sense of moment by the following lemma.

Lemma 3. *For any $\Delta \in (0, \Delta^*]$, $\hat{q} > 0$, it holds that*

$$\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^{\hat{q}} \leq C_{\hat{q}} \Delta^{\frac{\hat{q}}{2}} (\psi(\Delta))^{\hat{q}}, \quad \forall\, 0 \leq t \leq T. \tag{9}$$

Proof. We first consider the case of $\hat{q} \geq 2$. For equation

$$\mathrm{d}z_\Delta(s) = F\left(\bar{z}_\Delta(s), \int_0^t k_1(t,v)\bar{z}_\Delta(v)\mathrm{d}v\right)\mathrm{d}s + G(\bar{z}_\Delta(s))\mathrm{d}B(s),$$

through integrating both sides, it can be obtained that

$$z_\Delta(t) - z_\Delta(t_k) = \int_{t_k}^t F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s + \int_{t_k}^t G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s).$$

Thus, we can obtain that

$$\mathbb{E}|z_\Delta(t) - z_\Delta(t_k)|^{\hat{q}} = \mathbb{E}\left|\int_{t_k}^t F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s + \int_{t_k}^t G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s)\right|^{\hat{q}},$$

It can be derived from basic inequalities that

$$\mathbb{E}|z_\Delta(t) - z_\Delta(t_k)|^{\hat{q}} \leq 2^{\hat{q}-1}\left[\mathbb{E}\left|\int_{t_k}^t F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s\right|^{\hat{q}} + \mathbb{E}\left|\int_{t_k}^t G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s)\right|^{\hat{q}}\right].$$

Let

$$j_1 = \mathbb{E}\left|\int_{t_t}^t F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s\right|^{\hat{q}}, \quad j_2 = \mathbb{E}\left|\int_{t_t}^t G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s)\right|^{\hat{q}}.$$

Therefore,

$$j_1 \leq \mathbb{E}\Big[\int_{t_k}^{t}|F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))|\mathrm{d}s\Big]^{\hat{q}}.$$

It is derived from the Hölder inequality [19] that

$$\begin{aligned} j_1 &\leq \mathbb{E}\Big[\Big(\int_{t_k}^{t}|f_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))|^{\hat{q}}\mathrm{d}s\Big)^{\frac{1}{\hat{q}}}\Big(\int_{t_k}^{t}1^{1-\frac{1}{\hat{q}}}\mathrm{d}s\Big)^{1-\frac{1}{\hat{q}}}\Big]^{\hat{q}} \\ &= (t-t_k)^{\hat{q}-1}\mathbb{E}\int_{t_k}^{t}|f_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))|^{\hat{q}}\mathrm{d}s, \\ j_2 &\leq \Big(\frac{\hat{q}(\hat{q}-1)}{2}\Big)^{\frac{\hat{q}}{2}}(t-t_k)^{\frac{\hat{q}-2}{2}}\mathbb{E}\int_{t_k}^{t}|G_\Delta(\bar{x}_\Delta(s))|^{\hat{q}}\mathrm{d}s. \end{aligned}$$

Thus,

$$\mathbb{E}|z_\Delta(t)-z_\Delta(t_k)|^{\hat{q}} \leq C_{\hat{q}}\Big[\Delta^{\hat{q}-1}\mathbb{E}\int_{t_k}^{t}|F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))|^{\hat{q}}\mathrm{d}s+\Delta^{\frac{\hat{q}-2}{2}}\mathbb{E}\int_{t_k}^{t}|G_\Delta(\bar{z}_\Delta(s))|^{\hat{q}}\mathrm{d}s\Big].$$

Let

$$j_3=\Delta^{\hat{q}-1}\mathbb{E}\Big|\int_{t_k}^{t}F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))\Big|^{\hat{q}}\mathrm{d}s,\quad j_4=\Delta^{\frac{\hat{q}-2}{2}}\mathbb{E}\Big|\int_{t_k}^{t}G_\Delta(\bar{z}_\Delta(s))\mathrm{d}s\Big|^{\hat{q}}.$$

Therefore,

$$j_3 \leq \Delta^{\hat{q}-1}\mathbb{E}\int_{t_k}^{t}\psi^{\hat{q}}(\Delta)\mathrm{d}s=\Delta^{\hat{q}-1}\psi^{\hat{q}}(\Delta)(t-t_k)=\Delta^{\hat{q}}\psi^{\hat{q}}(\Delta),$$

and

$$\begin{aligned} j_4 &\leq C\Delta^{\frac{\hat{q}-2}{2}}\mathbb{E}\int_{t_k}^{t}(1+|\bar{z}_\Delta(s)|^{\hat{q}})\mathrm{d}s \leq C\Delta^{\frac{\hat{q}-2}{2}}\int_{t_k}^{t}\mathbb{E}(1+|\bar{z}_\Delta(s)|^{\hat{q}})\mathrm{d}s \\ &\leq C\Delta^{\frac{\hat{q}}{2}}+C\Delta^{\frac{\hat{q}-2}{2}}\int_{t_k}^{t}\mathbb{E}|\bar{z}_\Delta(s)|^{\hat{q}}\mathrm{d}s. \end{aligned}$$

Due to

$$\sup_{0\leq r\leq t}\mathbb{E}|\bar{z}_\Delta(s)|^{\hat{q}} \leq \sup_{0\leq r\leq t}\mathbb{E}|z_\Delta(s)|^{\hat{q}},$$

and

$$z_\Delta(t)=z_0+\int_{t_k}^{t}F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))\mathrm{d}s+\int_{t_k}^{t}G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s),$$

we have

$$\begin{aligned} \mathbb{E}|z_\Delta(s)|^{\hat{q}} &\leq 3^{\hat{q}-1}\Big(|z_0|^{\hat{q}}+\mathbb{E}|\int_0^t F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))\mathrm{d}s|^{\hat{q}}+\mathbb{E}|\int_0^t G_\Delta(\bar{z}_\Delta(s))\mathrm{d}B(s)|^{\hat{q}}\Big) \\ &\leq 3^{\hat{q}-1}\Big(|z_0|^{\hat{q}}+T^{\hat{q}}(\psi(\Delta)^{\hat{q}})+\Big(\frac{\hat{q}(\hat{q}-1)}{2}\Big)^{\frac{\hat{q}}{2}}T^{\frac{\hat{q}-2}{2}}\mathbb{E}\int_0^t|G_\Delta(\bar{z}_\Delta(s))|^{\hat{q}}\mathrm{d}s\Big) \\ &\leq C_{\hat{q}}\Big(1+\psi(\Delta)^{\hat{q}}+\int_0^t\mathbb{E}|(\bar{z}_\Delta(s))|^{\hat{q}}\mathrm{d}s\Big), \end{aligned}$$

which means that

$$\sup_{0\leq r\leq t}\mathbb{E}|z_\Delta(s)|^{\hat{q}} \leq C_{\hat{q}}\Bigg(1+\psi(\Delta)^{\hat{q}}+\int_0^t\sup_{0\leq r\leq t}\mathbb{E}|\bar{z}_\Delta(s)|^{\hat{q}}\mathrm{d}s\Bigg).$$

It then follows from the Gronwall formula [19] that

$$\sup_{0\leq r\leq T}\mathbb{E}|\bar{z}_\Delta(s)|^{\hat{q}} \leq 1+(\psi(\Delta))^{\hat{q}}.$$

It can be concluded that

$$j_4 \leq C\Delta^{\frac{\hat{q}}{2}} + C\Delta^{\frac{\hat{q}-2}{2}}\Delta\Big(1 + (\psi(\Delta))^{\hat{q}}\Big) = C\Delta^{\frac{\hat{q}}{2}} + C\Delta^{\frac{\hat{q}-2}{2}}(\psi(\Delta))^{\hat{q}}.$$

Therefore, we have

$$\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^{\hat{q}} \leq C_{\hat{q}}(\Delta^{\hat{q}}(\psi(\Delta))^{\hat{q}} + \Delta^{\frac{\hat{q}}{2}} + \Delta^{\frac{\hat{q}}{2}}(\psi(\Delta))^{\hat{q}}) \leq C_{\hat{q}}\Delta^{\frac{\hat{p}}{2}}(\psi(\Delta))^{\hat{q}}.$$

When $0 < \hat{q} < 2$, according to the Hölder inequality, it can be concluded that

$$\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^{\hat{q}} \leq \Big(\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^{\hat{q}\cdot\frac{2}{\hat{q}}}\Big)^{\frac{\hat{q}}{2}} = \Big(\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^2\Big)^{\frac{\hat{q}}{2}},$$

It follows from $\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^2 \leq C\Delta\psi^2(\Delta)$ that

$$\mathbb{E}|z_\Delta(t) - \bar{z}_\Delta(t)|^2 \leq C_{\hat{q}}\Delta^{\frac{\hat{q}}{2}}(\psi(\Delta))^{\hat{q}}.$$

In summary, for every $\hat{q} > 0$, we have

$$E|z_\Delta(t) - \bar{z}_\Delta(t)|^{\hat{q}} \leq C_{\hat{q}}\Delta^{\frac{\hat{q}}{2}}(\psi(\Delta))^{\hat{q}}.$$

□

Lemma 4. *If Assumption A.3 and the conditions of Lemma 1 hold, then*

$$\sup_{0\leq\Delta\leq\Delta^*}\sup_{0\leq t\leq T}\mathbb{E}|z_\Delta(t)|^q \leq C, \quad \forall T > 0.$$

Proof. Using Itô's formula with (7), one has

$$\begin{aligned}\mathbb{E}|z_\Delta(t)|^q \leq& |z_0|^q + \mathbb{E}\int_0^t q|z_\Delta(s)|^{q-2}z_\Delta^T(s)F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))\mathrm{d}s \\ &+ \mathbb{E}\int_0^t \frac{q(q-1)}{2}|z_\Delta(s)|^{q-2}|G_\Delta(\bar{z}_\Delta(s))|^2\mathrm{d}s \\ =& M_1 + M_2,\end{aligned} \tag{10}$$

where

$$\begin{aligned}M_1 =& |z_0|^q + \mathbb{E}\int_0^t q|z_\Delta(s)|^{q-2}z_\Delta^T(s)F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s)\mathrm{d}s, \\ M_2 =& \mathbb{E}\int_0^t \frac{q(q-1)}{2}|z_\Delta(s)|^{q-2}|G_\Delta(\bar{z}_\Delta(s))|^2\mathrm{d}s.\end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned}M_1 \leq& Kq\mathbb{E}\int_0^t |z_\Delta(s)|^{q-2}(1 + |\bar{z}_\Delta(s)|^2 + |\bar{u}_\Delta(s)|^2)\mathrm{d}s \\ \leq& KT + Kq\mathbb{E}\int_0^t |z_\Delta(s)|^{q-2}\mathrm{d}s + Kq\mathbb{E}\int_0^t |z_\Delta(s)|^{q-2}|\bar{z}_\Delta(s)|^2\mathrm{d}s \\ &+ \mathbb{E}\int_0^t |z_\Delta(s)|^{q-2}|\bar{u}_\Delta(s)|^2\mathrm{d}s.\end{aligned}$$

According to Young's inequality [19],

$$a^{q-2}b \leq \frac{q-2}{2}a^q + \frac{2}{q}b^{\frac{q}{2}}, \quad \forall a, b \geq 0,$$

It can then be concluded that

$$M_1 \leq KT + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q ds + 2T + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\mathbb{E}\int_0^t |\bar{z}_\Delta(s)|^q \mathrm{d}s + Kq\mathbb{E}\int_0^t |z_\Delta(s)|^{q-2}|\bar{u}_\Delta(s)|^2 \mathrm{d}s.$$

Then, by combining

$$\mathbb{E}|\bar{u}_\Delta(s)|^q \leq \|k\|_\infty^q s^{q-1}\int_0^s \mathbb{E}|\bar{z}_\Delta(v)|^q \mathrm{d}v,$$

we have

$$\begin{aligned} M_1 \leq\ & KT + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2T + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\int_0^t \mathbb{E}|\bar{z}_\Delta(s)|^q \mathrm{d}s \\ & + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\mathbb{E}\int_0^t |\bar{u}_\Delta(s)|^q \mathrm{d}s \\ \leq\ & KT + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2T + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\int_0^t \mathbb{E}|\bar{z}_\Delta(s)|^q \mathrm{d}s \\ & + K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\int_0^t \mathbb{E}\Big|\int_0^s k(s,v)\bar{z}_\Delta(v)dv\Big|^q \mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} M_2 &\leq (q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\mathbb{E}\int_0^t |z_\Delta(s) - \bar{z}_\Delta(s)|^{\frac{q}{2}} |F_\Delta(\bar{z}_\Delta(s), \bar{u}_\Delta(s))|^{\frac{q}{2}} \mathrm{d}s \\ &\leq (q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\psi^{\frac{q}{2}}(\Delta)\mathbb{E}\int_0^T |z_\Delta(s) - \bar{z}_\Delta(s)|^{\frac{q}{2}} \mathrm{d}s \\ &\leq (q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2\psi^{\frac{q}{2}}(\Delta)\int_0^T \Big(\mathbb{E}|z_\Delta(s) - \bar{z}_\Delta(s)|^q\Big)^{\frac{1}{2}} \mathrm{d}s \\ &\leq (q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2^{\frac{q}{2}} T\psi^q(\Delta)\Delta^{\frac{q}{4}} \\ &\leq (q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2^{\frac{q}{2}} T. \end{aligned}$$

Substituting the estimations of M_1 and M_2 into (10) yields that

$$\begin{aligned} \mathbb{E}|z_\Delta(t)|^q \leq\ & 3KT + 4K(q-2)\mathbb{E}\int_0^t |z_\Delta(s)|^q \mathrm{d}s + 2K\int_0^t \mathbb{E}|z_\Delta(s)|^q \mathrm{d}s \\ & + 2K\int_0^t \Big|\int_0^s k(s,v)z_\Delta(v)\mathrm{d}v\Big|^q \mathrm{d}s + 2T2^{\frac{q}{2}} \\ \leq\ & C_1 + 8K\int_0^t \sup_{0\leq v\leq s} \mathbb{E}|z_\Delta(s)|^q \mathrm{d}v, \end{aligned}$$

Therefore, one can obtain that

$$\sup_{0\leq u\leq t} \mathbb{E}|z_\Delta(t)|^q \leq C_1 + 8K\int_0^t \sup_{0\leq v\leq s} \mathbb{E}|z_\Delta(v)|^q \mathrm{d}v.$$

It then follows from Gronwall's formula that

$$\sup_{0\leq t\leq T} \mathbb{E}|z_\Delta(t)|^q \leq C_1 + 8K\int_0^t C_1 e^{\int_v^t 1 d\tau} \mathrm{d}s = C, \quad \forall \Delta \in (0, \Delta^*],$$

where C is not dependent on the Δ. Therefore,

$$\sup_{0\leq\Delta\leq\Delta^*} \sup_{0\leq t\leq T} \mathbb{E}|z_\Delta(t)|^q \leq C, \quad \forall T > 0.$$

The proof is complete. □

When assumption A.1 and the conditions of Lemma 1 hold, we define the stopping time for any real number $R > |z_0|$ as follows:

$$\sigma_R = \inf\{t \geq 0 : |z(t) \geq R|\},$$

And when $\inf \varnothing = \infty$,

$$\mathbb{P}(\sigma_R \leq T) = \frac{C}{R^2}. \tag{11}$$

3.3. Convergence at Time T

To demonstrate that the truncated EM numerical solution $z_\Delta(t)$ converges to the exact solution $z(t)$ at a specific time T, we present the following theorem.

Theorem 1. *If assumptions A.1–A.4 hold, choose a real number $R > |z_0|$ and a small positive value $\Delta \in (0, \Delta^*]$ such that $\phi^{-1}(\psi(\Delta)) \geq R$. σ_R and $\rho_{\Delta,R}$ are identical to the definitions in Lemma 4 and Equation (11), respectively. For any constant $T > 0$, let*

$$\theta_{\Delta,R} = \sigma_R \wedge \rho_{\Delta,R}, \quad e_{\Delta,R}(t) = z(t) - z_\Delta(t), \quad 0 < t < T,$$

Then,

$$\mathbb{E}|e_{\Delta,R}(t \wedge \theta_{\Delta,R})| \leq \begin{cases} C\Big(\frac{1}{\ln \Delta^{-1}} + \Delta^{\frac{1}{4}}(\psi(\Delta)^{\frac{1}{2}})\Big), & \alpha = 0, \\ C\Big(\Delta^{\frac{1}{2}}(\psi(\Delta))\Big)^{2\alpha}, & 0 < \alpha < \frac{1}{2}. \end{cases} \tag{12}$$

Proof. Let $\delta > 1$, $\varepsilon > 0$. Since

$$\int_{\frac{\varepsilon}{\delta}}^{\varepsilon} \frac{1}{z} \mathrm{d}z = \ln \delta,$$

there exists a non-negative continuous function $\psi_{\delta\varepsilon}(z) \in [0, +\infty)$ that satisfies $\psi_{\delta\varepsilon}(z) = 0$ when $z < \frac{\varepsilon}{\delta}$ or $z > \varepsilon$, and

$$\int_{\frac{\varepsilon}{\delta}}^{\varepsilon} \psi_{\delta\varepsilon}(z)\mathrm{d}z = 1, \quad \psi_{\delta\varepsilon}(z) \leq \frac{2}{z \ln \delta}.$$

It should be noted that the function $\psi_{\delta\varepsilon}(z)$ was first proposed in [16] to deal with pathwise uniqueness for stochastic differential equations with Hölder continuous diffusion coefficients, and then it was generalized in [20] to study the convergence of the Euler–Maruyama method. Define

$$\phi_{\delta\varepsilon}(z) = \int_0^{|z|} \int_0^u \psi_{\delta\varepsilon}(s)\mathrm{d}s\mathrm{d}v, \quad z \in \mathbb{R}.$$

Then, for any $z \in \mathbb{R}$, we have

$$|z| \leq \phi_{\delta\varepsilon}(z) + \varepsilon, \quad 0 \leq |\phi'_{\delta\varepsilon}(z)| \leq 1, \phi''_{\delta\varepsilon}(z) = \psi_{\delta\varepsilon}(|z|) \leq \frac{2}{|z| \ln \delta} I_{[\frac{\varepsilon}{\delta} \leq |z| \leq \varepsilon]}.$$

By the condition

$$|z| \leq \phi_{\delta\varepsilon}(z) + \varepsilon,$$

we have

$$\mathrm{d}\phi_{\delta\varepsilon}(e_\Delta(s)) = [F(z(t), u(t)) - F_\Delta(\bar{z}_\Delta(t), \bar{u}_\Delta(t))]\mathrm{d}t + [G(z(t)) - G_\Delta(\bar{z}(t))]\mathrm{d}B(t).$$

Applying Itô's formula, we derive that

$$\begin{aligned}|e_\Delta(t\wedge\theta_{\Delta,R})| \leq& \varepsilon+\int_0^{t\wedge\theta_{\Delta,R}}\phi''_{\delta\varepsilon}(e_\Delta(s))F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))\mathrm{d}s\\ &+\frac{1}{2}\int_0^{t\wedge\theta_{\Delta,R}}\phi''_{\delta\varepsilon}(e_\Delta(s))[G(z(s))-G(\bar{z}_\Delta(s))]^2\mathrm{d}s\\ &+\int_0^{t\wedge\theta_{\Delta,R}}\phi'_{\delta\varepsilon}(e_\Delta(s))[G(z(t))-G_\Delta(\bar{z}(t))]\mathrm{d}B(t)\\ =&\varepsilon+I_1+I_2+I_3.\end{aligned} \tag{13}$$

According to $|\phi'_{\delta\varepsilon}(z)|\leq 1$, Assumption A.1, and Remark 1, we obtain

$$\begin{aligned}&\phi'_{\delta\varepsilon}(z-\bar{z})[F(z,u)-F(\bar{z},\bar{u})]\\ &=\begin{cases}\frac{\phi'_{\delta\varepsilon}(z-\bar{z})}{(z-\bar{z})}(z-\bar{z})[F(z,u)-F(\bar{z},u)]+\phi'_{\delta\varepsilon}(z-\bar{z})[F(\bar{z},u)-F(\bar{z},\bar{u})], & z\neq\bar{z}\\ 0, & z=\bar{z}\end{cases}\\ &\leq\begin{cases}\frac{\phi'_{\delta\varepsilon}(z-\bar{z})}{(z-\bar{z})}|z-\bar{z}|^2+\phi'_{\delta\varepsilon}|(z-\bar{z})||F(\bar{z},u)-F(\bar{z},\bar{u})|, & z\neq\bar{z}\\ 0, & z=\bar{z}\end{cases}\\ &\leq L_1|z-\bar{z}|+L_1(1+|\bar{z}|^\gamma+|u|^\gamma+|\bar{u}|^\gamma)|u-\bar{u}|.\end{aligned}$$

It thus follows from $|z|\vee|\bar{z}|\vee|u|\vee|\bar{u}|\leq R$ that

$$\phi'_{\delta\varepsilon}(z-\bar{z})[F(\bar{z},u)-F(\bar{z},\bar{u})]\leq K(|z-\bar{z}|+|u-\bar{u}|).$$

Therefore, one has

$$\begin{aligned}I_1=&\int_0^{t\wedge\theta_{\Delta,R}}\phi'_{\delta\varepsilon}(e_\Delta(s))[F(z(s),u(s))-F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))]\mathrm{d}s\\ \leq&\int_0^{t\wedge\theta_{\Delta,R}}\phi'_{\delta\varepsilon}(e_\Delta(s))[F(z(s),u(s))-F_\Delta(z_\Delta(s),u_\Delta(s))]\mathrm{d}s\\ &+\int_0^{t\wedge\theta_{\Delta,R}}\phi'_{\delta\varepsilon}(e_\Delta(s))[F(z_\Delta(s),u_\Delta(s))-F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))]\mathrm{d}s\\ \leq&L_1\int_0^{t\wedge\theta_{\Delta,R}}|e_\Delta(s)|\mathrm{d}s+\int_0^{t\wedge\theta_{\Delta,R}}\mid F(z_\Delta(s),u_\Delta(s))-F_\Delta(\bar{z}_\Delta(s),\bar{u}_\Delta(s))\mid\mathrm{d}s.\end{aligned} \tag{14}$$

For

$$I_2=\frac{1}{2}\int_0^{t\wedge\theta_{\Delta,R}}\phi''_{\delta\varepsilon}(e_\Delta(s))[G(z(s))-G(\bar{z}_\Delta(s))]^2\mathrm{d}s,$$

and according to Lemma 3, we have

$$\begin{aligned}I_2\leq&L_2^2\int_0^{t\wedge\theta_{\Delta,R}}\frac{1}{|e_\Delta(z)|\ln\delta}I_{[\frac{\varepsilon}{\delta}\leq|e_\Delta(s)|\leq\varepsilon]}\Big|z(s)-\bar{z}_\Delta(s)\Big|^{1+2\alpha}\mathrm{d}s\\ \leq&L_2^2\int_0^{t\wedge\theta_{\Delta,R}}\frac{1}{|e_\Delta(z)|\ln\delta}I_{[\frac{\varepsilon}{\delta}\leq|e_\Delta(s)|\leq\varepsilon]}\Big(|z(s)-z_\Delta(s)|^{1+2\alpha}+|z_\Delta(s)-\bar{z}_\Delta(s)|^{1+2\alpha}\Big)\mathrm{d}s\\ \leq&\frac{2^{2\alpha}L_2^2}{\ln\delta}\int_0^{t\wedge\theta_{\Delta,R}}e_\Delta^{2\alpha}(s)I_{[\frac{\varepsilon}{\delta}\leq|e_\Delta(s)|\leq\varepsilon]}\mathrm{d}s\\ &+\frac{2^{2\alpha}L_2^2}{\ln\delta}\int_0^t\frac{1}{e_\Delta(s)}|z_\Delta(s)-\bar{z}_\Delta(s)|^{1+2\alpha}I_{[\frac{\varepsilon}{\delta}\leq|e_\Delta(s)|\leq\varepsilon]}\mathrm{d}s\\ \leq&\frac{2^{2\alpha}L_2^2\varepsilon^{2\alpha}}{\ln\delta}+\frac{2^{2\alpha}L_2^2\delta}{\ln\delta}\int_0^t|z_\Delta(s)-\bar{z}_\Delta(s)|^{1+2\alpha}\mathrm{d}s.\end{aligned} \tag{15}$$

According to Equations (11) and (13)–(15), Assumption A.1, Lemma 4, and the Hölder inequality, we derive that

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \\
\le&\varepsilon + L_1\int_0^t \mathbb{E}e_{\Delta,R}(t\wedge\theta_{\Delta,R})\mathrm{d}s + \sqrt{H}\mathbb{E}\int_0^{t\wedge\theta_{\Delta,R}} |z_\Delta(s)-\bar{z}_\Delta(s)|(1+|z_\Delta(s)|^\gamma + |\bar{z}_\Delta(s)|^\gamma)^{\frac{1}{2}}\mathrm{d}s \\
&+2^{2\alpha}L_2^2\Big(\frac{\varepsilon^{2\alpha}T}{\ln\delta} + \frac{\delta}{\varepsilon\ln\delta}TC_{1+2\alpha}\Big(\Delta^{\frac{1}{2}}\psi(\Delta)\Big)^{1+2\alpha}\Big) \\
\le&\varepsilon + L_1\int_0^t \mathbb{E}|e_{\Delta,R}(t\wedge\theta_{\Delta,R})|\mathrm{d}s + C\Big(\Delta^{\frac{1}{2}}\psi(\Delta)\Big) + C\Big(\frac{\varepsilon^{2\alpha}T}{\ln\delta} + \frac{\varepsilon}{\ln\delta}\Big(\Delta^{\frac{1}{2}}\psi(\Delta)\Big)\Big).
\end{aligned}
$$

By the Gronwall inequality, it can be concluded that

$$
\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \le C\bigg(\varepsilon + \Delta^{\frac{1}{2}}H(\Delta) + \frac{\varepsilon^{2\alpha}}{\ln\delta} + \frac{\delta}{\varepsilon\ln\delta}(\Delta^{\frac{1}{2}}(H(\Delta))^{1+2\alpha})\bigg).
$$

If $\alpha=0$, let $\delta=\Delta^{-\frac{1}{8}}$ and $\varepsilon=-\frac{1}{\ln\Delta}$. Then,

$$
\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \le C\Big(-\frac{9}{\ln\Delta} + \Delta^{\frac{1}{4}}\Big(\psi(\Delta)^{\frac{1}{2}}\Big(\Delta^{\frac{1}{4}}(\psi(\Delta))^{\frac{1}{2}} + 8\Delta^{\frac{1}{8}}(\psi(\Delta))^{\frac{1}{2}}\Big)\Big).
$$

Noting that $\Delta^{\frac{1}{8}}(\psi(\Delta))^{\frac{1}{2}}\le 1$, $\Delta^{\frac{1}{4}}(\psi(\Delta))^{\frac{1}{2}}\le 1$, one has

$$
\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \le C\Big(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac{1}{4}}(\psi(\Delta)^{\frac{1}{2}})\Big).
$$

If $0<\alpha<\frac{1}{2}$, let $\delta=2$ and $\varepsilon=\Delta^{\frac{1}{2}}\psi(\Delta)$. We can then conclude that

$$
\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \le C\Big(\Delta^{\frac{1}{2}}(\psi(\Delta))\Big)^{2\alpha}.
$$

This completes the proof. □

Theorem 2. *If Assumptions A.1–A.4 and the conditions of Lemma 3 hold, and $q>1$,*

$$
\psi(\Delta) \ge \phi(\Delta^{\frac{1}{2}}(\psi(\Delta))^{\frac{-1}{p-1}}),
$$

Then, for any small $\Delta\in(0,\Delta^*]$, $T>0$, we have

$$
\mathbb{E}|z(T)-z_\Delta(T)| \le \begin{cases} C\Big(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac{1}{4}}(\psi(\Delta)^{\frac{1}{2}})\Big), & \alpha=0, \\ C\Big(\Delta^{\frac{1}{2}}(\psi(\Delta))\Big)^{2\alpha}, & 0<\alpha<\frac{1}{2}, \end{cases}
$$

Proof. When $\delta>0$, by Young's inequality, Theorem 1, and (11), we can obtain that for any $q>1$,

$$
\mathbb{E}\bigg[\sup_{0\le t\le T}|z(t)|^q\bigg] \le \infty, \tag{16}
$$

Then,

$$\begin{aligned}
\mathbb{E}|e_\Delta(T)| &= \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}\le T}| \\
&\le \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \frac{\delta}{p}\mathbb{E}|e_\Delta(T)|^q + \frac{q-1}{q\delta^{\frac{1}{q-1}}}q(\theta_{\Delta,R}\le T) \\
&\le \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \frac{C\delta}{q} + \frac{C(q-1)}{q\delta^{\frac{1}{q-1}}R^q}.
\end{aligned}$$

Since $\psi(\Delta) \ge \phi(\Delta^{\frac{1}{4}}(\psi(\Delta))^{\frac{-1}{q-1}})$, we obtain that

$$\phi^{-1}(\psi(\Delta)) \ge \left(\Delta^{\frac{1}{4}}(\psi(\Delta))\right)^{\frac{-1}{q-1}},$$

Let

$$\delta = \Delta^{\frac{1}{2}}\psi(\Delta), R = \left(\Delta^{\frac{1}{4}}(\psi(\Delta))\right)^{\frac{-1}{q-1}}.$$

Therefore, it can be concluded that

$$\mathbb{E}|e_\Delta(T)| = \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + C\Delta^{\frac{1}{2}}\psi(\Delta),$$

which, together with (11), can derive the results of the theorem. □

4. Conclusions

Based on the truncated Euler–Maruyama method, this paper studied SVIDEs with Hölder diffusion coefficients, in which the drift coefficient satisfies the local Lipschitz condition and Khasminskii condition. With the help of the truncated Euler–Maruyama method, the numerical soulutions of the SVIDEs were obtained. In addition, we revealed that the truncated Euler–Maruyama solutions are bounded in the sense of the pth moment and converge to the exact solutions at any fixed time T.

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