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Existence, Nonexistence and Multiplicity of Positive Solutions for Generalized Laplacian Problems with a Parameter

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Abstract: We investigate the homogeneous Dirichlet boundary value problem for generalized Laplacian equations with a singular, potentially non-integrable weight. By examining asymptotic behaviors of the nonlinear term near 0 and ∞, we establish the existence, nonexistence, and multiplicity of positive solutions for all positive values of the parameter λ. Our proofs rely on the fixed point theorem concerning cone expansion and compression of norm type and the Leray–Schauder's fixed point theorem.

Keywords: generalized Laplacian problems; multiplicity of positive solutions; singular weight function

MSC: 34B08; 34B15

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1. Introduction

Consider the following singular φ-Laplacian problem:

$$\begin{cases} (q(t)\varphi(u'(t)))' + \lambda h(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $q : [0,1] \to (0,\infty)$ is a continuous function, $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a parameter, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying $f(s) > 0$ for $s > 0$, and $h : (0,1) \to \mathbb{R}_+$ is a continuous function.

Throughout this paper, we assume the following hypotheses .

(A) There exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \le \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{2}$$

For the sake of convenience, we denote by $\mathcal{H}_\varrho$ the set

$$\left\{ L \in C((0,1), \mathbb{R}_+) : \int_0^1 \varrho^{-1}\left(\left| \int_s^{\frac{1}{2}} L(\tau)d\tau \right| \right) ds < \infty \right\}.$$

Here, $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing homeomorphism. Let us introduce the following notations:

$$f_0 := \lim_{s\to 0^+} \frac{f(s)}{\varphi(s)} \text{ and } f_\infty := \lim_{s\to\infty} \frac{f(s)}{\varphi(s)}.$$

As is well known, it follows from (B) that

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{3}$$

and

$$L^1(0,1) \cap C(0,1) \subsetneq \mathcal{H}_{\psi_1} \subseteq \mathcal{H}_{\varphi} \subseteq \mathcal{H}_{\psi_2}$$

(see, e.g., [1] or [2]).

It is a well-established fact that any function of the form

$$\varphi(s) = \sum_{k=1}^{n} |s|^{m_k-2}s$$

satisfies the assumption (B) with $\psi_1(s) = \min\{s^{m_1-1}, s^{m_n-1}\}$ and $\psi_2(s) = \max\{s^{m_1-1}, s^{m_n-1}\}$ for $s \in \mathbb{R}_+$ (see, e.g., [3,4]). Here, $n \in \mathbb{N}$, $m_k \in (1,\infty)$ for $1 \le k \le n$ and $m_j \le m_l$ for $1 \le j \le l \le n$. If $n = 1$, then $\varphi(s) = |s|^{m-2}s$ for some $m \in (1,\infty)$. In this case, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}_+$.

The existence of positive solutions to problem (1) has been extensively researched in recent decades. For instance, under various conditions on f_0 and f_∞, Agarwal, Lü and O'Regan [5] examined the existence and multiplicity of positive solutions to problem (1) with $\varphi(s) = |s|^{p-2}s$ and $q \equiv 1$. Among other results, the existence of two positive solutions to problem (1) was shown for λ, belonging to a certain open interval if either $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$. Subsequently, Wang [1] extended these results in [5] to generalized φ-Laplacian problems, assuming that φ satisfies (A) and $h \in C[0,1]$. More recently, Kim [6] further extended the results of [1,5] to singularly weighed φ-Laplacian problems, as demonstrated by the following theorem.

Theorem 1. *Assume that (A) and $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ hold.*

(i) *If $f_0 = f_\infty = 0$, then there exists $\lambda_1 > \bar{\lambda} > 0$ such that problem (1) has two positive solutions for $\lambda > \lambda_1$ and no positive solutions for $\lambda < \bar{\lambda}$.*

(ii) *If $f_0 = f_\infty = \infty$, then there exists $\underline{\lambda} > \lambda_2 > 0$ such that problem (1) has two positive solutions for $\lambda \in (0, \lambda_2)$ and no positive solutions for $\lambda > \underline{\lambda}$.*

However, in Theorem 1, there is no information for the existence of positive solutions on $[\bar{\lambda}, \lambda_1]$ or $[\lambda_2, \underline{\lambda}]$. When $q \equiv 1$, $h \in \mathcal{H}_{\psi_1}$ and $h \not\equiv 0$ on any subinterval in $(0,1)$, Xu and Lee [7] showed the existence, nonexistence, and multiplicity of positive solutions to problem (1) for all positive values of the parameter λ. For other interesting results, we refer the reader to [8–13] and the references therein.

Under the more general assumptions of q and h above, this paper aims to extend the results of previous studies [1,5–7]. The main result is stated as follows:

Theorem 2. *Assume that (A) and $h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ hold.*

(1) *If $f_0 = f_\infty = 0$, then there exists $\lambda_0^* \ge \lambda_*^0 > 0$ such that problem (1) has two positive solutions for $\lambda > \lambda_0^*$, one positive solution for $\lambda \in [\lambda_*^0, \lambda_0^*]$, and no positive solutions for $\lambda \in (0, \lambda_*^0)$.*

(2) *If $f_0 = f_\infty = \infty$, then there exists $\lambda_\infty^* \ge \lambda_*^\infty > 0$ such that problem (1) has two positive solutions for $\lambda \in (0, \lambda_*^\infty)$, one positive solution for $\lambda \in [\lambda_*^\infty, \lambda_\infty^*]$, and no positive solutions for $\lambda > \lambda_\infty^*$.*

In [2], the nonlinearity $f = f(t,s)$ was required to satisfy $f(t_0, 0) > 0$ for some $t_0 \in [0,1]$, so that all non-negative solutions are positive ones. By Theorem 1 in [2], the existence of an unbounded solution component was guaranteed, and further analysis of the behavior of f at ∞ showed the existence, nonexistence, and multiplicity of positive solutions to problem (1). Compared with the results in [2], the nonlinearity $f = f(s)$ considered in this paper may satisfy $f(0) = 0$, which allows for the trivial solution $u \equiv 0$ for every $\lambda \in \mathbb{R}_+$. While Theorem 1 in [2] provides the existence of an unbounded solution component to problem (1), it does not guarantee the existence of positive solutions. To address this limitation, we have utilized the fixed point theorem concerning cone expansion and compression of norm type and the Leray–Schauder's fixed point theorem in this paper.

The remainder of this paper is structured as follows. Section 2 summarizes relevant existing results without proof, providing a foundation for the subsequent proof of Theorem 2. Section 3 introduces auxiliary lemmas that are crucial for proving Theorem 2 and provides the proof of Theorem 2. Additionally, some examples are provided to illustrate the application of Theorem 2. Finally, Section 4 summarizes the main results, highlights limitations of our study, and outlines future research directions.

2. Preliminaries

Throughout this section, we assume that (A) and $h \in \mathcal{H}_\varphi \setminus \{0\}$ hold. The usual maximum norm in a Banach space $C[0,1]$ is denoted by

$$\|v\|_\infty := \max\{|v(t)| : t \in [0,1]\} \text{ for } v \in C[0,1].$$

Let $a_h := \inf\{t \in (0,1) : h(t) > 0\}$, $b_h := \sup\{t \in (0,1) : h(t) > 0\}$,

$$\bar{a}_h := \sup\{t \in (0,1) : h(z) > 0 \text{ for all } z \in (a_h, t)\},$$

$$\bar{b}_h := \inf\{t \in (0,1) : h(z) > 0 \text{ for all } z \in (t, b_h)\},$$

$$\gamma_h^1 := \frac{1}{4}(3a_h + \bar{a}_h), \gamma_h^2 := \frac{1}{4}(\bar{b}_h + 3b_h) \text{ and } \gamma_h^* := \frac{1}{2}(\gamma_h^1 + \gamma_h^2).$$

Then, since $h : (0,1) \to \mathbb{R}_+$ is a continuous function with $h \not\equiv 0$, we have two cases: either

$$(i)\ 0 \le a_h < \bar{a}_h \le \bar{b}_h < b_h \le 1$$

or

$$(ii)\ 0 \le a_h = \bar{b}_h < b_h \le 1 \text{ and } 0 \le a_h < \bar{a}_h = b_h \le 1.$$

Hence,

$$h(t) > 0 \text{ for } t \in (a_h, \bar{a}_h) \cup (\bar{b}_h, b_h), \text{ and } 0 \le a_h < \gamma_h^1 < \gamma_h^2 < b_h \le 1. \tag{4}$$

Let $q_h := q_1 \min\{\gamma_h^1, 1 - \gamma_h^2\} \in (0,1)$. Here,

$$q_0 := \min\{q(t) : t \in [0,1]\} > 0 \text{ and } q_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1].$$

Define $\mathcal{K}$ as the set of all non-negative continuous functions u satisfying

$$u(t) \ge q_h \|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2].$$

Then, $\mathcal{K}$ is a cone in $C[0,1]$. For $\alpha > 0$, let

$$\mathcal{K}_\alpha := \{u \in \mathcal{K} \ :\ \|u\|_\infty < \alpha\}, \partial\mathcal{K}_\alpha := \{u \in \mathcal{K} \ :\ \|u\|_\infty = \alpha\},$$

and $\overline{\mathcal{K}}_\alpha := \mathcal{K}_\alpha \cup \partial\mathcal{K}_\alpha$.

For $L \in \mathcal{H}_\varphi$, consider the following problem:

$$\begin{cases} (q(t)\varphi(v'(t)))' + L(t) = 0, & t \in (0,1), \\ v(0) = v(1) = 0. \end{cases} \tag{5}$$

Define the function $S : \mathcal{H}_\varphi \to C[0,1]$, for $L \in \mathcal{H}_\varphi \setminus \{0\}$, by

$$S(L)(t) = \begin{cases} \int_0^t \varphi^{-1}\left(\frac{1}{q(s)} \int_s^\sigma L(\tau)d\tau\right)ds, & \text{if } 0 \le t \le \sigma, \\ \int_t^1 \varphi^{-1}\left(\frac{1}{q(s)} \int_\sigma^s L(\tau)d\tau\right)ds, & \text{if } \sigma \le t \le 1, \end{cases} \tag{6}$$

where $\sigma = \sigma(L)$ is a constant satisfying

$$\int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma L(\tau)d\tau\right)ds = \int_\sigma^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_\sigma^s L(\tau)d\tau\right)ds. \tag{7}$$

For any $L \in \mathcal{H}_\varphi$ and any σ satisfying (7), $S(L)$ is monotonically increasing on $[0,\sigma)$ and monotonically decreasing on $(\sigma,1]$. Note that $\sigma = \sigma(L)$ is not necessarily unique, but $S(L)$ is invariant under the choice of σ satisfying (7) (see, e.g., [1]).

Lemma 1. ([2], Lemma 1 and Lemma 2) *Assume that (A) holds, and let $L \in \mathcal{H}_\varphi$ be given. Then,*
(1) $S(L)$ is a unique solution to problem (5) with the following property:

$$S(L)(t) \geq \min\{t, 1-t\}q_1\|S(L)\|_\infty \text{ for } t \in [0,1],$$

and thus, $S(L) \in \mathcal{K}$.
(2) If $L \not\equiv 0$, then there exists a subinterval $[\sigma_1, \sigma_2]$ of $(0,1)$ such that $(S(L))'(t) > 0$, $t \in (0,\sigma_1)$, $(S(L))'(t) = 0$ for $t \in [\sigma_1,\sigma_2]$ and $(S(L))'(t) < 0$, $t \in (\sigma_2, 1)$.

Define the function $G : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by

$$G(\lambda,u)(t) = \lambda h(t) f(u(t)) \text{ for } (\lambda,u) \in \mathbb{R}_+ \times \mathcal{K} \text{ and } t \in (0,1).$$

Obviously, $G(\lambda,u) \in \mathcal{H}_\varphi$ for all $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$.
Now, we introduce the operator $H : \mathbb{R}_+ \times \mathcal{K} \to C[0,1]$ by

$$H(\lambda,u) \equiv S(G(\lambda,u)) \text{ for } (\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}.$$

To be precise, for $(\lambda,u) \in \mathbb{R}_+ \times \mathcal{K}$,

$$H(\lambda,u)(t) = \begin{cases} \int_0^t \varphi^{-1}(\frac{1}{q(s)}\int_s^\sigma G(\lambda,u)(\tau)d\tau)ds, & \text{if } 0 \leq t \leq \sigma, \\ \int_t^1 \varphi^{-1}(\frac{1}{q(s)}\int_\sigma^s G(\lambda,u)(\tau)d\tau)ds, & \text{if } \sigma \leq t \leq 1, \end{cases}$$

where $\sigma = \sigma(\lambda,u)$ is a constant that satisfies

$$\int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma G(\lambda,u)(\tau)d\tau\right)ds = \int_\sigma^1 \varphi^{-1}\left(\frac{1}{q(s)}\int_\sigma^s G(\lambda,u)(\tau)d\tau\right)ds. \tag{8}$$

Remark 1. (1) *By Lemma 1 (1), $H(\mathbb{R}_+ \times \mathcal{K}) \subseteq \mathcal{K}$.*
(2) *It is evident that (1) has a solution if and only if $H(\lambda,\cdot)$ has a fixed point in $\mathcal{K}$.*
(3) *From $H(0,u) = 0$ for any $u \in \mathcal{K}$, it follows that 0 is a unique solution to problem (1) with $\lambda = 0$.*
(4) *Lemma 1 (2) ensures that if u is a nonzero solution to problem (1) with $\lambda > 0$, then u is a positive solution.*

Lemma 2. ([2], Lemma 4) *Assume that (A) holds, and let $h \in \mathcal{H}_\varphi$ hold. Then, the operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous, i.e., compact and continuous.*

Finally, we introduce the fixed point theorem concerning cone expansion and compression of norm type and the Leray–Schauder fixed point theorem.

Theorem 3. ([14]) *Let $(Y, \|\cdot\|)$ be a Banach space, and let P be a cone in Y. Assume that Γ_1 and Γ_2 are open subsets of Y with $0 \in \Gamma_1$ and $\overline{\Gamma}_1 \subset \Gamma_2$. Let $Q : P \cap (\overline{\Gamma}_2 \setminus \Gamma_1) \to P$ be a completely continuous operator, such that if either*
$\|Qu\| \leq \|u\|$ for $u \in P \cap \partial\Gamma_1$ and $\|Qu\| \geq \|u\|$ for $u \in P \cap \partial\Gamma_2$ or
$\|Qu\| \geq \|u\|$ for $u \in P \cap \partial\Gamma_1$ and $\|Qu\| \leq \|u\|$ for $u \in P \cap \partial\Gamma_2$,
then Q has a fixed point in $P \cap (\overline{\Gamma}_2 \setminus \Gamma_1)$.

Theorem 4. ([15]) *Let X be a Banach space, and let P be a closed, convex, and bounded set in X. Assume that $Q : P \to P$ is completely continuous. Then, Q has a fixed point in P.*

3. Proof of Main Results

Lemma 3. *Assume that $(A), h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ and $f_0 = f_\infty = 0$ hold. Let $I = [a, b]$ be a compact interval with $0 < a < b$. Then, there exist m_I and M_I such that $0 < m_I \le \|u\|_\infty \le M_I$ for any positive solution u to problem (1) with $\lambda \in I$.*

Proof. Let $w := (4b)^{-1}\psi_1(h_*^{-1}) > 0$. Here,

$$h_* := \max\left\{\int_0^{\frac{1}{2}} \psi_1^{-1}\left(q_0^{-1}\int_s^{\frac{1}{2}} h(\tau)d\tau\right)ds, \int_{\frac{1}{2}}^1 \psi_1^{-1}\left(q_0^{-1}\int_{\frac{1}{2}}^s h(\tau)d\tau\right)ds\right\} > 0.$$

It follows from $f_\infty = 0$ that there exists $s_w > 0$ such that $f(s) \le w\varphi(s)$ for $s \in [s_w, \infty)$. Let $C_w = \max\{f(s) : s \in [0, s_w]\} > 0$. Then,

$$f(s) \le C_w + w\varphi(s) \text{ for } s \in \mathbb{R}_+. \tag{9}$$

We begin by showing the existence of M_I satisfying $\|u\|_\infty \le M_I$ for any positive solution u to problem (1) with $\lambda \in I$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in I$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, for sufficiently large $N > 0$, $C_w \le w\varphi(\|u_N\|_\infty)$, and by (9),

$$f(u_N(t)) \le 2w\varphi(\|u_N\|_\infty) \text{ for } t \in [0, 1]. \tag{10}$$

Let σ_N denote a positive real number such that $\|u_N\|_\infty = u_N(\sigma_N)$. We restrict our attention to the case where $\sigma_N \le \frac{1}{2}$, because the case where $\sigma_N > \frac{1}{2}$ can be treated analogously. Then, by (3) and (10),

$$\begin{aligned}
\|u_N\|_\infty &= \int_0^{\sigma_N} \varphi^{-1}\left(\frac{1}{q(s)}\int_s^{\sigma_N} \lambda_N h(\tau) f(u_N(\tau))d\tau\right)ds \\
&\le \int_0^{\frac{1}{2}} \varphi^{-1}\left(q_0^{-1}\int_s^{\frac{1}{2}} h(\tau)d\tau 2bw\varphi(\|u_N\|_\infty)\right)ds \\
&\le \int_0^{\frac{1}{2}} \psi_1^{-1}\left(q_0^{-1}\int_s^{\frac{1}{2}} h(\tau)d\tau\right)ds\varphi^{-1}(2bw\varphi(\|u_N\|_\infty)) \\
&\le h_*\psi_1^{-1}(2bw)\|u_N\|_\infty.
\end{aligned}$$

Consequently, $w \ge (2b)^{-1}\psi_1(h_*^{-1})$. This contradicts the choice of w.

Next, we show the existence of m_I satisfying $\|u\|_\infty \ge m_I > 0$ for any positive solution u to problem (1) with $\lambda \in I$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in I$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Since $f_0 = 0$, there exists $\delta > 0$ such that $f(s) \le w\varphi(s)$ for $s \in [0, \delta]$. Since $\|u_n\|_\infty \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\|u_N\|_\infty < \delta$ and $f(u_N(t)) \le w\varphi(u_N(t)) \le w\varphi(\|u_N\|_\infty)$ for all $t \in [0, 1]$. By the same reasoning as above, we can easily show that the choice of w leads to a contradiction. Thus, the proof is complete. □

Lemma 4. *Assume that $(A), h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ and $f_\infty = 0$ hold. If (1) has a positive solution at $\lambda = \lambda_0$, then (1) has at least one positive solution for all $\lambda > \lambda_0$.*

Proof. Let u_0 be a positive solution to problem (1) with $\lambda = \lambda_0$ and let $\hat{\lambda} > \lambda_0$ be fixed. Consider the following modified problem:

$$\begin{cases} (q(t)\varphi(u'(t)))' + \hat{\lambda}h(t)f(\gamma_1(t, u(t))) = 0, \ \ t \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{11}$$

where $\gamma_1 : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined, for $(t,s) \in [0,1] \times \mathbb{R}_+$, by

$$\gamma_1(t,s) = \begin{cases} s, & \text{if } s \geq u_0(t), \\ u_0(t), & \text{if } 0 \leq s < u_0(t). \end{cases}$$

Define $T_1 : \mathcal{K} \to \mathcal{K}$ by $T_1(u) = S(F_1(u))$ for $u \in \mathcal{K}$, where $F_1(u)(t) = \hat{\lambda} h(t) f(\gamma_1(t, u(t)))$ for $u \in \mathcal{K}$ and $t \in (0,1)$. Since $F_1(u) \in \mathcal{H}_{\psi_1}$ for any $u \in \mathcal{K}$, by Lemma 1, T_1 is well defined. It is easy to see that T_1 is completely continuous on $\mathcal{K}$, and u is a solution to problem (11) if and only if $u = T_1 u$.

First, we show the existence of a solution to problem (11).

(i) Assume that f is bounded on $\mathbb{R}_+$. From the definition of γ_1 and the continuity of f, it follows that there exists $r > 0$ such that $||T_1(u)||_\infty < r$ for all $u \in \mathcal{K}$, and $T_1(\mathcal{K}_r) \subseteq \mathcal{K}_r$. Then, by Theorem 4, there exists $u \in \mathcal{K}_r$ such that $T_1(u) = u$, and consequently, problem (11) has a non-negative solution u.

(ii) Assume that f is unbounded on $\mathbb{R}_+$. Let $\epsilon \in (0, \hat{\lambda}^{-1} \psi_1(h_*^{-1}))$ be given. Here, h_* is the constant in the proof of Lemma 3. Since $f_\infty = 0$, there exists $r_1 > 0$ such that

$$f(s) \leq \epsilon \varphi(s) \text{ for all } (t,s) \in [0,1] \times [r_1, \infty). \tag{12}$$

Since f is unbounded on $\mathbb{R}_+$ and $\gamma_1(t,s) = s$ for $(t,s) \in [0,1] \times [\|u_0\|_\infty, \infty)$, there exists $r_2 > 0$ such that $r_2 > r_1$ and

$$f(\gamma_1(t,s)) \leq f(r_2) \text{ for all } (t,s) \in [0,1] \times [0, r_2]. \tag{13}$$

Let $u \in \mathcal{K}_{r_2}$ be given. Then, by (12) and (13),

$$f(\gamma_1(t, u(t))) \leq f(r_2) \leq \epsilon \varphi(r_2) \text{ for all } t \in [0,1]. \tag{14}$$

Let σ denote a positive constant satisfying $\|T_1(u)\|_\infty = T_1(u)(\sigma)$. We restrict our attention to the case where $\sigma \leq \frac{1}{2}$, since the case where $\sigma > \frac{1}{2}$ can be treated analogously. Then, by (3), (14), and the choice of ϵ,

$$\begin{aligned} \|T_1(u)\|_\infty &= \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)} \int_s^\sigma \hat{\lambda} h(\tau) f(\gamma_1(\tau, u(\tau))) d\tau\right) ds \\ &\leq \int_0^{\frac{1}{2}} \varphi^{-1}\left(q_0^{-1} \int_s^{\frac{1}{2}} h(\tau) d\tau \epsilon \hat{\lambda} \varphi(r_2)\right) ds \\ &\leq h_* \varphi^{-1}(\epsilon \hat{\lambda} \varphi(r_2)) \leq h_* \psi_1^{-1}(\epsilon \hat{\lambda}) r_2 < r_2. \end{aligned}$$

By Theorem 4, there exists $u \in \mathcal{K}_{r_2}$ such that $T_1(u) = u$, and consequently, problem (11) has a non-negative solution u.

Finally, we show that if u is a solution to problem (11), then $u(t) \geq u_0(t)$ for $t \in [0,1]$. If it is true, by the definition of γ_1, u is a positive solution to problem (1) with $\lambda = \hat{\lambda}$, and consequently, the proof is complete.

Assume on the contrary that there exists a solution u to problem (11) such that $u(t) \not\geq u_0(t)$ for $t \in [0,1]$. Since $u(0) = u(1) = u_0(0) = u_0(1) = 0$, there exists a subinterval $(t_1, t_2) \subseteq (0,1)$ such that $u_0(t) - u(t) > 0$ for $t \in (t_1, t_2)$ and $u_0(t_1) - u(t_1) = u_0(t_2) - u(t_2) = 0$. From the fact that $u_0 - u \in C[0,1]$, it follows that there exists $t^* \in (t_1, t_2)$ such that $u_0(t^*) - u(t^*) = \max\{u_0(t) - u(t) : t \in [t_1, t_2]\} > 0$ and $u_0'(t^*) = u'(t^*)$. For $t \in (t_1, t_2)$,

$$-(q(t)\varphi(u'(t)))' = \hat{\lambda} h(t) f(\gamma_1(t, u(t))) = \hat{\lambda} h(t) f(u_0(t)) \geq \lambda_0 h(t) f(u_0(t)) = -(q(t)\varphi(u_0'(t)))',$$

i.e.,

$$-(q(t)\varphi(u'(t)))' \geq -(q(t)\varphi(u_0'(t)))' \text{ for } t \in (t_1, t_2). \tag{15}$$

For $t \in (t_1, t^*)$, integrating (15) from t to t^*, $q(t)\varphi(u'(t)) \geq q(t)\varphi(u_0'(t))$. Since φ is increasing,

$$u'(t) \geq u_0'(t) \text{ for } t \in (t_1, t^*). \tag{16}$$

Integrating (16) from t_1 to t^*, $u_0(t^*) - u(t^*) \leq 0$, which contradicts the choice of t^*. Thus, the proof is complete. □

Lemma 5. *Assume that* $(A), h \in \mathcal{H}_{\psi_1} \setminus \{0\}$ *and* $f_0 = f_\infty = \infty$ *holds. Let* $I = [a, b]$ *be a compact interval with* $0 < a < b$*. Then, there exist* m_I *and* M_I *such that* $0 < m_I \leq \|u\|_\infty \leq M_I$ *for any positive solution* u *to problem* (1) *with* $\lambda \in I$.

Proof. First, we show the existence of M_I satisfying $\|u\|_\infty \leq M_I$ for any positive solution u to problem (1) with $\lambda \in I$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in I$ and $||u_n||_\infty \to \infty$ as $n \to \infty$.

Let $C^* \in (\psi_2(h_{**}^{-1}), \infty)$ be given. Here,

$$h_{**} = \min\left\{ \gamma_h^1 \psi_2^{-1}\left(a\|q\|_\infty^{-1} \int_{\gamma_h^1}^{\gamma_h^*} h(\tau)d\tau \right), (1-\gamma_h^2)\psi_2^{-1}\left(a\|q\|_\infty^{-1} \int_{\gamma_h^*}^{\gamma_h^2} h(\tau)d\tau \right) \right\}.$$

Recall that $\gamma_h^* = 2^{-1}(\gamma_h^1 + \gamma_h^1) > 0$ and note that $h_{**} > 0$ by (4). By $f_\infty = \infty$, there exists $K > 0$ such that $f(s) \geq C^*\varphi(s)$ for $s > K$. For all n, $u_n \in \mathcal{K}$, and $u_n(t) \geq q_h \|u_n\|_\infty$ for $t \in [\gamma_h^1, \gamma_h^2]$. For sufficiently large $N > 0$, $\|u_N\|_\infty > q_h^{-1}K$ and $u_N(t) \geq K$ for $t \in [\gamma_h^1, \gamma_h^2]$. Thus,

$$f(u_N(t)) \geq C^*\varphi(u_N(t)) \text{ for all } t \in [\gamma_h^1, \gamma_h^2]. \tag{17}$$

Let σ_N denote a positive real number such that $u_N(\sigma_N) = \|u_N\|_\infty$. We restrict our attention to the case where $\sigma_N \geq \gamma_h^*$, because the case where $\sigma_N < \gamma_h^*$ can be treated analogously. Since $u_N(t) \geq u_N(\gamma_h^1)$ for $t \in [\gamma_h^1, \sigma_N]$, by (17),

$$\lambda_N h(t) f(u_N(t)) \geq aC^* h(t)\varphi(u_N(\gamma_h^1)) \text{ for all } t \in [\gamma_h^1, \sigma_N].$$

Thus, by (3),

$$\begin{aligned}
u_N(\gamma_h^1) &= \int_0^{\gamma_h^1} \varphi^{-1}\left(\frac{1}{q(s)} \int_s^{\sigma_N} \lambda_N h(\tau) f(u_N(\tau)) d\tau \right) ds \\
&\geq \int_0^{\gamma_h^1} \varphi^{-1}\left(a\|q\|_\infty^{-1} \int_{\gamma_h^1}^{\gamma_h^*} h(\tau)d\tau C^*\varphi(u_N(\gamma_h^1)) \right) ds \\
&\geq \gamma_h^1 \psi_2^{-1}\left(a\|q\|_\infty^{-1} \int_{\gamma_h^1}^{\gamma_h^*} h(\tau)d\tau \right) \varphi^{-1}\left(C^*\varphi(u_N(\gamma_h^1)) \right) \\
&\geq h_{**}\psi_2^{-1}(C^*) u_N(\gamma_h^1),
\end{aligned}$$

which contradicts the choice of C^*.

Next, we show the existence of m_I satisfying $\|u\|_\infty \geq m_I > 0$ for any positive solution u to problem (1) with $\lambda \in I$. By contradiction, we assume that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in I$ and $\|u_n\|_\infty \to 0$ as $n \to \infty$. Since $f_0 = \infty$, there exists $\delta > 0$ such that $f(s) \geq C^*\varphi(s)$ for $s \in [0, \delta]$. Since $\|u_n\|_\infty \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\|u_N\|_\infty < \delta$ and $f(u_N(t)) \geq C^*\varphi(u_N(t))$ for all $t \in [0, 1]$. By the same reasoning as above, we can easily show that the choice of C^* leads to a contradiction. Thus, the proof is complete. □

Lemma 6. *Assume that* $(A), h \in \mathcal{H}_{\psi_1} \setminus \{0\}$*, and* $f_0 = \infty$ *hold. If* (1) *has a positive solution at* $\lambda = \lambda_0$*, then* (1) *has at least one positive solution for all* $\lambda \in (0, \lambda_0)$.

Proof. Let u_0 be a positive solution to problem (1) with $\lambda = \lambda_0$ and let $\hat{\lambda} \in (0, \lambda_0)$ be fixed. Consider the following modified problem:

$$\begin{cases} (q(t)\varphi(u'(t)))' + \hat{\lambda} h(t) f(\gamma_2(t, u(t))) = 0, \ \ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{18}$$

where $\gamma_2 : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined, for $(t,s) \in [0,1] \times \mathbb{R}_+$, by

$$\gamma_2(t,s) = \begin{cases} u_0(t), & \text{if } \ s > u_0(t), \\ s, & \text{if } \ 0 \le s \le u_0(t). \end{cases}$$

Define $T_2 : \mathcal{K} \to \mathcal{K}$ by $T_2(u) = S(F_2(u))$ for $u \in \mathcal{K}$, where $F_2(u)(t) = \hat{\lambda} h(t) f(\gamma_2(t, u(t)))$ for $u \in \mathcal{K}$ and $t \in (0,1)$. Since $F_2(u) \in \mathcal{H}_{\psi_1}$ for any $u \in \mathcal{K}$, by Lemma 1, T_2 is well defined. It is easy to see that T_2 is completely continuous on $\mathcal{K}$, and u is a solution to problem (18) if and only if $u = T_2 u$. By the definition of γ_2, $f(\gamma_2(t,s)) \le \max\{f(s) : 0 \le s \le \|u_0\|_\infty\} \in (0, \infty)$ for all $(t,s) \in [0,1] \times \mathbb{R}_+$ and there exists $r_1 > 0$ such that $||T_2(v)||_\infty < r_1$ for all $v \in \mathcal{K}$, which implies

$$\|T_2(v)\|_\infty \le \|v\|_\infty \text{ for } v \in \partial\mathcal{K}_{r_1}. \tag{19}$$

Let $\bar{u}_0 = \min\{u_0(t) : t \in [\gamma_h^1, \gamma_h^2]\} > 0$. Since $\gamma_2(t,s) = s$ for $(t,s) \in [\gamma_h^1, \gamma_h^2] \times [0, \bar{u}_0]$,

$$\lim_{s \to 0^+} \frac{\min\{f(\gamma_2(t,s)) : t \in [\gamma_h^1, \gamma_h^2]\}}{\varphi(s)} = \lim_{s \to 0^+} \frac{f(s)}{\varphi(s)} = f_0 = \infty.$$

For fixed

$$C_* = \|q\|_\infty \hat{\lambda}^{-1} \max\left\{ \psi_2\left(\left[q_h \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h^*} h(\tau)d\tau \right) ds \right]^{-1} \right), \psi_2\left(\left[q_h \int_{\gamma_h^*}^{\gamma_h^2} \psi_2^{-1}\left(\int_{\gamma_h^*}^{s} h(\tau)d\tau \right) ds \right]^{-1} \right) \right\},$$

there exists $r_* \in (0, \bar{u}_0)$ such that

$$f(\gamma_2(t,s)) \ge C_* \varphi(s) \text{ for all } (t,s) \in [\gamma_h^1, \gamma_h^2] \times (0, r_*).$$

Take $r_2 \in (0, \min\{r_1, r_*\})$, and let $v \in \partial\mathcal{K}_{r_2}$ be given. Then, $0 \le v(t) \le r_2 < r_*$ and

$$f(\gamma_2(t, v(t))) \ge C_* \varphi(v(t)) \ge C_* \varphi(q_h \|v\|_\infty) \text{ for all } t \in [\gamma_h^1, \gamma_h^2]. \tag{20}$$

Let σ_N denote a positive real number such that $T_2(v)(\sigma) = \|T_2(v)\|_\infty$. We have two cases: either $\sigma \ge \gamma_h^*$ or $\sigma < \gamma_h^*$. We restrict our attention to the case where $\sigma \ge \gamma_h^*$, because the case where $\sigma < \gamma_h^*$ can be treated analogously. By (3) and (20),

$$\begin{aligned} \|T_2(v)\|_\infty &= \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)} \int_s^\sigma \hat{\lambda} h(\tau) f(\gamma_2(\tau, v(\tau))) d\tau \right) ds \\ &\ge \int_{\gamma_h^1}^{\gamma_h^*} \varphi^{-1}\left(\|q\|_\infty^{-1} \hat{\lambda} C_* \varphi(q_h \|v\|_\infty) \int_s^{\gamma_h^*} h(\tau)d\tau \right) ds \\ &\ge \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau \right) ds \varphi^{-1}(\|q\|_\infty^{-1} \hat{\lambda} C_* \varphi(q_h \|v\|_\infty)) \\ &\ge \int_{\gamma_h^1}^{\gamma_h^*} \psi_2^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau \right) ds \psi_2^{-1}(\|q\|_\infty^{-1} \hat{\lambda} C_*) q_h \|v\|_\infty, \end{aligned}$$

which implies, by the choice of C_*,

$$\|T_2(v)\|_\infty \ge \|v\|_\infty \text{ for } v \in \partial\mathcal{K}_{r_2}. \tag{21}$$

By (19) and (21), in view of Theorem 3, problem (18) has a nonzero solution u with $\|r_2\|_\infty \le \|u\|_\infty \le \|r_1\|_\infty$. By Lemma 1 (2), u is a positive solution to problem (18).

We show that if u is a positive solution to problem (18), then $u(t) \le u_0(t)$ for $t \in [0,1]$. If it is true, by the definition of γ_2, u is a positive solution to problem (1) with $\lambda = \hat{\lambda}$, and thus, the proof is complete.

By contradiction, suppose that there exists a solution u to problem (18) such that $u(t) \not\le u_0(t)$ for $t \in [0,1]$. Since $u(0) = u(1) = u_0(0) = u_0(1) = 0$, there exists an interval $(t_1, t_2) \subseteq (0,1)$ such that $u(t) - u_0(t) > 0$ for $t \in (t_1, t_2)$ and $u(t_1) - u_0(t_1) = u(t_2) - u_0(t_2) = 0$. From the fact that $u - u_0 \in C[0,1]$, it follows that there exists $t^* \in (t_1, t_2)$ such that $u(t^*) - u_0(t^*) = \max\{u(t) - u_0(t) : t \in [t_1, t_2]\} > 0$ and $u'(t^*) = u_0'(t^*)$. For $t \in (t_1, t_2)$,

$$-(q(t)\varphi(u'(t)))' = \hat{\lambda}h(t)f(\gamma_2(t,u(t))) = \hat{\lambda}h(t)f(u_0(t)) \le \lambda_0 h(t)f(u_0(t)) = -(q(t)\varphi(u_0'(t)))',$$

i.e.,

$$-(q(t)\varphi(u'(t)))' \le -(q(t)\varphi(u_0'(t)))' \text{ for } t \in (t_1, t_2). \tag{22}$$

For $t \in (t_1, t^*)$, integrating (22) from t to t^*, $q(t)\varphi(u'(t)) \le q(t)\varphi(u_0'(t))$. Since φ is increasing,

$$u'(t) \le u_0'(t) \text{ for } t \in (t_1, t^*). \tag{23}$$

Integrating (23) from t_1 to t^*, $u(t^*) - u_0(t^*) \le 0$, which contradicts the choice of t^*. Thus, the proof is complete. □

Now, we give the proof of Theorem 2.

Proof of Theorem 2. (1) Let $\lambda_0^* := \inf\{\mu : (1)$ have at least two positive solutions for $\lambda > \mu\}$ and $\lambda_*^0 := \inf\{\lambda : (1)$ have at least one positive solution$\}$. By Theorem 1 (i), λ_0^* and λ_*^0 are well defined and $\lambda_0^* \ge \lambda_*^0 \ge \bar{\lambda} > 0$. From Lemma 4, it follows that problem (1) has two positive solutions for $\lambda > \lambda_0^*$, one positive solution for $\lambda > \lambda_*^0$, and no positive solutions for $\lambda \in (0, \lambda_*^0)$. To complete the proof, it is enough to show that problem (1) has a positive solution for $\lambda = \lambda_*^0$. By the definition of λ_*^0 and Theorem 1 (i), there exists a sequence $\{(\lambda_n, u_n)\}$ such that $\bar{\lambda} \le \lambda_*^0 < \lambda_n \le \lambda_*^0 + n^{-1}$ and u_n is a positive solution to problem (1) with $\lambda = \lambda_n$. Then, $\lambda_n \to \lambda_*^0$ as $n \to \infty$, and by Lemma 3, there exists $m, M > 0$ such that $m \le \|u_n\|_\infty \le M$ for all n. Since $\{u_n\}$ is bounded and $H = H(\lambda, u)$ is compact, there exists a subsequence $\{H(\lambda_{n_k}, u_{n_k})\}$ of $\{H(\lambda_n, u_n)\}$ and $u_* \in \mathcal{K}$ such that $H(\lambda_{n_k}, u_{n_k}) \to u_*$ as $n_k \to \infty$. Since $H(\lambda_n, u_n) = u_n$, $u_{n_k} \to u_*$ as $n_k \to \infty$. Since H is continuous, $u_* = \lim_{n_k \to \infty} u_{n_k} = \lim_{n_k \to \infty} H(\lambda_{n_k}, u_{n_k}) = H(\lambda_*, u_*)$. Since $\|u_n\|_\infty \ge m$ for all n, $u_* \not\equiv 0$. Thus, (1) has a positive solution u_* for $\lambda = \lambda_*^0$.

(2) Let $\lambda_*^\infty := \sup\{\mu : (1)$ have at least two positive solutions for $\lambda \in (0, \mu)\}$ and $\lambda_\infty^* := \sup\{\lambda : (1)$ have at least one positive solution$\}$. By Theorem 1 (ii), λ_∞^* and λ_*^∞ are well defined and $\lambda^* \ge \lambda_* \ge \underline{\lambda} > 0$. From Lemma 6, it follows that problem (1) has two positive solutions for $\lambda \in (0, \lambda_*^\infty)$, one positive solution for $\lambda \in (0, \lambda_\infty^*)$, and no positive solutions for $\lambda > \lambda_\infty^*$. To complete the proof, it is enough to show that problem (1) has a positive solution for $\lambda = \lambda_\infty^*$. By the same reasoning as in the proof of Theorem 2 (1), we can complete the proof. □

Finally, we give some examples to illustrate the main result (Theorem 2).

Example 1. *Consider the following problem:*

$$\begin{cases} \left(\dfrac{1}{1+t^2}\varphi(u'(t))\right)' + \lambda h(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{24}$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism defined by

$$\varphi(s) = s + s^2 \text{ for } s \in \mathbb{R}_+$$

and $h : (0,1) \to \mathbb{R}_+$ is a continuous function defined by

$$h(t) = 0 \text{ for } t \in [0, \frac{1}{18}] \text{ and } h(t) = (t - \frac{1}{18})(1-t)^{-\alpha} \text{ for } t \in (\frac{1}{18}, 1).$$

Consequently, taking

$$\psi_1(y) = \min\{y, y^2\} \text{ and } \psi_2(y) = \max\{y, y^2\} \text{ for } y \in \mathbb{R}_+,$$

the assumption (A) holds. Since $\psi_1^{-1}(s) = \max\{\sqrt{s}, s\}$ for $s \in \mathbb{R}_+$, $h \in \mathcal{H}_{\psi_1} \setminus L^1(0,1)$ for any $\alpha \in [1,2)$.

Let f_1 and f_2 be continuous functions on $\mathbb{R}_+$ defined by

$$f_1(s) = s^{\beta_1} \text{ for } s \in \mathbb{R}_+ \text{ and } f_2(s) = \begin{cases} s^{\beta_2}, & \text{for } s \in [0,1], \\ e^{s-1}, & \text{for } s \in (1, \infty). \end{cases}$$

Here, $\beta_1 \in (1,2)$ and $\beta_2 \in (0,1)$ are constants. Then,

$$(f_1)_0 = (f_1)_\infty = 0 \text{ and } (f_2)_0 = (f_2)_\infty = \infty.$$

Consequently, by Theorem 2, there exist $\lambda_0^* \geq \lambda_*^0 > 0$ and $\lambda_\infty^* \geq \lambda_*^\infty > 0$ such that problem (24) with $f = f_1$ has two positive solutions for $\lambda > \lambda_0^*$, one positive solution for $\lambda \in [\lambda_*^0, \lambda_0^*]$, and no positive solutions for $\lambda \in (0, \lambda_*^0)$, and problem (24) with $f = f_2$ two positive solutions for $\lambda \in (0, \lambda_*^\infty)$, one positive solution for $\lambda \in [\lambda_*^\infty, \lambda_\infty^*]$, and no positive solutions for $\lambda > \lambda_\infty^*$.

4. Conclusions

In this work, we investigated the existence, nonexistence, and multiplicity of positive solutions to problem (1) for all positive values of the parameter λ. Our analysis relied on the application of two key fixed point theorems: the cone expansion and compression of norm-type theorem, and the Leray–Schauder fixed point theorem.

While our findings contribute to the understanding of problem (1), there are still some questions that remain unanswered and opportunities for further research. Under the assumption that $f(0) > 0$ instead of $f_0 = \infty$ in Theorem 2 (2), we can show that $\lambda_\infty^* = \lambda_*^\infty$ (see [2]). It is still unknown whether $\lambda_\infty^* = \lambda_*^\infty$ or $\lambda_*^0 = \lambda_0^*$ in Theorem 2, even in the simplified case where $q \equiv 1$ and $\varphi(u) = u''$. Our investigation focused on problems with Dirichlet boundary conditions. As part of a natural extension of this work, future research could explore analogous problems with nonlocal boundary conditions. Such an extension would provide a more comprehensive understanding of how boundary conditions influence the existence of positive solutions.

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