mathematics

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Approximating Continuous Function by Smooth Functions on Orbit Spaces

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Abstract: In this paper, we study the approximation of continuous functions on a subclass of singular space—the subcartesian space. As is well known, the orbit space of the proper action of a Lie group on a smooth manifold is a subcartesian space. We prove that continuous functions on the orbit space can be approximated by smooth functions.

Keywords: subcartesian space; orbit space; function; approximation

MSC: 58A40; 57R12; 70H33

1. Introduction

There has long been perceived the need for an extension of the framework of smooth manifolds in differential geometry, which is too restrictive and does not admit certain basic geometric intuitions. Sikorski's [1] theory of differential spaces studies the differential geometry of a large class of singular spaces, which both contains the theory of manifolds and allows the investigation of singularities. Analogous to algebraic geometry, which is the investigation of geometry in terms of polynomials, the theory of differential space is the investigation of geometry in terms of differentiable functions.

Precisely, a differential structure on a topological space S is a family $C^\infty(S)$ of real-valued functions on S satisfying the following conditions:

1. The family
$$\{f^{-1}(I)|f \in C^\infty(S) \text{and } I \text{ is an open interval in } \mathbb{R}\}$$
is a subbasis for the topology of S.
2. If $f_1, \cdots, f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \cdots, f_n) \in C^\infty(S)$.
3. If $f : S \to \mathbb{R}$ is a function such that, for every $x \in S$, there exists an open neighborhood U of x, and a function $f_x \in C^\infty(S)$ satisfying
$$f_x|_U = f|_U,$$
then $f \in C^\infty(S)$. Here, the subscript vertical bar $|$ denotes a restriction. $(S, C^\infty(S))$ is said to be a differential space. Functions $f \in C^\infty(S)$ are called smooth functions on S.

It follows that a smooth manifold M can be characterized as a differential space $(M, C^\infty(M))$, with $C^\infty(M)$ being all smooth functions on the smooth manifold M, such that every point has a neighborhood U diffeomorphic to an open subset V of $\mathbb{R}^n$, where n is the dimension of the manifold, the differential structures on U and V are generated by restrictions of smooth functions of M and $\mathbb{R}^n$, respectively, and diffeomorphism is in the sense of differential space. This definition can be weakened by not requiring V to be open in $\mathbb{R}^n$ and allowing n to be an arbitrary non-negative integer.

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Citation: Zhou, L.; Xia, Q. Approximating Continuous Function by Smooth Functions on Orbit Spaces. *Mathematics* **2024**, *12*, 386. https://doi.org/10.3390/math12030386

Academic Editor: Jay Jahangiri

Received: 12 December 2023
Revised: 17 January 2024
Accepted: 18 January 2024
Published: 25 January 2024

Definition 1. *A differential space S is said to be subcartesian [2] if every point of S has a neighborhood U diffeomorphic to a subset of some Cartesian space $\mathbb{R}^n$. $(U, \Phi, \mathbb{R}^n)$ is said to be a local chart of p, where $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^n$ is the diffeomorphism.*

The theory of subcartesian spaces has been developed by Śniatycki et al. in recent years. See [2] for a systematic treatment of this topic. From the above definition, any subset of a Euclidean space endowed with the differential subspace structure is a subcartesian space. Another typical example of subcartesian space is the orbit space of the proper action of a connected Lie group on a smooth manifold [2–4].

In this paper, we investigate the following problem: given any continuous function f on a subcartesian space S, we ask whether it can be approximated by smooth functions on S.

It is well known that a continuous function on a smooth manifold can be approximated by a smooth function [5,6], as stated by the following theorem.

Theorem 1 ([5])**.** *Let M be a smooth manifold and $f : M \to \mathbb{R}$ be a continuous function. Then, for any $\delta > 0$, there exists a smooth function $h \in C^\infty(M)$, such that*

$$|h(x) - f(x)| < \delta,$$

for all $x \in M$.

However, for a subcartesian space S, we cannot infer that a continuous function on S can be approximated by smooth functions on S. The first obstruction is that, given $p \in S$, with $(U, \Phi, \mathbb{R}^n)$ being its local chart, we cannot infer that the continuous function f restricting to U can be extended to a continuous function $\tilde{f}$ on an open subset of $\mathbb{R}^n$ that contains $\Phi(U)$ such that $\tilde{f} \circ \Phi(x) = f(x)$, for each $x \in U$.

In this paper, we investigate a special class of subcartesian space—orbit space R of the proper action of a connected Lie group G on a smooth manifold M. We overcome the obstructions described above by taking advantage of the geometric structure of the orbit space, which is obtained by the reduction of the symmetry of smooth manifolds. Precisely, we first investigate the local approximation problem, which is defined on an open neighborhood of $p \in R$, and then study passages from local to global. Since continuous or smooth functions on R can be lifted to G-invariant continuous or smooth functions on M, respectively, the local problem can be solved by approximating G-invariant continuous functions by G-invariant smooth functions on M. This can be solved by using the geometry of the symmetry of smooth manifolds together with Theorem 1. For the problem of passages from local to global, the geometric structure of the symmetry of smooth manifolds also plays a central role. We obtain the following theorem that gives a positive answer to the problem proposed above.

Theorem 2. *Let $f : R \to \mathbb{R}$ be a continuous function on the orbit space R. Then, for any $\delta > 0$, there exists $h \in C^\infty(R)$, such that*

$$|h(y) - f(y)| < \delta,$$

for any $y \in R$.

To the best of our knowledge, this is the first result on the approximation of functions in subcartesian space. We have not seen any approximation theorem in subcartesian space in the existing literature.

The paper is organized as follows. In Section 2, we recall some basic definitions in the subcartesian space. In Section 3, we recall some basic facts about the orbit space. In Section 4, we prove our main results.

2. Subcartesian Space

Definition 2 ([2])**.** *Let S_1 and S_2 be two differential spaces. A map $\phi : S_1 \to S_2$ is C^∞ if $\phi^* f = f \circ \phi \in C^\infty(S_1)$ for every $f \in C^\infty(S_2)$. A C^∞ map ϕ between differential spaces is a diffeomorphism if it is invertible and its inverse is C^∞.*

An alternative means of constructing a differential structure on a set S is as follows. Let $\mathcal{F}$ be a family of real-valued functions on S. Endow S with the topology generated by a subbasis

$$\{f^{-1}(I) | f \in \mathcal{F} \text{ and } I \text{ is an open interval in } \mathbb{R}\}.$$

Define $C^\infty(S)$ by the requirement that $h \in C^\infty(S)$ if, for each $x \in S$, there exists an open subset U of S, functions $f_1, \cdots, f_n \in \mathcal{F}$, and $F \in C^\infty(\mathbb{R}^n)$ such that

$$h|_U = F(f_1, \cdots, f_n)|_U.$$

Clearly, $\mathcal{F} \in C^\infty(S)$. It is proven in [2] that $C^\infty(S)$ defined here is a differential structure on S. We refer to it as the differential structure on S generated by $\mathcal{F}$.

Let S be a differential space with a differential structure $C^\infty(S)$, and let T be an arbitrary subset of S endowed with the subspace topology (open sets in T are of the form $T \cap U$, where U is an open subset of S). Let

$$S(T) = \{f|_T | f \in C^\infty(S)\}.$$

Definition 3 ([2])**.** *The space $S(T)$ of restrictions to $T \subseteq S$ of smooth functions on S generates a differential structure $C^\infty(T)$ on T such that the differential-space topology of S coincides with its subspace topology. In this differential structure, the inclusion map $i : T \to S$ is smooth.*

In other words, $S(T)$ is the space of restrictions to T of smooth functions on S.

Now, consider an equivalence relation $\sim$ on a differential space S with differential structure $C^\infty(S)$. Let $R = S/\sim$ be the set of equivalence classes of $\sim$, and let $\rho : S \to R$ be the map assigning to each $x \in S$ its equivalence class $\rho(s)$.

Definition 4 ([2])**.** *The space of functions on R, given by*

$$C^\infty(R) = \{f : R \to \mathbb{R} | \rho^* f \in C^\infty(S)\},$$

is a differential structure on R. In this differential structure, the projection map $\rho : S \to R$ is smooth.

It should be emphasized that, in general, the quotient topology of $R = S/\sim$ is finer than the differential-space topology defined by $C^\infty(R)$.

A condition for the differential-space topology to coincide with the quotient topology is given below.

Proposition 1 ([2])**.** *The topology of R induced by $C^\infty(R)$ coincides with the quotient topology of R if, for each set U in R that is open in the quotient topology, and each $y \in U$, there exists a function $f \in C^\infty(R)$ such that $f(y) = 1$ and $f|_{R \setminus U} = 0$, where $R \setminus U$ denotes the complement of U in R.*

3. Orbit Space

Consider the smooth and proper action

$$\begin{aligned} \Phi : \quad & G \times M \to M \\ & (g, x) \to \Phi(g, x) = \Phi_g(x) = gx \end{aligned} \tag{1}$$

of a locally compact connected Lie group G on a manifold M. Recall that the action is proper if, for every convergent sequence (x_n) in M and a sequence (g_n) in G such that the sequence $(g_n x_n)$ is convergent, the sequence (g_n) has a convergent subsequence (g_{n_k}) and

$$\lim_{k\to\infty}(g_{n_k}x_{n_k}) = (\lim_{k\to\infty} g_{n_k})(\lim_{k\to\infty} x_{n_k}).$$

The isotropy group G_x of a point $x \in M$ is

$$G_x = \{g \in G | gx = x\}.$$

G_x is compact [2]. The orbit Gx of G through x is defined by $Gx = \{gx|g \in G\}$. The function f on M is said to be G-invariant, if $f(x) = f(gx)$, for any $x \in M$ and $g \in G$. The subset $A \subseteq M$ is said to be G-invariant if $gx \in A$, for any $x \in A$, $g \in G$.

We endow the orbit space $R = M/G$ with the quotient topology. In other words, a subset V of R is open if $U = \rho^{-1}(V)$ is open in M, where $\rho : M \to R$ is the canonical projection (the orbit map). Let

$$C^\infty(R) = \{f : R \to \mathbb{R} | \rho^* f \in C^\infty(M)\}.$$

$C^\infty(R)$ is a differential structure on R.

Proposition 2 ([2])**.** *The topology of R induced by $C^\infty(R)$ coincides with the quotient topology.*

In the following, we introduce the definition of a slice, which plays a central role in the geometric structure of the symmetry of smooth manifolds.

Definition 5 ([2])**.** *A slice through $x \in M$ for an action of G on M is a submanifold S_x of M containing x such that*

1. *S_x is transverse and complementary to the orbit Gx of G through x. In other words,*

$$T_x M = T_x S_x \oplus T_x(Gx).$$

Specifically,

$$T_x M = T_x S_x + T_x Gx, \quad T_x S_x \cap T_x Gx = \{0\}.$$

2. *For every $x' \in S_x$, the manifold S_x is transverse to the orbit Gx'; in other words,*

$$T_{x'} M = T_{x'} S_x + T_{x'}(Gx').$$

3. *S_x is G_x-invariant. Specifically, $gy \in S_x$ for any $g \in G_x$ and $y \in S_x$.*
4. *Let $x' \in S_x$. If $gx' \in S_x$, then $g \in G_x$.*

Given a G-invariant Riemannian metric k on M, we denote by verTM the generalized distribution on M consisting of vectors tangent to G-orbits in M, and by horTM the k-orthogonal complement of verTM. The existence of a slice through $x \in M$ is ensured by the following result.

Proposition 3 ([2])**.** *There is an open ball B in horT_xM centered at 0 such that $S_x = \exp_x B$ is a slice through x for the action of G on M, where $\exp_x v$ is the value at 1 of the geodesics of the G-invariant Riemannian metric originating from x in the direction v. Further, the set $GS_x = \{gq | g \in G \text{ and } q \in S_x\}$ is a G-invariant open neighborhood of x in M.*

Let $H = G_x$. By construction, $S_x = \exp_x B$, where $\exp_x$ is an H-equivariant map from a neighborhood of 0 in T_xM to a neighborhood of x in M, and B is a ball in horT_xM invariant under a linear action of H centered at the origin. The action of H on T_xM is linear, and it leaves $\mathrm{hor}T_xM$ invariant. Hence, it gives rise to a linear action of H on $\mathrm{hor}T_xM$. Moreover, the restriction of $\exp_x$ to B gives a diffeomorphism $\psi : B \to S_x$, which intertwines the linear action of H on $\mathrm{hor}T_xM$ and the action of H on S_x.

Since B is an H-invariant open subset of $\mathrm{hor}T_xM$ and the action of H on $\mathrm{hor}T_xS_x$ is linear, via the theorem of G. W. Schwarz [7], smooth H-invariant functions on S_x are smooth functions of algebraic invariants of the action of H on $\mathrm{hor}T_xM$. Let $\mathbb{R}[\mathrm{hor}T_xM]^H$ denote the algebra of H-invariant polynomials on $\mathrm{hor}T_xM$. Hilbert's Theorem [8] ensures that $\mathbb{R}[\mathrm{hor}T_xM]^H$ is finitely generated. Let $\sigma_1, \cdots, \sigma_n$ be a Hilbert basis for $\mathbb{R}[\mathrm{hor}T_xM]^H$ consisting of homogeneous polynomials. The corresponding Hilbert map

$$\sigma : \mathrm{hor}T_xM \to \mathbb{R}^n : v \to \sigma(v) = (\sigma_1(v), \cdots, \sigma_n(v)) \tag{2}$$

induces a monomorphism $\tilde{\sigma} : (\mathrm{hor}T_xM)/H \to \mathbb{R}^n : Hv \to \sigma(v)$, where Hv is the orbit of H through $v \in \mathrm{hor}T_xM$ treated as a point in $(\mathrm{hor}T_xM)H$. Let Q be the range of σ. By the Tarski–Seidenberg Theorem [9], Q is a semi-algebraic set in $\mathbb{R}^n$. Let

$$\phi : (\mathrm{hor}T_xM)/H \to Q \subseteq \mathbb{R}^n \tag{3}$$

be the bijection induced by $\tilde{\sigma}$. ϕ is a diffeomorphism [2].

Since B is an H-invariant open neighborhood of 0 in $\mathrm{hor}T_xS_x$, it follows that B/H is open in $(\mathrm{hor}T_xM)/H$. Hence, B/H is in the domain of the diffeomorphism $\phi : (\mathrm{hor}T_xM)/H \to Q$, which induces a diffeomorphism of B/H onto $\phi(B/H) \subseteq Q \subseteq \mathbb{R}^n$. Thus, B/H is diffeomorphic to a subset of $\mathbb{R}^n$. However, B/H is diffeomorphic to S_x/H, and S_x/H is diffeomorphic to GS_x/G. Therefore, GS_x/G is diffeomorphic to a subset of $\mathbb{R}^n$. Hence, we have the following.

Theorem 3 ([2])**.** *The orbit space $R = M/G$ of a proper action of G on M with the differential structure $C^\infty(R)$ is subcartesian.*

4. Approximating Continuous Function on Orbit Space by Smooth Functions

In this section, we prove Theorem 2. We first study the local approximation problem.

Lemma 1 ([5])**.** *Let M be a smooth manifold and f be a continuous function on M. Given $\epsilon > 0$, then there exists $h \in C^\infty(M)$ such that $|h(p) - f(p)| < \epsilon$, for $p \in M$.*

Lemma 2 ([10])**.** *Let U, V be two open subsets of the smooth manifold M satisfying that $cl(U)$ is compact and $cl(U) \cap cl(V) = \emptyset$, where $cl(U)$ and $cl(V)$ denote the closure of U and V. Then, there exists a smooth function $f \in C^\infty(M)$ such that*

$$\begin{aligned} &f(x) = 1, x \in cl(U), \\ &0 < f(x) < 1, x \in M \backslash (cl(V) \cup cl(U)), \\ &f(x) = 0, x \in cl(V). \end{aligned} \tag{4}$$

Now, consider the subcartesian space $(R = M/G, C^\infty(R))$. The following result provides a positive solution to the local approximation of a continuous function on R by a smooth function.

Lemma 3. *For each $y_0 \in R$, there exists a local neighborhood V of y_0 satisfying that, for any continuous function f on V and any $\delta > 0$, there exists a smooth function $f_1 \in C^\infty(V)$, where $(V, C^\infty(V))$ is a differential subspace of R, such that*

$$|f_1(y) - f(y)| < \delta,$$

for any $y \in V$.

Proof. Let $x_0 \in M$ such that $\rho(x_0) = y_0$. Let H be the isotropy group of x_0 and $S_{x_0} = \exp_{x_0} B$ be a slice through $x_0 \in M$, where $\exp_{x_0}$ is an H-equivariant map from a neighborhood of 0 in $T_{x_0}M$ to a neighborhood of x_0 in M, and B is a ball in $\text{hor}T_{x_0}M$ invariant under the linear action of H centered at the origin.

Then, for any continuous function f on $\rho(GS_{x_0})$, it follows that $\exp^*_{x_0}(\rho^* f|_{S_{x_0}})$ is a continuous function on B, where $\rho : M \to \mathbb{R}$ is the orbit map. From Lemma 1, we know that for any $\delta > 0$, there exists a smooth function $h \in C^\infty(B)$, such that

$$|h(v) - \exp^*_{x_0}(\rho^* f|_{S_{x_0}})(v)| < \delta,$$

for any $v \in B$.

Now, consider the smooth function $h \circ \exp^{-1}_{x_0}$ on S_{x_0}, which satisfies that $|h \circ \exp^{-1}_{x_0}(v) - \rho^* f|_{S_{x_0}}(v)| < \delta$, for any $x \in \exp_{x_0} B$. Since H is compact, we may average $h \circ \exp^{-1}_{x_0}$ over H, obtaining a H-invariant function

$$\tilde{h} = \int_H \Phi^*_g(h \circ \exp^{-1}_{x_0}) d\mu(g),$$

where $d\mu(g)$ is the Haar measure on H normalized so that $\text{vol}H = 1$.

The set GS_{x_0} is a G-invariant open neighborhood of x_0 in M. We can define a G-invariant smooth function $\tilde{f}_1$ on GS_{x_0} as follows. For each $x'' \in GS_{x_0}$, there exists $g \in G$ such that $x'' = gx'$ for $x' \in S_x$, and we set

$$\tilde{f}_1(x'') = \tilde{h}(x').$$

$\tilde{f}_1$ is well defined. Let $x'' = g_1 x_1$, where $g_1 \in G$ and $x_1 \in S_x$. From the above definition, we have $\tilde{f}_1(x'') = \tilde{h}(x_1)$. On the other hand, since $g_1 x_1 = gx'$, we have $g_1^{-1} g x' = x_1$. Since $x', x_1 \in S_x$, it follows from Definition 5 and Proposition 3 that $g_1^{-1} g \in H$. Hence, $\tilde{h}(x_1) = \tilde{h}(g_1^{-1} g x') = \tilde{h}(x')$ since $\tilde{h}$ is H-invariant; this yields that $\tilde{f}_1$ is well defined. From the definition of $\tilde{f}_1$, we know that $\tilde{f}_1$ is G-invariant, which descends to a function f_1 on $\rho(S_{x_0})$ such that $\rho^* f_1 = \tilde{f}_1$. Moreover, for each $y \in \rho(\exp_{x_0} B)$, we have

$$\begin{aligned}
|f_1(y) - f(y)| &= |\tilde{h}(x) - \rho^* f(x)| \\
&= |\int_H (\Phi^*_g(h \circ \exp^{-1}_{x_0}))(x) d\mu(g) - \int_H (\Phi^*_g(\rho^* f))(x) d\mu(g)| \\
&= |\int_H (\Phi^*_g(h \circ \exp^{-1}_{x_0}) - \Phi^*_g(\rho^* f))(x) d\mu(g)| \\
&\leq \int_H |(\Phi^*_g(h \circ \exp^{-1}_{x_0}) - \Phi^*_g(\rho^* f))(x)| d\mu(g) \\
&< \int_H \delta d\mu(g) \\
&= \delta,
\end{aligned} \tag{5}$$

where $x \in \exp_{x_0} B$ satisfies that $\rho(x) = y$. We claim that $f_1 \in C^\infty(\rho(S_{x_0}))$, where $(\rho(S_{x_0}), C^\infty(\rho(S_{x_0})))$ is the differential subspace of $(R, C^\infty(R))$. For any $x \in GS_{x_0}$, let $0 \in W \subseteq V \subseteq \text{hor}T_x M$ be G_x-invariant open subsets of $\text{hor}T_x M$ such that $\exp_x(V) \subseteq GS_{x_0}$. It follows from Lemma 2 that there exists a smooth function $\eta : \text{hor}T_x M \to \mathbb{R}$ such that

$$\begin{aligned}
&\eta(x) = 1, x \in cl(W), \\
&0 < \eta(x) < 1, x \in V \backslash cl(W), \\
&\eta(x) = 0, x \in \text{hor}T_x M \backslash V,
\end{aligned} \tag{6}$$

which yields a smooth function $\eta \circ \exp_x^{-1}$ on S_x. Since V, W are H-invariant, then by averaging $\eta \circ \exp_x^{-1}$ over S_x, we obtain a G_x-invariant smooth function $\tilde{\eta}$ on S_x satisfying that

$$\begin{aligned} &\tilde{\eta}(x) = 1, x \in \exp_x(cl(W)), \\ &0 < \tilde{\eta}(x) < 1, x \in \exp_x(V \backslash cl(W)), \\ &\tilde{\eta}(x) = 0, x \in S_x \backslash \exp_x(V), \end{aligned} \tag{7}$$

which can be extended to a smooth G-invariant function $\tilde{\eta}_1$ on M. Now, consider the function $\tilde{\eta}_1 \tilde{f}_1$ on M. Since $\exp_x V \subseteq GS_{x_0}$, it follows that $\tilde{\eta}_1 \tilde{f}_1$ is a smooth G-invariant function on M satisfying that $\tilde{\eta}_1 \tilde{f}_1|_{G \exp_x(cl(W))} = \tilde{f}_1|_{G \exp_x(cl(W))}$. Since $\tilde{\eta}_1 \tilde{f}_1$ descends to $\eta_1 f_1 \in C^\infty(R)$, it follows that $\eta_1 f_1|_{\rho(\exp_x(cl(W)))} = f_1|_{\rho(\exp_x(cl(W)))}$, where $\rho(\exp_x W)$ is an open neighborhood of $\rho(x)$ in R. Since x is arbitrary in S_{x_0}, it follows that $f_1 \in C^\infty(\rho(GS_x))$, where $(\rho(GS_x), C^\infty(\rho(GS_x)))$ is a differential subspace of R. This completes the proof of the claim.

Hence, for $y_0 \in R$ and for $x \in \rho^{-1}(y_0)$, there exists a local neighborhood $\rho(GS_x)$ of y_0 satisfying that, for any continuous function f on $\rho(GS_x)$ and any $\delta > 0$, there exists a smooth function $f_1 \in C^\infty(\rho(GS_x))$, where $(\rho(GS_x), C^\infty(\rho(GS_x)))$ is a differential subspace of R, such that

$$|f_1(y) - f(y)| < \delta,$$

for any $y \in \rho(GS_x)$. Hence, the result follows. □

In the following, we investigate passages from local to global for the approximation problem on R.

Lemma 4. *Let $x \in M$ and let $0 \in W \subseteq V \subseteq U \subseteq B$ be H-invariant open subsets of $horT_xM$ such that $cl(W) \subseteq V$ and $cl(V)$ are compact, where B satisfies that $\exp_x B = S_x$, and $cl(W)$ and $cl(V)$ denote the closure of W and V. Let T be an open subset of R and $(\rho(\exp_x U), \psi)$ be the local coordinate for R induced by the Hilbert map (2). Let $f : R \to \mathbb{R}$ be a continuous map satisfying that $f|_T \in C^\infty(T)$, where $(T, C^\infty(T))$ is a differential subspace of $(R, C^\infty(R))$. Then, for any $\delta > 0$, there exists a continuous map $h : R \to \mathbb{R}$, such that*

(1) $h(y) = f(y)$, for any $y \in R \backslash \rho(\exp_x(V))$;
(2) $h|_{T \cup \rho(\exp_x W)} \in C^\infty(T \cup \rho(\exp_x W))$;
(3) $|h(y) - f(y)| < \delta$, for all $y \in R$.

Proof. It follows from Lemma 2 that there exists smooth function $\eta : \mathrm{hor}T_xM \to \mathbb{R}$ such that

$$\begin{aligned} &\eta(x) = 1, x \in cl(W), \\ &0 < \eta(x) < 1, x \in V \backslash cl(W), \\ &\eta(x) = 0, x \in \mathrm{hor}T_xM \backslash V, \end{aligned} \tag{8}$$

which yields a smooth function $\eta \circ \exp_x^{-1}$ on S_x. Since V, W are H-invariant, then by averaging $\eta \circ \exp_x^{-1}$ over S_x, we obtain a G_x-invariant smooth function $\tilde{\eta}$ on S_x satisfying that

$$\begin{aligned} &\tilde{\eta}(x) = 1, x \in \exp_x(cl(W)), \\ &0 < \tilde{\eta}(x) < 1, x \in \exp_x(V \backslash cl(W)), \\ &\tilde{\eta}(x) = 0, x \in S_x \backslash \exp_x(V), \end{aligned} \tag{9}$$

which can be extended to a smooth G-invariant function on M. Hence, we obtain a function $\bar{\eta} \in C^\infty(R)$ satisfying that

$$\begin{aligned} &\bar{\eta}(y) = 1, y \in \rho(\exp_x(cl(W))), \\ &0 < \bar{\eta}(y) < 1, y \in \rho(\exp_x(V \backslash cl(W))), \\ &\bar{\eta}(y) = 0, y \in R \backslash \rho(\exp_x(V)). \end{aligned} \tag{10}$$

It follows from Lemma 3 that the function $f|_{\rho(GS_x)}$ can be approximated by smooth functions on $\rho(GS_x)$. In other words, for any $\delta > 0$, there exists a smooth function $h_0 \in C^\infty(\rho(\exp_x(U)))$ such that $|h_0(y) - f(y)| < \delta$, for $y \in \rho(GS_x)$.

Since $f = (1 - \bar{\eta})f + \bar{\eta}f$, we define

$$h = (1 - \bar{\eta})f + \bar{\eta}h_0.$$

Since $\bar{\eta}(y) = 0, y \in R\backslash\rho(\exp_x(V))$, it follows that $h(y) = f(y)$, for any $y \in R\backslash\rho(\exp_x(V))$; Since $\tilde{\eta}h_0 \in C^\infty(R)$ and $f|_T \in C^\infty(T)$, it follows that $h|_T \in C^\infty(T)$. Since $\bar{\eta}(y) = 1$, $y \in \rho(\exp_x(cl(W)))$, it follows that $h|_{\rho(\exp_x W)} = \tilde{\eta}h_0|_{\rho(\exp_x W)} \in C^\infty(\rho(\exp_x W))$. Hence, $h|_{T\cup\rho(\exp_x W)} \in C^\infty(T \cup \rho(\exp_x W))$, since both T and $\rho(\exp_x W)$ are open in R. Since $h(y) - f(y) = \tilde{\eta}h_0(y) - \tilde{\eta}h(y)$, it follows immediately that $|h(y) - f(y)| < \delta$, for all $y \in R$. Then, the result follows. □

Lemma 5 ([5]). *Let X be a second, countable, locally compact Hausdorff topological space. Then, there exist countable many sets $G_1, G_2, \cdots, G_k, \cdots$ satisfying that*

(1) $cl(G_j)$ is compact, $j = 1, 2, \cdots$;
(2) $cl(G_j) \subseteq G_{j+1}, j = 1, 2, \cdots$;
(3) $\cup G_j = \cup cl(G_j) = X$,

where $cl(G_j)$ denotes the closure of $G_j, j = 1, 2, \cdots$.

Lemma 6. *There exist locally finite open covers $(U_j)_{j\in\mathbb{Z}_{>0}}$, $(V_j)_{j\in\mathbb{Z}_{>0}}$, $(W_j)_{j\in\mathbb{Z}_{>0}}$ of R such that $cl(U_j) \subseteq V_j, cl(V_j) \subseteq W_j$, and $cl(U_j), cl(V_j), cl(W_j)$ are compact, for each $j > 0$, where $(W_j, \mathbb{R}^{n_j}, \phi_j)$ is a local chart of R induced by the Hilbert map (2).*

Proof. From Lemma 5, we know that there exist countable open sets $G_1, \cdots, G_k, \cdots$ on R satisfying conditions (1), (2) and (3) in Lemma 5. It follows that $cl(G_h)\backslash G_{h-1}$ is compact, $G_{h+1}\backslash cl(G_{h-2})$ is open and $cl(G_h)\backslash G_{h-1} \subseteq G_{h+1}\backslash cl(G_{h-2})$. On the other hand, we know that the local charts induced by the Hilbert map σ (see (2)) of R form an open cover of R. Then, for $y \in \mathrm{cl}(G_h)\backslash G_{h-1}$, there exists a local chart (V, ϕ) of y induced by the Hilbert map σ. Consider the H invariant open set $\sigma^{-1}(\phi(G_{h+1}\backslash cl(G_{h-2})) \cap V))$ in $\mathrm{hor}T_xM$, where $\rho(x) = y$. There exists an open ball B_ϵ such that $cl(B_\epsilon) \subseteq \sigma^{-1}(\phi(G_{h+1}\backslash cl(G_{h-2})) \cap V))$ centered at 0. Let $W = \phi^{-1} \circ \sigma(B_\epsilon)$. Hence, W is an open subset containing y such that $cl(W) \subseteq (G_{h+1}\backslash cl(G_{h-2})) \cap V$.

It follows that

(1) $y \in W \subseteq (G_{h+1}\backslash cl(G_{h-2})) \cap V$; (2) $\phi(y) = 0$ and $\phi(W) \subsetneqq \phi(V)$.

Since $cl(W) = \sigma(cl(B_\epsilon))$ and $cl(B_\epsilon)$ is compact, it follows that $cl(W)$ is compact.

Let $W_1 = \sigma(B_{\epsilon_1})$, where $0 < \epsilon_1 < \epsilon$. Then, W_1 is an open set containing y such that $cl(W_1) \subseteq W$. Denote by $V = W_1$. Moreover, let $W_2 = \phi^{-1} \circ \sigma(B_{\epsilon_2})$, where $0 < \epsilon_2 < \epsilon_1$. Then, W_2 is an open set containing y such that $cl(W_2) \subseteq V$. Denote by $U = W_2$. Then, we have $cl(U) \subseteq V$ and $cl(V) \subseteq W$.

Since $cl(G_h)\backslash G_{h-1}$ is compact, there exist finitely many points $y_{h,1}, y_{h,2}, \cdots, y_{h,k_h} \in cl(G_h)\backslash G_{h-1}$, such that the corresponding open sets $U_{h,1}, U_{h,2}, \cdots, U_{h,k_h}$ form an open cover of $cl(G_h)\backslash G_{h-1}$. We claim that the corresponding open sets

$$\{U_{1,1}, U_{1,2}, \cdots, U_{1,k_1}; U_{2,1}, U_{2,2}, \cdots, U_{2,k_2}; \cdots\},$$

and

$$\{V_{1,1}, V_{1,2}, \cdots, V_{1,k_1}; V_{2,1}, V_{2,2}, \cdots, V_{2,k_2}; \cdots\},$$

and

$$\{W_{1,1}, W_{1,2}, \cdots, W_{1,k_1}; W_{2,1}, W_{2,2}, \cdots, W_{2,k_2}; \cdots\},$$

satisfy the conditions in the lemma. We only need to prove the local finiteness of $\{W_{i,j}\}$. Given $y \in R$, assume that $y \in G_r$, for some $r \in \{1, 2, \cdots\}$; then, it follows from the above construction that there exist many finite $W_{i,j}$ that intersect G_r. In fact,

$$W_{i,j} \cap G_r = \emptyset, i \geq r+2, 1 \leq j \leq k_i.$$

This completes the proof of the lemma. □

Now, we begin to prove Theorem 2.

Proof. From Lemma 6, we know that there exist locally finite open covers $(U_j)_{j\in\mathbb{Z}_{>0}}$, $(V_j)_{j\in\mathbb{Z}_{>0}}$, $(W_j)_{j\in\mathbb{Z}_{>0}}$ of R such that $cl(U_j) \subseteq V_j$, $cl(V_j) \subseteq W_j$, and $cl(U_j)$, $cl(V_j)$, $cl(W_j)$ are compact, for each $j > 0$, where $(W_j, \mathbb{R}^{n_j}, \phi_j)$ is a local chart of R induced by the Hilbert map (2).

Set $W_0 = \emptyset, f_0 = f$. Assume that we have continuous function f_k on R such that $f_k|_{G_k}$ is smooth, where

$$G_k = \cup_{j=0}^{k} W_j.$$

Then, it follows from Lemma 4 that there exists continuous function f_{k+1} on R, such that $f_{k+1}|_{G_{k+1}}$ is smooth, where

$$G_{k+1} = \cup_{j=0}^{k+1} W_j,$$

is a subset of R.

Moreover, $f_{k+1}|_{M\setminus V_{k+1}} = f_k$, and

$$|f_{k+1}(y) - f_k(y)| < \frac{\delta}{2^{k+1}}, \tag{11}$$

for all $y \in R$.

Hence, let

$$h(y) = \lim_{k\to\infty} f_k(y).$$

It follows from (11) that, for fixed $y \in R$, $\{f_k(y)\}$ is a Cauchy sequence in $\mathbb{R}$. Hence, h is well defined. Moreover,

$$|h(y) - f(y)| < \delta,$$

for any $y \in R$.

We claim that $h \in C^\infty(R)$. For $y \in R$, there exists $l \in \mathbb{Z}_{>0}$, such that $y \in W_l$. Now, consider the functions $f_l, f_{l+1}, \cdots$, which are smooth on W_l. It follows that $\rho^* f_l, \rho^* f_{l+1}, \cdots$ are smooth functions on the open subsets $\rho^{-1}(W_l)$ of M, which satisfies

$$|\rho^* f_{k+1}(x) - \rho^* f_k(x)| < \frac{\delta}{2^{k+1}}, \tag{12}$$

for $k \geq l$ and $x \in \rho^{-1}(W_l)$. It follows that $\lim_{k\to\infty} \rho^* f_k \in C^\infty(\rho^{-1}(W_l))$. Moreover, since $\rho^* f_k(gx) = \rho^* f_k(x)$, for any $x \in \rho^{-1}(W_l)$, $g \in G$, it follows that $\lim_{k\to\infty} \rho^* f_k(gx) = \lim_{k\to\infty} \rho^* f_k(x)$. Hence, we find that $\lim_{k\to\infty} \rho^* f_k$ is G-invariant and hence descends to g on W. Hence, $h|_{W_l} \in C^\infty(W_l)$. Since y is arbitrary and W_l is open, it follows from condition 3 in the definition of differential space that $h \in C^\infty(R)$. This completes the proof. □

Theorem 4. *Let $f : R \to \mathbb{R}^n$ be a continuous function on R. Then, for any $\epsilon > 0$, there exists $h \in C^\infty(R; \mathbb{R}^n)$, such that*

$$|h(y) - f(y)| < \epsilon,$$

for any $y \in R$. Moreover, h is homotopic to f.

Proof. Let $\delta = \epsilon/\sqrt{n}$. It follows from Theorem 2 that there exist smooth functions $h_1, \cdots, h_n \in C^\infty(R)$ such that

$$|h_i(y) - f_i(y)| < \delta,$$

for any $y \in R$, where $f = (f_1, \cdots, f_n)$. Consider the smooth map $h = (h_1, \cdots, h_n)$. We have that

$$|h(y) - f(y)| < \epsilon,$$

for any $y \in R$. Moreover, define

$$F(t, y) = (1-t)h(y) + tf(y),$$

for $(t, y) \in I \times R$. It is obvious that F defines a homotopy from h to f. Hence, the result follows immediately. □

5. Conclusions

In this paper, we have considered the problem of approximating continuous functions by smooth functions on a subclass of singular spaces—subcartesian spaces. We have investigated a special class of subcartesian spaces—the orbit space of the proper action of a Lie group on a smooth manifold. By taking advantage of the geometric structure of the symmetry of the smooth manifold, we have shown that continuous functions on the orbit space can be approximated by smooth functions. In the future, we would like to investigate more subclasses of subcartesian spaces on which the approximation theorem holds.

Author Contributions: Conceptualization, Q.X.; methodology, Q.X.; software, L.Z. and Q.X.; validation, Q.X.; formal analysis, Q.X.; investigation, Q.X.; resources, L.Z. and Q.X.; data curation, L.Z. and Q.X.; writing—original draft preparation, Q.X.; writing—review and editing, Q.X.; visualization, L.Z. and Q.X.; supervision, Q.X.; project administration, L.Z. and Q.X. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by NSFC (grant number: 61703211).

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Acknowledgments: The authors are grateful to the reviewers, who gave valuable remarks, comments, and advice that improved the quality of the paper.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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