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On Global Solutions of Hyperbolic Equations with Positive Coefficients at Nonlocal Potentials

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Abstract: We study hyperbolic equations with positive coefficients at potentials undergoing translations with respect to the spatial independent variable. The qualitative novelty of the investigation is that the real part of the symbol of the differential-difference operator contained in the equation is allowed to change its sign. Earlier, only the case where the said sign is constant was investigated. We find a condition relating the coefficient at the nonlocal term of the investigated equation and the length of the translation, guaranteeing the global solvability of the investigated equation. Under this condition, we explicitly construct a three-parametric family of smooth global solutions of the investigated equation.

Keywords: differential-difference operators; hyperbolic equations; nonlocal potentials; smooth solutions

MSC: 35L10; 35R10

1. Introduction

The fact that reasonable models of mathematical physics cannot be exhausted by *differential* equations and, therefore, equations containing other operators apart from differential ones (they are called *functional-differential* equations) are to be studied as well, which is known to researchers over many decades. Unlike differential operators, those operators might be bounded, but the main novelty of this enhancement approach is that such functional-differential equations link values of the desired function at different points. This *nonlocal* nature of functional-differential equations qualitatively causes new properties of their solutions, provides a possibility to use them in applications not covered by the classical theory of differential equations, and demonstrates close relations to the theory of nonlocal problems, which is quite important for various applications as well (see, e.g., [1,2]).

Ordinary functional-differential equations are studied at least from the middle of the previous century (see [3] and references therein). The theory of partial functional-differential equations is relatively younger: one can refer to [4] as a pioneering work. It should be noted that even the term "elliptic" is to be carefully clarified in the functional-difference case because the traditional classification of partial *differential* equations does not work in the *functional-differential* case. Thus, the notion of elliptic functional-differential equations is not trivial at all. For the current state of the general theory of elliptic functional-differential equations, we refer to [5,6].

Though the present paper is devoted to hyperbolic functional-differential equations, the general elliptic theory is important within its framework due to the following circumstance: once we are able to define elliptic functional-differential operators, we can say that each equation of the kind $u_t - Lu = f$ is parabolic and each equation of the kind $u_{tt} - Lu = f$ is hyperbolic, where L is an arbitrary elliptic functional-differential operator. In this paper, we concentrate our attention on the important class of functional-differential equations, called *differential-difference* ones. They are equations containing translation operators (apart from differential ones). It is important that translation operators are

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Fourier multipliers and, therefore, there is a natural way to introduce the ellipticity of such operators, basing on the signs of the real parts of their symbols.

In this paper, we study hyperbolic differential-difference equations containing sums of differential operators and (spatial) translation operators (in other words, hyperbolic equations with nonlocal potentials). Such equations are investigated since [7], where the prototype equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = au(x+h,t). \tag{1}$$

is studied under the assumption that the coefficient a is negative (this restriction for the sign of the coefficient means the positivity of the real part of the symbol of the differential-difference operator with respect to the spatial variable). In [8–10], the investigation is successively extended to more general hyperbolic equations with nonlocal potentials, but the requirement for the sign of the said real part is still preserved.

Here, we take this restriction off, allowing the coefficient a to be positive. Now, the real part of the symbol of the differential-difference operator might change its sign, but we are still able to construct smooth global solutions of the investigated equation.

2. Results and Proofs

Let $a > 0$ and $h \in \mathbb{R}^1$. In $\mathbb{R}^2$, consider Equation (1). Assuming that $ah^2 < 2$, consider the function $f(\xi) := \xi^2 - a\cos h\xi$. Its derivative is equal to

$$2\xi + ah\sin h\xi = 2\xi\left(1 + \frac{ah^2}{2}\frac{\sin h\xi}{h\xi}\right),$$

i.e., the continuous function f monotonously increases on $(0, +\infty)$. Further, $f(0) = -a < 0$ and $f(\xi) > 0$ for each ξ exceeding $\sqrt{a}$. Thus, the function f has one and only one positive zero, and it belongs to $(0, \sqrt{a}\,]$. Denote this zero by ξ_0.

If $\sin h\xi_0 \neq 0$, then the function

$$\varphi(\xi) = \frac{1}{2}\arctan\frac{a\sin h\xi}{|\xi^2 - a\cos h\xi|} \tag{2}$$

can be defined (as a continuous function) at the point ξ_0: its left-side and right-side limits at that point are equal to $\frac{\pi}{4}\operatorname{sgn}(\sin h\xi_0)$.

If $\sin h\xi_0 = 0$, then ξ_0 satisfies the following system of equations:

$$\begin{cases} \sin h\xi_0 = 0 \\ a\cos h\xi_0 = \xi_0^2, \end{cases}$$

which means that $\sqrt{1 - \frac{\xi_0^4}{a^2}} = 0$, i.e., the only positive root of the above system is $\sqrt{a}$. To compute the limit of the function φ at the point $\sqrt{a}$, compute the value of the fraction

$$\frac{(\sin h\xi)'}{(\xi^2 - a\cos h\xi)'} = \frac{h\cos h\xi}{2\xi + ah\sin h\xi}$$

at the said point. It is equal to $\frac{h}{2\sqrt{a}}$ and, therefore, the function φ can be redefined (as a continuous function) at the point $\sqrt{a}$ by the value $\frac{1}{2}\arctan\frac{h\sqrt{a}}{2}$. Taking into account the oddness of function (2), we conclude that it is continuous in $(-\infty, \infty)$.

Note that the function

$$\rho(\xi) = \left(\xi^4 - 2a\xi^2\cos h\xi + a^2\right)^{\frac{1}{4}} \tag{3}$$

is well defined in $(-\infty, \infty)$ because the radicand is not exceeded by

$$\xi^4 - 2a\xi^2 + a^2 = (\xi^2 - a)^2 > 0.$$

The following assertions are valid.

Theorem 1. *If the inequality*

$$ah^2 < 2 \tag{4}$$

holds, then each function

$$\alpha F(x,t;\xi) + \beta H(x,t;\xi), \tag{5}$$

where

$$F(x,t;\xi) = e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right], \tag{6}$$

$$H(x,t;\xi) = e^{-tG_1(\xi)} \sin\left[tG_2(\xi) - \varphi(\xi) - x\xi\right], \tag{7}$$

and

$$G_{\left\{ {1 \atop 2} \right\}}(\xi) = \rho(\xi) \left\{ {\sin \atop \cos} \right\} \varphi(\xi), \tag{8}$$

is an infinitely smooth function satisfying Equation (1) for each real α, each real β, and each ξ from $(-\infty, -\xi_0) \cup (\xi_0, \infty)$.

Proof. Let $\xi \in (-\infty, -\xi_0) \cup (\xi_0, \infty)$ and, therefore, $\varphi(\xi) = \dfrac{1}{2} \arctan \dfrac{a \sin h\xi}{\xi^2 - a\cos h\xi}$. Substitute function (6) in Equation (1):

$$\frac{\partial F}{\partial t} = G_1(\xi)e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] + G_2(\xi)e^{tG_1(\xi)} \cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right],$$

$$\begin{aligned}\frac{\partial^2 F}{\partial t^2} &= G_1^2(\xi)e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &+ G_1(\xi)G_2(\xi)e^{tG_1(\xi)} \cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &+ G_1(\xi)G_2(\xi)e^{tG_1(\xi)} \cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &- G_2^2(\xi)e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &= \left[G_1^2(\xi) - G_2^2(\xi)\right] e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &+ 2G_1(\xi)G_2(\xi)e^{tG_1(\xi)} \cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right],\end{aligned}$$

and

$$\frac{\partial^2 F}{\partial x^2} = -\xi^2 e^{tG_1(\xi)} \sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right].$$

Further, $G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\sin^2\varphi(\xi) - \rho^2(\xi)\cos^2\varphi(\xi) = -\rho^2(\xi)\cos 2\varphi(\xi)$ and $2G_1(\xi)G_2(\xi) = \rho^2(\xi)\sin 2\varphi(\xi)$.

Now, we note that $-\dfrac{\pi}{4} < \varphi(\xi) < \dfrac{\pi}{4}$ on $\mathbb{R}^1$ by definition. Then, $2\varphi(\xi) \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ on $\mathbb{R}^1$ and, therefore, the function $\cos 2\varphi(\xi)$ is positive everywhere. Then,

$$\cos 2\varphi(\xi) = \frac{1}{\sqrt{1 + \tan^2 2\varphi(\xi)}} = \left[1 + \frac{a^2 \sin^2 h\xi}{(\xi^2 - a\cos h\xi)^2}\right]^{-\frac{1}{2}} = \sqrt{\frac{(\xi^2 - a\cos h\xi)^2}{\xi^4 - 2a\xi^2 \cos h\xi + a^2}}.$$

Note that the denominator of the last fraction can vanish only under the assumption that $\cos h\xi = 1$, but the numerator is equal to the denominator in that case. Thus, the last relation has a sense under the assumptions of the theorem. Further, the denominator of the last fraction is equal to $\rho^4(\xi)$. Therefore, $\cos 2\varphi(\xi) = \frac{\sqrt{(\xi^2 - a\cos h\xi)^2}}{\rho^2(\xi)}$.

If $\xi \in (-\infty, -\xi_0) \cup (\xi_0, \infty)$, then the function $\xi^2 - a\cos h\xi$ is positive. Hence,

$$\cos 2\varphi(\xi) = \frac{\xi^2 - a\cos h\xi}{\rho^2(\xi)}$$

and, therefore,

$$\sin 2\varphi(\xi) = \tan 2\varphi(\xi)\cos 2\varphi(\xi) = \frac{a\sin h\xi}{\xi^2 - a\cos h\xi}\frac{\xi^2 - a\cos h\xi}{\rho^2(\xi)} = \frac{a\sin h\xi}{\rho^2(\xi)}.$$

Thus,

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} &= -\rho^2(\xi)\cos 2\varphi(\xi)e^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &\quad +\rho^2(\xi)\sin 2\varphi(\xi)e^{tG_1(\xi)}\cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &\quad +\xi^2 e^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &= \left(a\cos h\xi - \xi^2\right)e^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &\quad +a\sin h\xi e^{tG_1(\xi)}\cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &\quad +\xi^2 e^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &= ae^{tG_1(\xi)}\Big(\cos h\xi\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi\right] \\ &\quad +\sin h\xi\cos\left[tG_2(\xi) + \varphi(\xi) + x\xi\right]\Big) \\ &= ae^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + x\xi + h\xi\right] \\ &= ae^{tG_1(\xi)}\sin\left[tG_2(\xi) + \varphi(\xi) + (x+h)\xi\right] = aF(x+h, t), \end{aligned} \tag{9}$$

i.e., function (6) satisfies (in the classical sense) Equation (1) in the half-space $\mathbb{R}^{n+1}$ for each ξ from $(-\infty, -\xi_0) \cup (\xi_0, \infty)$.

Now, substitute function (7) in Equation (1):

$$\begin{aligned} \frac{\partial H}{\partial t} &= -G_1(\xi)e^{-tG_1(\xi)}\sin\left[tG_2(\xi) - \varphi(\xi) - x\xi\right] \\ &\quad +G_2(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi) - \varphi(\xi) - x\xi\right], \end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 H}{\partial t^2} &= G_1^2(\xi)e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-G_1(\xi)G_2(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-G_1(\xi)G_2(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-G_2^2(\xi)e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&= \left[G_1^2(\xi)-G_2^2(\xi)\right]e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-2G_1(\xi)G_2(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&= -\rho^2(\xi)\cos 2\varphi(\xi)e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-\rho^2(\xi)\sin 2\varphi(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&= \left(a\cos h\xi-\xi^2\right)e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-a\sin h\xi e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right],
\end{aligned}$$

and

$$\frac{\partial^2 H}{\partial x^2} = -\xi^2 e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right].$$

Hence,

$$\begin{aligned}
\frac{\partial^2 H}{\partial t^2}-\frac{\partial^2 H}{\partial x^2} &= ae^{-tG_1(\xi)}\Big(\cos h\xi\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\
&-\sin h\xi\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right]\Big) \\
&= ae^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi-h\xi\right] \\
&= ae^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-(x+h)\xi\right] = aH(x+h,t),
\end{aligned} \tag{10}$$

i.e., function (7) is a classical solution of Equation (1) for each ξ from $(-\infty,-\xi_0)\cup(\xi_0,\infty)$. □

Theorem 2. *If inequality (4) holds, then each function (5), where $F(x,t;\xi)$ and $H(x,t;\xi)$ are defined by relations (6) and (7), respectively, while*

$$G_{\left\{\substack{1\\2}\right\}}(\xi) = \rho(\xi)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\varphi(\xi), \tag{11}$$

is an infinitely smooth solution of Equation (1) for each real α, each real β, and each ξ from $(-\xi_0,\xi_0)$.

Proof. Assuming that $\xi\in(-\xi_0,\xi_0)$ and, therefore, $\varphi(\xi)=\frac{1}{2}\arctan\frac{a\sin h\xi}{a\cos h\xi-\xi^2}$, we conclude that $\varphi(\xi)\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$, i.e., $2\varphi(\xi)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ on $\mathbb{R}^1$ and, therefore, the function $\cos 2\varphi(\xi)$ is positive everywhere. Therefore,

$$\cos 2\varphi(\xi) = \frac{1}{\sqrt{1+\tan^2 2\varphi(\xi)}} = \left[1+\frac{a^2\sin^2 h\xi}{(a\cos h\xi-\xi^2)^2}\right]^{-\frac{1}{2}} = \sqrt{\frac{(a\cos h\xi-\xi^2)^2}{\xi^4-2a\xi^2\cos h\xi+a^2}}.$$

The denominator of the last fraction can vanish only if $\cos h\xi=1$. However, its numerator is equal to its denominator in that case and, therefore, it vanishes as well. This is possible only under the assumption that $\xi = \xi_0$, but this contradicts the assumptions of the theorem. Thus,

$$\begin{aligned} \cos 2\varphi(\xi) &= \frac{1}{\sqrt{1+\tan^2 2\varphi(\xi)}} = \frac{a\cos h\xi - \xi^2}{\sqrt{\xi^4 - 2a\xi^2\cos h\xi + a^2}} = \frac{a\cos h\xi - \xi^2}{\rho^2(\xi)}, \\ \sin 2\varphi(\xi) &= \tan 2\varphi(\xi)\cos 2\varphi(\xi) = \frac{a\sin h\xi}{a\cos h\xi - \xi^2}\frac{a\cos h\xi - \xi^2}{\rho^2(\xi)} = \frac{a\sin h\xi}{\rho^2(\xi)}. \end{aligned} \tag{12}$$

Now, taking into account that, unlike the case of Theorem 1, $G_1^2(\xi) - G_2^2(\xi) = \rho^2(\xi)\cos 2\varphi(\xi)$, substitute function (6) in Equation (1):

$$\begin{aligned} \frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} &= \rho^2(\xi)\cos 2\varphi(\xi)e^{tG_1(\xi)}\sin\left[tG_2(\xi)+\varphi(\xi)+x\xi\right] \\ &+\rho^2(\xi)\sin 2\varphi(\xi)e^{tG_1(\xi)}\cos\left[tG_2(\xi)+\varphi(\xi)+x\xi\right] + \xi^2 e^{tG_1(\xi)}\sin\left[tG_2(\xi)+\varphi(\xi)+x\xi\right] \\ &= \left(a\cos h\xi - \xi^2\right)e^{tG_1(\xi)}\sin\left[tG_2(\xi)+\varphi(\xi)+x\xi\right] \\ &+a\sin h\xi e^{tG_1(\xi)}\cos\left[tG_2(\xi)+\varphi(\xi)+x\xi\right] + \xi^2 e^{tG_1(\xi)}\sin\left[tG_2(\xi)+\varphi(\xi)+x\xi\right], \end{aligned}$$

which coincides with (9). Therefore, function (6) satisfies Equation (1) (in the classical sense).

Function (7) is substituted in Equation (1) in the same way: as above,

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} &= \left[G_1^2(\xi) - G_2^2(\xi)\right]e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\ &-2G_1(\xi)G_2(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right], \end{aligned}$$

but $G_1^2(\xi) - G_2^2(\xi) = \cos 2\varphi(\xi)$ now (under the assumptions of Theorem 2). Therefore,

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} &= \rho^2(\xi)\cos 2\varphi(\xi)e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\ &-\rho^2(\xi)\sin 2\varphi(\xi)e^{-tG_1(\xi)}\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right]. \end{aligned}$$

Now, taking into account relations (12), we conclude that

$$\begin{aligned} \frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial x^2} &= e^{-tG_1(\xi)}\Big((a\cos h\xi - \xi^2)\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] \\ &-a\sin h\xi\cos\left[tG_2(\xi)-\varphi(\xi)-x\xi\right]\Big) \\ &+\xi^2 e^{-tG_1(\xi)}\sin\left[tG_2(\xi)-\varphi(\xi)-x\xi\right] = aH(x+h,t) \end{aligned}$$

in the same way as in (10).

Thus, function (7) satisfies Equation (1) (in the classical sense) as well. □

3. Heuristic Considerations

As we see above, Theorems 1 and 2 are proven directly: we just take function (5) and substitute in Equation (1). Such a proof is strict and clear, but it does not explain how to find the solution. In this section, we show how to apply the well-known Gel'fand–Shilov operational scheme (see, e.g., [11], Section 10) in the considered case.

Formally applying the Fourier transformation with respect to the variable x to Equation (1), which is a *partial differential-difference* equation, we obtain the following *ordinary differential* equation depending on the parameter ξ:

$$\frac{d^2\widehat{u}}{dt^2} + \left(|\xi|^2 - a\cos h\xi - ia\sin h\xi\right)\widehat{u} = 0. \tag{13}$$

Its general solution (up to arbitrary constants depending on the parameter ξ) is equal to

$$\frac{1}{\rho(\xi)}\Big(e^{-t\,G_1(\xi)}e^{i[t\,G_2(\xi)-\varphi(\xi)]}-e^{t\,G_1(\xi)}e^{-i[t\,G_2(\xi)+\varphi(\xi)]}\Big),$$

where the functions $G_{\left\{\substack{1\\2}\right\}}(\xi)$, $\varphi(\xi)$, and $\rho(\xi)$ are defined by relations (8), (2), and (3), respectively.

Now, it remains to (formally) apply the inverse Fourier transformation, to eliminate terms with odd integrands, and to choose the arbitrary constants depending on the parameter ξ such that purely imaginary terms are to be eliminated. Note that those remaining actions of the Gel'fand–Shilov procedure cannot be performed in our case because no convergence of the arising improper integral with respect to ξ is guaranteed. However, if we truncate the Gel'fand–Shilov procedure before integration with respect to the dual variable ξ and treat that variable as a parameter, then the obtained function, which is represented by (5), satisfies Equation (1). To verify this, we substitute it in Equation (1) in Section 2 above.

4. Novelty Nature Notes

As we note above, the crucial novelty of the presented results is as follows: earlier, the coefficient at the nonlocal potential was always assumed to be negative. In more general cases (for instance, if there are more than one nonlocal term or the dimension of the spatial variable is greater than one), that restriction is more complicated, but its sense is the same: the real part of the symbol of the differential-difference operator acting with respect to spatial variables is required to be of a constant sign. In no way this is a technical restriction: in the classical case of *differential* equations, the sign-constancy of the symbol is the criterion of the ellipticity. However, the said classical case is much more simple: symbols of differential operators are just polynomials. Moreover, second-order homogeneous differential operators reduced to the canonical form are most frequently studied in the classical case; their symbols are just quadratic forms. In the case of an elliptic operator (i.e., if the operator has no real characteristics), the corresponding quadratic form has a constant sign. If the operator is hyperbolic, then the sign of the corresponding quadratic form varies and its sign-constancy sets are divided from each other by the conical surface (a pair of lines in the case of a one-dimensional spatial variable) that is the only real characteristic of the operator.

In the differential-difference case studied here, the symbol is not a polynomial anymore, and this causes a qualitatively greater diversity: we can omit the sign-constancy requirement for the symbol.

However, under the assumptions imposed in this paper, we actually have the monotonicity of the specified symbol $\xi^2 - a\cos h\xi$. This deserves a special attention because no symbol monotonicity arises in the classical theory of differential equations (as far as the author is aware). It is clear that the monotonicity is a scalar property. Therefore, only the monotonicity of symbols of operators acting with respect to the selected (coordinate) directions might arise in the general multidimensional case, but this seems to be reasonable because differential-difference equations are actually differential equations with translations acting along selected directions.

5. Conclusions

In this paper, we continue the investigation of differential-difference hyperbolic equations with nonlocal potentials, taking off restrictions for signs of real (or imaginary) parts of symbols of operators contained in the investigated equations. The prototype case of Equation (1) is considered and the coefficient a at the nonlocal potential is assumed to be positive; earlier, only the case of negative values of the said coefficient was considered.

For Equation (1), we impose Condition (4), which neither means the smallness of the coefficient at the nonlocal term nor the smallness of its translation. Under the said assumption, we explicitly construct the following three-parameter family of smooth global solutions of Equation (1):

$$u(x,t) = \alpha F(x,t;\xi) + \beta H(x,t;\xi),$$

where

$$\left\{ \begin{matrix} F \\ H \end{matrix} \right\}(x,t;\xi) = e^{\pm tG_1(\xi)} \sin\left[tG_2(\xi) \pm \varphi(\xi) \pm x\xi\right],$$

$$G_{\left\{ \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \right\}}(\xi) = \begin{cases} \rho(\xi)\left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}\varphi(\xi) \text{ for } |\xi| > \xi_0, \\ \rho(\xi)\left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\}\varphi(\xi) \text{ for } |\xi| < \xi_0, \end{cases}$$

$$\varphi(\xi) = \frac{1}{2}\begin{cases} \arctan \frac{a \sin h\xi}{\xi^2 - a\cos h\xi} \text{ for } |\xi| > \xi_0, \\ \arctan \frac{a \sin h\xi}{a\cos h\xi - \xi^2} \text{ for } |\xi| < \xi_0, \end{cases}$$

$$\rho(\xi) = \left(\xi^4 - 2a\xi^2 \cos h\xi + a^2\right)^{\frac{1}{4}},$$

ξ_0 is the only positive root of the equation $\xi^2 = a\cos h\xi$, α and β are arbitrary real constants, and ξ is an arbitrary value from $\mathbb{R}^1 \setminus \{-\xi_0, \xi_0\}$.

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