mathematics

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Order Properties Concerning Tsallis Residual Entropy

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Abstract: With the help of Tsallis residual entropy, we introduce Tsallis quantile entropy order between two random variables. We give necessary and sufficient conditions, study closure and reversed closure properties under parallel and series operations and show that this order is preserved in the proportional hazard rate model, proportional reversed hazard rate model, proportional odds model and record values model.

Keywords: Tsallis entropy; Tsallis quantile entropy; Tsallis residual entropy; Tsallis quantile entropy order

MSC: 60E15; 60K10; 62B10; 62N05; 90B25; 94A17

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Citation: Sfetcu, R.-C.; Preda, V. Order Properties Concerning Tsallis Residual Entropy. *Mathematics* **2024**, *12*, 417. https://doi.org/10.3390/math12030417

Academic Editor: Jean Pierre Boon

Received: 27 November 2023
Revised: 19 January 2024
Accepted: 24 January 2024
Published: 27 January 2024

1. Introduction

The concept of entropy, defined mathematically by Shannon in [1], measures the uncertainty of a physical system and has applications in many scientific and technological areas such as physics, probability theory, statistics, communication theory and economics. This notion appeared from thermodynamics and statistical mechanics. In this theory, for a data communication system, we have three elements: a receiver, a communication channel and a source of data. Based on the signal that is received through the channel, Shannon tried to identify what sort of data were generated. Many methods on how to encode, compress and transmit the message were considered. In Shannon's source coding theorem, also known as Shannon's first theorem, an error-free encoding is established. This result is generalized especially for noisy channel in Shannon's noisy coding theorem. In the last couple of years, Shannon entropy was intensively studied and many generalizations have appeared (Tsallis entropy, Rényi entropy, Varma entropy, Kaniadakis entropy, relative entropy, weighted entropy, cumulative entropy, etc.).

In [2], Tsallis used another formula instead of the classical algorithm which appears in Shannon entropy, defining, in this way, what we call today, Tsallis entropy. There are many applications of this new entropy, especially in physics, and, more precisely: superstatistics (see [3]), spectral statistics (see [4]), earthquakes (see [5–7]), stock exchanges (see [8,9]), plasma (see [10]), income distribution (see [11]), non-coding human DNA (see [12]), internet (see [13]), and statistical mechanics (see [2,14]). For more information about Tsallis entropy, we recommend reading [15].

Among the applications of other entropies (Rényi entropy, Varma entropy, Kaniadakis entropy, relative entropy, weighted entropy, etc.), we can list the following: Markov chains (see [16–18]), model selection (see [19,20]), combinatorics (see [21,22]), finance (see [23–25]), Lie symmetries (see [26,27]), and machine learning (see [28,29]).

There are several papers in which the authors compare random variables from the point of view of residual entropies: for Shannon residual entropy, see [30–32]; for Rényi residual entropy, see [33,34]; for Varma residual entropy, see [35]; and for Awad-Varma residual entropy, see [36]. Other orders between random variables can be found in [37–44].

Rao et al. [45] introduced an alternative measure to Shannon entropy, known as the cumulative residual entropy (CRE), by considering the survival function instead of the probability density function. Because the survival function is more regular than the probability density function, CRE is considered to be more stable and has more mathematical properties. Moreover, the distribution function exists even if the probability density function does not exist (see, e.g., generalized lambda, power-Pareto and Govindarajulu distributions). Sati and Gupta [46] introduced a cumulative residual Tsallis entropy and extended it to its dynamic form based on the residual lifetime. Rajesh and Sunoj [47] introduced an alternative form of the cumulative residual Tsallis entropy and proved some results with applications in reliability. Toomaj and Atabay [48] elaborated some further consequences of the alternative cumulative residual Tsallis entropy, introduced by Rajesh and Sunoj [47], including stochastic ordering, expressions and bounds and proposed a normalized version of the cumulative residual Tsallis entropy, which can be used as a dispersion measure in place of coefficient of variation. Kumar [49] obtained characterization results based on the dynamic cumulative residual Tsallis entropy. In many realistic situations, uncertainty is not necessarily related to the future and can refer to the past as well. For instance, if at time t, a system which is observed only at certain preassigned inspection times is found to be down, then the uncertainty of the system life relies on the past, i.e., on which instant in $(0, t)$ it has failed. A wide variety of research is available on entropy measures and its applications in past lifetime. For more detail, one can refer to Di Crescenzo and Longobardi [50], Di Crescenzo and Longobardi [51], Sachlas and Papaioannou [52] and Di Crescenzo and Toomaj [53]. Also, a study on the cumulative Tsallis entropy for past lifetime is available in Nair et al.'s work [54], Calì et al.'s work [55], Khammar and Jahanshahi's work [56], Sunoj et al.'s study [57] and Alomani and Kayid's study [58]. Baratpour and Khammar [59] studied Tsallis entropy of order statistics. The quantile-based approach has some advantages: it provides an alternative methodology in deriving the cumulative Tsallis entropy in past lifetime and facilitates the extension of domain of application of the cumulative Tsallis entropy in past lifetime to many flexible quantile functions which serve as useful lifetime models and which possess no probability distribution function.

The paper is organized as follows. After this *Introduction* section, in Section 2, *Background and Notations*, we present the main notions and notations used throughout the article. In Section 3, *Fundamental Results*, we present the main theorem (Theorem 1), which is used in all of our results. In this section, we also prove that the dispersive order and the convex transform order apply to the Tsallis quantile entropy order. In Section 4, *Closure and Reversed Closure Properties*, we show the closure and reversed closure properties of Tsallis quantile entropy order under parallel and series operations. In the last four sections, we show the preservation of Tsallis quantile entropy in some stochastic models: the proportional hazard rate model (Section 5—*Preservation of Tsallis Quantile Entropy Order in the Proportional Hazard Rate Model*), the proportional reversed hazard rate model (Section 6—*Preservation of Tsallis Quantile Entropy Order in the Proportional Reversed Hazard Rate Model*), the proportional odds model (Section 7—*Preservation of Tsallis Quantile Entropy Order in the Proportional Odds Model*) and the proportional record values model (Section 8—*Preservation of Tsallis Quantile Entropy Order in the Record Values Model*).

2. Background and Notations

Throughout this paper, we assume that all expectations are finite and all ratios and powers are well defined. For information on notions of probability theory, we recommend [60].

We consider X a non-negative random variable with an absolutely continuous cumulative distribution function F_X, a survival function $\overline{F}_X \overset{def}{=} 1 - F_X$ and a probability density function f_X (X represents a living thing or the lifetime of a device).

Shannon entropy of X is given by

$$H_X = -\mathbb{E}_Z(\log f_X(Z)),$$

where "log" is the natural logarithm function and Z is a non-negative random variable identically distributed like X.

Let $\alpha \in \mathbb{R} \setminus \{1\}$. Tsallis logarithm is given via

$$\log^T(x) = \begin{cases} \dfrac{x^{\alpha-1}-1}{\alpha-1} & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

From this point onward, we assume that $\alpha > 0$.

Tsallis entropy of X is defined by

$$H_X^T = -\mathbb{E}_Z\left(\log^T(f_X(Z))\right).$$

In this paper, we work with Tsallis residual entropy, defined via

$$H_X^T(t) = -\mathbb{E}_Z\left(\frac{1}{\overline{F}_X(t)}\log^T\left(\frac{f_X(Z)}{\overline{F}_X(t)}\right)\Big|[Z > t]\right) \text{ for any } t \geq 0.$$

We recall that the quantile function of X is given by

$$Q_X(u) \stackrel{def}{=} F_X^{-1}(u) = \inf\{x \in [0,\infty) \mid F_X(x) \geq u\} \text{ for any } u \in [0,1].$$

We have $F_X(Q_X(u)) = u$ for any $u \in [0,1]$. Differentiating both sides of this equality with respect to u, we obtain $F_X'(Q_X(u))Q_X'(u) = 1$ for any $u \in [0,1]$. With the notation $q_X(u) = Q_X'(u)$ for any $u \in [0,1]$, it follows that $q_X(u)f_X(Q_X(u)) = 1$ for any $u \in [0,1]$.

Let $\Psi_X^T(u) = H_X^T(Q_X(u))$ for any $u \in [0,1]$.

For any $u \in [0,1]$, we obtain

$$\Psi_X^T(u) = -\mathbb{E}_Z\left(\frac{1}{1-u}\log^T\left(\frac{f_X(Z)}{1-u}\right)\Big|[Z > Q_X(u)]\right) =$$
$$-\mathbb{E}_U\left(\frac{1}{1-u}\log^T\left(\frac{f_X(Q_X(U))}{1-u}\right)\Big|[u < U < 1]\right),$$

where U is a random variable uniformly distributed on $[0,1]$.

In this paper, we are concerned about comparing two absolutely continuous non-negative random variables from the point of view of Tsallis residual entropy. More precisely, if X and Y are absolutely continuous non-negative random variables, we compare $\Psi_X^T(u)$ and $\Psi_Y^T(u)$ for any $u \in [0,1]$.

In the proofs, we will make use of the lemma below.

Lemma 1 (see [33]). *Let $g : [0,\infty) \to [0,\infty)$ an increasing function and $h : [0,1] \times [0,\infty) \to \mathbb{R}$ such that*

$$\mathbb{E}_U\left(h(u,U)\Big|[u < U < 1]\right) \geq 0 \text{ for any } u \in [0,1].$$

Then

$$\mathbb{E}_U\left(h(u,U)g(U)\Big|[u < U < 1]\right) \geq 0 \text{ for any } u \in [0,1].$$

3. Fundamental Results

Definition 1. *We say that X is smaller than Y in Tsallis quantile entropy order (and denote by $X \leq_T Y$) if $\Psi_X^T(u) \leq \Psi_Y^T(u)$ for any $u \in [0,1]$.*

In the last several years, stochastic orders and inequalities have been used intensively in many areas of probability and statistics, like reliability theory, queuing theory, survival analysis, biology, economics, insurance, actuarial science, operations research and management science. The simplest way of comparing two distribution functions is by a comparison of the associated means. Because this comparison is based on only two numbers (the means), it is sometimes not very informative. Moreover, the means to not exist is possible. In many applications, we have more detailed information concerning the comparison of two distribution functions than just the two means. If we compare two distribution functions with the same mean (or that are centered about the same value), we can compare the dispersion of these distributions. The simplest way of doing this is by the comparison of the associated standard deviations. But, again, the comparison depends on only two numbers (the standard deviations), which are at times not very informative. As mentioned above, it is possible for the standard deviations to not exist. The concept of stochastic orders plays a major role in the theory and practice of statistics. It generally refers to a set of relations that may hold between a pair of distributions of random variables. In reliability, the stochastic orders which compare life distributions based on different characteristics are used to study aging properties, to develop bounds on reliability functions, to compare the performance of policies and systems and to derive new inference procedures. Many of such orders are defined in terms of concepts based on distribution functions.

The theorem below is the main result of this paper.

Theorem 1. *The following assertions are equivalent:*

1. $X \leq_T Y$.
2. $\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\dfrac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Bigg|[Z>t]\right)\geq 0$ *for any* $t\geq 0$.

Proof. From Definition 1, $X \leq_T Y$ if and only if

$$-\mathbb{E}_U\left(\frac{1}{1-u}\log^T\left(\frac{f_X(Q_X(U))}{1-u}\right)\Bigg|[u<U<1]\right)\leq$$
$$-\mathbb{E}_U\left(\frac{1}{1-u}\log^T\left(\frac{f_Y(Q_Y(U))}{1-u}\right)\Bigg|[u<U<1]\right) \text{ for any } u\in[0,1].$$

If we take $Q_X(U)=Z$ in the preceding inequality, the following equivalences are valid for any $u\in[0,1]$:

$$X\leq_T Y \Longleftrightarrow \mathbb{E}_Z\left(\log^T\left(\frac{f_X(Z)}{\overline{F}_X(F_X^{-1}(u))}\right)\Bigg|[Z>F_X^{-1}(u)]\right)\geq$$
$$\mathbb{E}_Z\left(\log^T\left(\frac{f_Y(Q_Y(F_X(Z)))}{\overline{F}_X(F_X^{-1}(u))}\right)\Bigg|[Z>F_X^{-1}(u)]\right)\Longleftrightarrow$$
$$\mathbb{E}_Z\left(\log^T\left(\frac{f_X(Z)}{\overline{F}_X(F_X^{-1}(u))}\right)-\log^T\left(\frac{f_Y(Q_Y(F_X(Z)))}{\overline{F}_X(F_X^{-1}(u))}\right)\Bigg|[Z>F_X^{-1}(u)]\right)\geq 0\Longleftrightarrow$$
$$\mathbb{E}_Z\left(\left(f_Y(F_Y^{-1}(F_X(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))}\right)\Bigg|[Z>F_X^{-1}(u)]\right)\geq 0.$$

In order to obtain the conclusion, it is sufficient to denote $F_X^{-1}(u)=t$. □

Definition 2 (see [61]). *We say that:*

1. *X is smaller than Y in the dispersive order (and write $X\leq_{disp} Y$) if*

$$f_X(x)\geq f_Y(F_Y^{-1}(F_X(x))) \text{ for any } x\geq 0.$$

2. *X is smaller than Y in the convex transform order (and write $X\leq_c Y$) if the function*

$$[0,\infty) \ni x \longmapsto \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))} \text{ is non-negatively increasing.}$$

The dispersive order is a basic concept for comparing spread among probability distributions, with applications to order statistics, spacings and convolutions of independent random variables. The convex transform order is used to make precise comparisons between the skewness of probability distributions on the real line. From the point of view of the aging interpretation, this order can be seen as identifying aging rates in a way that also works when lifetimes do not start simultaneously (for more details concerning these two orders, the reader can consult [61]).

Theorem 2. *If $X \leq_{disp} Y$, then $X \leq_T Y$.*

Proof. Assume that $X \leq_{disp} Y$. Then $f_X(x) \geq f_Y(F_Y^{-1}(F_X(x)))$ for any $x \geq 0$, hence $\log^T\left(\frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))}\right) \geq 0$ for any $x \geq 0$ and the conclusion follows Theorem 1. □

Theorem 3. *If $X \leq_c Y$ and $f_Y(0) \leq f_X(0)$, then $X \leq_T Y$.*

Proof. Assume that $X \leq_c Y$. Then the function $[0,\infty) \ni x \longmapsto \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))}$ is non-negatively increasing; hence

$$\frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))} \geq \frac{f_X(0)}{f_Y(0)} \geq 1 \text{ for any } x \geq 0.$$

With Theorem 1, we obtain the conclusion. □

4. Closure and Reversed Closure Properties

We consider $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ to be independent and identically distributed (i.i.d.) copies of X and Y, respectively, and

$$X_{1:n} = \min\{X_1, \ldots, X_n\}, X_{n:n} = \max\{X_1, \ldots, X_n\},$$

$$Y_{1:n} = \min\{Y_1, \ldots, Y_n\}, Y_{n:n} = \max\{Y_1, \ldots, Y_n\}.$$

Theorem 4. *If $X \leq_T Y$, then $X_{n:n} \leq_T Y_{n:n}$.*

Proof. Because $X \leq_T Y$, we can determine with Theorem 1 that

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1} \log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right) \Big| [Z > t]\right) \geq 0 \text{ for any } t \geq 0. \quad (1)$$

It can be seen that, for any $x \geq 0$,

$$F_{X_{n:n}}(x) = (F_X(x))^n,$$

$$F_{Y_{n:n}}(x) = (F_Y(x))^n,$$

$$f_{X_{n:n}}(x) = n(F_X(x))^{n-1} f_X(x),$$

$$f_{Y_{n:n}}(x) = n(F_Y(x))^{n-1} f_Y(x),$$

$$F_{Y_{n:n}}^{-1}(F_{X_{n:n}}(x)) = F_Y^{-1}(F_X(x)),$$

$$f_{Y_{n:n}}(F_{Y_{n:n}}^{-1}(F_{X_{n:n}}(x))) = n(F_X(x))^{n-1} f_Y(F_Y^{-1}(F_X(x)))$$

and

$$\frac{f_{X_{n:n}}(x)}{f_{Y_{n:n}}(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(x)))} = \frac{f_X(x)}{f_Y(F^{-1}_Y(F_X(x)))}.$$

Then

$$\mathbb{E}_Z\left(\left(f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{n:n}}(Z)}{f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(Z))\right)}\right)\Big|[Z>t]\right) =$$
$$\mathbb{E}_Z\left(\left(n(F_X(Z))^{n-1}\right)^{\alpha}\left(f_Y\left(F^{-1}_Y(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F^{-1}_Y(F_X(Z))\right)}\right)\Big|[Z>t]\right) \text{ for any } t \geq 0.$$

Because the function

$$[0,\infty) \ni x \longmapsto \left(n(F_X(x))^{n-1}\right)^{\alpha} \text{ is non-negatively increasing,}$$

it follows, via inequality (1) and Lemma 1, that

$$\mathbb{E}_Z\left(\left(f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{n:n}}(Z)}{f_{Y_{n:n}}\left(F^{-1}_{Y_{n:n}}(F_{X_{n:n}}(Z))\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0.$$

The relationship $X_{n:n} \leq_T Y_{n:n}$ follows Theorem 1. □

Theorem 5. *If $X_{1:n} \leq_T Y_{1:n}$, then $X \leq_T Y$.*

Proof. Because $X_{1:n} \leq_T Y_{1:n}$, we have, by Theorem 1, that

$$\mathbb{E}_Z\left(\left(f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{1:n}}(Z)}{f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(Z))\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0. \quad (2)$$

We can see that, for any $x \geq 0$,

$$\overline{F}_{X_{1:n}}(x) = \left(\overline{F}_X(x)\right)^n,$$
$$\overline{F}_{Y_{1:n}}(x) = \left(\overline{F}_Y(x)\right)^n,$$
$$f_{X_{1:n}}(x) = n\left(\overline{F}_X(x)\right)^{n-1}f_X(x),$$
$$f_{Y_{1:n}}(x) = n\left(\overline{F}_Y(x)\right)^{n-1}f_Y(x),$$
$$F^{-1}_{Y_{1:n}}\left(F_{X_{1:n}}(x)\right) = F^{-1}_Y(F_X(x)),$$
$$f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x))) = n\left(\overline{F}_X(x)\right)^{n-1}f_Y(F^{-1}_Y(F_X(x)))$$

and

$$\frac{f_{X_{1:n}}(x)}{f_{Y_{1:n}}(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(x)))} = \frac{f_X(x)}{f_Y(F^{-1}_Y(F_X(x)))}.$$

Then

$$\mathbb{E}_Z\left(\left(f_Y\left(F^{-1}_Y(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F^{-1}_Y(F_X(Z))\right)}\right)\Big|[Z>t]\right) =$$
$$\mathbb{E}_Z\left(\left(n\overline{F}_X(Z))^{n-1}\right)^{-\alpha}\left(f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{1:n}}(Z)}{f_{Y_{1:n}}\left(F^{-1}_{Y_{1:n}}(F_{X_{1:n}}(Z))\right)}\right)\Big|[Z>t]\right)$$
$$\text{for any } t \geq 0.$$

Because the function

$$[0,\infty) \ni x \longmapsto \left(n\overline{F}_X(x))^{n-1}\right)^{-\alpha} \text{ is non-negatively increasing,}$$

it follows, via inequality (2) and Lemma 1, that

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0.$$

By applying Theorem 1, we obtain that $X \leq_T Y$. □

The natural step is to generalize the preceding two theorems from a finite number n to a random variable N.

We consider $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ as sequences of independent and identically distributed copies of X and Y, respectively. Let N be a positive integer random variable with the probability mass function $p_N(n) = P(N = n)$, $n = 1, 2, \ldots$ and such that N is independent of $X_i's$ and $Y_i's$. Take

$$X_{1:N} = \min\{X_1, \ldots, X_N\}, X_{N:N} = \max\{X_1, \ldots, X_N\}$$

and

$$Y_{1:N} = \min\{Y_1, \ldots, Y_N\}, Y_{N:N} = \max\{Y_1, \ldots, Y_N\}.$$

Theorem 6. *If $X \leq_T Y$, then $X_{N:N} \leq_T Y_{N:N}$.*

Proof. Because $X \leq_T Y$, we can determine by Theorem 1 that

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0. \quad (3)$$

One can see that, for any $x \geq 0$,

$$F_{X_{N:N}}(x) = \sum_{n=1}^{\infty}(F_X(x))^n p_N(n),$$

$$F_{Y_{N:N}}(x) = \sum_{n=1}^{\infty}(F_Y(x))^n p_N(n),$$

$$f_{X_{N:N}}(x) = \left[\sum_{n=1}^{\infty} n(F_X(x))^{n-1}p_N(n)\right] \cdot f_X(x)$$

and

$$f_{Y_{N:N}}(x) = \left[\sum_{n=1}^{\infty} n(F_Y(x))^{n-1}p_N(n)\right] \cdot f_Y(x).$$

It was proven in [61] that

$$F_{Y_{N:N}}^{-1}\left(F_{X_{N:N}}(x)\right) = F_Y^{-1}(F_X(x)) \text{ for any } x \geq 0.$$

Hence, for any $x \geq 0$,

$$f_{Y_{N:N}}(F_{Y_{N:N}}^{-1}(F_{X_{N:N}}(x))) = \left[\sum_{n=1}^{\infty} n(F_X(x))^{n-1}p_N(n)\right] \cdot f_Y(F_Y^{-1}(F_X(x)))$$

and

$$\frac{f_{X_{N:N}}(x)}{f_{Y_{N:N}}(F_{Y_{N:N}}^{-1}(F_{X_{N:N}}(x)))} = \frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))}.$$

Then

$$\mathbb{E}_Z\left(\left(f_{Y_{N:N}}\left(F^{-1}_{Y_{N:N}}(F_{X_{N:N}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{N:N}}(Z)}{f_{Y_{N:N}}\left(F^{-1}_{Y_{N:N}}(F_{X_{N:N}}(Z))\right)}\right)\Big|[Z>t]\right)=$$

$$\mathbb{E}_Z\left(\left(\sum_{n=1}^{\infty}n(F_X(Z))^{n-1}p_N(n)\right)^{\alpha}\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\text{ for any } t\geq 0.$$

Because the function

$$[0,\infty)\ni x\longmapsto\left(\sum_{n=1}^{\infty}n(F_X(x))^{n-1}p_N(n)\right)^{\alpha}\text{ is non-negatively increasing,}$$

it follows, via inequality (3) and Lemma 1, that

$$\mathbb{E}_Z\left(\left(f_{Y_{N:N}}\left(F^{-1}_{Y_{N:N}}(F_{X_{N:N}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{N:N}}(Z)}{f_{Y_{N:N}}\left(F^{-1}_{Y_{N:N}}(F_{X_{N:N}}(Z))\right)}\right)\Big|[Z>t]\right)\geq 0\text{ for any } t\geq 0.$$

The conclusion thus follows Theorem 1. □

Theorem 7. *If $X_{1:N}\leq_T Y_{1:N}$, then $X\leq_T Y$.*

Proof. Because $X_{1:N}\leq_T Y_{1:N}$, we can determine by Theorem 1 that

$$\mathbb{E}_Z\left(\left(f_{Y_{1:N}}\left(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_{1:N}}(Z)}{f_{Y_{1:N}}\left(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(Z))\right)}\right)\Big|[Z>t]\right)\geq 0\text{ for any } t\geq 0. \quad (4)$$

We can see that, for any $x\geq 0$,

$$\overline{F}_{X_{N:N}}(x)=\sum_{n=1}^{\infty}\left(\overline{F}_X(x)\right)^n p_N(n),$$

$$\overline{F}_{Y_{N:N}}(x)=\sum_{n=1}^{\infty}\left(\overline{F}_Y(x)\right)^n p_N(n),$$

$$f_{X_{1:N}}(x)=\left[\sum_{n=1}^{\infty}n\left(\overline{F}_X(x)\right)^{n-1}p_N(n)\right]\cdot f_X(x)$$

and

$$f_{Y_{1:N}}(x)=\left[\sum_{n=1}^{\infty}n\left(\overline{F}_Y(x)\right)^{n-1}p_N(n)\right]\cdot f_Y(x).$$

It was proven in [61] that

$$F^{-1}_{Y_{1:N}}\left(F_{X_{1:N}}(x)\right)=F_Y^{-1}(F_X(x))\text{ for any } x\geq 0.$$

Hence, for any $x\geq 0$,

$$f_{Y_{1:N}}(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(x)))=\left[\sum_{n=1}^{\infty}n\left(\overline{F}_X(x)\right)^{n-1}p_N(n)\right]\cdot f_Y(F_Y^{-1}(F_X(x)))$$

and

$$\frac{f_{X_{1:N}}(x)}{f_{Y_{1:N}}(F^{-1}_{Y_{1:N}}(F_{X_{1:N}}(x)))}=\frac{f_X(x)}{f_Y(F_Y^{-1}(F_X(x)))}.$$

Then

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)=$$

$$\mathbb{E}_Z\left(\frac{\left(f_{Y_{1:N}}\left(F_{Y_{1:N}}^{-1}\left(F_{X_{1:N}}(Z)\right)\right)\right)^{\alpha-1}}{\left(\sum_{n=1}^{\infty}n(\overline{F}_X(Z))^{n-1}p_N(n)\right)^{\alpha}}\log^T\left(\frac{f_{X_{1:N}}(Z)}{f_{Y_{1:N}}\left(F_{Y_{1:N}}^{-1}\left(F_{X_{1:N}}(Z)\right)\right)}\right)\Big|[Z>t]\right)\text{ for any } t\geq 0.$$

Because the function

$$[0,\infty)\ni x\longmapsto\left(\sum_{n=1}^{\infty}n(\overline{F}_X(x))^{n-1}p_N(n)\right)^{-\alpha}\text{ is non-negatively increasing,}$$

it follows, via inequality (4) and Lemma 1, that

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\geq 0\text{ for any } t\geq 0.$$

By Theorem 1, we conclude that $X \leq_T Y$. □

5. Preservation of Tsallis Quantile Entropy Order in the Proportional Hazard Rate Model

We consider the following proportional hazard rate model (see [61]), namely for any $\theta > 0$, for which we take $X(\theta)$ and $Y(\theta)$ as two absolutely continuous non-negative random variables with the survival functions $(\overline{F}_X)^\theta$ and $(\overline{F}_Y)^\theta$, respectively.

Theorem 8.

1. *If $\theta \geq 1$ and $X \leq_T Y$, then $X(\theta) \leq_T Y(\theta)$.*
2. *If $0 < \theta \leq 1$ and $X(\theta) \leq_T Y(\theta)$, then $X \leq_T Y$.*

Proof. For any $x \geq 0$, we can obtain:

$$\overline{F}_{X(\theta)}(x)=\left(\overline{F}_X(x)\right)^\theta,$$

$$\overline{F}_{Y(\theta)}(x)=\left(\overline{F}_Y(x)\right)^\theta,$$

$$f_{X(\theta)}(x)=\theta\left(\overline{F}_X(x)\right)^{\theta-1}f_X(x),$$

$$f_{Y(\theta)}(x)=\theta\left(\overline{F}_Y(x)\right)^{\theta-1}f_Y(x),$$

$$F_{Y(\theta)}^{-1}\left(F_{X(\theta)}(x)\right)=F_Y^{-1}(F_X(x)),$$

$$f_{Y(\theta)}\left(F_{Y(\theta)}^{-1}\left(F_{X(\theta)}(x)\right)\right)=\theta\left(\overline{F}_X(x)\right)^{\theta-1}f_Y\left(F_Y^{-1}(F_X(x))\right)$$

and

$$\frac{f_{X(\theta)}(x)}{f_{Y(\theta)}\left(F_{Y(\theta)}^{-1}\left(F_{X(\theta)}(x)\right)\right)}=\frac{f_X(x)}{f_Y\left(F_Y^{-1}(F_X(x))\right)}.$$

Then:

$$\mathbb{E}_Z\left(\left(f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)}\right)\Big|[Z>t]\right)=$$
$$\mathbb{E}_Z\left(\left(\theta\left(\overline{F}_X(Z)\right)^{\theta-1}\right)^{\alpha}\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\geq 0 \text{ for any } t\geq 0.$$

1. If $0<\theta\leq 1$ and $X\leq_T Y$, then the function
$$[0,\infty)\ni x\longmapsto\left(\theta\left(\overline{F}_X(x)\right)^{\theta-1}\right)^{\alpha} \text{ is non-negatively increasing}$$
and
$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\geq 0 \text{ for any } t\geq 0.$$
Using Lemma 1, we can determine that $X(\theta)\leq_T Y(\theta)$.
2. If $\theta\geq 1$ and $X(\theta)\leq_T Y(\theta)$, then the function
$$[0,\infty)\ni x\longmapsto\left(\theta\left(\overline{F}_X(x)\right)^{\theta-1}\right)^{-\alpha} \text{ is non-negatively increasing}$$
and
$$\mathbb{E}_Z\left(\left(f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)}\right)\Big|[Z>t]\right)\geq 0 \text{ for any } t\geq 0.$$
Using Lemma 1, we can determine that $X\leq_T Y$.

□

6. Preservation of Tsallis Quantile Entropy Order in the Proportional Reversed Hazard Rate Model

We consider the following proportional reversed hazard rate model (see [61]), namely for any $\theta>0$, for which we take $X(\theta)$ and $Y(\theta)$ as two absolutely continuous non-negative random variables with the distribution functions $(F_X)^{\theta}$ and $(F_Y)^{\theta}$, respectively.

Theorem 9.

1. *If $\theta\geq 1$ and $X\leq_T Y$, then $X(\theta)\leq_T Y(\theta)$.*
2. *If $0<\theta\leq 1$ and $X(\theta)\leq_T Y(\theta)$, then $X\leq_T Y$.*

Proof. We can determine for any $x\geq 0$:
$$F_{X(\theta)}(x)=(F_X(x))^{\theta},$$
$$F_{Y(\theta)}(x)=(F_Y(x))^{\theta},$$
$$f_{X(\theta)}(x)=\theta(F_X(x))^{\theta-1}f_X(x),$$
$$f_{Y(\theta)}(x)=\theta(F_Y(x))^{\theta-1}f_Y(x),$$
$$F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right)=F_Y^{-1}(F_X(x)),$$
$$f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right)\right)=\theta(F_X(x))^{\theta-1}f_Y\left(F_Y^{-1}(F_X(x))\right)$$
and

$$\frac{f_{X(\theta)}(x)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(x)\right)\right)}=\frac{f_X(x)}{f_Y\left(F_Y^{-1}(F_X(x))\right)}.$$

Then:

$$\mathbb{E}_Z\left(\left(f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)}\right)\Big|[Z>t]\right)=$$
$$\mathbb{E}_Z\left(\left(\theta(F_X(Z))^{\theta-1}\right)^{\alpha}\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\geq 0 \text{ for any } t\geq 0.$$

1. If $\theta\geq 1$ and $X\leq_T Y$, then the function

$$[0,\infty)\ni x\longmapsto\left(\theta(F_X(x))^{\theta-1}\right)^{\alpha}\text{ is non-negatively increasing}$$

and

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)\geq 0\text{ for any } t\geq 0.$$

Using Lemma 1, we can determine that $X(\theta)\leq_T Y(\theta)$.

2. If $0<\theta\leq 1$ and $X(\theta)\leq_T Y(\theta)$, then the function

$$[0,\infty)\ni x\longmapsto\left(\theta(F_X(x))^{\theta-1}\right)^{-\alpha}\text{ is non-negatively increasing}$$

and

$$\mathbb{E}_Z\left(\left(f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X(\theta)}(Z)}{f_{Y(\theta)}\left(F^{-1}_{Y(\theta)}\left(F_{X(\theta)}(Z)\right)\right)}\right)\Big|[Z>t]\right)\geq 0 \text{ for any } t\geq 0.$$

Using Lemma 1, we can determine that $X\leq_T Y$.

□

7. Preservation of Tsallis Quantile Entropy Order in the Proportional Odds Model

We work with the following proportional odds model (see [62]), namely for any $\theta>0$, for which we take the proportional odds random variables X_p and Y_p, defined by the survival functions $\overline{F}_{X_p}(x)=\dfrac{\theta\overline{F}_X(x)}{1-(1-\theta)\overline{F}_X(x)}$ and $\overline{F}_{Y_p}(x)=\dfrac{\theta\overline{F}_Y(x)}{1-(1-\theta)\overline{F}_Y(x)}$, respectively, for any $x\geq 0$.

Theorem 10.

1. *If $\theta\geq 1$ and $X\leq_T Y$, then $X_p\leq_T Y_p$.*
2. *If $0<\theta\leq 1$ and $X_p\leq_T Y_p$, then $X\leq_T Y$.*

Proof. For any $x\geq 0$ we have

$$\overline{F}_{X_p}(x)=\frac{\theta\overline{F}_X(x)}{1-(1-\theta)\overline{F}_X(x)},$$

$$\overline{F}_{Y_p}(x)=\frac{\theta\overline{F}_Y(x)}{1-(1-\theta)\overline{F}_Y(x)},$$

$$f_{X_p}(x)=\frac{\theta}{\left(1-(1-\theta)\overline{F}_X(x)\right)^2}\cdot f_X(x),$$

$$f_{Y_p}(x) = \frac{\theta}{\left(1-(1-\theta)\overline{F}_Y(x)\right)^2} \cdot f_Y(x),$$

$$F_{Y_p}^{-1}\left(F_{X_p}(x)\right) = F_Y^{-1}(F_X(x)),$$

$$f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(x)\right)\right) = \frac{\theta}{\left(1-(1-\theta)\overline{F}_X(x)\right)^2} \cdot f_Y\left(F_Y^{-1}(F_X(x))\right)$$

and

$$\frac{f_{X_p}(x)}{f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(x)\right)\right)} = \frac{f_X(x)}{f_Y\left(F_Y^{-1}(F_X(x))\right)}.$$

Then

$$\mathbb{E}_Z\left(\left(f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_p}(Z)}{f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(Z)\right)\right)}\right)\Big|[Z>t]\right) =$$

$$\mathbb{E}_Z\left(\left(\frac{\theta}{\left(1-(1-\theta)\overline{F}_X(Z)\right)^2}\right)^{\alpha}\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right)$$

for any $t \geq 0$.

1. Assume that $X \leq_T Y$ and $\theta \geq 1$. Then

$$\mathbb{E}_Z\left(\left(f_Y\left(F_Y^{-1}(F_X(Z))\right)\right)^{\alpha-1}\log^T\left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0$$

and the function

$$[0,\infty) \ni x \longmapsto \left(\frac{\theta}{\left(1-(1-\theta)\overline{F}_X(x)\right)^2}\right)^{\alpha} \text{ is non-negatively increasing.}$$

Hence, by Lemma 1, we obtain $X_p \leq_T Y_p$.

2. Assume that $X_p \leq_T Y_p$ and $0 < \theta \leq 1$. Then

$$\mathbb{E}_Z\left(\left(f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(Z)\right)\right)\right)^{\alpha-1}\log^T\left(\frac{f_{X_p}(Z)}{f_{Y_p}\left(F_{Y_p}^{-1}\left(F_{X_p}(Z)\right)\right)}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0$$

and the function

$$[0,\infty) \ni x \longmapsto \left(\frac{\theta}{\left(1-(1-\theta)\overline{F}_X(x)\right)^2}\right)^{-\alpha} \text{ is non-negatively increasing.}$$

Hence, by Lemma 1, we obtain $X \leq_T Y$.

□

8. Preservation of Tsallis Quantile Entropy Order in the Record Values Model

Let $\{X_i \mid i \geq 1\}$ and $\{Y_i \mid i \geq 1\}$ be sequences of i.i.d. random variables from the random variables X and Y, respectively, with survival functions $\overline{F}_X$ and $\overline{F}_Y$, respectively, and density functions f_X and f_Y, respectively. We consider the nth record times T_n^X and T_n^Y, respectively, defined via $T_1^X = 1$ and $T_{n+1}^X = \min\{j > T_n^X \mid X_j > X_{T_n^X}\}$ for any $n \geq 1$ and $T_1^Y = 1$ and $T_{n+1}^Y = \min\{j > T_n^Y \mid Y_j > Y_{T_n^Y}\}$, respectively.

We denote $X_{T_n^X} \overset{def}{=} R_n^X$ and $Y_{T_n^Y} \overset{def}{=} R_n^Y$, respectively, and call them the nth record values (see [63]).

For any $x \geq 0$, we can obtain

$$\overline{F}_{R_n^X}(x) = \overline{F}_X(x) \sum_{j=0}^{n-1} \frac{(\Lambda_X(x))^j}{j!} = \overline{\Gamma}_n(\Lambda_X(x)),$$

$$\overline{F}_{R_n^Y}(x) = \overline{F}_Y(x) \sum_{j=0}^{n-1} \frac{(\Lambda_Y(x))^j}{j!} = \overline{\Gamma}_n(\Lambda_Y(x)),$$

$$f_{R_n^X}(x) = \frac{1}{\Gamma(n)} \Lambda_X^{n-1}(x) f_X(x)$$

and

$$f_{R_n^Y}(x) = \frac{1}{\Gamma(n)} \Lambda_Y^{n-1}(x) f_Y(x),$$

where $\overline{\Gamma}_n$ is the survival function of a Gamma random variable with the shape parameter n and the scale parameter 1, $\Lambda_X(x) = -\log \overline{F}_X(x)$ is the cumulative failure rate function of X and $\Lambda_Y(x) = -\log \overline{F}_Y(x)$ is the cumulative failure rate function of Y.

Theorem 11. *Let $m, n \in \mathbb{N} \stackrel{def}{=} \{1, 2, \dots\}$.*
1. *If $X \leq_T Y$, then $R_n^X \leq_T R_n^Y$.*
2. *If $n > m \geq 1$ and $R_m^X \leq_T R_m^Y$, then $R_n^X \leq_T R_n^Y$.*

Proof.
1. If $X \leq_T Y$, then

$$\mathbb{E}_Z \left(\left(f_Y \left(F_Y^{-1}(F_X(Z)) \right) \right)^{\alpha-1} \log^T \left(\frac{f_X(Z)}{f_Y\left(F_Y^{-1}(F_X(Z))\right)} \right) \Bigg| [Z > t] \right) \geq 0 \text{ for any } t \geq 0.$$

We have, for any $x \geq 0$,

$$F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right) = F_Y^{-1}(F_X(x)),$$

$$f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right)\right) = \frac{1}{\Gamma(n)} \Lambda_X^{n-1}(x) f_Y\left(F_Y^{-1}(F_X(x))\right)$$

and

$$\frac{f_{R_n^X}(x)}{f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right)\right)} = \frac{f_X(x)}{f_Y\left(F_Y^{-1}(F_X(x))\right)}.$$

Then

$$\mathbb{E}_Z \left(\left(f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(Z)\right)\right) \right)^{\alpha-1} \log^T \left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))} \right) \Bigg| [Z > t] \right) =$$
$$\mathbb{E}_Z \left(\left(\frac{1}{\Gamma(n)} \Lambda_X^{n-1}(Z) \right)^{\alpha} \left(f_Y\left(F_Y^{-1}(F_X(Z))\right) \right)^{\alpha-1} \log^T \left(\frac{f_X(Z)}{f_Y(F_Y^{-1}(F_X(Z)))} \right) \Bigg| [Z > t] \right)$$
$$\text{for any } t \geq 0.$$

Because the function

$$[0, \infty) \ni x \longmapsto \left(\frac{1}{\Gamma(n)} \Lambda_X^{n-1}(x) \right)^{\alpha} \text{ is non-negatively increasing,}$$

we obtain via Lemma 1 that

$$\mathbb{E}_Z \left(\left(f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(Z)\right)\right) \right)^{\alpha-1} \log^T \left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))} \right) \Bigg| [Z > t] \right) \geq 0 \text{ for any}$$
$$t \geq 0,$$

i.e., $R_n^X \leq_T R_n^Y$.

2. If $R_m^X \leq_T R_m^Y$, then

$$\mathbb{E}_Z\left(\left(f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0.$$

For any $x \geq 0$, we can determine

$$F_{R_m^Y}^{-1}\left(F_{R_m^X}(x)\right) = F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right) = F_Y^{-1}(F_X(x)),$$

$$\frac{f_{R_m^X}(x)}{f_{R_m^Y}\left(F_{R_m^Y}^{-1}\left(F_{R_m^X}(x)\right)\right)} = \frac{f_{R_n^X}(x)}{f_{R_n^Y}\left(F_{R_n^Y}^{-1}\left(F_{R_n^X}(x)\right)\right)} = \frac{f_X(x)}{f_Y\left(F_Y^{-1}(F_X(x))\right)}$$

and

$$f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(x))) = \frac{\Gamma(m)}{\Gamma(n)}(\Lambda_X(x))^{n-m} f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(x))).$$

Then

$$\mathbb{E}_Z\left(\left(f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))}\right)\Big|[Z>t]\right) =$$
$$\mathbb{E}_Z\left(\left(\frac{\Gamma(m)}{\Gamma(n)}(\Lambda_X(Z))^{n-m}\right)^{\alpha}\left(f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))}\right)\Big|[Z>t]\right)$$
for any $t \geq 0$.

Because the function

$$[0,\infty) \ni x \longmapsto \left(\frac{\Gamma(m)}{\Gamma(n)}(\Lambda_X(x))^{n-m}\right)^{\alpha} \text{ is non-negatively increasing}$$

and

$$\mathbb{E}_Z\left(\left(f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_{R_m^X}(Z)}{f_{R_m^Y}(F_{R_m^Y}^{-1}(F_{R_m^X}(Z)))}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0,$$

using Lemma 1, we obtain that

$$\mathbb{E}_Z\left(\left(f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))\right)^{\alpha-1}\log^T\left(\frac{f_{R_n^X}(Z)}{f_{R_n^Y}(F_{R_n^Y}^{-1}(F_{R_n^X}(Z)))}\right)\Big|[Z>t]\right) \geq 0 \text{ for any } t \geq 0,$$

i.e., $R_n^X \leq_T R_n^Y$.

□

9. Conclusions

We introduced Tsallis quantile entropy order between two random variables, found necessary and sufficient conditions for it and proved closure and reversed closured properties of this order under parallel and series operations. We also showed that Tsallis quantile entropy order is preserved in some stochastic models, like proportional hazard rate model, proportional reversed hazard rate model, proportional odds model and record values model. In this way, there are generalized results from other papers working with Tsallis residual entropy instead of Shannon residual entropy (which is used in [30–32]), Rényi residual entropy (which is used in [33,34]), Varma residual entropy (which is used in [35]) and Awad-Varma residual entropy (which is used in [36]). The difference is that we work with Tsallis residual entropy instead of other residual entropies considered in the aforementioned papers.

Author Contributions: Conceptualization, R.-C.S. and V.P.; methodology, R.-C.S. and V.P.; software, R.-C.S. and V.P.; validation, R.-C.S. and V.P.; formal analysis, R.-C.S. and V.P.; investigation, R.-C.S. and V.P.; writing—original draft, R.-C.S. and V.P.; writing—review and editing, R.-C.S. and V.P.; visualization, R.-C.S. and V.P.; supervision, R.-C.S. and V.P.; project administration, R.-C.S. and V.P. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors are very much indebted to the anonymous referees and to the editors for their most valuable comments and suggestions which improved the quality of the paper.

Conflicts of Interest: The authors declare no conflicts of interest.

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