mathematics

Article

The Friedrichs Extension of Elliptic Operators with Conditions on Submanifolds of Arbitrary Dimension

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Abstract: We describe the Friedrichs extension of elliptic symmetric pseudodifferential operators on a closed smooth manifold with the domain consisting of functions vanishing on a given submanifold. In summary, the Friedrichs extension is an elliptic Sobolev problem defined in terms of boundary and coboundary operators, and the number of boundary and coboundary conditions in the problem depends on the order of the operator and the codimension of the submanifold. In this paper, the discreteness of the spectrum is proved, and singularities of eigenfunctions are described.

Keywords: Friedrichs extension; Sobolev problems; boundary conditions on submanifolds

MSC: 35P05; 58J50

1. Introduction

Boundary value problems with conditions on submanifolds of arbitrary dimension were introduced by Sobolev [1] for polyharmonic equations. Such problems arise in mechanics when studying oscillations of bars and plates with hinged fastening. For arbitrary pseudodifferential operators, the theory of boundary value problems with conditions on submanifolds of arbitrary dimensions was developed by Sternin and his co-authors (see [2–5]). Such problems are stated in terms of boundary operators (restriction to submanifold) and dual coboundary operators. In local coordinates, the coboundary operator takes functions on a submanifold to distributions on the ambient manifold obtained by multiplying by the Dirac δ-function in the directions normal to the submanifold. Unlike classical boundary value problems, the number of boundary conditions in Sobolev problems depends on the smoothness exponent of the Sobolev space, in which the problem is considered.

Similar problems arise for the Schrödinger equation with the potential containing δ-functions supported on some submanifolds (see [6–14]). From the spectral theory point of view, in such problems, one has to describe self-adjoint extensions of second-order operators with domains consisting of functions vanishing at a submanifold. The domains of self-adjoint extensions in these problems are effectively described in terms of coboundary operators (see [15,16]).

Although self-adjoint extensions for second-order operators are considered in many papers, operators of arbitrary order have been studied much less. In [17,18], results on the completeness of eigenfunctions and asymptotics of the counting function of eigenvalues for some self-adjoint Sobolev problems are stated without proof for elliptic differential operators of arbitrary order. Our aim in this work is to study one particular self-adjoint extension, namely, the Friedrichs extension [19], for operators or arbitrary order with conditions on submanifolds of arbitrary dimensions. We note that the Friedrichs extension is a canonical extension for semibounded operators, and it is one of the most studied extensions for differential operators in geometry (e.g., see [20,21]) with numerous applications (e.g., see [22]).

Citation: Savin, A. The Friedrichs Extension of Elliptic Operators with Conditions on Submanifolds of Arbitrary Dimension. *Mathematics* **2024**, *12*, 418. https://doi.org/10.3390/math12030418

Academic Editor: Denis Borisov

Received: 23 December 2023
Revised: 22 January 2024
Accepted: 23 January 2024
Published: 27 January 2024

On a compact, smooth, closed manifold, we consider an elliptic symmetric nonnegative operator with the domain consisting of functions vanishing to a certain order on a given submanifold. For this operator, a self-adjoint Friedrichs extension is defined, and the main result of the work is an explicit description of the Friedrichs extension. It turns out that this extension is a Sobolev problem, i.e., it contains boundary and coboundary operators. We show that the spectrum of the Friedrichs extension, in this case, is discrete and study the smoothness of the eigenfunctions. Note that our results agree with the model case considered in [23] of the biharmonic operator on the 3-torus, with the domain consisting of functions vanishing on a circle. We expect our results to have applications in hyperbolic problems (e.g., wave equation for the biharmonic operator with conditions on a submanifold) and spectral invariants and asymptotics for pairs (manifold, submanifold).

The author is grateful to the referees for their useful remarks.

2. Sobolev–Dirichlet Operators

Let us recall basic definitions from relative elliptic theory; see [2–5].

Consider the pair (M, X) of closed smooth manifolds with the codimension of the submanifold X denoted by $\nu \geq 1$ and the embedding $X \subset M$ denoted by i.

Denote by $N \to X$ the normal bundle of X and by N^* its dual. We suppose in this paper that the normal bundle is trivial and denote $y_1, \dots, y_\nu$ as some normal coordinate system in a neighborhood of X such that X is defined by the equations $\{y_1 = 0, \dots, y_\nu = 0\}$. As local coordinates on M in a neighborhood of $m \in X$, we use $(x_1, \dots, x_{n-\nu}, y_1, \dots, y_\nu)$, where $(x_1, \dots, x_{n-\nu})$ stand for local coordinates on X.

Given integer $l \geq 0$, the *Sobolev–Dirichlet boundary operator* is defined as

$$i_l^* : H^s(M) \longrightarrow \bigoplus_{|j| \leqslant l} H^{s-\nu/2-|j|}(X, \mathbb{C}^{n_j}) \equiv \mathcal{H}_l^{s-\nu/2}(X) \tag{1}$$

$$i_l^* f = \left\{ \frac{\partial^{|j|} f}{\partial y_j}\Big|_X \right\}_{|j| \leqslant l},$$

and it takes functions on the ambient manifold to the restriction on X of their jets in the normal variables. Here, $j = (j_1, \dots, j_\nu)$ is a multiindex, $|j| = j_1 + \dots + j_\nu$, while n_j stands for the dimension of the space of homogeneous symmetric polynomials of degree $|j|$ in ν variables. Finally, H^s stands for Sobolev spaces with smoothness exponent s on the corresponding manifolds. The operator (1) is well-defined and bounded provided that $s - \nu/2 - l > 0$.

Sobolev–Dirichlet coboundary operator

$$i_*^l : \mathcal{H}_l^{-(s-\nu/2)}(X) \longrightarrow H^{-s}(M) \tag{2}$$

is defined by duality. Namely, the operators (1) and (2) are duals of each other

$$\langle i_l^* u, v \rangle = \langle u, i_*^l v \rangle, \quad \text{for all } u \in C^\infty(M), v \in \mathcal{H}_l^{-(s-\nu/2)}(X)$$

with respect to the pairing $\langle \ , \ \rangle$ of smooth functions and distributions. In the special case, when $M = \mathbb{R}^n$ and $X = \mathbb{R}^{n-\nu} \times \{0\}$, the operator (2) is equal to

$$i_*^l \left(\{w_j\}_{|j| \leq l} \right) = \sum_{|j| \leq l} (-1)^{|j|} w_j(x) \frac{\partial^{|j|}}{\partial y_j} \delta(y).$$

The following proposition is a generalization of Theorems 3.2.4 and 4.3.1 from [24] (cf. [25] Theorem 2.3.5) for Sobolev spaces on manifolds.

Proposition 1.

1. *The boundary operator* (1) *is surjective. Moreover, it has a continuous right inverse operator, which is independent of s.*
2. *The coboundary operator* (2) *is injective. Moreover, it has a left inverse operator, which is independent of s.*
3. *The range of the coboundary operator* (2) *for* $l = [s - \nu/2]$ *is equal to the subspace of distributions in* $H^{-s}(M)$ *with support in* $X \subset M$*. Here,* $[x] \in \mathbb{Z}$ *is the maximal integer, which is less than* x*.*

The proof repeats the proof from the cited monograph, and we omit it here.

3. Sobolev–Dirichlet Problem

We fix a volume form on M and consider the corresponding Hilbert space $L^2(M)$.

Let $A : C^\infty(M) \to C^\infty(M)$ be an elliptic symmetric positive definite pseudodifferential operator of order $d > 0$ on M. Below, we also consider the standard action of A on distributions and denote this action by $\mathcal{A}$.

We consider A as an unbounded operator on $L^2(M)$ with the dense domain

$$\mathcal{D}(A) = \{u \in C^\infty(M) \mid u|_X = 0\} \equiv C_0^\infty(M, X).$$

Our aim in this paper is to describe the Friedrichs extension of this operator. To this end, we first describe the adjoint operator A^* and its domain $\mathcal{D}(A^*)$.

Theorem 1. *One has*

$$\mathcal{D}(A^*) = \begin{cases} H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X), & \text{if } d > \nu/2, \\ H^d(M), & \text{if } d \le \nu/2, \end{cases} \tag{3}$$

where for $d \le \nu/2$*, we have* $A^* u = \mathcal{A} u$ *for all* $u \in H^d(M)$*, while for* $d > \nu/2$*, we have*

$$A^*(u + v) = \mathcal{A} u, \quad \text{for all } u \in H^d(M), v \in \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X).$$

Proof. 1. Consider the case when $d \le \nu/2$. Given $u \in \mathcal{D}(A^*)$, we have

$$(Av, u) = (v, A^* u) \qquad \text{for all } v \in \mathcal{D}(A),$$

where $(\ ,\ )$ denotes the inner product in $L^2(M)$. Since A is symmetric by assumption, we obtain

$$\langle \overline{v}, \mathcal{A} u - A^* u \rangle = 0, \tag{4}$$

where $\langle\ ,\ \rangle$, as above, denotes the pairing of smooth functions and distributions, while

$$\mathcal{A} u \in H^{-d}(M), \quad u, A^* u \in L^2(M), \quad v \in \mathcal{D}(A).$$

Hence, (4) implies that the difference $\mathcal{A} u - A^* u \in H^{-d}(M)$ is a distribution supported on X. It follows from Proposition 1 above that such distributions are equal to zero, whenever $d \le \nu/2$. Hence, we obtain the desired equality $A^* u = \mathcal{A} u$ and also $u = \mathcal{A}^{-1}(A^* u) \in H^d(M)$. Thus, we proved that $\mathcal{D}(A^*) \subset H^d(M)$ and $A^* = \mathcal{A}$ on this domain. Let us prove the converse inclusion $H^d(M) \subset \mathcal{D}(A^*)$. Indeed, given $u \in H^d(M)$, we have

$$(Av, u) = \langle \overline{Av}, u \rangle = \langle \overline{v}, \mathcal{A} u \rangle = (v, \mathcal{A} u), \qquad \forall v \in \mathcal{D}(A), \tag{5}$$

where we first expressed the inner product as the pairing with the corresponding distributions, then used the definition of action of pseudodifferential operators on distributions, and in the last equality, noted that $\mathcal{A} u \in L^2(M)$ and expressed the pairing in terms of the inner product in L^2. Now, (5) gives the desired facts: $u \in H^d(M)$ and $A^*u = \mathcal{A}u$. This gives the proof of Theorem 1 in the special case $d \le \nu/2$.

2. Now suppose that $d > \nu/2$. Let $u \in \mathcal{D}(A^*)$. Similar to the previous case, we conclude that the difference $\mathcal{A}u - A^*u \in H^{-d}(M)$ is a distribution supported on X. Then Proposition 1 implies that this difference is in the range of some coboundary operator

$$\mathcal{A}u - A^*u = i_*^l(\varphi), \qquad l = [d - \nu/2],$$

where $\varphi = (\varphi_0, ..., \varphi_l) \in \mathcal{H}_l^{-d+\nu/2}(X)$ is a distribution on X. Substituting this in Equation (4), we obtain

$$0 = \langle \overline{v}, \mathcal{A}u - A^*u \rangle = \langle \overline{v}, i_*^l \varphi \rangle = \langle \overline{i_l^*(v)}, \varphi \rangle. \tag{6}$$

Since i_l^* is surjective and $i_0^* v = 0$ for all $v \in \mathcal{D}(A)$, it follows that φ_j in (6) is zero whenever $j \ge 1$. Hence

$$A^*u = \mathcal{A}u - i_*^0 \varphi_0, \quad \varphi_0 \in H^{-d+\nu/2}(X).$$

This shows that u lies in the desired subspace (see (3))

$$u = \mathcal{A}^{-1}(A^*u) + \mathcal{A}^{-1} i_*^0 \varphi_0 \in H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X).$$

It remains to prove the inclusions $H^d(M), \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X) \subset \mathcal{D}(A^*)$. Clearly, one has $H^d(M) \subset \mathcal{D}(A^*)$ and $A^*u = \mathcal{A}u$ for all $u \in H^d(M)$. Let us now show that

$$\mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X) \subset \mathcal{D}(A^*) \tag{7}$$

and this subspace is in the kernel of A^*. Indeed, given $u \in \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X)$, we conclude that $\mathcal{A}u = i_*^0 \varphi$ for some $\varphi \in H^{-d+\nu/2}(X)$. Note also that $u \in L^2(M)$. We have to show that $u \in \mathcal{D}(A^*)$ and $A^*u = 0$. By the definition of the adjoint operator, this is equivalent to proving the equality

$$(Av, u) = 0 \qquad \text{for all } v \in \mathcal{D}(A).$$

Indeed, we have

$$(Av, u) = \langle \overline{v}, \mathcal{A}u \rangle = \langle \overline{v}, i_*^0 \varphi \rangle = \langle \overline{i_0^* v}, \varphi \rangle = \langle 0, \varphi \rangle = 0.$$

Here we used duality between boundary and coboundary operators and the fact that A is symmetric. Hence, we obtain (7).

The proof of Theorem 1 is now complete. □

Denote by A_F the Friedrichs extension of A. Recall that (e.g., see [26–28])

$$\mathcal{D}(A_F) = H_1 \cap \mathcal{D}(A^*), \tag{8}$$

where H_1 is the completion of $\mathcal{D}(A)$ with respect to the norm

$$\|u\|_1 = (Au, u)^{1/2}.$$

Since A is elliptic and positive definite, the norm $\|\cdot\|_1$ is equivalent to the norm in $H^{d/2}(M)$.

Theorem 2. *We have*

$$\mathcal{D}(A_F) = \begin{cases} (H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d/2+\nu/2}(X)) \cap \ker i_0^*, & \text{if } d > \nu, \\ H^d(M), & \text{if } d \le \nu, \end{cases} \tag{9}$$

$$A_F = A^*|_{\mathcal{D}(A_F)}.$$

Proof. Let us describe the space H_1 in (8).

Lemma 1. *We have*

$$H_1 = \begin{cases} H^{d/2}(M) \cap \ker i_0^* & \text{if } d > \nu, \\ H^{d/2}(M) & \text{if } d \le \nu. \end{cases} \quad (10)$$

Proof. 1. Consider the case $d \le \nu$. To prove (10), it suffices (cf. [24] Theorem 5.1.14) to show that $\mathcal{D}(A) = C_0^\infty(M, X)$ is dense in $C^\infty(M)$ with respect to the norm in $H^{d/2}(M)$. By locality, it suffices to prove this statement in the special case $M = \mathbb{T}^n$, $X = \mathbb{T}^{n-\nu}$, where $\mathbb{T}^k = (\mathbb{S}^1)^k$ is the k-dimensional torus.

First, for all $\varepsilon > 0$, we construct smooth functions $\chi = \chi_\varepsilon \in C^\infty(\mathbb{T}^\nu)$ with the properties

$$\chi(0) = 1 \qquad \text{and} \qquad \|\chi\|_{H^{\nu/2}(\mathbb{T}^\nu)} < \varepsilon. \quad (11)$$

We define the desired function as a Fourier series

$$\chi(y) = \sum_{n \in \mathbb{Z}^\nu} u_n e^{iny}.$$

We set $u_0 = 1 - \sum\limits_{n \neq 0} u_n$ (hence, condition $\chi(0) = 1$ will be satisfied automatically) and define u_n, $n \neq 0$, as

$$u_n = \begin{cases} \dfrac{\delta}{|n|^\nu \ln(|n|+1)} & \text{if } 1 \le |n| \le N \\ 0 & \text{if } |n| > N. \end{cases}$$

Note that the series $\sum_n (|n|^\nu \ln(|n|+1))^{-1}$ is divergent, while the sums over $n : |n| = k$ enjoy the estimate

$$\sum_{|n|=k} (|n|^\nu \ln(|n|+1))^{-1} = C^k_{\nu+k-1} (k^\nu \ln(k+1))^{-1} \le C(k \ln(k+1))^{-1}.$$

Here and below C denotes some constants, while C_n^k denotes binomial coefficients. These two facts imply that there exists $N = N(\delta)$ such that

$$1 - \delta < \sum_{1 \le |n| \le N} u_n < 1 + \delta.$$

Using this choice of u_n, we obtain the estimates

$$\|\chi\|^2_{H^{\nu/2}(\mathbb{T}^\nu)} = |u_0|^2 + \sum_{0<|n|\le N} |u_n|^2 |n|^\nu \le C\delta^2 + \delta^2 \sum_{0<|n|} |n|^{\nu-2\nu} \frac{1}{\ln^2(|n|+1)} \le$$
$$\le C\delta^2 \sum_{k \ge 2} |k|^{\nu-2\nu+\nu-1} \frac{1}{\ln^2 k} \le C\delta^2$$

and if we take δ small enough, then the last expression will be $\le \varepsilon^2$. Thus, we constructed function χ with properties (11).

Let us now take $u \in C^\infty(\mathbb{T}^n)$ and set $w(x,y) = u(x,y) - u(x,0)\chi_\varepsilon(y)$. Then, we have $w \in C_0^\infty(M, X)$, and we can estimate the norm of the difference $u - w$ as follows

$$\|u - w\|_{H^{\nu/2}(\mathbb{T}^n)} = \|u(x,0)\chi_\varepsilon(y)\|_{H^{\nu/2}(\mathbb{T}^n)} \le C\|\chi_\varepsilon(y)\|_{H^{\nu/2}(\mathbb{T}^n)} \le C\|\chi_\varepsilon(y)\|_{H^{\nu/2}(\mathbb{T}^\nu)} \le C\varepsilon. \quad (12)$$

Estimate (12) implies that $C_0^\infty(M, X) \subset C^\infty(M)$ is dense with respect to the $H^s(M)$-norm whenever $s \le \nu/2$.

2. Let now $d > \nu$. To prove (10), it suffices to show that

$$C_0^\infty(M, X) \subset H^{d/2}(M) \cap \ker i_0^*$$

is dense with respect to the $H^{d/2}(M)$-norm. Indeed, if $u \in H^{d/2}(M) \cap \ker i_0^*$, then there exists a sequence $u_n \in C^\infty(M)$ with the properties

$$u_n \longrightarrow u \text{ in } H^{d/2}(M)\text{-norm}, \quad \text{while} \quad i_0^* u_n \longrightarrow 0 \text{ in } H^{d/2-\nu/2}(X)\text{-norm}. \tag{13}$$

Let $\beta : H^{d/2-\nu/2}(X) \to H^{d/2}(M)$ denote the right inverse operator to operator i_0^* in Proposition 1 (i.e., $i_0^*\beta = \mathrm{Id}$).

Consider the sequence

$$w_n = u_n - \beta i_0^* u_n.$$

Then $w_n \in C_0^\infty(M, X)$, since $i_0^* w_n = i_0^* u_n - i_0^* \beta i_0^* u_n = i_0^* u_n - i_0^* u_n = 0$. We have the following estimates

$$\|w_n - u\|_{H^{d/2}(M)} \le \|u_n - u\|_{H^{d/2}(M)} + \|\beta i_0^* u_n\|_{H^{d/2}(M)} \le \|u_n - u\|_{H^{d/2}(M)} + C\|i_0^* u_n\|_{H^{d/2-\nu/2}(X)}.$$

This and Equation (13) give the desired statement $w_n \to u$ in $H^{d/2}(M)$ as $n \to \infty$.

The proof of Lemma 1 is now complete. □

Theorem 1 and Lemma 1 (see Equations (3), (8), (10)) give the desired Equation (9). Indeed, we have

(1) if $d \le \nu/2$, then we have $H_1 \cap \mathcal{D}(A^*) = H^{d/2}(M) \cap H^d(M) = H^d(M)$;

(2) if $\nu/2 < d \le \nu$, then we have

$$\begin{aligned} H_1 \cap \mathcal{D}(A^*) &= H^{d/2}(M) \cap (H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X)) \\ &= H^d(M) + \left(H^{d/2}(M) \cap \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X)\right) \\ &= H^d(M) + \mathcal{A}^{-1}\left(H^{-d/2}(M) \cap i_*^0 H^{-d+\nu/2}(X)\right) \\ &= H^d(M), \end{aligned}$$

where $H^{-d/2}(M) \cap i_*^0 H^{-d+\nu/2}(X) = 0$, since nonzero distributions in $i_*^0 H^{-d+\nu/2}(X)$ are not in $H^{-\nu/2}(M)$ (hence, also not in its subset $H^{-d/2}(M)$);

(3) if $d > \nu$, we have

$$\begin{aligned} H_1 \cap \mathcal{D}(A^*) &= H^{d/2}(M) \cap \ker i_0^* \cap (H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d+\nu/2}(X)) \\ &= \ker i_0^* \cap (H^d(M) + \mathcal{A}^{-1}(H^{-d/2}(M) \cap i_*^0 H^{-d+\nu/2}(X))) \\ &= \ker i_0^* \cap \left(H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d/2+\nu/2}(X)\right). \end{aligned}$$

where we have $H^{-d/2}(M) \cap i_*^0 H^{-d+\nu/2}(X) = i_*^0 H^{-d/2+\nu/2}(X)$ by Proposition 1.

This gives the desired Equation (9) and completes the proof of Theorem 2. □

Remark 1. *The statement of Theorem 2 remains true if instead of positive definiteness of A we require that A is only semibounded from below. In this case, one has to replace $\mathcal{A}^{-1}$ by an arbitrary pseudodifferential parametrix $\mathcal{R}$, $\mathrm{ord} R = -d$, of $\mathcal{A}$ such that $\mathcal{R}\mathcal{A} - 1$ is of order $\le -d/2$. We claim that the domain of the Friedrichs extension does not depend on the choice of such parametrix. Indeed, we have $\mathcal{R} = \mathcal{A}^{-1} + \mathcal{C}$, where $\mathcal{C}$ is a pseudodifferential operator of order $\le -3d/2$. This implies $\mathcal{C} i_*^0 H^{-d/2+\nu/2}(X) \subset H^{-d/2-\mathrm{ord}\mathcal{C}}(M) \subset H^d(M)$. Hence, we obtain*

$$H^d(M) + \mathcal{R} i_*^0 H^{-d/2+\nu/2}(X) = H^d(M) + \mathcal{A}^{-1} i_*^0 H^{-d/2+\nu/2}(X).$$

This equality implies the desired independence of $\mathcal{D}(A_F)$ of the choice of parametrix.

Remark 2. *We note that the above results can be generalized to the case of Sobolev–Dirichlet problems with jets of order $l \geq 1$ on the submanifold. More precisely, consider the operator A as above with the domain*

$$\mathcal{D}(A) = \ker i_l^* \subset C^\infty(M). \tag{14}$$

Then, one can describe the adjoint operator and the Friedrichs extension as follows

$$\mathcal{D}(A^*) = H^d(M) + \mathcal{A}^{-1} i_*^{l'}(\mathcal{H}_{l'}^{-d+\nu/2}(X)), \quad \textit{where } l' = \min(l, [d-\nu/2]),$$

$$A^*(u+v) = \mathcal{A}u.$$

$$\mathcal{D}(A_F) = \left(H^d(M) + \mathcal{A}^{-1} i_*^{l''}(\mathcal{H}_{l''}^{-d/2+\nu/2}(X))\right) \cap \ker i_{l''}^* \quad \textit{where } l'' = \min(l, [d/2-\nu/2])$$

$$A_F = A^*|_{\mathcal{D}(A_F)}.$$

The proof of these formulas is similar to the proof of Theorems 1 and 2 above and is omitted.

4. Examples

1. Consider operator $A = \partial_x^4 : C^\infty(\mathbb{S}^1) \to C^\infty(\mathbb{S}^1)$ with the domain $\mathcal{D}(A) = \{u(x) \mid u(0) = 0\}$. Then, we have $d = 4 > 1 = \nu$ and we use Theorem 2 to describe the domain of the Friedrichs extension of A:

$$\mathcal{D}(A_F) = H^4(\mathbb{S}^1) + \partial_x^{-4}\mathbb{C}\delta(x) \simeq H^4[0,2\pi] \cap \{u(0) = u(2\pi), u'(0) = u'(2\pi), u''(0) = u''(2\pi)\},$$

while $A_F u = u^{IV}$. Here, we used the fact that $\partial_x^{-4}\delta(x) = |x|^3/12$ modulo smooth functions.

Similar boundary conditions arise when studying oscillations of bars with hinged fastening at $x = 0$.

2. Consider the biharmonic operator $A = \Delta^2 : C^\infty(\mathbb{T}^3) \to C^\infty(\mathbb{T}^3)$, where $\Delta = \partial_{x_1}^2 + \partial_{y_1}^2 + \partial_{y_2}^2$ with the domain $\mathcal{D}(A) = C_0^\infty(\mathbb{T}^3, \mathbb{T}^1) = \{u(x_1, y_1, y_2) \mid u(x_1, 0, 0) = 0\}$. Then, we have $d = 4 > 2 = \nu$, and we use Theorem 2 to describe the domain of the Friedrichs extension of A:

$$\mathcal{D}(A_F) \simeq \{u \in H^2(\mathbb{T}^3) \mid \Delta^2 u \in L^2(\mathbb{T}^3) + i_*^0 H^{-1}(\mathbb{T}^1), u|_{\mathbb{T}^1} = 0\}$$

while $A_F u = \Delta^2 u \bmod \mathbb{T}^1$. This extension was considered in [23]. We note that the domain of the Friedrichs extension will not change if we perturb Δ^2 by an operator of order ≤ 2.

3. Note that the Friedrichs extension of the Laplacian $A = \Delta : C^\infty(M) \to C^\infty(M)$ with the domain $\mathcal{D}(A) = \{u \mid u|_X = 0\}$ recovers the Laplacian on distributions with the domain $\mathcal{D}(A_F) = H^2(M)$ whenever $\nu \geq 2$. In the codimension 1 case, the Friedrichs extension is isomorphic to the Dirichlet problem for the Laplacian on the smooth manifold $\overline{M}$ obtained by cutting M along X. This manifold has boundary $\partial\overline{M} \simeq X \cup X$.

5. Properties of Eigenvalues and Eigenfunctions

Theorem 3. *The spectrum of the Friedrichs extension A_F in Theorem 2 is discrete and consists of real eigenvalues with finite multiplicities, which tend to $+\infty$. If $u \in \mathcal{D}(A_F)$ is an eigenfunction of A_F with eigenvalue λ, then $u \in (\mathcal{A} - \lambda)^{-1} i_*^0 C^\infty(X)$ modulo smooth functions, where $(\mathcal{A} - \lambda)^{-1}$ stands for a parametrix of elliptic operator $\mathcal{A} - \lambda$ modulo smoothing operators. In particular, u is smooth on $M \setminus X$.*

Proof. We consider the case $d > \nu$.

1. Let us prove the discreteness of the spectrum. To this end, we compute the inverse operator $A_F^{-1} : L^2(M) \to L^2(M)$. A direct computation shows that the equation $A_F u = f$, $u \in \mathcal{D}(A_F)$, $f \in L^2(M)$ is equivalent to the system

$$\begin{cases} u = v + \mathcal{A}^{-1} i_*^0 w, \quad u \in H^d(M), w \in H^{-d/2+\nu/2}(X), \\ \mathcal{A}v = f, \\ i_0^*(v + \mathcal{A}^{-1} i_*^0 w) = 0. \end{cases}$$

Then $v = \mathcal{A}^{-1} f$, $w = -\Delta^{-1} i_0^* v$, where

$$\Delta = i_0^* \mathcal{A}^{-1} i_*^0 : H^{-d/2+\nu/2}(X) \longrightarrow H^{d/2-\nu/2}(X)$$

is a pseudodifferential operator of order $-d+\nu$ on X (see [2]). We claim that this operator is elliptic and positive definite. Indeed, its principal symbol is equal to

$$\sigma(i_0^* \mathcal{A}^{-1} i_*^0)(x,\xi) = \int_{\mathbb{R}^\nu} \sigma(A)(x,0,\xi,\eta)^{-1} d\eta.$$

Here the integral is absolutely convergent since the integrand is $O(|\eta|^{-d})$. The integral is a positive function since the integrand enjoys this property for all η. Similarly, one can prove the positive definiteness of the operator $i_0^* \mathcal{A}^{-1} i_*^0$.

Hence, we obtain the following expression for the inverse operator

$$u = (A_F)^{-1} f = (1 - \mathcal{A}^{-1} i_*^0 \Delta^{-1} i_0^*) \mathcal{A}^{-1} f.$$

It follows that the self-adjoint operator $(A_F)^{-1}$ has Sobolev order $-d$. Hence, it is compact as an operator in L^2. Thus, by the spectral theorem for compact self-adjoint operators $(A_F)^{-1}$ has a discrete spectrum with eigenvalues of finite multiplicities, and the eigenvalues tend to zero. This gives the desired properties for the eigenvalues of A_F.

2. Note that $u \in \mathcal{D}(A_F)$ is an eigenfunction of A_F with eigenvalue λ if and only if $(\mathcal{A}-\lambda)u = i_*^0 w$, $i_0^* u = 0$, $w \in H^{-d/2+\nu/2}(X)$. These conditions are equivalent to

$$u = u_0 + (\mathcal{A}-\lambda)_0^{-1} i_*^0 w, \quad i_0^* u_0 + i_0^* (\mathcal{A}-\lambda)_0^{-1} i_*^0 w = 0, \quad u_0 \in \ker(\mathcal{A}-\lambda),$$

where $(\mathcal{A}-\lambda)_0^{-1}$ is the inverse operator on the orthogonal complement of the kernel. Since $i_0^*(\mathcal{A}-\lambda)_0^{-1} i_*^0$ is an elliptic operator on X, all the solutions w are smooth. Thus, we obtain the desired property:

$$u = u_0 + (\mathcal{A}-\lambda)_0^{-1} i_*^0 w \in C^\infty(M) + (\mathcal{A}-\lambda)_0^{-1} i_*^0 C^\infty(X).$$

The proof of Theorem 3 is now complete. □

6. Conclusions

In this paper, we described the Friedrichs extension of elliptic symmetric operators defined on subspaces of functions vanishing on a submanifold of arbitrary dimension and showed that this extension is defined by boundary and coboundary operators. We showed that the spectrum of the Friedrichs extension is discrete and described the singularities of the eigenfunctions on the submanifold. This work paves the way for more detailed studies of spectral properties of such problems (e.g., spectral asymptotics and their relation with the geometry of the submanifold) and corresponding nonstationary problems. Let us also mention that there is an interesting problem of describing extensions of operators with general boundary conditions on the submanifold.

Funding: This research was funded by RFBR grant number 19-01-00574.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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