mathematics

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A Study of Some New Hermite–Hadamard Inequalities via Specific Convex Functions with Applications

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Abstract: Convexity plays a crucial role in the development of fractional integral inequalities. Many fractional integral inequalities are derived based on convexity properties and techniques. These inequalities have several applications in different fields such as optimization, mathematical modeling and signal processing. The main goal of this article is to establish a novel and generalized identity for the Caputo–Fabrizio fractional operator. With the help of this specific developed identity, we derive new fractional integral inequalities via exponential convex functions. Furthermore, we give an application to some special means.

Keywords: exponential convex function; fractional integrals; Hölder's inequality; power-mean inequality

MSC: 26A33; 26D07; 26D10; 26D15

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Citation: Junjua, M.-u.-D.; Qayyum, A.; Munir, A.; Budak, H.; Saleem, M.M.; Supadi, S.S. A Study of Some New Hermite–Hadamard Inequalities via Specific Convex Functions with Applications. *Mathematics* **2024**, *12*, 478. https://doi.org/10.3390/math12030478

Received: 14 November 2023
Revised: 10 December 2023
Accepted: 12 December 2023
Published: 2 February 2024

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1. Introduction

Fractional calculus has gained significant recognition and application in various areas of mathematics. These contemporary developments in fractional calculus reflect the growing interest in exploring fractional derivatives and integrals to address complex problems in various fields. Fractional derivatives have become a powerful mathematical tool for researchers, enabling them to formulate more accurate models [1–3]. As a result, researchers across various scientific and engineering disciplines continue to rely on both Caputo and Riemann Liouville fractional derivatives to better understand and model complex phenomena (see [4]). The use of innovative fractional general operators of distinct, local, and nonlocal kernels has also been studied by other authors [5].

Moreover, fractional integrals have found practical applications across a diverse range engineering and science fields, such as electromagnetic studies, photoelasticity, fluid mechanics, electrochemistry, biological population modeling, optics, and signal processing. On the other side, the theory of convexity has been proven to have many potential applications in a wide range of research fields including, coding theory, machine learning, and data science. Convex mapping is arguably the most fundamental and significant mapping method in the theory of mathematical inequality since it has vast applications in mechanics [6], statistics [7], pure and applied mathematics [8], and economics [9]. The well-known definition of a convex function [10] is given below.

$$\hat{G}(\Lambda\psi + (1-\Lambda)\pi) \leq \Lambda\hat{G}(\psi) + (1-\Lambda)\hat{G}(\pi),$$

for all $\psi, \pi \in I$ and $\Lambda \in [0,1]$.

Theorem 1 ([11])**.** *Suppose the function $\hat{G} : I \subset \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on I, where $\psi, \pi \in I$ with $\psi < \pi$, then*

$$\hat{G}\left(\frac{\psi+\pi}{2}\right) \leq \frac{1}{\pi-\psi}\int_{\psi}^{\pi} \hat{G}(x)dx \leq \frac{\hat{G}(\psi)+\hat{G}(\pi)}{2}.$$

This inequality, known as the Hermite–Hadamard inequality, was established by C. Hermite and J. Hadamard. For more information concerning the Hermite–Hadamard inequality, see [12]. Hermite–Hadamard and trapezoidal inequalities have recently been established by Sarikaya et al. [13] using Riemann–Liouville fractional integrals. Several authors have used different classes of function to generalizes the Hermite–Hadmard-type inequality. Additionally, a number of mathematicians have developed Hermite–Hadmard-type inequality inequalities for differentiable convex mappings [14], s-convex functions [15], m-convex mappings [16], and Green's functions [17]. Numerous scholars have presented applications for fractional operators, see, for example [18,19]. Kadakal and Iscan gave a new description for the exponential-type convex function [20]. Additionally, they again demonstrated the Hermite–Hadamard inequalities in [20] using the definition given below.

Definition 1 ([20])**.** *$\hat{G} : I \to \mathbb{R}$ is said to be an exponential convex function if the inequality*

$$\hat{G}(\Lambda\psi + (1-\Lambda)\pi) \leq \left(e^{\Lambda}-1\right)\hat{G}(\psi) + \left(e^{1-\Lambda}-1\right)\hat{G}(\pi),$$

holds for all $\psi, \pi \in I$ and $\Lambda \in [0,1]$.

Definition 2 ([21])**.** *The function $\hat{G} : I \subseteq R \to R_0 = [0,\infty)$ is said to be an s-convex function if the inequality*

$$\hat{G}(\Lambda\psi + (1-\Lambda)\pi) \leq \Lambda^s\hat{G}(\psi) + (1-\Lambda)^s\hat{G}(\pi),$$

holds for all $\psi, \pi \in I$, $s \in (0,1]$ and $\Lambda \in [0,1]$.

Theorem 2. *The function $\hat{G} : [\psi, \pi] \to \mathbb{R}$ is an exponential function. If $\psi < \pi$ and $\hat{G} \in L[\psi, \pi]$, then*

$$\frac{1}{(2\sqrt{e}-1)}\hat{G}\left(\frac{\psi+\pi}{2}\right) \leq \frac{1}{\pi-\psi}\int_{\psi}^{\pi} \hat{G}(x)dx \leq (e-2)\big(\hat{G}(\psi)+\hat{G}(\pi)\big).$$

Hölder's inequality is indeed a fundamental inequality in mathematics, particularly in the context of L_p spaces. İşcan introduced a new form of Hölder's inequality in [22] which is given below. Suppose $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If $\hat{G}$ and g are real functions defined on $[\psi, \pi]$ and if $|\hat{G}|^p$ and $|g|^q$ are integrable on $[\psi, \pi]$, we have

$$\begin{aligned}\int_{\psi}^{\pi}|\hat{G}(x)g(x)|dx \quad &\leq \quad \frac{1}{\pi-\psi}\Bigg[\left(\int_{\psi}^{\pi}(\pi-x)|\hat{G}(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\psi}^{\pi}(\pi-x)|g(x)|^q dx\right)^{\frac{1}{q}} \\ &\quad + \left(\int_{\psi}^{\pi}(x-\psi)|\hat{G}(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\psi}^{\pi}(x-\psi)|g(x)|^q dx\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

The development of the Hölder–İşcan inequality and its use in obtaining better upper bounds demonstrates the ongoing progress and innovation in mathematical inequalities.

Different authors have used fractional operators to generalize the Hermite–Hadamard inequality. In this context, we confine our focus to the Caputo–Fabrizio fractional integral operator. The distinguishing factors between these operators lie in their singularities and local characteristics, with the operator's kernel expression involving functions such as the power law, the exponential function, or a Mittag–Leffler function. Notably, the Caputo–Fabrizio operator stands out due to its kernel-lacking singularity. The key characteristic of this operator is best described as a real power transformed into an integer through the Laplace transformation, thereby facilitating the straightforward derivation of exact solutions for various problems. Xiaobin Wang et al. [23] presented the Hermite–Hadmard-type inequality for modified h-convex functions utilizing a Caputo–Fabrizio integral operator. Butt et al. [24] exponentially obtained different inequalities for s- and (s,m)-convex functions using Caputo fractional integrals and derivatives. Furthermore, Abbasi et al. [25] constructed new variants of the Hermite–Hadamard-type inequalities for s-convex functions via a Caputo–Fabrizio integral operator. Li et al. [26] proved analogous inequalities for strongly convex functions. In 2015, Caputo and Fabrizio introduced the Caputo–Fabrizio fractional operator as follows:

Definition 3 ([27])**.** *Let $H^1(\psi,\pi)$ be the Sobolev space of order one defined as*

$$H^1(\psi,\pi) = \left\{\hat{G} \in L^2(\psi,\pi) : \hat{G}' \in L^2(\psi,\pi)\right\},$$

where

$$L^2(\psi,\pi) = \left\{\hat{G}(z) : \left(\int_\psi^\pi \hat{G}^2(z)dz\right)^{\frac{1}{2}} < \infty\right\}.$$

Let $\hat{G} \in H^1(\psi,\pi)$, where $\psi < \pi$ and $\alpha \in [0,1]$; the nth notion of left derivative in the sense of Caputo–Fabrizio is defined as

$$\left({}^{CFD}_{\psi}D^\alpha \hat{G}\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_\psi^x \hat{G}'(\Lambda)e^{\frac{-\alpha(x-\Lambda)^\alpha}{1-\alpha}}d\Lambda,$$

$x > \alpha$ and the associated integral operator is

$$\left({}^{CF}_{\psi}I^\alpha \hat{G}\right)(x) = \frac{1-\alpha}{\beta(\alpha)}\hat{G}(x) + \frac{\alpha}{\beta(\alpha)}\int_\psi^x \hat{G}(\Lambda)d\Lambda,$$

where $\beta(\alpha) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\alpha = 0$ and $\alpha = 1$, the left derivative is, respectively, defined as follows

$$\begin{aligned} \left({}^{CFD}_{\psi}D^0 \hat{G}\right)(x) &= \hat{G}'(x) \\ \left({}^{CFD}_{\psi}D^1 \hat{G}\right)(x) &= \hat{G}(x) - \hat{G}(\psi). \end{aligned}$$

For the right derivative operator

$$\left({}^{CFD}_{\pi}D^\alpha \hat{G}\right)(x) = \frac{\beta(\alpha)}{1-\alpha}\int_x^\pi \hat{G}'(\Lambda)e^{\frac{-\alpha(\Lambda-x)^\alpha}{1-\alpha}}d\Lambda,$$

$x < \pi$ and the associated integral operator is

$$\left({}^{CF}I^\alpha_\pi \hat{G}\right)(x) = \frac{1-\alpha}{\beta(\alpha)}\hat{G}(x) + \frac{\alpha}{\beta(\alpha)}\int_x^\pi \hat{G}(\Lambda)d\Lambda.$$

In [10], Nasir et al. presented the following trapezoidal-type inequalities for the Caputo–Fabrizio fractional operator.

Theorem 3. *The function $\hat{G} : [\psi, \pi] \to \mathbb{R}$ is a differentiable function on I, where $\psi, \pi \in I$ with $\psi < \pi$. If $|\hat{G}'|$ is s-convex on $[\psi, \pi]$ and some $s \in (0,1]$, $\Lambda \in [0,1]$, then the following inequality holds*

$$\begin{aligned} &\Bigg| \frac{\hat{G}(\psi) + \hat{G}(\pi)}{2} + \frac{4(1-\alpha)}{\beta(\alpha)} \\ &- \frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left(\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{\psi+\pi}{2}}\hat{G}\right)(k)\right) + \left({}^{CF}_{\frac{\psi+\pi}{2}}I^{\alpha}\hat{G}\right)(k)\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]\Bigg| \\ \leq\ & \frac{\pi-\psi}{4}\left(\frac{2se^{\ln(2)s}+1}{2^s(s+1)(s+2)} + \frac{\beta(s+1,s)}{2^s}\right)\left[|\hat{G}'(\pi)| + |\hat{G}'(\psi)|\right]. \end{aligned}$$

In [28], Sahoo obtained new error bounds for the midpoint-type inequality via the s-convex function given below:

Theorem 4. *The function $\hat{G} : [\psi, \pi] \to \mathbb{R}$ is a differentiable function on I, where $\psi, \pi \in I$ with $\psi < \pi$. If $|\hat{G}'|$ is s-convex on $[\psi, \pi]$ and some $s \in (0,1]$, $\Lambda \in [0,1]$, then*

$$\begin{aligned} &\left|\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\frac{\psi+\pi}{2}-}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\frac{\psi+\pi}{2}+}\hat{G}\right)(k)\right] - \hat{G}\left(\frac{\psi+\pi}{2}\right) - \frac{(1-\alpha)}{\beta(\alpha)}(\hat{G}(\psi)+\hat{G}(\pi))\right| \\ \leq\ & \frac{\pi-\psi}{4}\left(\frac{\hat{G}'(\psi)+\hat{G}'(\pi)}{2}\right). \end{aligned}$$

Motivated by ongoing research in recent years on the generalizations of Hermite–Hadamard-type inequalities for different convex functions and the Caputo–Fabrizio fractional integral operator. We established a new identity for the Caputo-Fabrizio fractional integral operator and the functions whose absolute value of the second derivative is convex. By using this identity, we obtained several new Hermite–Hadmard-type inequalities to derive several new fractional inequalities for exponential convex functions. This paper is structured as follows: In Section 1, we delve into the established definitions and outcomes pertaining to the Caputo–Fabrizio fractional integral. Section 2 introduces novel Hermite–Hadamard-type inequalities concerning the fractional operator. Moving on to Section 3, we explore intriguing applications linked to special means. Lastly, Section 4 encompasses the conclusion along with prospects for future research.

2. Main Results

Here, we first establish a general identity for the famous fractional operator (Caputo–Fabrizio) and then use the following identity which plays a central role to develop new inequalities. It is expected that the obtained inequality in this section will point to novel developments in the field of fractional integrals. In addition, by putting specific values of $\vartheta = 1$ in the auxiliary result, we will obtain a variety of valuable results which were previously obtained.

Lemma 1. *The function $\hat{G} : I^o \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function on I^o, where $\psi, \pi \in I^o$ with $\psi < \pi$ and $\vartheta \in N$. If $\hat{G}'' \in L[\psi, \pi]$ and $\Lambda \in [0,1]$, then*

$$\begin{aligned} &\sum_{i=0}^{\vartheta-1}\frac{1}{2\vartheta}\left[\hat{G}\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right) + \hat{G}\left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right] \\ &- \frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k) \\ =\ & \sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{2\vartheta^3}\left[\int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta} + (1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda\right]. \end{aligned} \tag{1}$$

Proof. Using integration by parts, we obtain

$$\begin{aligned}
\Omega_i &= \int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda \\
&= \frac{\vartheta}{\psi-\pi}\left(\Lambda-\Lambda^2\right)\hat{G}'\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\Big|_0^1 \\
&\quad -\frac{\vartheta}{\psi-\pi}\int_0^1(1-2\Lambda)\hat{G}'\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda \\
&= \frac{\vartheta}{\pi-\psi}\int_0^1(1-2\Lambda)\hat{G}'\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda \\
&= \frac{\vartheta}{\pi-\psi}\left(\frac{\vartheta(1-2\Lambda)}{\psi-\pi}\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\Big|_0^1\right. \\
&\quad \left.+\frac{2\vartheta}{\psi-\pi}\int_0^1\hat{G}'\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda\right) \\
&= \frac{\vartheta^2}{(\pi-\psi)^2}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\frac{2\vartheta^2}{(\pi-\psi)^2}\int_0^1\hat{G}'\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)d\Lambda \\
&= \frac{\vartheta^2}{(\pi-\psi)^2}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right)-\frac{2\vartheta^3}{(\pi-\psi)^3}\int_{\frac{(\vartheta-i)\psi+i\pi}{\vartheta}}^{\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}}\hat{G}(u)du.
\end{aligned} \tag{2}$$

By multiplying by $\frac{(\pi-\psi)^3\alpha}{2\vartheta^3\beta(\alpha)}$ with the equality in (2) and subtracting $\frac{2(1-\alpha)}{\Lambda\beta(\alpha)}\hat{G}(k)$, we obtain

$$\begin{aligned}
&\Omega_i\frac{(\pi-\psi)^3\alpha}{2\vartheta^3\beta(\alpha)}-\frac{2(1-\alpha)}{\Lambda\beta(\alpha)}\hat{G}(k) \\
&= \frac{\vartheta^2}{(\pi-\psi)^2}\frac{(\pi-\psi)^3\alpha}{2\vartheta^3\beta(\alpha)}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\frac{\alpha}{\beta(\alpha)}\int_{\frac{(\vartheta-i)\psi+i\pi}{\vartheta}}^{\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}}\hat{G}(u)du-\frac{2(1-\alpha)}{\Lambda\beta(\alpha)}\hat{G}(k) \\
&= \frac{(\pi-\psi)\alpha}{2\vartheta\beta(\alpha)}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\frac{\alpha}{\beta(\alpha)}\int_{\frac{(\vartheta-i)\psi+i\pi}{\vartheta}}^{\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}}\hat{G}(u)du-\frac{2(1-\alpha)}{\Lambda\beta(\alpha)}\hat{G}(k).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\sum_{i=0}^{\vartheta-1}\Omega_i\frac{(\pi-\psi)^3\alpha}{2\vartheta^3\beta(\alpha)}-\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k) \\
&= \sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)\alpha}{2\vartheta\beta(\alpha)}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\frac{\alpha}{\beta(\alpha)}\int_\psi^\pi\hat{G}(u)du-\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k) \\
&= \sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)\alpha}{2\vartheta\beta(\alpha)}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\left(\frac{\alpha}{\beta(\alpha)}\int_\psi^k\hat{G}(u)du-\frac{(1-\alpha)}{\beta(\alpha)}\hat{G}(k)+\frac{\alpha}{\beta(\alpha)}\int_k^\pi\hat{G}(u)du-\frac{(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right) \\
&= \sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)\alpha}{2\vartheta\beta(\alpha)}\left(\hat{G}\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right) \\
&\quad -\left[\left({}_{\psi}^{CF}I^\alpha\hat{G}\right)(k)+\left({}^{CF}I_\pi^\alpha\hat{G}\right)(k)\right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\sum_{i=0}^{\mathring{\vartheta}-1} \frac{1}{2\mathring{\vartheta}} \left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi + i\pi}{\mathring{\vartheta}} \right) + \hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi + (i+1)\pi}{\mathring{\vartheta}} \right) \right] \\
&- \frac{\beta(\alpha)}{\alpha(\pi-\psi)} \left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G} \right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G} \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)} \hat{G}(k) \\
= &\sum_{i=0}^{\mathring{\vartheta}-1} \frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3} \left[\int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda \frac{(\mathring{\vartheta}-i)\psi + i\pi}{\mathring{\vartheta}} + (1-\Lambda)\frac{(\mathring{\vartheta}-i-1)\psi + (i+1)\pi}{\mathring{\vartheta}} \right) d\Lambda \right].
\end{aligned}$$

Thus, the proof of Lemma 1 is complete. □

Corollary 1. *If we set $\mathring{\vartheta} = 1$ into Lemma 1, we then obtain*

$$\begin{aligned}
&\frac{\hat{G}(\psi) + \hat{G}(\pi)}{2} - \frac{\beta(\alpha)}{\alpha(\pi-\psi)} \left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G} \right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G} \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)} \hat{G}(k) \\
= &\frac{(\pi-\psi)^2}{2} \int_0^1 \Lambda(1-\Lambda)\hat{G}''(\Lambda\psi + (1-\Lambda)\pi) d\Lambda.
\end{aligned} \tag{3}$$

Remark 1. *If we set $\alpha = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 1, we then obtain*

$$\frac{\hat{G}(\psi) + \hat{G}(\pi)}{2} - \frac{1}{\pi-\psi} \int_{\psi}^{\pi} \hat{G}(x)dx = \frac{(\pi-\psi)^2}{2} \int_0^1 \Lambda(1-\Lambda)\hat{G}''(\Lambda\psi + (1-\Lambda)\pi) d\Lambda,$$

which was obtained by Alomari et al. [11].

Corollary 2. *If we set $\mathring{\vartheta} = 2$ into Lemma 1, we then have*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\hat{G}(\psi) + \hat{G}(\pi)}{2} + \hat{G}\left(\frac{\psi+\pi}{2} \right) \right) \\
&- \frac{\beta(\alpha)}{\alpha(\pi-\psi)} \left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G} \right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G} \right)(k) \right] + \frac{2(1-\alpha)}{\beta(\alpha)} \hat{G}(k) \\
= &\frac{(\pi-\psi)^2}{16} \left[\int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\psi + (1-\lambda)\frac{\psi+\pi}{2} \right) d\Lambda \right. \\
&\left. + \int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\frac{\psi+\pi}{2} + (1-\Lambda)\pi \right) d\Lambda \right].
\end{aligned} \tag{4}$$

Remark 2. *If we set $\alpha = 1$ and $\beta(0) = \beta(1) = 1$ in Corollary 2, we then have*

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\hat{G}(\psi) + \hat{G}(\pi)}{2} + \hat{G}\left(\frac{\psi+\pi}{2} \right) \right) - \frac{1}{\pi-\psi} \int_{\psi}^{\pi} \hat{G}(x)dx \\
= &\frac{(\pi-\psi)^2}{16} \left[\int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\psi + (1-\Lambda)\frac{\psi+\pi}{2} \right) d\Lambda \right. \\
&\left. + \int_0^1 \Lambda(1-\Lambda)\hat{G}''\left(\Lambda\frac{\psi+\pi}{2} + (1-\Lambda)\pi \right) d\Lambda \right],
\end{aligned}$$

which was obtained by B.Y. et al. [29]. The above inequalities can also be proven by other fractional integral operators and convexity, for example, quasi-convex function, strongly quasi-convex function, etc.

Theorem 5. *Under the assumption of Lemma 1, if $|\hat{G}''|$ is an exponential convex function on $[\psi,\pi]$, then the following fractional inequality holds*

$$\begin{aligned}
&\Bigg|\sum_{i=0}^{\vartheta-1}\frac{1}{2\vartheta}\left[\hat{G}\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)+\hat{G}\left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right]\\
&-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\Bigg| \qquad (5)\\
\leq\;&\sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{\vartheta^3}\left(\frac{17}{6}-e\right)A\left(\hat{G}''\left|\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right|,\hat{G}''\left|\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right|\right),
\end{aligned}$$

where $A(\psi,\pi)$ is the arithmetic mean.

Proof. By using the Lemma 1, since $|\hat{G}''|$ is an exponential convex function, we have

$$\begin{aligned}
&\Bigg|\sum_{i=0}^{\vartheta-1}\frac{1}{2\vartheta}\left[\hat{G}\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)+\hat{G}\left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right]\\
&-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\Bigg|\\
\leq\;&\sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{2\vartheta^3}\left[\int_0^1|\Lambda(1-\Lambda)|\left|\hat{G}''\left(\Lambda\frac{(\vartheta-i)\psi+i\pi}{\vartheta}+(1-\Lambda)\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right|d\Lambda\right]\\
=\;&\sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{2\vartheta^3}\left[\int_0^1\left|\Lambda-\Lambda^2\right|\left\{\left(e^{\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)\right|\right|\right.\\
&\left.+\left|\left(e^{1-\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right|\right\}\right]d\Lambda\\
=\;&\sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{2\vartheta^3}\left[\int_0^1\left|\Lambda-\Lambda^2\right|\left(e^{\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)\right|d\Lambda\right.\\
&\left.+\int_0^1\left|\Lambda-\Lambda^2\right|\left(e^{1-\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)\right|d\Lambda\right]\\
\leq\;&\sum_{i=0}^{\vartheta-1}\frac{(\pi-\psi)^2}{\vartheta^3}\left(\frac{17}{6}-e\right)A\left(\hat{G}''\left|\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right|,\hat{G}''\left|\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right|\right).
\end{aligned}$$

This completes the proof. □

Corollary 3. *If we set $\vartheta=1$ in Theorem 5, the following trapezoidal-type inequality holds*

$$\begin{aligned}
&\left|\frac{\hat{G}(\psi)+\hat{G}(\pi)}{2}-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\
\leq\;&\frac{(\pi-\psi)^2}{2}\left(\frac{17}{6}-e\right)\left(|\hat{G}''(\psi)|+|\hat{G}''(\pi)|\right)
\end{aligned}$$

Corollary 4. *If we set $\vartheta=2$ in Theorem 5, the following Bullen-type inequality holds*

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left(\hat{G}(\psi)+2\hat{G}\left(\frac{\psi+\pi}{2}\right)+\hat{G}(\pi)\right)\\
&-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\Bigg|\\
\leq\;&\frac{(\pi-\psi)^2}{8}\left(\frac{17}{6}-e\right)\left(\frac{|\hat{G}''(\pi)|+|\hat{G}''(\psi)|}{2}+\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|\right)
\end{aligned}$$

Theorem 6. *Using the assumption in Lemma 1, if $|\hat{G}''|^q$ is an exponential convex function on $[\psi,\pi]$ and $q>1$, then*

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\
\leq\ &\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2^{1-\frac{1}{q}}\mathring{\vartheta}^3}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}}\\
&\times A^{\frac{1}{q}}\left(\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q,\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right).
\end{aligned}\tag{6}$$

Proof. By employing Lemma 1, the modulus properties, and the Hölder inequality, since $\left|\hat{G}''\right|^q$ is an exponential convex function, we obtain

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\
\leq\ &\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\int_0^1|\Lambda(1-\Lambda)|\left|\hat{G}''\left(\Lambda\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}+(1-\Lambda)\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|d\Lambda\right]\\
\leq\ &\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\left(\int_0^1\left|\Lambda-\Lambda^2\right|^p d\Lambda\right)^{\frac{1}{p}}\left\{\left(e^{\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q\right.\right.\\
&\left.\left.+\left(e^{1-\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right\}^{\frac{1}{q}}d\Lambda\right]\\
\leq\ &\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2^{1-\frac{1}{q}}\mathring{\vartheta}^3}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}}\\
&\times A^{\frac{1}{q}}\left(\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q,\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right).
\end{aligned}$$

This completes the proof. □

Corollary 5. *If we set $\mathring{\vartheta}=1$ in Theorem 6, the following trapezoidal-type inequality holds*

$$\begin{aligned}
&\left|\frac{\hat{G}(\psi)+\hat{G}(\pi)}{2}-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\
\leq\ &\frac{(\pi-\psi)^2}{2^{1-\frac{1}{q}}}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}}\left(\frac{\left|\hat{G}''(\psi)\right|^q+\left|\hat{G}''(\pi)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 6. *If we set $\mathring{\vartheta}=2$ in Theorem 6, the following Bullen-type inequality holds*

$$\begin{aligned}
&\left|\frac{1}{4}\left(\hat{G}(\psi)+2\hat{G}\left(\frac{\psi+\pi}{2}\right)+\hat{G}(\pi)\right)\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\
\leq\ &\frac{(\pi-\psi)^2}{2^{4-\frac{1}{q}}}\beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}}\\
&\times\left[\left(\frac{\left|\hat{G}''(\psi)\right|^q+\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q+\left|\hat{G}''(\pi)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 7. *Using the assumption in Lemma 1, if $\left|\hat{G}''\right|^q$ is an exponential convex function on $[\psi,\pi]$ and $q\geq 1$, then*

$$\begin{aligned}&\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\\leq&\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{12^{1-\frac{1}{q}}\mathring{\vartheta}^3}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\times A^{\frac{1}{q}}\left(\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q,\left|\left(\hat{G}''\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right).\end{aligned}\tag{7}$$

Proof. By utilizing Lemma 1, modulus properties, and the power-mean inequality, since $\left|\hat{G}''\right|^q$ is an exponential convex function, we obtain

$$\begin{aligned}&\left|\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right.\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\\leq&\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\int_0^1|\Lambda(1-\Lambda)|\left|\hat{G}''\left(\Lambda\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}+(1-\Lambda)\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|d\Lambda\right]\\\leq&\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\left(\int_0^1\left|\Lambda-\Lambda^2\right|d\Lambda\right)^{1-\frac{1}{q}}\left\{\int_0^1\left|\Lambda-\Lambda^2\right|\left(\left(e^{\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q\right.\right.\right.\\&\left.\left.\left.+\left(e^{1-\Lambda}-1\right)\hat{G}''\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^q\right)d\Lambda\right\}^{\frac{1}{q}}\right]\\=&\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\int_0^1\left|\Lambda-\Lambda^2\right|\left(e^{\Lambda}-1\right)\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q d\Lambda\right.\\&\left.+\int_0^1\left|\Lambda-\Lambda^2\right|\left(e^{1-\Lambda}-1\right)\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^q d\Lambda\right]^{\frac{1}{q}}\\=&\sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{12^{1-\frac{1}{q}}\mathring{\vartheta}^3}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\times A^{\frac{1}{q}}\left(\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q,\left|\left(\hat{G}''\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right).\end{aligned}$$

This completes the proof. □

Remark 3. *Under the assumption in Theorem 7, if we set $q=1$, we can obtain Theorem 5.*

Corollary 7. *If we set $\mathring{\vartheta}=1$ in Theorem 7, the following trapezoidal-type inequality holds*

$$\begin{aligned}&\left|\frac{\hat{G}(\psi)+\hat{G}(\pi)}{2}-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\\leq&\frac{(\pi-\psi)^2}{12^{1-\frac{1}{q}}}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\left(\frac{\left|\hat{G}''(\psi)\right|^q+\left|\hat{G}''(\pi)\right|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 8. *If we set $\mathring{\vartheta}=2$ in Theorem 7, the following Bullen-type inequality holds*

$$\begin{aligned}&\left|\frac{1}{4}\left(\hat{G}(\psi)+2\hat{G}\left(\frac{\psi+\pi}{2}\right)+\hat{G}(\pi)\right)\right.\\&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|\\\leq&\frac{(\pi-\psi)^2}{12^{1-\frac{1}{q}}\times 8}\left(\frac{17}{6}-e\right)^{\frac{1}{q}}\left[\left(\frac{\left|\hat{G}''(\psi)\right|^q+\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left|\left(\frac{\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q+\left|\hat{G}''(\pi)\right|^q}{2}\right)^{\frac{1}{q}}\right|\right].\end{aligned}$$

Theorem 8. *Using the assumption in Lemma 1, if $\left|\hat{G}''\right|^q$ is s-convex function on $[\psi, \pi]$ and some $s \in (0,1]$, and $q > 1$, then the following holds*

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1} \frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1} \frac{\pi-\psi}{2\mathring{\vartheta}^3}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\times\left(\hat{G}''\left|\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right|^q,\ \hat{G}''\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^q\right)^{\frac{1}{q}}
\end{aligned} \tag{8}$$

Proof. By employing Lemma 1, modulus properties, and the Hölder inequality, since $\left|\hat{G}''\right|^q$ is an s-convex function, we obtain

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1} \frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1} \frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\int_0^1|\Lambda(1-\Lambda)|\left|\hat{G}''\left(\Lambda\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}+(1-\Lambda)\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|d\Lambda\right] \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1} \frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left[\left(\int_0^1\left|\Lambda-\Lambda^2\right|^p d\Lambda\right)^{\frac{1}{p}}\left\{\Lambda^s\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q\right.\right. \\
&\left.\left.+(1-\Lambda)^s\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right\}^{\frac{1}{q}}d\Lambda\right] \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1} \frac{\pi-\psi}{2\mathring{\vartheta}^3}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\times\left(\hat{G}''\left|\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right|^q,\ \hat{G}''\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, the proof of Theorem 8 is complete. □

Corollary 9. *If we set $s = 1$ in Theorem 8, then*

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1} \frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right. \\
&\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1} \frac{\pi-\psi}{2^{1+\frac{1}{q}}\mathring{\vartheta}^3}\beta^{\frac{1}{p}}(p+1,p+1)\left(\hat{G}''\left|\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right|^q,\ \hat{G}''\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

holds.

Corollary 10. *If we set $\mathring{\vartheta} = 2$ in Theorem 8, we then obtain the following Bullen-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{4}\left(\hat{G}(\psi) + 2\hat{G}\left(\frac{\psi+\pi}{2}\right) + \hat{G}(\pi)\right)\right. \\
&\left. - \frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \frac{\pi-\psi}{16}\beta^{\frac{1}{p}}(p+1,p+1)\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\left[\left(\frac{\left|\hat{G}''(\psi)\right|^{q} + \left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}} + \left|\left(\frac{\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^{q} + \left|\hat{G}''(\pi)\right|^{q}}{2}\right)^{\frac{1}{q}}\right|\right].
\end{aligned}$$

Theorem 9. *Using the assumption in Lemma 1, if $\left|\hat{G}''\right|^{q}$ is an s-convex function on $[\psi,\pi]$ and some $s \in (0,1]$, and $q \geq 1$, then*

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right) + \hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right. \\
&\left. - \frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^{2}}{2\mathring{\vartheta}^{3}}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+2)(s+3)}\right)^{\frac{1}{q}}\left[\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^{q} + \left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^{q}\right]^{\frac{1}{q}}
\end{aligned} \tag{9}$$

holds.

Proof. By utilizing Lemma 1, modulus properties of, and the power-mean inequality, since $\left|\hat{G}''\right|^{q}$ is an s-convex function, we obtain

$$\begin{aligned}
&\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right) + \hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right. \\
&\left. - \frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}^{CF}_{\psi}I^{\alpha}\hat{G}\right)(k) + \left({}^{CF}I^{\alpha}_{\pi}\hat{G}\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right| \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^{2}}{2\mathring{\vartheta}^{3}}\left[\int_{0}^{1}|\Lambda(1-\Lambda)|\left|\hat{G}''\left(\Lambda\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}} + (1-\Lambda)\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|d\Lambda\right] \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^{2}}{2\mathring{\vartheta}^{3}}\left[\left(\int_{0}^{1}\left|\Lambda-\Lambda^{2}\right|d\Lambda\right)^{1-\frac{1}{q}}\left\{\int_{0}^{1}\left|\Lambda-\Lambda^{2}\right|\left(\Lambda^{s}\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^{q}\right.\right.\right. \\
&\left.\left.\left. + (1-\Lambda)^{s}\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^{q}\right)d\Lambda\right\}^{\frac{1}{q}}\right] \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^{2}}{2\mathring{\vartheta}^{3}}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left[\int_{0}^{1}\left|\Lambda-\Lambda^{2}\right|\Lambda^{s}\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^{q}d\Lambda\right. \\
&\left. + \int_{0}^{1}\left|\Lambda-\Lambda^{2}\right|(1-\Lambda)^{s}\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^{q}d\Lambda\right]^{\frac{1}{q}} \\
\leq\ & \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^{2}}{2\mathring{\vartheta}^{3}}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{1}{(s+2)(s+3)}\right)^{\frac{1}{q}}\left[\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^{q} + \left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^{q}\right]^{\frac{1}{q}}.
\end{aligned}$$

Thus, the proof of Theorem 9 is complete. □

Corollary 11. *If we set $s = 1$ in Theorem 9, then*

$$\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{2\mathring{\vartheta}}\left[\hat{G}\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)+\hat{G}\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|$$
$$\leq \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2}{2\mathring{\vartheta}^3}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)\right|^q+\left|\hat{G}''\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)\right|^q\right]^{\frac{1}{q}}$$

holds.

Corollary 12. *If we set $\mathring{\vartheta}=2$ in Corollary 11, we then obtian the following Bullen-type inequality*

$$\left|\frac{1}{4}\left(\hat{G}(\psi)+2\hat{G}\left(\frac{\psi+\pi}{2}\right)+\hat{G}(\pi)\right)\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\pi-\psi)}\left[\left({}_{\psi}^{CF}I^{\alpha}\hat{G}\right)(k)+\left({}^{CF}I_{\pi}^{\alpha}\hat{G}\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\hat{G}(k)\right|$$
$$\leq \frac{(\pi-\psi)^2}{16}\left(\frac{1}{6}\right)^{\frac{1}{q}}\left[\left(\frac{\left|\hat{G}''(\psi)\right|^q+\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left|\left(\frac{\left|\hat{G}''\left(\frac{\psi+\pi}{2}\right)\right|^q+\left|\hat{G}''(\pi)\right|^q}{2}\right)^{\frac{1}{q}}\right|\right].$$

3. Applications to Special Means

(a) Arithmetic mean:

$$A=A(\psi,\pi):=\frac{\psi+\pi}{2},\ \psi,\pi\in\mathbb{R};$$

(b) Logarithmic mean:

$$L=L(\psi,\pi):=\frac{\pi-\psi}{\ln\pi-\ln\psi},\ \psi,\pi\in\mathbb{R},\ \psi\neq\pi;$$

(c) The generalized logarithmic mean:

$$L_r=L_r(\psi,\pi):=\left[\frac{\pi^{r+1}-\psi^{r+1}}{(r+1)(\pi-\psi)}\right]\ r\in\mathbb{R}\backslash\{-1,0\},\ \psi,\pi\in\mathbb{R},\ \psi\neq\pi;$$

Proposition 1. *Let $\psi,\pi\in\mathbb{R}$, $0<\psi<\pi$ and $\Lambda\in\mathbb{N}$, $\Lambda\geq 3$, then we have*

$$\left|\sum_{i=0}^{\mathring{\vartheta}-1}\frac{1}{\mathring{\vartheta}}A\left(\left(\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right)^{\Lambda},\left(\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right)^{\Lambda}\right)-L_{\Lambda}^{\Lambda}(\psi,\pi)\right|$$
$$\leq \sum_{i=0}^{\mathring{\vartheta}-1}\frac{(\pi-\psi)^2\Lambda(\Lambda-1)}{\mathring{\vartheta}^3}\left(\frac{17}{6}-e\right)A\left(\left|\frac{(\mathring{\vartheta}-i)\psi+i\pi}{\mathring{\vartheta}}\right|^{\Lambda-2},\left|\frac{(\mathring{\vartheta}-i-1)\psi+(i+1)\pi}{\mathring{\vartheta}}\right|^{\Lambda-2}\right).$$

Proof. The assertion follows from Theorem 5, applying $\hat{G}(x)=x^{\Lambda}$, $x\in[\psi,\pi]$. □

Proposition 2. *Let* $\psi, \pi \in \mathbb{R}$, $0 < \psi < \pi$ *and* $\Lambda \in \mathbb{N}$, $\Lambda \geq 3$, *we have*

$$\begin{aligned}
&\left|\sum_{i=0}^{\vartheta-1} \frac{1}{\vartheta} A\left(\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)^{\Lambda}, \left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)^{\Lambda}\right) - L_{\Lambda}^{\Lambda}(\psi,\pi)\right| \\
\leq\ & \sum_{i=0}^{\vartheta-1} \frac{(\pi-\psi)^2\Lambda(\Lambda-1)}{2^{1-\frac{1}{q}}\vartheta^3} \beta^{\frac{1}{p}}(p+1,p+1)(e-2)^{\frac{1}{q}} \\
&\times A^{\frac{1}{q}}\left(\left|\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right|^{(\Lambda-2)q}, \left|\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right|^{(\Lambda-2)q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from Theorem 6, applying $\hat{G}(x) = x^{\Lambda}$, $x \in [\psi, \pi]$. □

Proposition 3. *Let* $\psi, \pi \in \mathbb{R}$, $0 < \psi < \pi$ *and* $\Lambda \in \mathbb{N}$, $\Lambda \geq 3$, *we have*

$$\begin{aligned}
&\left|\sum_{i=0}^{\vartheta-1} \frac{1}{\vartheta} A\left(\left(\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right)^{\Lambda}, \left(\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right)^{\Lambda}\right) - L_{\Lambda}^{\Lambda}(\psi,\pi)\right| \\
&\sum_{i=0}^{\vartheta-1} \frac{(\pi-\psi)^2\Lambda(\lambda-1)}{12^{1-\frac{1}{q}}\vartheta^3}\left(\frac{17}{6}-e\right) \times A^{\frac{1}{q}}\left(\left|\frac{(\vartheta-i)\psi+i\pi}{\vartheta}\right|^{(\Lambda-2)q}, \left|\frac{(\vartheta-i-1)\psi+(i+1)\pi}{\vartheta}\right|^{(\Lambda-2)q}\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. The assertion follows from Theorem 7, applying $\hat{G}(x) = x^{\Lambda}$, $x \in [\psi, \pi]$. □

4. Conclusions

The investigation of fractional Hermite Hadamard-type inequalities is an informative and active area of research that reflects the growing importance of fractional calculus in modern science and engineering. Here, we established a new lemma (Lemma 1) and produced new Hermite–Hadamard-type inequalities for the exponential convex function. Additionally, several types of fractional integral inequalities were obtained based on this identity (different outcomes were found for various values of ϑ, $\vartheta \in \mathbb{N}$). From the developed corollaries, Corollaries 1 and 2, one can observe that by taking specific values of $\vartheta = 1$ for the factors, all the existing results were reduced to the results obtained by Alomari et al. [11] and B.Y et al. [29]. In future, authors can apply these new techniques and useful ideas, e.g., coordinates, and other fractional operators, produced in this paper. Furthermore, one can obtain likewise, parameterized inequalities via the Caputo–Fabrizio fractional integral operator for convex functions using quantum calculus. Additionally, our findings could potentially have specific implementations in numerical integration, optimization, and other related areas.

Author Contributions: Conceptualization, M.-u.-D.J. and A.M.; investigation, A.Q. and H.B.; methodology, M.M.S. and S.S.S.; validation, M.-u.-D.J. and M.M.S.; visualization, H.B.; writing—original draft, A.Q. and A.M.; writing—review and editing, S.S.S. and H.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Conflicts of Interest: The authors declare no conflict of interest.

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