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Inverse Spectrum and Structure of Topological Metagroups

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Abstract: In this article, a structure of topological metagroups is scrutinized. Their inverse spectra are studied. This also permits us to construct abundant families of topological metagroups and quasigroups. Specific features of the topological quasigroups structure are found in comparison with topological groups, and are discussed.

Keywords: inverse spectrum; structure; topological metagroup; locally compact

MSC: 46H70; 20N05; 22A30; 43-99; 54D45; 54H10

1. Introduction

A topological group structure plays very important role in mathematics, particularly in noncommutative analysis, abstract harmonic analysis and their applications [1–5]. Topologies on groupoids, semigroups, other algebraic structures attract great attention. There are also interesting nonassociative metagroups, which appear naturally in noncommutative analysis, noncommutative geometry, operator theory and PDEs. Topological groups are rather well studied, but little is known about their nonassociative analogs, such as topological metagroups and quasigroups. In this article, specific features of topological metagroups and quasigroups are scrutinized.

In particular, analysis of octonions and generalized Cayley–Dickson algebra developed quickly in recent years (see [6–19] and the references therein). It appears that a multiplicative law of their canonical bases is nonassociative and leads to a more general notion of a metagroup instead of a group [9,20,21]. They were used in [20–24] for investigations of partial differential operators and other unbounded operators over quaternions and octonions, and also for automorphisms, derivations and cohomologies of generalized C^*-algebras over **R** or **C**. They certainly have a lot of specific features in their derivations and (co)homology theory [20,21]. It was shown in [24] that an analog of the Stone theorem for one-parameter groups of unitary operators for the generalized C^*-algebras over quaternions and octonions becomes more complicated and multiparameter. The generalized C^*-algebras arise naturally, while there are decompositions of PDEs or systems of PDEs of higher orders into PDEs or their systems of order not higher than two [11,12,25,26], which permits integrating them subsequently or simplifying their analysis.

Recently, nonassociative algebras near to quasigroups were utilized in investigations of slave boson decompositions in superconductors [27] and in nonassociative quantum mechanics [28]. They were also actively used in gauge theories and Green–Schwartz superstrings [29,30]. Nonassociative algebras of such types are connected with quasi-hopf deformations in nonassociative quantum mechanics [31]. Nonassociative algebras near to quasigroups served as one of the main tools during studies of De Sitter representations of a curved space-time [32], in the great unification theory, and for studies of Yang–Mills fields [12,33]. The family of such nonassociative algebras was utilized for an analysis of Yang–Baxter PDEs with applications for the great unification theory (see [34–36] and the references therein). Quasigroups have found other applications in informatics and coding theory, because they open new opportunities in comparison to groups [37–40].

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In [41], different types of products of metagroups were studied such as smashed products and smashed twisted wreath products. Topologies of the smashed twisted wreath products of metagroups were studied in [42]. There conditions were investigated, providing topological metagroups. Examples were given of large families of topological metagroups in the articles [41–43]. That also permitted constructing of their abundant classes, which are different from topological groups. On the other hand, topologies of metagroups and their homomorphisms were not investigated. Their definition is recalled in Section 1.1.

Notice also that a loop in algebra (i.e., a unital quasigroup) is a quite different object than a loop group considered in geometry or mathematical physics. Note that metagroups are commonly nonassociative, and having many specific features in comparison with groups and quasigroups. On the other hand, if a loop G is simple, then a subloop generated by all elements of the form $((ab)c)/(a(bc))$ for all a, b, c in G coincides with G [17,44]. Metagroups are intermediate between groups and quasigroups.

We recall that, according to Chapter 2 and Sections 4.6, 4.10, 4.13 of [4] and Section 6 of [3], the compact connected T_0 topological group G can be presented as the limit of an inverse spectrum (that is, a projective limit of a homomorphism system) $G = \lim\{G_j, \pi^j_k, \Omega\}$ of compact finite-dimensional Lie groups of manifolds over $\mathbf{R}$, where Ω is a directed set, $\pi^j_k : G_j \to G_k$ is a continuous homomorphism for each $j > k$ in Ω, π^j_j is the identity map, $\pi^j_j(g_j) = g_j$ for each $g_j \in G_j$, $\pi^k_l \circ \pi^j_k = \pi^j_l$ for each $l < k < j$ in Ω.

This raises questions for a subsequent research. Does a nonassociative analog of a topological group have this property or not? How weak may a nonassociative structure be that does not satisfy this property? This article answers these questions. In it, analogs of topological groups are scrutinized with a rather mild nonassociative metagroup structure.

The methods used by Gleason, Montgomery and Zippin were based on analysis of one-parameter subgroups. In this article, quite different approaches are used. They are based on the previous works of the author, and use smashed twisted wreath products of topological metagroups (see also above). With the help of them the nonassociative analog of the Hilbert's fifths, the problem for topological metagroups is solved in Section 2.

In this article topologies on metagroups and quasigroups are studied. They have specific features in comparison with topological groups because of nonassociativity in general of topological metagroups or topological quasigroups. Necessary definitions are recalled. Transversal sets are studied in smashed twisted products of topological metagroups in Theorem 9, and Corollaries 8, 9, and 10. Their inverse spectra are investigated in Theorem 10 and Remark 4. Specific features of the topological quasigroup structures are found in comparison with topological groups and discussed.

1.1. Basic Facts on Metagroups

Necessary facts about metagroups are recalled in this subsection, though a reader familiar with previous works [41,42] can skip it.

Definition 1. *Let G be a set with a single-valued binary operation (multiplication) $G^2 \ni (a,b) \mapsto ab \in G$ defined on G, and satisfying the conditions:*

(i) For each a and b in G, there is a unique $x \in G$ with $ax = b$;

(ii) A unique $y \in G$ exists satisfying $ya = b$, which is denoted by $x = a \setminus b = Div_l(a,b)$ and $y = b/a = Div_r(a,b)$, correspondingly;

(iii) There exists a neutral (i.e., unit) element $e_G = e \in G$: $eg = ge = g$ for each $g \in G$.

If the set G with a single-valued multiplication satisfies conditions (i) and (ii), then it is called a quasigroup. If the quasigroup G also satisfies condition (iii), then it is called an algebraic loop (or a unital quasigroup or, more shortly, a loop).

The set of all elements $h \in G$ commuting and associating with G are:

(iv) $Com(G) := \{a \in G : \forall b \in G,\ ab = ba\}$;

(v) $N_l(G) := \{a \in G : \forall b \in G, \forall c \in G,\ (ab)c = a(bc)\}$;

(vi) $N_m(G)$: = $\{a \in G : \forall b \in G, \forall c \in G,\ (ba)c = b(ac)\}$;
(vii) $N_r(G)$: = $\{a \in G : \forall b \in G, \forall c \in G,\ (bc)a = b(ca)\}$;
(viii) $N(G)$: = $N_l(G) \cap N_m(G) \cap N_r(G)$.

$\mathcal{C}(G)$: = $Com(G) \cap N(G)$ is called the center $\mathcal{C}(G)$ of G.

We call G a metagroup if a set G possesses a single-valued binary operation and satisfies conditions (i)–(iii) and

(ix) $(ab)c = t(a,b,c)a(bc)$*—for each a, b and c in G, where* $t(a,b,c) = t_G(a,b,c) \in \mathcal{C}(G)$.

Then, the metagroup G will be called a central metagroup, if it also satisfies the condition:

(x) $ab = \mathrm{t}_2(a,b)ba$*—for each a and b in G, where* $\mathrm{t}_2(a,b) \in \mathcal{C}(G)$.

If H is a submetagroup (or a unital subquasigroup) of the metagroup G (or the unital quasigroup G) and

(xi) *$gH = Hg$ for each $g \in G$, then H will be called almost invariant (or algebraically almost normal). If, in addition,*

(xii) *$(gH)k = g(Hk)$ and $k(gH) = (kg)H$, for each g and k in G, then H will be called an invariant (or algebraically normal) submetagroup (or unital subquasigroup, respectively).*

Elements of a metagroup G will be denoted by small letters, and subsets of G will be denoted by capital letters. If A and B are subsets in G, then $A - B$ means the difference of them, $A - B = \{a \in A :\ a \notin B\}$. Henceforward, maps and functions of metagroups are supposed to be single-valued, if nothing else is specified.

If $\mathcal{T}_G$ is a topology on the metagroup (or quasigroup) G such that multiplication, Div_l and Div_r are (jointly) continuous from $G \times G$ into G, then $(G, \mathcal{T}_G)$ is called a topological metagroup (or quasigroup, respectively).

Remark 1 ([41])**.** *Let A and B be two metagroups and let $\mathcal{C}$ be a commutative group such that*

$$\mathcal{C}_m(A) \hookrightarrow \mathcal{C},\ \mathcal{C}_m(B) \hookrightarrow \mathcal{C},\ \mathcal{C} \hookrightarrow \mathcal{C}(A) \text{ and } \mathcal{C} \hookrightarrow \mathcal{C}(B), \tag{1}$$

where $\mathcal{C}_m(A)$ denotes a minimal subgroup in $\mathcal{C}(A)$ containing $t_A(a,b,c)$ for every a, b and c in A.

Using direct products, it is always possible to extend either A or B to obtain such a case. In particular, either A or B may be a group. On $A \times B$, an equivalence relation Ξ is considered such that

$$(\gamma v, b)\Xi(v, \gamma b) \text{ and } (\gamma v, b)\Xi\gamma(v, b) \text{ and } (\gamma v, b)\Xi(v, b)\gamma \tag{2}$$

for every v in A, b in B and γ in $\mathcal{C}$.

$$\text{Let } \phi : A \to \mathcal{A}(B) \text{ be a single-valued mapping,} \tag{3}$$

where $\mathcal{A}(B)$ denotes a family of all bijective surjective single-valued mappings of B onto B, subject to conditions (4)–(7) given below. If $a \in A$ and $b \in B$, then it will be written shortly as b^a instead of $\phi(a)b$, where $\phi(a) : B \to B$. Also let

$$\eta_{A,B,\phi} :\ A \times A \times B \to \mathcal{C},\ \kappa_{A,B,\phi} : A \times B \times B \to \mathcal{C}$$

$$\text{and } \xi_{A,B,\phi} : ((A \times B)/\Xi) \times ((A \times B)/\Xi) \to \mathcal{C}$$

be single-valued mappings written shortly as η, κ, and ξ, correspondingly, such that

$$(b^u)^v = b^{vu}\eta(v,u,b),\ e^u = e,\ b^e = b; \tag{4}$$

$$\eta(v,u,\gamma b) = \eta(v,u,b); \tag{5}$$

$$(cb)^u = c^u b^u \kappa(u,c,b); \tag{6}$$

$$\kappa(u,\gamma c,b) = \kappa(u,c,\gamma b) = \kappa(u,c,b) \tag{7}$$

and $\kappa\ (u, \gamma\ ,b) = \kappa(u,b,\gamma) = e$;

$$\xi((\gamma u, c),(v, b)) = \xi((u, c),(\gamma v, b)) = \xi((u, c),(v, b))$$

and

$$\xi((\gamma, e),(v, b)) = e \text{ and } \xi((u, c),(\gamma, e)) = e \tag{8}$$

for every u *and* v *in* A, b, c *in* B, γ *in* $\mathcal{C}$, *where* e *denotes the neutral element in* $\mathcal{C}$ *and in* A *and* B.

We write

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, \xi((a_1, b_1),(a_2, b_2))b_1b_2^{a_1}) \tag{9}$$

for each a_1, a_2 *in* A, b_1 *and* b_2 *in* B.

The Cartesian product $A \times B$ *supplied with such a binary operation* (9) *will be denoted by* $A \bigotimes^{\phi,\eta,\kappa,\xi} B$.

Then, we write

$$(a_1, b_1) \star (a_2, b_2) = (a_1a_2, \xi((a_1, b_1),(a_2, b_2))b_2^{a_1}b_1) \tag{10}$$

for each a_1, a_2 *in* A, b_1 *and* b_2 *in* B.

The Cartesian product $A \times B$ *supplied with a binary operation* (10) *will be denoted by* $A \star^{\phi,\eta,\kappa,\xi} B$.

Theorem 1 (Theorem 4 in [43])**.** *Let* G_j *be a family of topological metagroups, where* $j \in J$, J *is a set. Then, their direct product* $G = \prod_{j\in J} G_j$ *relative to the Tychonoff product topology is a topological metagroup, and*

$$\mathcal{C}(G) = \prod_{j\in J} \mathcal{C}(G_j).$$

Theorem 2 (Theorem 3 in [41])**.** *Let the conditions of Remark 1 be fulfilled. Then, the Cartesian product* $A \times B$ *supplied with a binary operation* (9) *is a metagroup. Moreover, there are embeddings of* A *and* B *into* $A \bigotimes^{\phi,\eta,\kappa,\xi} B = C_1$, *such that* B *is an almost normal submetagroup in* C_1. *If, in addition,* $\mathcal{C}_m(C_1) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, *then* B *is a normal submetagroup.*

Theorem 3 (Theorem 4 in [41])**.** *Suppose that the conditions of Remark 1 are satisfied. Then, the Cartesian product* $A \times B$ *supplied with a binary operation* (10) *is a metagroup. Moreover, there exist embeddings of* A *and* B *into* $A \star^{\phi,\eta,\kappa,\xi} B = C_2$, *such that* B *is an almost normal submetagroup in* C_2. *If, additionally,* $\mathcal{C}_m(C_2) \subseteq \mathcal{C}_m(B) \subseteq \mathcal{C}$, *then* B *is a normal submetagroup.*

Definition 2 ([41])**.** *We call the metagroup* $A \bigotimes^{\phi,\eta,\kappa,\xi} B$ *provided by Theorem 2 (or* $A \star^{\phi,\eta,\kappa,\xi} B$ *by Theorem 3) a smashed product (or a smashed twisted product, correspondingly) of metagroups* A *and* B *with smashing factors* ϕ, η, κ *and* ξ.

Lemma 1 (Lemma 5 in [41], Lemma 1.1 in [45])**.** *(a) Let* D *be a metagroup, and* A *be a submetagroup in* D. *Then, there exists a subset* V *in* D *such that* D *is a disjoint union of* vA, *where* $v \in V$; *that is,*

$$D = \bigcup_{v\in V} vA$$

and $(\forall v_1 \in V, \forall v_2 \in V, v_1 \neq v_2) \Rightarrow (v_1A \cap v_2A) = \emptyset)$.

(b) *If* G *is a left quasigroup, and* H *is a left subquasigroup in* G, *such that* $(ab)H = a(bH)$ *for each* a *and* b *in* G, *then there exists a transversal set* $V_{G,H}$ *for* H *in* G.

Definition 3. *A set* V *from Lemma 1 is called a transversal set of* A *in* D.

Corollary 1 ([41])**.** *Let* D *be a metagroup,* A *be a submetagroup in* D, *and* V *a transversal set of* A *in* D. *Then,*

$$\forall a \in D,\ \exists_1 s \in A,\ \exists_1 b \in V,\ a = sb \text{ for a given triple } (A, D, V). \tag{11}$$

Remark 2 (Remark 3 in [41]). *We denote b in the decomposition (11) by $b = \tau(a) = a^\tau$ and $s = \psi(a) = a^\psi$, where τ and ψ is a shortened notation of $\tau_{A,D,V}$ and $\psi_{A,D,V}$, respectively. That is, there are single-valued maps*

$$\tau : D \to V \text{ and } \psi : D \to A.$$

Remark 3 (Remark 4 in [41]). *Let B and D be metagroups, A be a submetagroup in D, and V be a transversal of A in D. Also let Conditions (1)–(8) be satisfied for A and B. We write*

$$(a^\tau)^{[c]} := (a^\tau c)^\tau \text{ for each } a \text{ and } c \text{ in } D. \tag{12}$$

(see also Remark 3 in [41] or 2 above). By Theorem 2, there exists a metagroup
$F = B^V$, where $B^V = \prod_{v\in V} B_v,\ B_v = B$ for each $v \in V$.
It contains a submetagroup

$$F^* = \{f \in F :\ card(\sigma(f)) < \aleph_0\},$$

where $\sigma(f) = \{v \in V :\ f(v) \neq e\}$ is a support of $f \in F$, and $card(\Omega)$ denotes the cardinality of a set Ω.

Let $T_h f = f^h$ for each $f \in F$ and $h : V \to A$. We write

$$\hat{S}_d(T_h f J) = T_{hS_d^{-1}} f S_d J,$$

where J: $V \times F \to B,\ J(f, v) = fJv,\ S_d Jv = Jv^{[d\backslash e]}$ for each $d \in D$, $f \in F$ and $v \in V$. Then, for each $f \in F$, $d \in D$, we write

$$f^{\{d\}} = \hat{S}_d(T_{g_d} f E),$$

where

$$s(d, v) = e/(v/d)^\psi,\ g_d(v) = s(d, v),$$

$fEv = f(v)$ for each $v \in V$, (see also (11) and (12)).

Definition 4 ([41]). *Suppose that the conditions of Remark 3 are satisfied, and on the Cartesian product $C = D \times F$ (or $C^* = D \times F^*$), a binary operation is given by the following formula:*

$$(d_1, f_1)(d, f) = (d_1 d, \xi((d_1^\psi, f_1), (d^\psi, f)) f_1 f^{\{d_1\}}), \tag{13}$$

where $\xi((d_1^\psi, f_1), (d^\psi, f))(v) = \xi((d_1^\psi, f_1(v)), (d^\psi, f(v)))$ for every d and d_1 in D, f and f_1 in F (or F^*, respectively), $v \in V$.

Theorem 4 (Theorem 5 in [41]). *Let C, C^*, D, F, F^* be the same as in Definition 4. Then, C and C^* are loops, and there are natural embeddings $D \hookrightarrow C$, $F \hookrightarrow C$, $D \hookrightarrow C^*$, $F^* \hookrightarrow C^*$, such that F (or F^*) is an almost normal subloop in C (or C^*, respectively).*

Definition 5 ([41]). *Product (13) in loop C (or C^*) of Theorem 4 is called a smashed twisted wreath product of D and F (or a restricted smashed twisted wreath product of D and F^*, respectively) with smashing factors ϕ, η, κ, ξ and it will be denoted by $C = D\Delta^{\phi,\eta,\kappa,\xi}F$ (or $C^* = D\Delta^{\phi,\eta,\kappa,\xi}F^*$, respectively). The loop C (or C^*) is also called a smashed splitting extension of F (or of F^*, respectively) by D.*

Theorem 5 (Theorem 6 in [41]). *Let the conditions of Remark 3 be satisfied, and $\mathcal{C}_m(D) \subseteq \mathcal{C}$, where $\mathcal{C}$ is as in (1). Then, C and C^* supplied with the binary operation (13) are metagroups.*

Theorem 6 (Theorem 2.1 in [45]). *Assume that G is a topological T_1 quasigroup with a topology $\mathcal{T}_G$. Assume also that H is a closed subquasigroup, such that $a(bH) = (ab)H$, $(aH)b = a(Hb)$, $H(ab) = (Ha)b$ for each a and b in G. Then, for each x, b in G, the family $\{\pi(xU) : b \in U \in \mathcal{T}_G\}$ is a local base for $G/_cH$ at $(xb)H \in G/_cH$, where $G/_cH$ is supplied with the quotient topology* *with respect to the quotient map $\pi : G \to G/_c H$. Moreover, the map π is continuous and open, and $G/_c H$ is a homogeneous T_1-space.*

Theorem 7 (Theorem 2.3 in [45])**.** *If the conditions of Theorem 6 are satisfied, then the quotient space $G/_c H$ is regular.*

Example 1. *If the conditions of Corollary 1 in [41] or above are satisfied, either $(G = D, H = \mathcal{AC}_1)$ or $(G = \mathcal{AC}_1, H = A)$, then the conditions $a(bH) = (ab)H$, $(aH)b = a(Hb)$, and $H(ab) = (Ha)b$ for each a and b in G are satisfied for these pairs.*

Theorem 8 (Theorem 2.4 in [45])**.** *Assume that G is a topological T_1 unital quasigroup, and H is a compact unital subquasigroup in G, satisfying the conditions of Theorem 6. Then, the quotient map $\pi : G \to G/_c H$ is perfect.*

Corollary 2 (Corollary 2.4 in [45])**.** *Suppose that the conditions of Theorem 8 are satisfied, and $G/_c H$ is compact. Then, G is compact.*

Corollary 3 (Corollary 2.5 in [45])**.** *Assume that the conditions of Theorem 8 are satisfied, and the quotient space $G/_c H$ is compact. Let $V = V_{G,H}$ be a transversal set for H in G, and let V be supplied with a topology $\mathcal{T}(V) = \mathcal{T}(G) \cap V$ inherited from G. Then, V can be chosen compact and closed in G.*

Corollary 4 (Corollary 2.6 in [45])**.** *If the conditions of Corollary 3 are satisfied, then the transversal set V and the transversal mapping $\tau = \tau^G_H$ can be chosen, such that $\tau : G \to V$ and $\psi : G \to H$ are continuous relative to topologies $\mathcal{T}(H) = \mathcal{T}(G) \cap H$ and $\mathcal{T}(V) = \mathcal{T}(G) \cap V$ on H and V, correspondingly, inherited from G.*

Corollary 5. *Let the conditions of Corollary 1 in [41] or above be satisfied, and let $\mathcal{C}_1$ and A and $\mathcal{AC}_1$ be closed in G; then, $D/_c(\mathcal{AC}_1)$ and $(\mathcal{AC}_1)/_c A$ are homogeneous $T_1 \cap T_3$ spaces, and the quotient maps $\pi^D_{\mathcal{AC}_1} : D \to D/_c(\mathcal{AC}_1)$ and $\pi^{\mathcal{AC}_1}_A : \mathcal{AC}_1 \to (\mathcal{AC}_1)/_c A$ are open.*

Proof. This follows from Theorems 2.1 and 2.3 in [45], or Theorems 6 and 7 above, as their particular case.

The following corollaries, together with the assertions above, can serve for constructions of suitable examples. □

Corollary 6. *Let the conditions of Remark 2 in [42] be satisfied, and let $(G, \mathcal{T}_{G,A})$ be compact and $(G, \mathcal{T}_G)$ be T_1 as the topological quasigroup. Then, $\mathcal{T}_{G,A} = \mathcal{T}_G$.*

Proof. Since $(G, \mathcal{T}_G)$ is the T_1 topological quasigroup, then it is regular. In view of Corollary 3.1.14 in [46], $\mathcal{T}_{G,A} = \mathcal{T}_G$. □

Corollary 7. *Assume that the conditions of Remark 4 in [42] are satisfied, and $(D, \mathcal{T}_{D,\mathcal{AC}_1,A})$ is compact. Then, $\mathcal{T}_D = \mathcal{T}_{D,\mathcal{AC}_1,A}$; moreover, A and $\mathcal{AC}_1$ are compact relative to the topologies $\mathcal{T}_D \cap A$ and $\mathcal{T}_D \cap (\mathcal{AC}_1)$, respectively, inherited from G.*

Proof. Corollary 6 implies that $\mathcal{T}_D = \mathcal{T}_{D,\mathcal{AC}_1,A}$. The maps τ^D_A and $\tau^D_{\mathcal{AC}_1}$ are continuous by the conditions of Remark 4 in [42]. On the other hand, $A = (\tau^D_A)^{-1}(e)$ and $\mathcal{AC}_1 = (\tau^D_{\mathcal{AC}_1})^{-1}(e)$; consequently, A and $\mathcal{AC}_1$ are closed in D; hence, A and $\mathcal{AC}_1$ are compact by Theorem 3.1.2 in [46]. □

Example 2. *In particular, as pairs of A and B can be taken as the special orthogonal group $A = SO(n, \mathbf{R})$ of the Euclidean space $\mathbf{R}^n$, the special linear group $B = SL(m, \mathbf{R})$ of the Euclidean space is $\mathbf{R}^m$, where $1 < n \leq m \in \mathbf{N}$, A and B are supplied with topologies induced by the operator norm topology. Then, their central extensions can be taken, or semidirect products or smashed products with connected commutative groups. Then, using smashed products and smashed twisted* *wreath products, new metagroups are subsequently constructed using the theorems and corollaries above or given in the references.*

Example 3. *Let l_2 be the separable Hilbert space over the complex field $\mathbf{C}$, where $\mathbf{C}$ is supplied with the standard multiplicative norm topology. We consider the unitary group $A = U(l_2)$ and the general linear group $B = GL(l_2)$ of l_2, where A and B are considered in the topologies inherited form the operator norm topology. Then, metagroups are constructed similarly to Example 2.*

Other examples are 2–4 in [43].

Possible applications and further developments are discussed in Section 3.

2. Inverse Spectrum and Structure of Topological Metagroups

Theorem 9. *Assume that the conditions of Remark 1 in [41] or above are satisfied, and $G = A \star^{\phi,\eta,\kappa,\xi} B$ is a smashed twisted product of metagroups A and B with smashing factors ϕ, η, κ, ξ. Then, embeddings $\theta^G_A : A \hookrightarrow G$ and $\theta^G_B : B \hookrightarrow G$ exist, and $\theta^G_B(B)$ in G is invariant. Moreover, a transversal set $V_{G,B}$ exists such that $V_{G,B} = \theta^G_A(A)$.*

Proof. We shortly denote θ^G_A as θ_A, because G is specified, and we write $\theta_A(a) = (a, e)$ with $e = e_B$ for each $a \in A$; $\theta_B(b) = (e, b)$ with $e = e_A$ for each $b \in B$. From Formula (37) in [41] or (10 above, it follows that $(e, b)(a, e) = (a, b\xi((e, b),(a, e)))$ for each $a \in A$ and $b \in B$. Therefore, for each $g = (a_1, b_1)$ in G, there exist unique $a \in A$ and $b \in B$, such that

$$(e, b)(a, e) = g \text{ with } a = a_1 \text{ and } b = b_1/\xi((e, b_1),(a_1, e)), \tag{14}$$

since $\xi((e, b),(a, e)) = \xi((e, b_1),(a, e))$ by (35) in [41] or (8) above. Certainly, the maps $a_1 \mapsto a$ and $(a_1, b_1) \mapsto b$ provided by (14) are single-valued.

For each $g_1 = (e, b_1) \in G$, $g_2 = (a_2, b_2) \in G$, $g_3 = (a_3, b_3) \in G$, we deduce that $I_1 = (g_1g_2)g_3 = (a_2a_3, b_3^{a_2}(b_2b_1)\xi((e, b_1),(a_2, b_2))\xi((a_2, b_2b_1),(a_3, b_3)))$ and

$$I_2 = g_1(g_2g_3) = (a_2a_3,(b_3^{a_2}b_2)\xi((a_2, b_2),(a_3, b_3))b_1\xi((e, b_1),(a_2a_3, b_3^{a_2}b_2)))$$

by Conditions (31), (32), and (34) in [41], or (4), (5), and (7) above. Hence, $I_1 = tI_2$ with $t = t(g_1, g_2, g_3) \in \theta_B(\mathcal{C})$; consequently, $\theta_B(B)$ satisfies $a(bH) = (ab)H$ for each a and b in G, since $\mathcal{C} \hookrightarrow \mathcal{C}(B)$ by Remark 1 in [41] or above.

In view of Lemma 1 and Formula (14), the transversal set $V_{G,B} = \theta_A(A)$ and the maps $\psi = \psi^G_B : G \to \theta_B(B)$ and $\tau = \tau^G_B : G \to \theta_A(A)$ exist, such that

$$g = g^{\psi}g^{\tau} \tag{15}$$

with

$$g^{\psi} = (e, b) \text{ and } g^{\tau} = (a, e) \text{ for each } g = (a_1, b_1) \in G, \tag{16}$$

where

$$a = a_1 \text{ and } b = b_1/\xi((e, b_1),(a_1, e)). \tag{17}$$

It remains to prove that $\theta_B(B)$ is invariant in G. For this, it is sufficient to prove that

$$g_1\theta_B(B) = \theta_B(B)g_1 \tag{18}$$

and

$$(g_1\theta_B(B))g_2 = g_1(\theta_B(B)g_2) \text{ for each } g_1 \text{ and } g_2 \text{ in } G, \tag{19}$$

since Properties (18) and (19) imply that $(g_1g_2)\theta_B(B) = g_1(g_2\theta_B(B))$ for each g_1 and g_2 in G.

For each $g_1 = (a_1, b_1)$ in G and $b_2 \in B$, $b_3 \in B$ we obtain

$(a_1, b_1)(e, b_2) = (a_1, b_2^{a_1} b_1 \xi((a_1, b_1), (e, b_2)))$ and

$$(e, b_3)(a_1, b_1) = (a_1, b_1 b_3 \xi((e, b_3), (a_1, b_1)))$$

according to (37) in [41] or (10) above. The following equation,

$b_2^{a_1} b_1 \xi((a_1, b_1), (e, b_2)) = b_1 b_3 \xi((e, b_3), (a_1, b_1))$ has a unique solution

$b_3 = [b_1 \setminus (b_2^{a_1} b_1)] \xi((a_1, b_1), (e, b_2)) / \xi((e, [b_1 \setminus (b_2^{a_1} b_1)]), (a_1, b_1))$ for given $g_1 = (a_1, b_1)$ and $g_2 = (e, b_2)$, since ξ satisfies Condition (35) in [41] or (8) above. From $\xi(g_1, g_3) \in \mathcal{C}$ for each g_1 and g_3 in G, and $\mathcal{C} \hookrightarrow \mathcal{C}(B) \subset B$, it follows that $b_3 \in B$. Thus, G satisfies Condition (18).

Then, we consider $I_1 = (g_1(e, b_2))g_3$ and $\tilde{I}_2 = g_1((e, \tilde{b}_2)g_3)$ for any g_1 and g_3 in G, b_2 and $\tilde{b}_2$ in B. Then, we infer that

$I_1 = (a_1 a_3, b_3^{a_1}(b_2^{a_1} b_1) \xi((a_1, b_1), (e, b_2)) \xi((a_1, b_2^{a_1} b_1), (a_3, b_3)))$ and

$$\tilde{I}_2 = (a_1 a_3, (b_3^{a_1} \tilde{b}_2^{a_1}) \kappa(a_1, b_3, \tilde{b}_2) \xi((e, \tilde{b}_2), (a_3, b_3)) b_1 \xi((a_1, b_1), (a_3, b_3 \tilde{b}_2)))$$

by (33) and (37) in [41], or (6) and (10) above. The following equation, $I_1 = \tilde{I}_2$, is satisfied if and only if

$b_2^{a_1} b_1 \gamma = \tilde{b}_2^{a_1} b_1 \tilde{\alpha} p(b_3^{a_1}, \tilde{b}_2^{a_1}, b_1 \tilde{\alpha})$, with

$$\gamma = \xi((a_1, b_1), (e, b_2)) \xi((a_1, b_2^{a_1} b_1), (a_3, b_3));$$

$$\tilde{\alpha} = \kappa(a_1, b_3, \tilde{b}_2) \xi((e, \tilde{b}_2), (a_3, b_3)) \xi((a_1, b_1), (a_3, b_3 \tilde{b}_2))$$

by (i) and (ii) in Definition 1. Using (34), (35) and Lemma 2 in [41], or (7) and (8) above, we deduce that there exists a unique solution,

$$\tilde{b}_2^{a_1} = (b_2^{a_1} b_1 \gamma) / (b_1 \delta), \text{ for the given } g_1 = (a_1, b_1) \text{ and } g_3 = (a_3, b_3) \text{ in } G, b_2 \in B \tag{20}$$

with $\delta = \alpha p(b_3^{a_1}, b_2^{a_1}, b_1 \alpha)$, and

$$\alpha = \kappa(a_1, b_3, b_2) \xi((e, b_2), (a_3, b_3)) \xi((a_1, b_1), (a_3, b_3 b_2)).$$

Since $\gamma \in \mathcal{C}$ and $\delta \in \mathcal{C}$, $\mathcal{C} \hookrightarrow \mathcal{C}(B)$, then $\tilde{b}_2^{a_1} \in B$. From (31) and (33) in [41], or (4) and (6) above, it follows that

$$\tilde{b}_2 = (\tilde{b}_2^{a_1})^{e/a_1} / \eta(e/a_1, a_1, (\tilde{b}_2^{a_1})^{e/a_1}). \tag{21}$$

Hence, (20) and (21) imply that $\tilde{b}_2 \in B$; consequently, G satisfies Condition (19). Thus, $\theta_B(B)$ is the invariant submetagroup in G. □

Corollary 8. *If the conditions of Remark 1 in [41] or above are satisfied, A and B are topological T_1 metagroups, the topology on G is induced by the Tychonoff product topology on $A \times B$, and the smashing factors ϕ, η, κ, ξ are (jointly) continuous, then the maps $\psi : G \to \theta_B(B)$ and $\tau : G \to V_{G,B} = \theta_A(A)$ are continuous relative to the topology $\mathcal{T}_G$ on the topological metagroup $G = A \star^{\phi,\eta,\kappa,\xi} B$.*

Proof. This follows from Formulas (15)–(17) and the (joint) continuity of the smashing factors ϕ, η, κ, ξ, and hence of Div_r and t_G on $(G, \mathcal{T}_G)$, where the topology $\mathcal{T}_G$ on G is induced by the Tychonoff product topology on $A \times B$. □

Corollary 9. *For pairs of metagroups, let A_j, B_j (the conditions of Remark 1 in [41] or above) be satisfied for each $j \in \{1,2\}$, where $B_1 = B_2$, such that $\mathcal{C}_m(A_j) \subset \mathcal{C} \hookrightarrow B_j \hookrightarrow \mathcal{C}(A_j)$ for each $j \in \{1,2\}$. Let $\phi_2(a)b = id(b) = b$ for each $a \in \mathcal{C}$ and $b \in B_2$, and $\xi_2((a,e), (e,b)) = \xi_2((e,b),(a,e))$ for each $a \in A_2$ and $b \in B_2$. Let $A' = A_1 \star^{\phi_1,\eta_1,\kappa_1,\xi_1} B_1$ and $B = A_2 \star^{\phi_2,\eta_2,\kappa_2,\xi_2} B_2$, and let ϕ_3, η_3, κ_3, ξ_3 for the pair (A', B) with $\mathcal{C}'_1 = \theta^B_{B_2}(B_2)$ satisfy the conditions of Remark 1 in [41] or above (with $\mathcal{C}'_1$ instead of $\mathcal{C}$), and let $D = A' \star^{\phi_3,\eta_3,\kappa_3,\xi_3} B$, where $\theta^B_{B_2} : B_2 \hookrightarrow B$ is the embedding provided by Theorem 9. Then, there are embeddings $\theta_{A_j} : A_j \hookrightarrow D$, $\theta_{B_j} : B_j \hookrightarrow D$ for each $j \in \{1,2\}$, $\theta_B : B \hookrightarrow D$, such that D with $A = \theta_2(A_2)$ and $\mathcal{C}_1 = \theta_{B_2}(B_2)$ satisfy Condition (28) in [41] or (1) above, and $xA = Ax$ for each $x \in \mathcal{C}_1$.*

Proof. By virtue of Theorem 4 in [41], or Theorem 3 above, B, A' and D are metagroups and there are embeddings $\theta_{A_j} : A_j \hookrightarrow D$, $\theta_{B_j} : B_j \hookrightarrow D$ for each $j \in \{1,2\}$, $\theta_B : B \hookrightarrow D$, such that $\theta_B(B) = \theta_{A_2}(A_2)\theta_{B_2}(B_2)$, since $B_2 \hookrightarrow \mathcal{C}(A_2)$.

For each $(e,b) \in B$, $(a,e) \in B$ and $(a_2,e) \in B$ with $b \in B_2$, $a \in A_2$, $a_2 \in A_2$, we deduce that

$(a,e)(e,b) = (a, b^a\xi((a,e),(e,b)))$ and

$$(e,b)(a_2,e) = (a_2, b\xi((e,b),(a_2,e))).$$

Therefore, $(a,e)(e,b) = (e,b)(a_2,e)$ if and only if $a = a_2$ and $b^a\xi((a,e),(e,b)) = b\xi((e,b),(a_2,e))$. From (31) and (35) in [41], or (4) and (8) above, and the conditions of this corollary, it follows that $xA = Ax$ for each $x \in \mathcal{C}_1$, since $\phi_2(a)b = b^a = b$ for each $a \in A_2$ and $b \in B_2$. In view of Theorem 9, the subgroup $\mathcal{C}_1$ is invariant in $\theta_B(B)$ and $\theta_{A'}(A')$.

Certainly, $\theta_{B_1}(B_1)$ and $\theta_{B_2}(B_2)$ are isomorphic subgroups in D, since $B_1 = B_2$. Hence, each $d \in D$ can be presented in the following form: $d = a_1a_2b$ with $a_1 \in \theta_{A_1}(A_1)$, $a_2 \in \theta_{A_2}(A_2)$ and $b \in \theta_{B_1}(B_1)$. From $a_1\theta_{B_2}(B_2) = \theta_{B_2}(B_2)a_1$ and $a_2\theta_{B_2}(B_2) = \theta_{B_2}(B_2)a_2$, it follows that $d\theta_{B_2}(B_2) = \theta_{B_2}(B_2)d$ for each $d \in D$. On the other hand, $\mathcal{C}_m(D) \subset \mathcal{C}_1$, since $\mathcal{C}_m(A_j) \subset \mathcal{C} \hookrightarrow B_j \hookrightarrow \mathcal{C}(A_j)$ for each $j \in \{1,2\}$. Consequently, the subgroup $\theta_{B_2}(B_2)$ is invariant in D. □

Corollary 10. *Assume that the conditions of Corollary 9 are satisfied, A_j, B_j are T_1 topological metagroups for each $j \in \{1,2\}$, and ϕ_i, η_i, κ_i, ξ_i are jointly continuous for each $i \in \{1,2,3\}$. Then, D, A, $\mathcal{C}_1$, provided by Corollaries 8 and 9, are $T_1 \cap T_3$ topological metagroups and satisfy the conditions of Theorem 6 in [41] or Theorem 5 above, and $\mathcal{C}_1$ is closed in D.*

Proof. This follows from Theorem 4 in [41], or Theorem 3 above, and Corollaries 8 and 9 above. □

Definition 6. *Let Λ be a directed set, G_j be a topological metagroup (or quasigroup), and $\pi^j_i : G_j \to G_i$ be a continuous homomorphism for each $i \le j$ in Λ, such that $\pi^j_i \circ \pi^k_j = \pi^k_i$ for each $i \le j \le k$ in Λ, and $\pi^i_i = id_{G_i}$ for each $i \in \Lambda$, where $id_{G_i}(g_i) = g_i$ for each $g_i \in G_i$. Then, $S = \{G_j, \pi^j_i, \Lambda\}$ is called an inverse spectrum of topological metagroups (or quasigroups, respectively). If a topological metagroup G is a limit of S, $G = \varprojlim S$, then it is said that G is decomposed into S.*

Theorem 10. *There exists an infinite family $\mathcal{F}$, where each $G \in \mathcal{F}$ is a topological T_1 metagroup, such that G is compact, locally connected and can not be decomposed into the inverse spectrum $S_G = \{G_j, \pi^j_i, \Lambda\}$ of topological metagroups G_j with $dim(G_j) < \infty$ for each $j \in \Lambda$.*

Proof. We take any locally connected T_1 compact metagroups A, B, and their invariant closed subgroup $\mathcal{C}$ with positive covering dimensions $dim(A) > 0$, $dim(B) > 0$, $dim(A/_c\mathcal{C}) > 0$, $dim(B/_c\mathcal{C}) > 0$, such that the conditions of Theorem 6 in [41] or Theorem 5 above are satisfied. Evidently, such triples $(A,B,\mathcal{C})$ exist, and their family is infinite. Indeed, in particular, they may be direct products $A = K_1 \times \mathcal{C}$, $B = P_1 \times \mathcal{C}$ or semidirect products $A = K_1 \times^s \mathcal{C}$, $B = P_1 \times^s \mathcal{C}$ with topological T_1 metagroups K_1, P_1, and a topological T_1 group $\mathcal{C}$; or, in particular, A, B may be topological T_1 groups (see also examples (α)–(γ) in Remark 2 [42]). $\square$

Therefore, $\theta_B(B)$ is invariant in the smashed twisted product $G = A \star^{\phi,\eta,\kappa,\xi} B$, such that G is a topological T_1 metagroup, and a transversal set exists $V_{G,B} = \theta_A(A)$ by Corollary 1 in [41] or above and Theorem 9. By virtue of Corollary 8, the maps $\psi : G \to \theta_B(B)$ and $\tau : G \to V_{G,B}$ are continuous. For compact A and B, the metagroup G is compact by the Tychonoff Theorem 3.2.4 in [46].

This implies that there are triples $(A_1, B_1, \mathcal{C})$ and $(A_2, B_2, \mathcal{C})$ satisfying the conditions of Corollary 10 with locally connected T_1 compact metagroups A_1, A_2, $B_1 = B_2$, and their invariant closed subgroup $\mathcal{C}$, with positive covering dimensions $dim(A_1) > 0$, $dim(A_2) > 0$, $dim(B_1) > 0$, $dim(A_1/_c\mathcal{C}) > 0$, $dim(A_2/_c\mathcal{C}) > 0$, $dim(B_1/_c\mathcal{C}) > 0$. Then, D, A, $\mathcal{C}_1$, provided by Corollaries 8 and 9, satisfy the conditions of Corollary 1 in [41] or Theorem 9, such that $A = \theta_{A_2}(A_2)$. By virtue of Theorem 6 in [41] or Theorem 5 above, Theorem 1 in [43], Theorem 2.4 in [45], or Theorem 8 and Corollary 10 above, D, A, B are locally connected T_1 compact metagroups with a closed invariant subgroup $\mathcal{C}_1 = \theta_{B_2}(B_2)$ and $dim(D) > 0$, $dim(A) > 0$, $dim(B) > 0$. Moreover, $V_{D,A\mathcal{C}_1} = \theta_{A'}(A')$, with $A' = A_1 \star^{\phi_1,\eta_1,\kappa_1,\xi_1} B_1$, and there is a bijection from $V_{A\mathcal{C}_1,A} = V_{\mathcal{C}_1,\mathcal{C}_{1,A}}$ onto $(A\mathcal{C}_1)/_cA$ by Remark 3 in [42] and $V_{\mathcal{C}_1,\mathcal{C}_{1,A}} V_{D,A\mathcal{C}_1} = V_{D,A}$ by Formula (3) in [42]. Therefore, $V_{\mathcal{C}_1,\mathcal{C}_{1,A}}$ and $V_{D,A\mathcal{C}_1}$ can be chosen to be compact; consequently, $V_{D,A}$ is compact by the Tychonoff Theorem 3.2.4 in [46]. Corollary 5 and Corollary 8 imply that the maps $\psi^D_A : D \to A$ and $\tau^D_A : D \to V_{D,A}$ are continuous relative to the topology $\mathcal{T}_D$.

In view of Theorems 3 and 4 in [42], there exists a $T_1 \cap T_3$ compact metagroup $C_0 = D\Delta^{\phi,\eta,\kappa,\xi}_A F_0$, where F_0 is closed in $(C(V, B), \mathcal{T}_{\mathcal{W}})$, where $F_0 \subset C(V, B) \subset F = B^V$, $V = V_{D,A}$. Hence, $dim(C_0) > 0$ and C_0 is locally connected. In view of Theorem 3.1.9 in [46], D, A, B, $\mathcal{C}_1$, C_0 are $T_1 \cap T_4$ topological spaces. By the construction above, $card(V) \geq \aleph_0$.

On the other hand, a family $Hom_{c,\mathcal{C}} = Hom_{c,\mathcal{C}}((A \times F_0) \times (A \times F_0), \mathcal{C}_1)$ of all continuous homomorphisms from $(A \times F_0) \times (A \times F_0)$ into $\mathcal{C}_1$, satisfying (35) in [41] or (8) above, is a proper closed subset in a family $C_{\mathcal{C}} = C_{\mathcal{C}}((A \times F_0) \times (A \times F_0), \mathcal{C}_1)$ of all continuous maps from $(A \times F_0) \times (A \times F_0)$ into $\mathcal{C}_1$, satisfying (35) in [41] or (8) above. Since $dim(B) > 0$ and $dim(V) > 0$, then $dim(C(V, B)) = \infty$, where $C(V, B)$ is in the $\mathcal{T}_{\mathcal{W}}$ topology.

We choose F_0 with $dim(F_0) = \infty$ and the map $\xi \in C_{\mathcal{C}} - Hom_{c,\mathcal{C}}$ with values in $\mathcal{C}_1$ such that $\xi((d^\psi_1, f_1), (d^\psi, f))(v)$ depends nontrivially on infinite number of coordinates $v \in V$ for an infinite family of $(f_1, f) \in F_0 \times F_0$ for each $d^\psi_1 \neq e$ and $d^\psi \neq e$, where d and d_1 belong to D, $f \in F_0$, $f_1 \in F_0$, since $dim(\mathcal{C}_1) > 0$ and $dim(A_2) > 0$. This implies that there exists the topological metagroup $G = C_0$ with $dim(G) = \infty$, which cannot be decomposed into the inverse spectrum $S_G = \{G_j, \pi^j_i, \Lambda\}$ of topological metagroups G_j with $dim(G_j) < \infty$ for each $j \in \Lambda$. From the proof above, it follows that the family of such topological metagroups is infinite.

Remark 4. *If, instead of Theorem 6, we use Theorem 5 in [41], or instead of Theorem 5, we use Theorem 4 above, then Theorem 10 will be for topological unital quasigroups (loops) G.*

Thus, Theorem 10 illustrates a principal structural distinction between topological groups and topological metagroups.

3. Conclusions

All the primary results of this article are obtained for the first time. They can be used for further studies of noncommutative analysis and noncommutative harmonic analysis [15,16,47,48], operator theory, generalized C^*-algebras [20,22], topological quasigroups, and representations of topological quasigroups [3,18,47], in noncommutative geometry [13,18], PDEs [11,25,34], nonassociative quantum field theory [27,28], nonassociative quantum mechanics and quantum gravity [31,32], gauge theory, the great unification theory [12,29,30,33–36], informatics, and coding theory [37–40,49].

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