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Qualitative Properties of the Solutions to the Lane–Emden Equation in the Cylindrical Setup

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Abstract: We analyze the Lane–Emden equations in the cylindrical framework. Although the explicit forms of the solutions (which are also called polytropes) are not known, we identify some of their qualitative properties. In particular, possible critical points and zeros of the polytropes are investigated and discussed, leading to possible improvements in the approximation methods which are currently employed. The cases when the critical parameter is odd and even are separately analyzed. Furthermore, we propose a technique to evaluate the distance between a pair of polytropes in small intervals.

Keywords: Lane–Emden equation; polytrope; ODEs

MSC: 45J05; 44A45; 76E20; 85A99

1. Introduction

One of the most fascinating open problems in Applied Mathematics is the Lane–Emden equation, together with its variation, the Emden–Fowler equation, which was initially proposed in 1870 by Jonathan Homer Lane [1] and subsequently extended by Robert Emden [2] in 1907, who aimed to model the dynamic behaviour of a non-rotating fluid subject to internal pressure and self-gravity. In order to briefly introduce the physical setting, the Lane–Emden equation originates from the combination of Poisson's equation and a generic polytropic equation of state, $P = K_N \rho^\gamma$, where P and ρ, respectively, are the pressure and the density of a fluid, K_N is a positive constant, and $\gamma = 1 + \frac{1}{N}$ is the ratio of specific heats (see [3,4] for more details). After some simple manipulations, the Lane–Emden equation is derived as having $\theta(x) = \left(\frac{\rho(x)}{\rho_c}\right)^{1/N}$, where ρ_c is the central density.

The Lane–Emden equation has encountered wide success, especially in the 1930s, both in physics, where Sir Ralph Howard Fowler [5,6] found and generalized further results and gave birth to the Emden–Fowler equation, and in astrophysics, where Chandrasekhar established the related spherical solutions in [7], the first edition of which was published in 1939 and then subsequently reprinted in 1967. Furthermore, Chandrasekhar and Fermi applied the Lane–Emden equation to isothermal filaments [8], some years later.

Successively, many contributions have been published on the equation, its several modified versions, and its applications. Christodoulou and Kazanas [9] derived exact asymmetric solutions of the Lane–Emden equation under rotation. A major result had already been provided by Jeremiah Paul Ostriker [10] in 1964; he was able to determine the solutions to the equation in closed form for cylindrical polytropes for the parameters $N = 0$ (i.e., liquid cylinders), $N = 1$, and $N = \infty$ (i.e., cylinders with an isothermal perfect gas). In the astrophysics literature, a solution to the Lane–Emden equation is often called a polytrope. We will also use this denomination throughout this paper. For those who are

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willing to develop an extensive knowledge of polytropes, the main textbook on this subject was published by Horedt [11] in 2004.

The present study focuses on Lane–Emden equations in the cylindrical setup. The reason for this focus is that quite recent astronomical observations, particularly the ones obtained by the Herschel space observatory (see [12]), show that star-forming regions occur preferentially in long thin cylindrical filaments of gas. The radial density distribution within the filaments can be well approximated with the solutions of appropriate Lane–Emden equations, but the specific profiles differ considerably from the expected profile for the isothermal case mentioned above. There seems to also be clear evidence that the polytropic index, N, can change within a single filament. Theoretical studies show that this behaviour could be due to temperature gradients within a filament (see [3]). Moreover, star-forming regions are highly dynamic, and, hence, the properties of star-forming filaments change over time. These arguments show, on the one hand, the relevance of analytical solutions for the Lane–Emden equations, and, on the other hand, show that differences between analytical solutions with different values of N are important tools to obtain insights about real-world phenomena. We would like to outline, however, that a detailed comparison with astrophysical observations is out of the scope of this work. We wish here to lay down purely mathematical ideas, which will be further developed in follow-up papers (see Section 6).

In recent years, the Lane–Emden equation has been widely studied in several versions, although it can be solved in closed form in only a few cases. An approach based on operational calculus, initially introduced by Adomian [13], was outlined by Bengochea et al. in some recent works [14–16]. In particular, in [14], a procedure is derived which is based on a linear operator acting on the set of all formal series, which turns out to be helpful in solving several kinds of differential equations with variable coefficients, fractional differential equations, and difference equations as well. Such an approach is adopted in [15], to determine an algebraic solution to a specific version of the Lane–Emden equation. More recently, a numerical approximation algorithm was proposed by [17]. Furthermore, some novel computational strategies were implemented by [18,19].

Our approach is substantially different from the ones adopted in the above-mentioned literature. Firstly, we focus only on the cylindrical setup, which can be identified by the use of the following class of equations for each integer $N \in \mathbb{Z}$:

$$\theta''(x) + \frac{\theta'(x)}{x} + \theta^N(x) = 0$$

Actually, we might also extend this analysis to $N \in \mathbb{R}$, although the most intuitive application of Lane–Emden theory is to integer numbers. We are trying to establish a number of qualitative properties of the related solutions, in order to better visualize their behaviour. In our analysis, what emerges about these solutions is a slight difference between the cases where N is odd and where N is even. To the best of our knowledge, no previous study has ever captured such a qualitative distinction. The most recent studies, such as [17–19], rely on well-structured numerical methods for approximation or on novel techniques that are implemented in wider classes of problems (see [14–16,20]). We stress that our attempt is somewhat more specific, because a deep qualitative analysis may lead to new ideas to refine the search for the closed form solutions that are still unknown.

The following is a basic summary of the present work.

- We reconstruct the extended derivations of the basic Lane–Emden equations in the basic scenarios;
- We outline the current state of the art, including explicit solutions, solution methods, and cases in which the Lane–Emden equation is still unsolved;
- We identify a sequence of qualitative properties of the solutions in the cylindrical scenario. In particular, two distinct analyses are carried out, depending on whether the critical exponent, M, is either odd or even;
- Finally, we expose a relation which may be helpful in evaluating the distance between a pair of solutions in a small interval.

The remainder of this paper is structured as follows. In Section 2, we introduce the Lane–Emden differential equation, together with some of its variations, and in Section 3 we present an overview of some known solutions, emphasizing the cases where $N = 0$ and $N = 1$. Our main analytical results are collected in Section 4, where some qualitative properties are stated and demonstrated. In Section 5, the results are summarized and discussed. Finally, our concluding remarks and some ideas regarding future developments can be read in Section 6.

2. The Standard Lane–Emden Equation

Firstly, we introduce the Lane–Emden equation in its well-known form, as follows:

$$\theta''(x) + k\frac{\theta'(x)}{x} + \theta^N(x) = 0, \tag{1}$$

Based on the value of k, either we have the following cylindrical setting, if $k = 1$:

$$\theta''(x) + \frac{\theta'(x)}{x} + \theta^N(x) = 0, \tag{2}$$

or we have the following spherical setting, if $k = 2$:

$$\theta''(x) + 2\frac{\theta'(x)}{x} + \theta^N(x) = 0. \tag{3}$$

2.1. Initial Conditions and an Analysis of the Singularity

The standard boundary conditions that form a Cauchy problem with a Lane–Emden equation are the specifications of the values of θ and θ' at 0, i.e., $\theta(0) = 1$ and $\theta'(0) = 0$. Namely, the value of θ at 0 is due to its definition, whereas the vanishing of its derivative at 0 indicates the absence of gravity in the cylinder's axis (see [10,21] for more explanation on the related physical motivations).

It is worthwhile to discuss the singularity topic, because the initial conditions of the Lane–Emden problem are typically taken to be $x = 0$. In (1), there seems to be a singularity at 0; hence, a specific strategy is necessary to overcome such a critical characteristic of the equation. Namely, (1) can be also written as follows:

$$\frac{\theta''(x) + \theta^N(x)}{k} = -\frac{\theta'(x)}{x}.$$

Since the initial condition of the related Cauchy problem is $\theta'(0) = 0$, meaning that we are analysing the problem in the neighbourhood of 0, we can note that an indeterminate form appears in the right-hand side of the above expression. By applying De L'Hospital's Theorem (as we will also do when carrying out the qualitative analysis in Section 4), we deduce that the limit is finite, i.e.,

$$\lim_{x \longrightarrow 0}\left(-\frac{\theta'(x)}{x}\right) = -\lim_{x \longrightarrow 0}\theta''(x).$$

A further result is established as well; taking the limits for x and tending to 0 on both sides also provides the value of the second order derivative at 0, i.e., $\theta''(0) = -\frac{1}{k+1}$, which can be easily verified based on the form of the known exact solutions in Section 3. Additionally, the results that we will expose in Section 4 will clarify that such a singularity always disappears when computing the limits at $x = 0$, and that, consequently, the analysis of the Lane–Emden equation does not suffer from this critical aspect.

2.2. Modified Versions of the Lane–Emden Equation

Some researchers extend the form of (1) to establish the definitions of other classes of Lane–Emden equations. For example, in [22], (1) is referred to as a Lane–Emden equation of the first kind (see [20]), whereas the Lane–Emden equation of the second kind has the following formulation:

$$\theta''(x) + k\frac{\theta'(x)}{x} + e^{\theta(x)} = 0. \tag{4}$$

Moreover, in [20], a further version is mentioned, originating from a change of variable in (4), whose form is

$$\theta''(x) + k\frac{\theta'(x)}{x} + e^{-\theta(x)} = 0$$

and whose initial conditions are replaced with $\theta(0) = \theta'(0) = 0$, which turns out to be the profile of isothermal cylinders (see [10] for the derivation of the hydrostatic problem).

A whole class of Lane–Emden problems can be established by employing the most general form, as follows:

$$\theta''(x) + k\frac{\theta'(x)}{x} + f(\theta(x)) = 0, \tag{5}$$

where $f(\cdot)$ is a sufficiently regular function of $\theta(x)$.

It is also interesting to remark that (5) can be reformulated as an integro-differential equation. By multiplying the left hand side by x^k, we have

$$x^k\theta''(x) + kx^{k-1}\theta'(x) + x^k f(\theta(x)) = 0,$$

which is equivalent to the equation

$$\Big(x^k\theta'(x)\Big)' = -x^k f(\theta(x)),$$

which can be integrated on both sides, entailing

$$\theta'(x) = -\frac{1}{x^k}\int_0^x t^k f(\theta(t))dt, \tag{6}$$

where the initial condition turns out to be

$$\lim_{x\longrightarrow 0}\frac{1}{x^k}\int_0^x t^k f(\theta(t))dt = 0,$$

which holds if and only if

$$\lim_{x\to 0} x f(\theta(x)) = 0,$$

by De L'Hospital's Theorem.

Form (6) is commonly used for numerical approximations of the solutions (see, for example, [22]). Perhaps the most relevant modification of the Lane–Emden equation is the Emden–Fowler equation, as follows (see Chandrasekhar [7] or Fowler's contributions [5,6]):

$$\frac{d}{dx}\left(x^\rho\frac{dy}{dx}\right) + x^\alpha y^\tau(x) = 0, \qquad x \geq 0, \tag{7}$$

where $\rho, \alpha \in \mathbb{R}$, $\tau \in \mathbb{R}_+$. Many papers contain a number of results for (7); a survey outlining the results found up to 1975 is discussed in [23], whereas subsequent relevant papers include [24–26], as well as many others.

Such equations can be transformed into the following modified form:

$$y''(x) - h(x)y^\tau(x) = 0, \qquad x \geq 0, \tag{8}$$

where $h(x)$ is a continuous and non-negative function.

3. Exact Solutions

In the literature, the known solutions to (2) in closed forms are only available for $N = 0$, $N = 1$, and $N = \infty$. In particular, since our main interest lies in the solution of the Lane–Emden equations for specific values of N (see Section 4), the case of $N = \infty$ must, necessarily, be neglected. Physically, as we mentioned in the introduction, the case $N = \infty$ does not fit astrophysical observations of star-forming filaments.

We will proceed to briefly outlining the related polytropes and solution procedures for $N = 0$ and $N = 1$.

3.1. Polytropes for N = 0

The easiest case occurs when $N = 0$, and we can trivially solve this via the separation of the variables (this case is far from reality, in that $N = \frac{1}{\gamma-1}$. Despite this, we will outline the polytropes for completeness). In fact, in this case, a generalization of (3) and (2) can be solved as well.

Proposition 1. *All generalized Lane–Emden equations of the following kind:*

$$\begin{cases} \dfrac{1}{x^k}\dfrac{d}{dx}\left(x^k\dfrac{d\theta}{dx}\right) + \theta^N(x) = 0 \\ \theta(0) = 1 \\ \theta'(0) = 0 \end{cases} \tag{9}$$

can be solved for all $k \geq 0$ when $N = 0$, and the solution is the following family of parabolas:

$$\theta_k^*(x) = 1 - \frac{x^2}{2(k+1)}. \tag{10}$$

Proof. When $N = 0$, (9) amounts to:

$$\frac{d}{dx}\left(x^k\frac{d\theta}{dx}\right) = -x^k.$$

Then, after integrating both sides, we have:

$$x^k\frac{d\theta}{dx} = -\frac{x^{k+1}}{k+1} + C_0 \iff \cdots \iff \theta(x) = -\frac{x^2}{2(k+1)} + \frac{C_0}{(-k+1)x^{k-1}} + C_1$$

Then, applying the boundary conditions yields $C_0 = 0$ and $C_1 = 1$, leading to the following family of parabolas, indexed by k: $\theta_k^*(x) = 1 - \dfrac{x^2}{2(k+1)}$. □

The respective polytropes for (2) and (3) are as follows:

$$\theta_1^*(x) = 1 - \frac{x^2}{4}, \qquad \theta_2^*(x) = 1 - \frac{x^2}{6}.$$

3.2. Polytropes for N = 1

When $N = 1$, the polytrope of (3) is known as well. Expanding Equation (3) yields the following:

$$\frac{1}{x^2}\left(2x\theta'(x) + x^2\theta''(x)\right) + \theta(x) = 0 \iff \theta''(x) + \frac{2}{x}\theta'(x) + \theta(x) = 0. \tag{11}$$

On the other hand, expanding the form (2) yields the following:

$$\frac{1}{x}\left(\theta'(x) + x\theta''(x)\right) + \theta(x) = 0 \iff \theta''(x) + \frac{1}{x}\theta'(x) + \theta(x) = 0. \tag{12}$$

In order to solve these, we assume a power series solution of the following kind (where $a_0 = 1$ because $\theta(0) = 1$):

$$\theta(x) = 1 + \sum_{j=1}^{\infty} a_j x^j. \tag{13}$$

Plugging (13) into (11) leads to

$$\sum_{j=2}^{\infty}(j-1)ja_j x^{j-2} + 2\sum_{j=1}^{\infty} ja_j x^{j-2} + 1 + \sum_{j=1}^{\infty} a_j x^j = 0 \iff$$

$$\iff \frac{2a_1}{x} + 2a_2 + 4a_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + 2j)a_j + a_{j-2}]x^{j-2} = 0,$$

whose coefficients are supposed to verify

$$a_1 = 0, \qquad a_2 = -\frac{1}{6}, \qquad a_j = -\frac{a_{j-2}}{j(j+1)};$$

hence, the polytrope is as follows:

$$\theta^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{(2j+1)!} = \frac{\sin x}{x}. \tag{14}$$

An analogous procedure can be carried out to solve (12). Plugging (13) into (12) yields

$$\sum_{j=2}^{\infty}(j-1)ja_j x^{j-2} + \sum_{j=1}^{\infty} ja_j x^{j-2} + 1 + \sum_{j=1}^{\infty} a_j x^j = 0 \iff$$

$$\iff \frac{a_1}{x} + 2a_2 + 2a_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + j)a_j + a_{j-2}]x^{j-2} = 0,$$

whose coefficients are

$$a_1 = 0, \qquad a_2 = -\frac{1}{4}, \qquad a_j = -\frac{a_{j-2}}{j^2},$$

leading to the following polytrope:

$$\theta^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{((2j)!!)^2}. \tag{15}$$

The next proposition intends to generalize the above findings, as in Proposition 1.

Proposition 2. *All generalized Lane–Emden equations of the following kind:*

$$\begin{cases} \dfrac{1}{x^k}\dfrac{d}{dx}\left(x^k \dfrac{d\theta}{dx}\right) + \theta^N(x) = 0 \\ \theta(0) = 1 \\ \theta'(0) = 0 \end{cases}$$

can be solved for all $k \geq 0$ when $N = 1$, and the solution is the following family of power series:

$$\theta_k^*(x) = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j x^{2j}}{(2j)!!(2j-1+k)!!}. \tag{16}$$

Proof. Expanding the equation leads to the following:

$$\theta''(x) + \frac{k}{x}\theta'(x) + \theta(x) = 0. \tag{17}$$

Employing the above method, we obtain the following:

$$\frac{ka_1}{x} + 2a_2 + 2ka_2 + 1 + \sum_{j=3}^{\infty}[((j-1)j + kj)a_j + a_{j-2}]x^{j-2} = 0.$$

By exploiting the above relation, all the coefficients can be calculated explicitly. For example, the three terms not containing x yield $2a_2 + 2ka_2 + 1 = 0$ imply that $a_2 = -1/(2(k+1))$. Considering all the terms in the identity, we can also obtain a recurrence relation to generate all coefficients for $j \geq 3$, as follows:

$$a_1 = 0, \qquad a_2 = -\frac{1}{2+2k}, \qquad a_j = -\frac{a_{j-2}}{j(j+k-1)},$$

Consequently, the polytrope is (16). □

4. Analytical Properties in the Cylindrical Scenario

In this Section, we will establish some qualitative properties of the solutions to (2). We begin from some elementary analytical results, and then proceed to provide some insights regarding the graph of the involved functions. From now on, we will indicate, with $\theta_M^*(x)$, the solution to the cylindrical Lane–Emden equation for $N = M$.

Proposition 3. *For all $M \geq 0$, we have that $(\theta_M^*)''(0) = -\frac{1}{2}$.*

Proof. It is elementary to collect the terms in (2) as follows:

$$(\theta_M^*)''(x) + \frac{(\theta_M^*)'(x)}{x} + (\theta_M^*)^M(x) = 0 \qquad \Longleftrightarrow \qquad \frac{\left((\theta_M^*)'(x)x\right)'}{x} = -(\theta_M^*)^M(x).$$

Now we call $F_M(x) = (\theta_M^*)'(x)x$, whose derivatives, respectively, are as follows:

$$F_M'(x) = (\theta_M^*)''(x)x + (\theta_M^*)'(x), \qquad F_M''(x) = (\theta_M^*)'''(x)x + 2(\theta_M^*)''(x).$$

Since $F_M'(0) = (\theta_M^*)'(0) = 0$, and by the initial condition $\theta_M^*(0) = 1$, we can deduce the following:

$$\lim_{x \longrightarrow 0} \frac{F_M'(x)}{x} = -1$$

However, the above limit is equal to $F_M''(0)$ by De L'Hospital's Theorem, hence the following:

$$F_M''(0) = 2(\theta_M^*)''(0) = -1,$$

which implies that $(\theta_M^*)''(0) = -\frac{1}{2}$. □

It is simple to check that the same procedure illustrated in Proposition 3 can be extended to calculate the higher order derivatives of the solution at zero. Although we will not be further developing this argument in this paper, the implementation of this method might provide an approximation series of the solution in a neighbourhood of the origin.

As an illustrative example of the method, we can check the value of the third derivative at 0.

Since $F''(x) = x(\theta_M^*)'''(x) + 2(\theta_M^*)''(x)$, differentiating the right-hand side as well yields the following:

$$x(\theta_M^*)'''(x) + 2(\theta_M^*)''(x) = -(\theta_M^*)^M(x) - Mx(\theta_M^*)^{M-1}(x)(\theta_M^*)'(x)$$

which can be reformulated as follows:

$$(\theta_M^*)'''(x) = -\frac{2(\theta_M^*)''(x) + (\theta_M^*)^M(x)}{x} - M(\theta_M^*)^{M-1}(x)(\theta_M^*)'(x).$$

Subsequently, evaluating both sides at 0 entails the following:

$$(\theta_M^*)'''(0) = \lim_{x\longrightarrow 0}\left(-\frac{2(\theta_M^*)''(x) + (\theta_M^*)^M(x)}{x} - M(\theta_M^*)^{M-1}(x)(\theta_M^*)'(x)\right)$$

Then, by De L'Hospital's Theorem, we obtain $(\theta_M^*)'''(0) = -2(\theta_M^*)'''(0)$, implying that $(\theta_M^*)'''(0) = 0$.

Furthermore, Proposition 3 establishes that all M, $\theta_M^*(x)$ is concave in a neighbourhood of 0. As a matter of fact, the solution that we explicitly know for $M = 0$ is a parabola, with decreasing and concave behaviour for $x > 0$. As is well-known, if the function admits no inflection points for $x > 0$, this is a sufficient condition to guarantee the existence of a zero x_M^*. When $M = 0$, $x_0^* = 2$.

The following results intend to establish some further qualitative properties of $\theta_M^*(x)$, which are verified for all $M \geq 1$.

Proposition 4. *If $\theta_M^*(x)$ admits at least a positive zero for $M \geq 1$, and x_M is the smallest zero of $\theta_M^*(x)$, then one of the following conditions holds:*

1. $\theta_M^*(x_M) = (\theta_M^*)'(x_M) = (\theta_M^*)''(x_M) = \cdots = (\theta_M^*)^{(k)}(x_M) = 0$ *for all* $k \in \mathbb{Z}_+$;
2. *The function $\theta_M^*(x)$ admits at least one inflection point $F(x_F, y_F)$, such that $0 < x_F < x_M$.*

Proof. If we call $F_M(x) = (\theta_M^*)'(x)x$, it is easy to note that $F_M'(x_M) = 0$ by construction. Since $F_M'(x_M) = (\theta_M^*)''(x_M)x_M + (\theta_M^*)'(x_M) = 0$, two cases may occur. In the first case, both the first and second derivatives of $\theta_M^*(x)$ vanish at x_M, but this, necessarily, implies that all the derivatives of any order vanish at x_M, which is the least interesting scenario for qualitative analysis. In the second case, we have that $x_M = -\dfrac{(\theta_M^*)'(x_M)}{(\theta_M^*)''(x_M)}\theta_M^*(x_M)$, which can only hold if the second order derivative changed its sign in the interval $(0, x_M)$, meaning that the graph has an inflection point at $x_F < x_M$. □

Now, we will provide further insights on the behaviour of the solution by separating two circumstances, specifically where M is odd and where M is even, because some relevant differences occur. The role of possible inflection points, zeros, and stationary points will be analysed in detail.

4.1. Qualitative Behaviour if M Is Odd

The presence of a stationary point, i.e., either a maximum or a minimum point when M is odd, is an interesting issue. If we suppose that $\theta_M^*(x)$ admits one stationary point, x^*, such that $(\theta_M^*)'(x^*) = 0$, in the main equation we would have the following:

$$(\theta_M^*)''(x^*) = -(\theta_M^*)^M(x^*).$$

If $\theta_M^*(x^*) > 0$, this point can only be a local maximum, by the negativity of the second order derivative. Vice versa, if $\theta_M^*(x^*) < 0$, it is a local minimum, and clearly $x_F < x_M < x^*$.

The above considerations establish that, if M is odd, $(\theta_M^*)(x)$ can only have a maximum point by having both positive coordinates. On the other hand, any local minimum has a negative image; hence, there is always at least one inflection point and a zero between each maximum and minimum point.

4.2. Qualitative Behaviour if M Is Even

If M is even, i.e., a positive integer greater than or equal to 2, the results are slightly different, with respect to the previous case. Suppose that $x^* > 0$ is the first stationary point for $\theta_M^*(x)$. If $\theta_M^*(x^*) > 0$, the negativity of the second order derivative implies that such a point is a local maximum, but this holds true even if $\theta_M^*(x^*) < 0$. A stationary point can only be a maximum point; therefore, there can only be one maximum point, after which the solution decreases asymptotically. No oscillating behaviour is feasible in this case, unlike in the easiest case we have seen, where $M = 0$ and the polytrope is monotonically decreasing. There may be some changes in the convexity/concavity form of the graph, but the behaviour is unambiguously decreasing.

4.3. Evaluation of the Difference between the Two Solutions

If we call θ_M^* and θ_P^* the solutions for any $M, P \in \mathbb{Z}_+$, where $M \neq P$, we posit that $k = 1$; i.e., we are in the cylindrical setup. With (1), we have the following:

$$\begin{cases} (\theta_M^*)''(x) + \dfrac{(\theta_M^*)'(x)}{x} + (\theta_M^*)^M(x) = 0 \\ \\ (\theta_P^*)''(x) + \dfrac{(\theta_P^*)'(x)}{x} + (\theta_P^*)^P(x) = 0 \end{cases} \iff$$

$$\iff \quad \begin{cases} \dfrac{\left((\theta_M^*)'(x)x\right)'}{x} = -(\theta_M^*)^M(x) \\ \\ \dfrac{\left((\theta_P^*)'(x)x\right)'}{x} = -(\theta_P^*)^P(x) \end{cases}.$$

Now, if we call $F_M(x) = (\theta_M^*)'(x)x$ and $F_P(x) = (\theta_P^*)'(x)x$, we obtain the following dynamic system:

$$\begin{cases} F_M'(x) = -x(\theta_M^*)^M(x) \\ \\ F_P'(x) = -x(\theta_P^*)^P(x) \end{cases}$$

which is endowed with the initial conditions $F_M(0) = 0$ and $F_P(0) = 0$. By subtracting the left-hand sides, we obtain the following:

$$(F_M(x) - F_P(x))' = x\left[(\theta_P^*)^P(x) - (\theta_M^*)^M(x)\right]$$

from which we then obtain, after integrating both sides, the following:

$$F_M(x) - F_P(x) = \int_0^x t\left[(\theta_P^*)^P(t) - (\theta_M^*)^M(t)\right]dt,$$

i.e.,

$$(\theta_M^*(x) - \theta_P^*(x))' = \frac{\int_0^x t\left[(\theta_P^*)^P(t) - (\theta_M^*)^M(t)\right]dt}{x};$$

then, by integrating both sides again, we obtain the following:

$$\theta_M^*(x) - \theta_P^*(x) = \int_0^x \left[\frac{\int_0^t s\left[(\theta_P^*)^P(s) - (\theta_M^*)^M(s)\right]ds}{t}\right]dt. \tag{18}$$

In (18), the difference between solutions is on the left-hand side, whereas the difference between their powers is in the double integral on the right-hand side. This relation can be employed to identify an approximation method for the polytropes, with the help of the above considerations on the qualitative behaviour of the solutions.

5. Summary of the Main Results

In this paper, we identified some properties of the solutions of the Lane–Emden equation (i.e., the polytropes) in a cylindrical framework, especially taking into account the critical points and their possible positions in the graph of the functions. The qualitative properties that we exposed may be helpful in either constructing possible new explicit forms for the polytropes or implementing methods for approximation. Moreover, the results obtained in this paper (in particular in Section 4) can help shed light on the properties of important astrophysical objects, such as filamentary star-forming regions.

Such results can potentially be extended and improved. Eventually, important follow-up work will be devoted to an accurate study of the relevance of our mathematical analysis and the results of the study of star-forming filaments. In particular, our method for approximating solutions with different polytropic indices, as in Equation (18), can be used to study the temporal evolution of the density profile of filaments in specific regions. Since star formation changes the thermal properties of a filament, this, in turn, leads to temporal variations of the polytropic index, which can be described by our solutions if we can identify P and M in (18) as values of N at different moments in time.

Finally, some new elements for a better understanding of the behaviour of the solutions may emerge from the comparison between polytropes, which can also be viewed as a double integral and can, therefore, be numerically approximated by traditional methods.

6. Concluding Remarks and Discussion

The findings in the present work can be further extended in several different ways. A future development of the present work may concern the realization and computational optimization of a suitable algorithm to constructively approximate the real solutions in their explicit forms.

Moreover, we will also use existing numerical schemes, in follow-up papers, to approximate the solution of the Lane–Emden equations in relevant cases. One simple method that can be employed is a Runge–Kutta scheme (see [4]), but we will also employ modern techniques based in higher order Haar wavelet methods, tailored to the Lane–Emden equations [18,19].

Another possible extension of this work might specifically be focused on Emden–Fowler equations of several kinds. In this respect, some studies may be carried out in future that are based on the recent methodology introduced by Rufai and Ramos for third-order Emden–Fowler equations [25] in 2023.

More generally, the qualitative analysis of the Emden–Fowler equations' solutions is a complex and stimulating issue.

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