Article

Poncelet Porisms and Loci of Centers in the Isotropic Plane

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Abstract: Any triangle in an isotropic plane has a circumcircle u and incircle i. It turns out that there are infinitely many triangles with the same circumcircle u and incircle i. This one-parameter family of triangles is called a poristic system of triangles. We study the trace of the centroid, the Feuerbach point, the symmedian point, the Gergonne point, the Steiner point and the Brocard points for such a system of triangles. We also study the traces of some further points associated with the triangles of the poristic family, and we prove that the vertices of the contact triangle, tangential triangle and anticomplementary triangle move on circles while the initial triangle traverses the poristic family.

Keywords: isotropic plane; Poncelet porism; centroid; Feuerbach point; symmedian point; Gergonne point; Steiner point; Brocard points

MSC: 51N25

1. Introduction

A porism is a theorem about a closure property of a geometric figure or construction, as explained in [1] (pp. 412–417). A well-known example comes from the elementary triangle geometry. Any triangle in the Euclidean plane has a circumcircle u and incircle i. It turns out that there are infinitely many triangles with the same circumcircle u and incircle i. In other words, if we pick a point P_1 on u and draw a tangent t_1 from P_1 to i, then t_1 intersects u in points P_2 and P_1. Now, we draw a tangent $t_2 \neq t_1$ from P_2 to i that intersects u in P_3 and P_2. Finally, we draw a tangent $t_3 \neq t_2$ from P_3 to i that intersects u in P_4 and P_3. It turns out that $P_4 = P_1$. The one-parameter family of triangles with a common incircle and a common circumcircle is usually called a porisitic system of triangles. In the Euclidean plane, the poristic family of triangles interscribed in between the incircle and the circumcircle is called Chapple's porism. The fact that such families exist even in some non-Euclidean planes is due to the projective nature of the problem. Hence, it is not necessary to prove the existence of a closed triangular path between two isotropic circles (Euclidean parabolas with parallel axes). Nevertheless, in Section 3.1, in order to obtain a parametrization of the poristic family, we offer the proof that the described property is also valid in an isotropic plane.

In [2], the author showed that many triangle centers move on circles while the triangle traces the poristic family. Some centers move on conics, and some centers move on curves of a higher degree. In this paper, in Section 3.2, we study the loci of some triangle centers in the isotropic plane. A further investigation of the loci of triangle centers in the Euclidean plane is given in [3], where the authors studied a poristic family defined by two confocal ellipses.

In Section 3.3, we study the traces of the vertices of the contact triangle, tangential triangle and anticomplementary triangle associated with the triangle of the poristic family in the isotropic plane.

2. Methods

An isotropic plane is a projective plane with a distinct line f and a distinct point $F \in f$. Lines incident with the absolute point F are called isotropic lines, and points incident with

the absolute line f are called isotropic points. Two lines are said to be parallel if their intersection point is an isotropic point, while two points are said to be parallel if they are joined by an isotropic line.

The basic facts about the isotropic plane can be found in [4]. Here, we highlight the ones that are important for our study.

The standard affine model of the isotropic plane, where the coordinates of points are given by $x = \dfrac{x_1}{x_0}$ and $y = \dfrac{x_2}{x_0}$, is obtained by setting $f : x_0 = 0$ and $F = (0,0,1)$. In this model, the isotropic lines have equations of the form $x = c$ with $c \in \mathbb{R}$. The isotropic distance $d(P,Q)$ of a pair of two non-parallel points $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ is defined by $d(P,Q) = x_Q - x_P$, while two non-parallel lines p and q, given by the equations $y = k_p x + l_p$ and $y = k_q x + l_q$, respectively, form the isotropic angle defined by $\angle(p,q) = k_q - k_p$. The midpoint of points P and Q is defined in a natural way as the point with coordinates $\left(\frac{1}{2}(x_P + x_Q), \frac{1}{2}(y_P + y_Q) \right)$, while the bisector of lines p and q is the line with the equation $y = \frac{1}{2}(k_p + k_q)x + \frac{1}{2}(l_p + l_q)$.

According to their position with respect to the absolute figure, conics are ellipses, hyperbolas, special hyperbolas, parabolas and circles, as explained in [4]. The most interesting conics for us are circles, conics that touch f at F. Therefore, a circle has an equation of the form $y = ax^2 + bx + c$ with $a, b, c \in \mathbb{R}$.

A curve in the isotropic plane is said to be circular if it passes through the absolute point F. If the curve intersects f only at F, then it is entirely circular, as explained in [5]. Thus, circles are entirely circular conics.

It was shown in [6] that any allowable triangle ABC (triangle whose sides are non-isotropic lines) in the isotropic plane can be set in the so-called standard position by choosing an appropriate affine coordinate system and having the circumcircle u with the equation

$$u \quad \dots \quad y = x^2, \tag{1}$$

and vertices

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{2}$$

with $a + b + c = 0$.

Let σ_1, σ_2 and σ_3 be elementary symmetric functions in three variables a, b and c; in other words, let

$$\sigma_1 = a + b + c = 0, \quad \sigma_2 = ab + bc + ca, \quad \sigma_3 = abc. \tag{3}$$

The centroid, the symmedian point, the Gergonne point, the Feuerbach point and the Steiner point of the triangle ABC are labeled X_2, X_6, X_7, X_{11} and X_{99}, respectively, according to Kimberling's encyclopedia [7].

The coordinates of the centroid X_2 of the triangle ABC are

$$X_2 = \left(\frac{a+b+c}{3}, \frac{a^2+b^2+c^2}{3} \right) = \left(0, -\frac{2}{3}\sigma_2 \right). \tag{4}$$

It was shown in [8] that the incircle i

$$i \quad \dots \quad y = \frac{1}{4}x^2 - \sigma_2 \tag{5}$$

and the Euler circle

$$e \quad \dots \quad y = -2x^2 - \sigma_2 \tag{6}$$

touch each other externally at the Feuerbach point

$$X_{11} = (0, -\sigma_2). \tag{7}$$

The symmedians are the reflections of medians in the bisectors. According to [9], they intersect at the symmedian point X_6 of the triangle ABC having the coordinates

$$X_6 = \left(\frac{3\sigma_3}{2\sigma_2}, -\frac{\sigma_2}{3} \right). \tag{8}$$

The Gergonne point X_7 of the triangle ABC is the intersection point of three lines AA_i, BB_i and CC_i, where A_i, B_i and C_i are the contact points of the triangle and its incircle i. In [10], the coordinates of the Gergonne point X_7 of the standard triangle ABC were determined as follows:

$$X_7 = \left(-\frac{3\sigma_3}{\sigma_2}, -\frac{4\sigma_2}{3} \right). \tag{9}$$

The Steiner point X_{99} of the triangle ABC is defined as the fourth (in addition to A, B and C) common point X_{99} of the circumscribed circle u and circumscribed Steiner ellipse of that triangle, as explained in [11]. In the case of the standard triangle ABC, this point has the coordinate vector

$$X_{99} = \left(-\frac{3\sigma_3}{\sigma_2}, \frac{9\sigma_3^2}{\sigma_2^2} \right). \tag{10}$$

For every triangle in the isotropic plane, there is the first and the second Brocard point, and they are unique. The first Brocard point is defined as the point such that its connections with the vertices A, B and C form equal angles with the sides AC, BA and CB, respectively. Analogously, the second Brocard point is defined as the point such that its connection lines with the vertices A, B and C form equal angles with the sides AB, BC and CA, respectively. According to [12], the Brocard points B_1 and B_2 of the standard triangle are given by

$$B_1 = \left(\frac{\sigma_3 - p_1}{\sigma_2}, \frac{3p_1^2}{\sigma_2^2} - \frac{2}{9}\sigma_2 \right), \quad B_2 = \left(\frac{\sigma_3 - p_2}{\sigma_2}, \frac{3p_2^2}{\sigma_2^2} - \frac{2}{9}\sigma_2 \right), \tag{11}$$

where

$$p_1 = \frac{1}{3}\left(bc^2 + ca^2 + ab^2 \right), \quad p_2 = \frac{1}{3}\left(b^2c + c^2a + a^2b \right). \tag{12}$$

In Section 3.2, we will show that in the isotropic plane, all triangles in a poristic system share the centroid and the Feuerbach point. The symmedian point and the Gergonne point of all triangles move on straight lines. The Steiner point traces a circle, while Brocard points trace a quartic curve.

3. Results

3.1. Poncelet's Porism

In order to obtain an explicit parametrization of the poristic family, we assume that we are given a triangle ABC with the vertices in Equation (2), a circumcircle u with Equation (1) and an incircle i with Equation (5). Let $P_1 = (t, t^2)$ be a point on the circle u. The polar line p of P_1 with respect to i is

$$p \quad \dots \quad -4\sigma_2 - 2t^2 + tx - 2y = 0.$$

The intersections C_2 and C_3 of p and i are the points of contact of the tangents from P_1 to the incircle i:

$$C_2 = \left(t + W, \frac{tW - t^2 - 4\sigma_2}{2} \right), \quad C_3 = \left(t - W, \frac{-tW - t^2 - 4\sigma_2}{2} \right),$$

where $W = \sqrt{-4\sigma_2 - 3t^2}$. Lines P_1C_3 and P_1C_2 intersect the circumcircle u at points P_2 and P_3, respectively:

$$P_2 = \left(\frac{W-t}{2}, \frac{-tW - t^2 - 2\sigma_2}{2} \right), \quad P_3 = \left(\frac{-W-t}{2}, \frac{tW - t^2 - 2\sigma_2}{2} \right).$$

The connection line P_2P_3 has the equation

$$y = -tx - t^2 - \sigma_2$$

and touches i at the point

$$C_1 = \left(-2t, t^2 - \sigma_2 \right).$$

Thus, for any choice of the initial vertix P_1, the triangle $P_1P_2P_3$ closes (see Figure 1). The triangles ABC and $P_1P_2P_3$ share the centroid that has the coordinates

$$X_2 = \frac{1}{3}\left(t + \frac{W-t}{2} + \frac{-W-t}{2}, t^2 + \frac{-tW - t^2 - 2\sigma_2}{2} + \frac{tW - t^2 - 2\sigma_2}{2} \right) = \left(0, -\frac{2}{3}\sigma_2 \right). \tag{13}$$
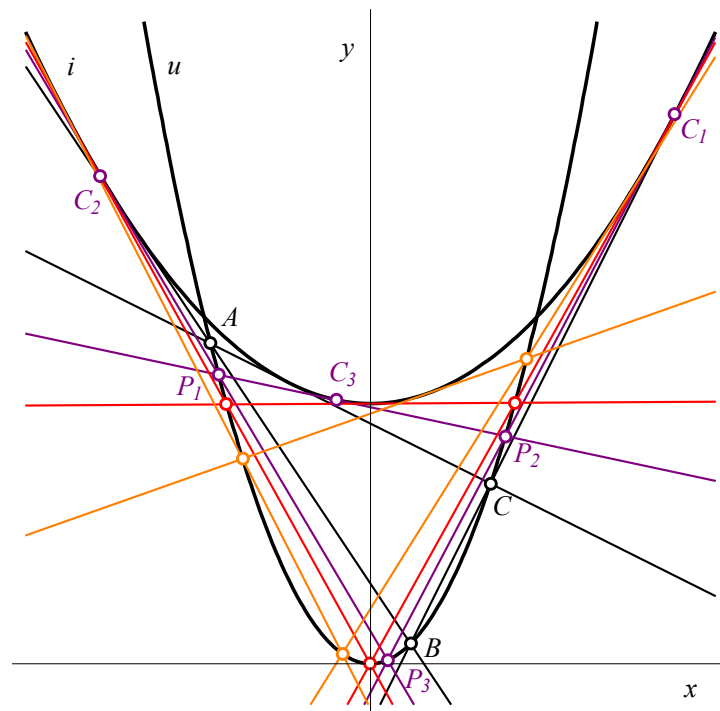

Figure 1. A porisitic system of triangles in the isotropic plane.

It follows from Equation (13) that all triangles in the poristic system are in the standard position, and therefore, Equations (3)–(12) for ABC can be applied to $P_1P_2P_3$. The points and values related to $P_1P_2P_3$ will be marked with a $\star$.

We should notice that all sides of the triangle $P_1P_2P_3$ will be real lines only if we chose the vertex P_1 as an outer of i, which is precisely the case when $t^2 \le -\frac{4}{3}\sigma_2$, and hence $W \in \mathbb{R}$.

3.2. Loci of Centers in the Poristic System of Triangles

Let $P_1P_2P_3$ be a triangle from the poristic system with vertices

$$P_1 = \left(t, t^2 \right), P_2 = \left(\frac{W-t}{2}, \frac{-tW - t^2 - 2\sigma_2}{2} \right), P_3 = \left(\frac{-W-t}{2}, \frac{tW - t^2 - 2\sigma_2}{2} \right), \tag{14}$$

where $t \in \mathbb{R}$.

The elementary symmetric functions related to $P_1 P_2 P_3$ are

$$
\begin{aligned}
\sigma_1^\star &= 0 = \sigma_1, \\
\sigma_2^\star &= t \cdot \frac{W-t}{2} + \frac{W-t}{2} \cdot \frac{-W-t}{2} + \frac{-W-t}{2} \cdot t = \frac{-W^2 - 3t^2}{4} \\
&= \frac{-(-4\sigma_2 - 3t^2) - 3t^2}{4} = \sigma_2, \\
\sigma_3^\star &= t \cdot \frac{W-t}{2} \cdot \frac{-W-t}{2} = -\frac{t}{4}\left(-4\sigma_2 - 4t^2\right) = t\left(\sigma_2 + t^2\right).
\end{aligned}
\tag{15}
$$

Theorem 1. *The centroid and the Feuerbach point of triangle $P_1 P_2 P_3$ are fixed while the triangle traces its poristic family.*

Proof. We have already shown that $X_2^\star = \left(0, -\frac{2}{3}\sigma_2^\star\right) = \left(0, -\frac{2}{3}\sigma_2\right) = X_2$. Similarly, $X_{11}^\star = (0, -\sigma_2^\star) = (0, -\sigma_2) = X_{11}$. □

Theorem 2. *The Gergonne point of triangle $P_1 P_2 P_3$ moves on a line while the triangle traverses the poristic family.*

Proof. The Gergonne point $X_7^\star$ of triangle $P_1 P_2 P_3$ has the coordinates $X_7^\star = \left(-\frac{3\sigma_3^\star}{\sigma_2^\star}, -\frac{4\sigma_2^\star}{3}\right) = \left(-\frac{3t(\sigma_2 + t^2)}{\sigma_2}, -\frac{4\sigma_2}{3}\right)$. Thus, the Gergonne points of all triangles from the poristic family lie on the line g with the equation

$$
y = -\frac{4\sigma_2}{3},
$$

the connection line of the intersections $\left(\pm\sqrt{-\frac{4}{3}\sigma_2}, -\frac{4}{3}\sigma_2\right)$ of the incircle i and circumcircle u. □

Theorem 3. *The symmedian point of triangle $P_1 P_2 P_3$ moves on a line if the triangle traverses the poristic family.*

Proof. The symmedian point $X_6^\star$ of triangle $P_1 P_2 P_3$ has the coordinates $X_6^\star = \left(-\frac{3\sigma_3^\star}{2\sigma_2^\star}, -\frac{\sigma_2^\star}{3}\right) = \left(\frac{3t(\sigma_2 + t^2)}{2\sigma_2}, -\frac{\sigma_2}{3}\right)$. Thus, the symmedian point of all triangles from the poristic family moves on the line k with the equation

$$
y = -\frac{\sigma_2}{3},
$$

the connection line of the intersections $\left(\pm\sqrt{-\frac{\sigma_2}{3}}, -\frac{\sigma_2}{3}\right)$ of the circumcircle u and common Euler circle e given by Equation (6). □

Remark 1. *The locus of the Gergonne point and the locus of the symmedian point are parallel lines (see Figure 2).*

Theorem 4. *The Steiner point of triangle $P_1 P_2 P_3$ moves on a circle when the triangle traverses the poristic family.*

Proof. The statement follows trivially from the fact that the Steiner point of a triangle is the intersection point of the circumcircle and the Steiner ellipse of the triangle. The Steiner point $X_{99}^\star = \left(-\frac{3\sigma_3^\star}{\sigma_2^{\star 2}}, \frac{9\sigma_3^{\star 2}}{\sigma_2^{\star 2}}\right) = \left(-\frac{3t(\sigma_2 + t^2)}{\sigma_2}, \frac{9t^2(\sigma_2 + t^2)^2}{\sigma_2^2}\right)$ of triangle $P_1 P_2 P_3$ obviously lies on the circle with the equation $y = x^2$ (i.e., the circumcircle u). □
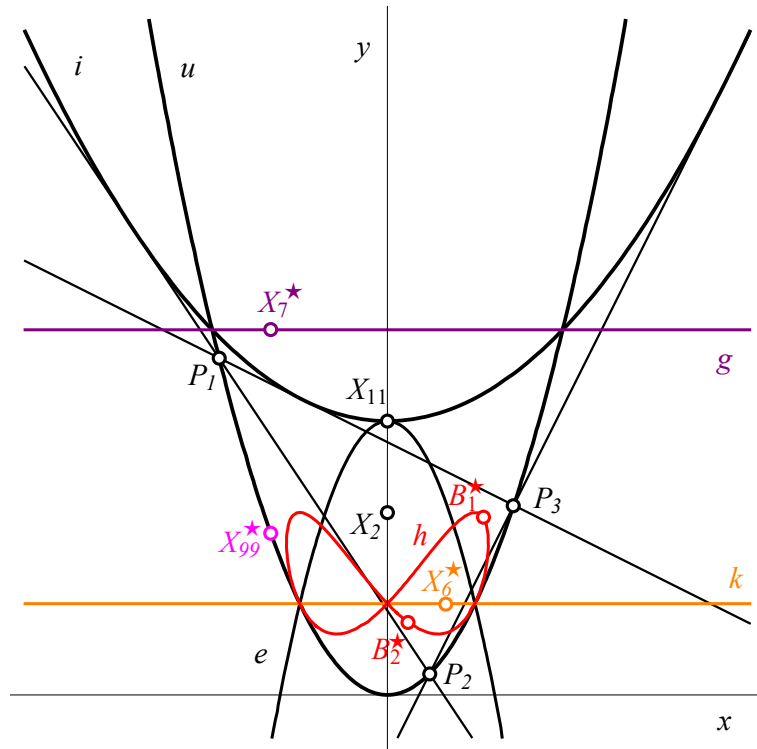
Figure 2. The loci of the symmedian point, the Gergonne point and the Brocard points for a porisitic system of triangles in the isotropic plane. The centroid X_2 and the Feuerbach point X_{11} are fixed points.

Theorem 5. *The Brocard points of triangle $P_1P_2P_3$ move on a quartic curve when the triangle traverses the poristic family.*

Proof. Let $B_1^\star$ be the first Brocard point of triangle $P_1P_2P_3$. Then, we have

$$B_1^\star = \left(\frac{\sigma_3^\star - p_1^\star}{\sigma_2^\star}, \frac{3p_1^{\star 2}}{\sigma_2^{\star 2}} - \frac{2}{9}\sigma_2^\star \right) = \left(\frac{t(\sigma_2 + t^2) - p_1^\star}{\sigma_2}, \frac{3p_1^{\star 2}}{\sigma_2^2} - \frac{2}{9}\sigma_2 \right),$$

where

$$
\begin{aligned}
p_1^\star &= \frac{1}{3}\left(\frac{W-t}{2} \cdot \frac{(-W-t)^2}{4} + \frac{-W-t}{2} \cdot t^2 + t\frac{(W-t)^2}{4} \right) \\
&= \frac{1}{24}\left(W(W^2 - 9t^2) + 3t(W^2 - t^2) \right) = -\frac{1}{6}\left(W(\sigma_2 + 3t^2) + 3t(\sigma_2 + t^2) \right)
\end{aligned}
$$

and

$$p_1^{\star 2} = \frac{1}{36}\left(-4\sigma_2^3 - 18\sigma_2^2 t^2 - 36\sigma_2 t^4 - 18t^6 + 6Wt(\sigma_2^2 + 4\sigma_2 t^2 + 3t^4) \right).$$

The coordinates of $B_1^\star$ are therefore given by

$$
\begin{aligned}
x &= \frac{9t(\sigma_2 + t^2) + W(\sigma_2 + 3t^2)}{6\sigma_2}, \\
y &= \frac{-3\sigma_2^2 t^2 - 6\sigma_2 t^4 - 3t^6 + Wt(\sigma_2^2 + 4\sigma_2 t^2 + 3t^4)}{2\sigma_2^2} - \frac{5\sigma_2}{9}.
\end{aligned}
$$

After inserting $W = \sqrt{-4\sigma_2 - 3t^2}$ and eliminating the parameter t from the above equations, we obtain the equation of a quartic curve h:

$$9x^4 - 9x^2y + 9y^2 + 6\sigma_2 y + \sigma_2^2 = 0, \tag{16}$$

In other words, we have

$$9x^2(x^2 - y) + (3y + \sigma_2)^2 = 0. \tag{17}$$

The same quartic curve h is obtained as the locus of the second Brocard points:

$$B_2^\star = \left(\frac{9t(\sigma_2 + t^2) - W(\sigma_2 + 3t^2)}{6\sigma_2}, \frac{-3\sigma_2^2 t^2 - 6\sigma_2 t^4 - 3t^6 - Wt(\sigma_2^2 + 4\sigma_2 t^2 + 3t^4)}{2\sigma_2^2} - \frac{5\sigma_2}{9} \right).$$

$\square$

It follows from Equations (1) and (17) that the locus h of Brocard points touches the circumcircle u at two points: the intersections $\left(\pm\sqrt{-\frac{\sigma_2}{3}}, -\frac{\sigma_2}{3} \right)$ of the circumcircle u and the Euler circle e.

Homogenizing $\left(x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0} \right)$ Equation (16) of h and setting $x_0 = 0$ yields the intersection with the absolute line f as $9x_1^4 = 0$. Thus, the absolute point F is the intersection point with an intersection multiplicity of four , and therefore, h is an entirely circular quartic.

3.3. Traces of Some Points

In [2], the author also studied the traces of some further points associated with triangles of the poristic family in the Eucildean plane. He showed that the trace of the midpoint of any side of a triangle traversing a poristic family is a Limaçon of Pascal and that the vertices of the tangential triangle of a triangle move on an ellipse. We will show that in the isotropic plane, both curves are isotropic circles.

We have already seen that while triangle $P_1P_2P_3$ traverses the poristic family, the midpoints of its sides trace the common Euler circle e, and the centroid is fixed. Now, we focus on the contact triangles and tangential triangles of the triangles from the poristic family (see Figure 3).

Theorem 6. *The vertices of the contact triangle of triangle $P_1P_2P_3$ move on a circle while $P_1P_2P_3$ traverses the poristic family. The trace of the midpoint of any side of the contact triangle is a circle, and the centroid of the contact triangle is fixed.*

Proof. The vertices C_1, C_2 and C_3 of the contact triangle of $P_1P_2P_3$ are in Section 3.1 and are determined as follows:

$$C_1 = \left(-2t, t^2 - \sigma_2 \right), C_2 = \left(t + W, \frac{tW - t^2 - 4\sigma_2}{2} \right), C_3 = \left(t - W, \frac{-tW - t^2 - 4\sigma_2}{2} \right),$$

The midpoints of its sides are

$$M_{C12} = \left(\frac{W-t}{2}, \frac{tW + t^2 - 6\sigma_2}{4} \right), M_{C13} = \left(\frac{-W-t}{2}, \frac{-tW + t^2 - 6\sigma_2}{4} \right),$$
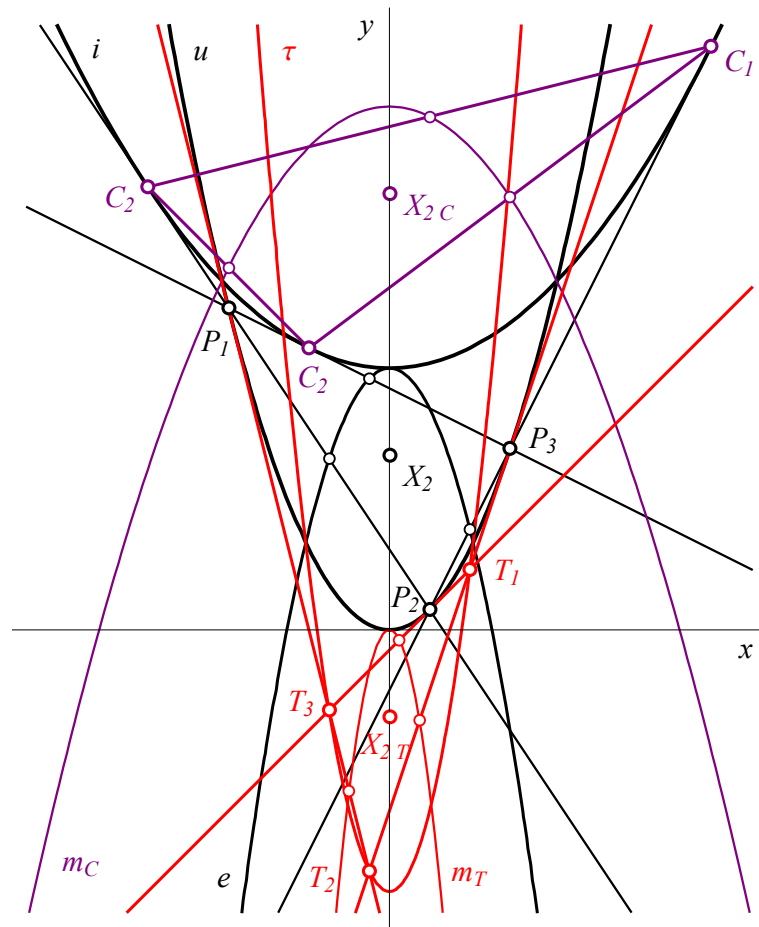
$$M_{C23} = \left(t, -\frac{t^2}{2} - 2\sigma_2 \right).$$

The point M_{C23} obviously traces the circle

$$m_C \quad \dots \quad y = -\frac{x^2}{2} - 2\sigma_2 \tag{18}$$

and it can be easily checked that M_{C12} and M_{C13} lie on the same circle.

The centroid of triangle $C_1C_2C_3$ is $X_{2C} = \left(0, -\frac{5}{3}\sigma_2\right)$. $\square$

Figure 3. When triangle $P_1P_2P_3$ traverses the poristic family, the midpoints of its sides traverse a circle e (black). The midpoints of the sides of its contact triangle $C_1C_2C_3$ traverse a circle m_C, and the centroid X_{2C} is fixed (purple). The vertices of the tangential triangle $T_1T_2T_3$ traverse a circle τ, the midpoints of $T_1T_2T_3$ traverse a circle m_T, and the centroid X_{2T} is fixed (red).

Theorem 7. *The vertices of the tangential triangle of triangle $P_1P_2P_3$ move on a circle while $P_1P_2P_3$ traverses the poristic family. The trace of the midpoint of any side of the tangential triangle is a circle, and the centroid of the tangential triangle is fixed.*

Proof. The polar line of the point $(\bar{x}, \bar{y})$ with respect to the circle $y = x^2$ has the equation $y + \bar{y} = 2\bar{x}x$. Therefore, the tangents to u at vertices P_1, P_2 and P_3 are given by the respective equations

$$
\begin{aligned}
t_1 & \quad \ldots \quad y = 2tx - t^2, \\
t_2 & \quad \ldots \quad y = (W - t)x - \frac{(W - t)^2}{4}, \\
t_3 & \quad \ldots \quad y = (-W - t)x - \frac{(W + t)^2}{4}.
\end{aligned}
$$

The vertices $T_1 = t_2 \cap t_3$, $T_2 = t_1 \cap t_3$ and $T_3 = t_1 \cap t_2$ of the tangential triangle are

$$
T_1 = \left(-\frac{t}{2}, \frac{t^2 - W^2}{4}\right), \quad T_2 = \left(\frac{t - W}{4}, \frac{-tW - t^2}{2}\right), \quad T_3 = \left(\frac{t + W}{4}, \frac{tW - t^2}{2}\right).
$$

The point T_1 traces the circle

$$\tau \quad \dots \quad y = 4x^2 + \sigma_2$$

since $W^2 = -4\sigma_2 - 3t^2$. T_2 and T_3 lie on the same circle. The midpoints of the sides of $T_1 T_2 T_3$ given by the coordinates

$$M_{T23} = \left(\frac{t}{4}, -\frac{t^2}{2} \right), M_{T12} = \left(\frac{-t - W}{8}, -\frac{(t + W)^2}{8} \right), M_{T13} = \left(\frac{-t + W}{8}, -\frac{(t - W)^2}{8} \right).$$

trace the circle

$$m_T \quad \dots \quad y = -8x^2.$$

The centroid of triangle $T_1 T_2 T_3$ is the point $X_{2T} = \left(0, \frac{\sigma_2}{3} \right)$. $\square$

Remark 2. *The centroid X_2 of triangle $P_1 P_2 P_3$, centroid X_{2C} of the contact triangle $C_1 C_2 C_3$ and centroid X_{2T} of the tangential triangle $T_1 T_2 T_3$ are parallel points.*

Remark 3. *The circle τ is polar to circle i with respect to circle u. Therefore, there is also a poristic family with u as the incircle and τ as the circumcircle.*

Theorem 8. *The vertices of the anticomplementary triangle of triangle $P_1 P_2 P_3$ move on a circle while $P_1 P_2 P_3$ traverses the poristic family. The centroid of the anticomplementary triangle is fixed.*

Proof. The anticomplementary triangle $A_1 A_2 A_3$ is the triangle which has triangle $P_1 P_2 P_3$ as its medial triangle (i.e., it is formed by the lines through the vertices of $P_1 P_2 P_3$ parallel to its sides). It is in perspective with $P_1 P_2 P_3$ at centroid X_2, as explained in [6]. The vertices of the anticomplementary triangle are

$$A_1 = \left(-2t, -2t^2 - 2\sigma_2 \right), A_2 = \left(t - W, tW + t^2 \right), A_3 = \left(t + W, -tW + t^2 \right),$$

and they lie on the circle m_C given by Equation (18). The triangle $P_1 P_2 P_3$ and its anticomplementary triangle $A_1 A_2 A_3$ share the centroid X_2. $\square$

4. Discussion

Any triangle in the isotropic plane has a circumcircle u and incircle i. We showed that there are infinitely many triangles with the same circumcircle u and incircle i. We also proved that all triangles in a poristic system share a centroid and Feuerbach point. The symmedian point and the Gergonne point trace straight lines, the Steiner point traces a circle, and the Brocard points trace a quartic curve.

To make this study simpler, we put the initial triangle ABC in the so-called standard position by choosing its vertices to be (a, a^2), (b, b^2) and (c, c^2), with $\sigma_1 = a + b + c = 0$. Assuming that $\sigma_1 \neq 0$, we would obtain the same results, but the calculations would be much more complicated, and the formulas and equations would be much longer.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflict of interest.

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