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Fuzzy Testing Method of Process Incapability Index

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Abstract: The process capability index is a tool for quality measurement and analysis widely used in the industry. It is also a good tool for the sales department to communicate with customers. Although the value of the process capability index can be affected by the accuracy and precision of the process, the index itself cannot be differentiated. Therefore, the process incapability index is directly divided into two items, accuracy and precision, based on the expected value of the Taguchi process loss function. In fact, accuracy and precision are two important reference items for improving the manufacturing process. Thus, the process incapability index is good for evaluating process quality. The process incapability index contains two unknown parameters, so it needs to be estimated with sample data. Since point estimates are subject to misjudgment incurred by the inaccuracy of sampling, and since modern businesses are in the era of rapid response, the size of sampling usually tends to be small. A number of studies have suggested that a fuzzy testing method built on the confidence interval be adopted at this time because it integrates experts and the experience accumulated in the past. In addition to a decrease in the possibility of misjudgment resulting from sampling error, this method can improve the test accuracy. Therefore, based on the confidence interval of the process incapability index, we proposed the fuzzy testing method to assess whether the process capability can attain a necessary level of quality. If the quality level fails to meet the requirement, then an improvement must be made. If the quality level exceeds the requirement, then it is equivalent to excess quality, and a resource transfer must be considered to reduce costs.

Keywords: process incapability index; Taguchi loss function; confidence interval; fuzzy testing method; sampling error

MSC: 62C05; 62C86

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1. Introduction

The process capability index is a tool for quality measurement and analysis widely used in the industry. It is also a tool for process engineers to self-examine the process quality and judge whether to make an improvement at any time [1–4]. Since the promotion of Industry 4.0, new technologies, including the Internet of Things (IOT) and Big Data analysis, have rapidly developed and evolved [5–7]. In this environment, numerous scholars have put forward various production data analyses and evaluation models by integrating the rapid collection of production data with process capability indicators. Then, a smart manufacturing environment with a comprehensive network is gradually formed so that various manufacturing industries step into the goal of smart manufacturing [8–10].

Suppose X (random variable) is normally distributed with μ (mean) and σ (standard deviation), expressed as $X \sim N(\mu, \sigma^2)$. Kane [11] proposed process capability indices C_P and C_{PK} of bilateral specifications as follows:

$$C_P = \frac{USL - LSL}{6\sigma} = \frac{d}{3\sigma} \tag{1}$$

and

$$C_{PK} = \frac{USL - LSL}{6\sigma} = \frac{d - |\mu - T|}{3\sigma}. \tag{2}$$

In the above equations, USL stands for "upper specification limit", LSL stands for "lower specification limit", $T = (USL + LSL)/2$ denotes the target value, and $d = (\text{USL} - \text{LSL})/2$ denotes half the specification interval. Since process capability index C_P excludes mean μ, it means that C_P fails to reflect the process when the process is shifted. Thus, the index C_{PK} is proposed to make up for this problem. Additionally, in order to solve the problem that process capability index C_P cannot reflect the process deviation, Chan et al. [12] proposed the second-generation process capability index C_{PM}, as expressed below:

$$C_{PM} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \tag{3}$$

where $d = (USL - LSL)/2$. Both the C_{PK} and the second-generation index C_{PM} belong to bilateral specifications used for processes that suit the nominal-the-best characteristic. In addition, the Taguchi loss function can be expressed as:

$$L(X) = k(X - T)^2. \tag{4}$$

The denominator of C_{PM} is the expected value of the Taguchi loss function containing $k = 1$. Similar to the index C_{PM}, Greenwich and Jahr-Schaffrath [13] came up with a process incapability index as follows:

$$C_{PP} = \left(\frac{\mu - T}{d}\right)^2 + \left(\frac{\sigma}{d}\right)^2. \tag{5}$$

Let the random variable be $Y = (X - T)/d$, where Y is normally distributed with δ and γ, denoted by $Y \sim N(\delta, \gamma^2)$. Also, its Taguchi loss function can be rewritten as $L(Y) = k'Y^2$. In fact, $\delta = (\mu - T)/d$ is seen as an index of accuracy while $\gamma = \sigma/d$ is viewed as an index of precision. Since incapability index C_{PP} is the square of the reciprocal of index C_{PM}, incapability index C_{PP} can be expressed as follows:

$$C_{PP} = (3\delta)^2 + (3\gamma)^2. \tag{6}$$

As a matter of fact, the incapability index C_{PP} is three times higher than the expected Taguchi Loss Function ($E[L(Y)] = \delta^2 + \gamma^2$). Obviously, the incapability index is suitable for evaluating the quality characteristic of the nominal-the-best type (*NTB*). According to the concept by Chen and Chen [14], there is a relation between the process yield (*Yield*%) and the index, expressed as follows:

$$Yield\% \geq 2\Phi\left(3/\sqrt{C_{PP}}\right) - 1 \text{ for } Q_{PP} < \left(\sqrt{3}\right)^{-1}. \tag{7}$$

Obviously, the incapability index C_{PP} can reflect the expected loss and the yield of the process, showing that it is a good indicator for the evaluation of process quality. In addition, the process incapability index is directly divided into two items, accuracy and precision, based on the expected value of the Taguchi process loss function. Accuracy and precision are two important reference items for improving the manufacturing process.

Thus, the process incapability index is good for evaluating process quality. According to some studies, the incapability index C_{PP} contains two unknown parameters, so it needs to be estimated with sample data. Since point estimates are subject to wrong judgments for sampling error [15–17], the size of the sample is normally small in the sampling test, given the consideration of costs and the emphasis on timeliness of the quick response [18,19]. In order to solve the above problems, we follow the suggestions made by numerous studies. A fuzzy testing method built on the confidence interval can be used at this time because this method integrates experts and their experiences accumulated in the past. This method can help lessen the occurrence of misjudgment resulting from sampling error as well as level up the test accuracy [20]. In the paper, we first deduce the corresponding relation between the process incapability index and the six-sigma quality level. Next, according to the engineers' requirements for the process quality level, a lower-confidence-limit-based fuzzy testing method is proposed to evaluate the quality level that is required by the process capability of the product. If the quality level fails to meet the requirement, then it is necessary to make an improvement on the process.

To follow the introduction in Section 1, we derive the lower confidence limit of the process incapability index in Section 2. Next, the lower confidence limit of the process incapability index is developed using the mathematical programming (MP) model in Section 3. In Section 4, a fuzzy testing model based on the confidence interval is established. In Section 5, an example is taken to demonstrate the application of the fuzzy testing method proposed in Section 4. Finally, the conclusions are made in Section 6.

2. Lower Confidence Limit of Incapability Index

Let $(X_1, \ldots, X_i, \ldots, X_n)$ be a set of random samples with the sample size n. Suppose X (random variable) is normally distributed with μ (mean) and σ^2 (variance), expressed as $X \sim N(\mu, \sigma^2)$. When the random variable is $Y = (X - T)/d$, $(Y_1, \ldots, Y_i, \ldots, Y_n)$ is a set of random samples of the random variable Y, and Y is distributed as $N(\delta, \gamma^2)$. Then, the mean of the samples is the estimator of δ, and their standard deviation is the estimator of γ, displayed as follows:

$$\delta^* = \frac{1}{n}\sum_{i=1}^{n} Y_i; \tag{8}$$

$$\gamma^* = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(Y_i - \delta^*)^2}. \tag{9}$$

According to Equations (8) and (9), the estimator of the incapability index C_{PP} is

$$C_{PP}^* = (3\delta^*)^2 + (3\gamma^*)^2. \tag{10}$$

Furthermore, this paper lets the random variable Z be expressed as follows:

$$Z = \frac{\sqrt{n}(\delta^* - \delta)}{\gamma}. \tag{11}$$

Under the assumption of normality, Z is regarded as a standard normal distribution, i.e., $Z \sim N(0, 1)$. Similarly, this paper lets the random variable K be expressed as follows:

$$K = \frac{n\gamma^{*2}}{\gamma^2}. \tag{12}$$

Suppose Z is normally distributed and viewed as a chi-square distribution having $n - 1$ degree of freedom, i.e., $K \sim \chi^2_{n-1}$. Thus,

$$p\left\{-Z_{0.5-\sqrt{1-\alpha}/2} \le Z \le Z_{0.5-\sqrt{1-\alpha}/2}\right\} = \sqrt{1-\alpha} \tag{13}$$

and

$$p\left\{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1} \le K \le \chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}\right\} = \sqrt{1-\alpha}, \tag{14}$$

where $Z_{0.5-\sqrt{1-\alpha}/2}$ means the upper $0.5-\sqrt{1-\alpha}/2$ quintile of $N(0,1)$, $\chi^2_{a/2;n-1}$ denotes the lower $a/2$ quintile of χ^2_{n-1}, and $\sqrt{1-\alpha}$ refers to the confidence level. δ^* and γ^{*2} are mutually independent with the normal distribution, and so are Z and K [21]. Thus, the equation can be further obtained below:

$$p\left\{-Z_{0.5-\sqrt{1-\alpha}/2} \le Z \le Z_{0.5-\sqrt{1-\alpha}/2}, \chi^2_{0.5-\sqrt{1-\alpha}/2;n-1} \le K \le \chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}\right\} = 1-\alpha. \tag{15}$$

Equivalently,

$$p\left\{\begin{array}{l} \delta^* - Z_{0.5-\sqrt{1-\alpha}/2} \times \left(\frac{\gamma}{\sqrt{n}}\right) \le \delta \le \delta^* + Z_{0.5-\sqrt{1-\alpha}/2} \times \left(\frac{\gamma}{\sqrt{n}}\right), \\ \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma^* \le \gamma \le \sqrt{\frac{n-1}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}\gamma^* \end{array}\right\} = 1-\alpha. \tag{16}$$

Suppose $(y_1, \ldots, y_i, \ldots, y_n)$ is the observed set of $(Y_1, \ldots, Y, \ldots, Y_n)$; δ_0^* and γ_0^*, the observed values of δ^* and γ^*, are expressed individually as follows:

$$\delta_0^* = \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{17}$$

and

$$\gamma_0^* = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \delta_0^*)^2}. \tag{18}$$

Then, the confidence region is expressed as follows:

$$CR = \{\,(\delta,\gamma)|\delta_0^* - E_\gamma \le \delta \le \delta_0^* + E_\gamma, \gamma_L \le \gamma \le \gamma_U\}, \tag{19}$$

where

$$E_\gamma = Z_{0.5-\sqrt{1-\alpha}/2} \times \left(\frac{\gamma}{\sqrt{n}}\right), \tag{20}$$

$$\gamma_L = \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*, \tag{21}$$

and

$$\gamma_U = \sqrt{\frac{n-1}{\chi^2_{0.5-\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*. \tag{22}$$

Clearly, the process incapability index C_{PP} is a function of mean δ and standard deviation γ. Thus, this study denotes the incapability index as $C_{PP}(\delta,\gamma)$, uses this incapability index as an objective function, and defines the confidence region $CR(\delta,\gamma)$ as a feasible solution area. For any $\gamma \ge \gamma_L$, then $C_{PP}(\delta,\gamma) \ge C_{PP}(\delta,\gamma_L)$. Based on the mathematical programming model, the lower confidence limit of the incapability index C_{PP} is presented as follows:

$$\left\{\begin{array}{l} LC_{PP} = min\ C_{PP}(\delta,\gamma_L) \\ subject\ to \\ \delta_L \le \delta \le \delta_U \end{array}\right. \tag{23}$$

where $\delta_L = \delta_0^* - E_{\gamma_L}$, $\delta_U = \delta_0^* + E_{\gamma_L}$, and $E_{\gamma_L} = Z_{0.5-\sqrt{1-\alpha}/2} \times \gamma_L/\sqrt{n}$. Based on the abovementioned information, the lower confidence limit of the process incapability index C_{PP} is deduced in the following section.

3. Develop Lower Confidence Limit of Process Incapability Index Using MP Model

In this section, this paper employs the method of statistical inference with the confidence interval and the mathematical programming model to derive the lower confidence limit of the process incapability index. We discover the lower confidence limit based on Situation 1 $0 \leq \delta_L$, Situation 2 $\delta_L \leq 0 \leq \delta_U$, and Situation 3 $\delta_U \leq 0$, respectively.

Situation 1: $0 \leq \delta_L$

In this situation, we can conclude that $\delta > 0$, and then the *MP* model can be rewritten as follows: $0 \leq \delta_L$

$$\begin{cases} LC_{PP} = \min C_{PP}(\delta, \gamma_L) \\ subject\ to \\ 0 < \delta_L \leq \delta \leq \delta_U \end{cases}. \quad (24)$$

According to Equation (24), for any $\delta \geq \delta_L$, then $C_{PP}(\delta, \gamma_L) \geq C_{PP}(\delta_L, \gamma_L)$. Therefore, the lower confidence limit of the process incapability index is defined below:

$$LC_{PP} = C_{PP}(\delta_L, \gamma_L) = 9\delta_L^2 + 9\gamma_L^2 = 9\left(\delta_0^* - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \times \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2 + 9\left(\sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2. \quad (25)$$

Situation 2: $\delta_L \leq 0 \leq \delta_U$

$\delta = 0$ is concluded in this situation. Then, the objective function of the *MP* model is $C_{PP} = (3\gamma)^2$, and the lower confidence limit of the process incapability index is written as:

$$LC_{PP} = C_{PP}(0, \gamma_L) = 9\gamma_L^2 = \frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}\gamma_0^{*2}. \quad (26)$$

Situation 3: $\delta_U \leq 0$

In this situation, we can conclude $\delta < 0$ and then rewrite the *MP* model as follows:

$$\begin{cases} LC_{PP} = \min C_{PP}(\delta, \gamma_L) \\ subject\ to \\ \delta_L \leq \delta \leq \delta_U < 0 \end{cases}. \quad (27)$$

According to Equation (27), for any $\delta \leq \delta_U$, then $\delta \leq \delta_U$. Accordingly, the lower confidence limit of the process incapability index is expressed as:

$$LC_{PP} = C_{PP}(\delta_L, \gamma_L) = 9\delta_U^2 + 9\gamma_L^2 = 9\left(\delta_0^* + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \times \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2 + 9\left(\sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2. \quad (28)$$

Based on the three situations and according to Equations (25), (26), and (28), the $100 \times (1-\alpha)\%$ lower confidence limit of process incapability index is defined as:

$$LC_{PP}(\alpha) = \begin{cases} 9\left(\delta_0^* - \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \times \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2 + 9\left(\sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2, 0 < \delta_L \\ 9\left(\sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2, \delta_L \leq 0 \leq \delta_U \\ 9\left(\delta_0^* + \frac{Z_{0.5-\sqrt{1-\alpha}/2}}{\sqrt{n}} \times \sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2 + 9\left(\sqrt{\frac{n-1}{\chi^2_{0.5+\sqrt{1-\alpha}/2;n-1}}}\gamma_0^*\right)^2, \delta_U < 0 \end{cases}. \quad (29)$$

According to Equation (29), this paper adopts the lower confidence limit of the process incapability index to build the fuzzy testing method of the incapability index C_{PP} built on the confidence interval in the next section.

4. Confidence-Interval-Based Fuzzy Testing Method of Incapability Index

Based on Motorola's requirements defined by Harry [22] and Chen et al. [23], a process is called "excellent" if the process incapability index is no more than 0.25 ($C_{PP} \leq 0.25$). Analogously, the Motorola requirements can be met by transforming the process capability index $C_{PK} \geq 1.5$ from $C_P \geq 2.0$ to $C_{PP} \leq 0.81$, serving as a process capability standard.

Furthermore, a few studies have also indicated that the process quality level can be said to have reached 6 sigma when the mean deviates from the target value by 1.5 sigma and the standard deviation is d/6 in the process. In other words, when $d = |\mu - T|/d \leq 1.5/k$ and $\sigma = d/\text{k}$, the process quality level gets to k sigma [24]. Thus, the relation of the process incapability index to the k-sigma quality level can be denoted as:

$$C_{PP} = (3\delta)^2 + (3\gamma)^2 \leq (4.5/k)^2 + (3/k)^2 = 9 \times \frac{3.25}{k^2} \tag{30}$$

According to Equation (30), the value of the process incapability index corresponding to the k-sigma quality level is shown in the following Table 1:

Table 1. Quality levels and their corresponding index values.

Quality Level	Value of the Incapability Index
4.0 sigma	$C_{PP} \leq 1.83$
4.5 sigma	$C_{PP} \leq 1.44$
5.0 sigma	$C_{PP} \leq 1.17$
5.5 sigma	$C_{PP} \leq 0.97$
6.0 sigma	$C_{PP} \leq 0.81$

When quality engineers or customers request the quality level of the product to reach k sigma, the value of the process incapability index must be C at most. The values of C are shown in Table 1. Thus, the null hypothesis and the alternative hypothesis for the test are written as:

Null hypothesis is $H_0 : C_{PP} \leq \text{C}$;

Alternative hypothesis is $H_1 : C_{PP} > \text{C}$.

The index has unknown parameters. Consequently, if we want to judge whether the process quality level can satisfy the requirement of the quality level, we can use the lower confidence limit of the process incapability index to perform the above hypothesis test. Thus, we obtained the following statistical testing rules including a significance level at 0.01:

(1) If $LC_{PP} \leq \text{C}$, then do not reject H_0 and conclude $C_{PP} \leq$.

(2) If $LC_{PP} > \text{C}$, then reject H_0 and conclude $C_{PP} > \text{C}$.

LC_{PP} is the lower confidence limit of the process incapability index with $\alpha = 0.01$.

According to Equation (29), the α-cuts of the half-triangular-shaped fuzzy number $\widetilde{L}C_{PP}$ are displayed below [22]:

$$\widetilde{L}C_{PP}[\alpha] = [LC_{PP}(\alpha), LC_{PP}(1)], \text{ for } 0.01 \leq \alpha \leq 1.00 \tag{31}$$

It is recalled that for $0 < \alpha < 0.01$, the α-cuts of $\widetilde{L}C_{PP}\ [\alpha]$ equal $\widetilde{L}C_{PP}$. Thus, the half-triangular-shaped fuzzy number $\Delta\widetilde{L}C_{PP} = (LC_{PPL},\ LC_{PPM})$, where $LC_{PPM} = LC_{PPM}$ (1) and $LC_{PPL} = LC_{PP}$ (0.01) (see Equation (29)).

This paper used the variable x to denote the lower confidence limit of the process incapability index LC_{PP}, that is, $x = LC_{PP}$. Then, we defined the fuzzy membership function of x as:

$$\eta(x) = \begin{cases} 0 \ if \ \ x < LC_{PPL} \\ \alpha \ if \ \ LC_{PPL} \leq x < LC_{PPM} \\ 1 \ if \ \ x = LC_{PPM} \\ 0 \ if \ \ LC_{PPM} < x \end{cases}, \tag{32}$$

where α is determined by $LC_{PP}(\alpha) = x$. Figure 1 presents a diagram of the membership function $\eta(x)$ with the vertical line $x = C$.

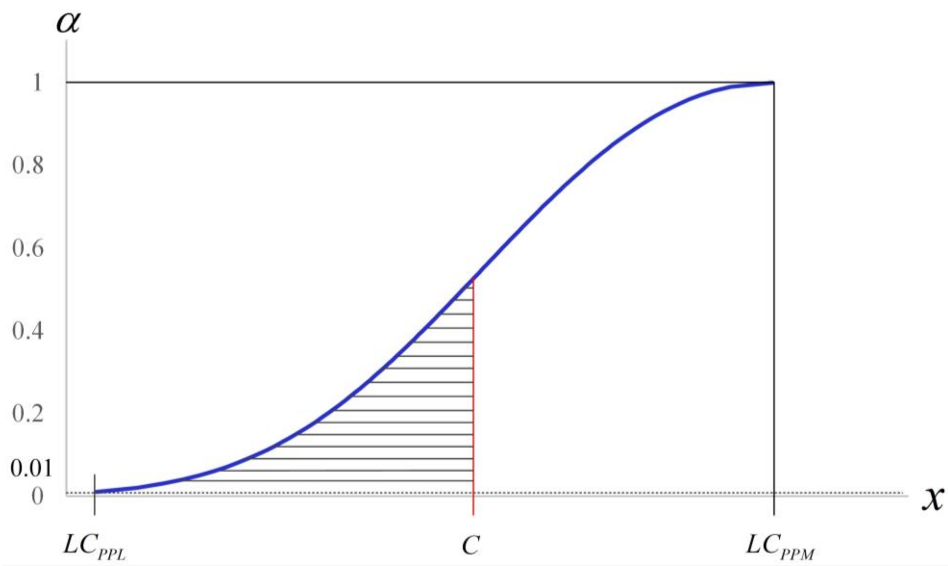

Figure 1. Fuzzy membership function $\eta(x)$ with vertical line $x = C$.

Based on Chen et al. [21], let set A_T be the area in the graph of $\eta(x)$ and set A_R be the area in the graph of $\eta(x)$ but to the left of the vertical line $x = \mathrm{C}$. Set A_T and set A_R are presented below:

$$A_T = \{ (x, \alpha) | LC_{PP}(\alpha) \leq x \leq LC_{PP}(1), 0 \leq \alpha \leq 1\} \tag{33}$$

and

$$A_R = \{ (x, \alpha) | LC_{PP}(\alpha) \leq x \leq C, 0 \leq \alpha \leq a\}, \tag{34}$$

where α = a such that $LC_{PP}(\alpha) = C$. According to Equation (33), the bottom length d_R of set A_R is equal to C minus LC_{PPL}, that is, $d_R = C - LC_{PPL}$. Similarly, according to Equation (32), the bottom length d_T of set A_T is equal to LC_{PPM} minus LC_{PPL}, that is, $d_T = LC_{PPM} - LC_{PPL}$. According to Chen and Lin [20], suppose $0 < \phi \leq 0.5$, and ϕ is determined by either production data or expert experience accumulated in the past [21,24]. The fuzzy evaluation and decision rules are explained below:

(1) If $d_R/2d_T \leq \phi$, then reject H_0 and conclude $C_{PP} > C$.
(2) If $\phi < d_R/2d_T$ <0.5, then do not reject H_0 and conclude $C_{PP} \leq C$.

5. A Practical Application

Plenty of studies have suggested that Taiwan's machinery and machine tool industry was ranked No. 5 in export and No. 7 in production worldwide. In addition to machine tools, including lathes, milling machines, drilling machines, grinders, and broaching machines, hand tools as well as woodworking machinery are also produced [25,26]. According to numerous studies, the central region in Taiwan is the core of the machine tool industry, and numerous component processing factories are located in the surrounding area. Given the concept of the Taguchi loss function, the cost expenditure and carbon emissions caused by fault maintenance will increase after the product is sold if the quality level of the component manufacturing process is poor and the size of the processed product deviates too far from the target value [27–29]. Moreover, machine tools will also be unable to continue working due to component failures, which will most likely lead to insufficient production and failure to deliver as scheduled, resulting in cost losses. In an attempt to boost the process quality level of these components, the outer diameter of the shaft processed by a machining factory in Central Taiwan is taken as an example to describe how to apply the confidence-interval-based fuzzy testing model suggested by this paper. When the quality engineers require that the product quality level should reach 6 sigma, it is equivalent to requiring that the process incapability index must be 0.81 at most. Thus, we rewrite the null hypothesis and alternative hypothesis applied to the test as follows:

Null hypothesis is $H_0 : C_{PP} \leq 0.81$;

Alternative hypothesis is $H_1 : C_{PP} > 0.81$.

As noted above, since the index contains unknown parameters, we can use the lower confidence limit of the process incapability index to perform the above hypothesis test if we want to decide whether the process quality level meets the requirement. Aiming to perform the above fuzzy test, 20 samples were randomly chosen from the product processed by a certain machining shaft. These 20 samples are described in detail below:

$$\begin{array}{llll} x_1 = 1.225, & x_2 = 1.214, & x_3 = 1.215, & x_4 = 1.216, \\ x_5 = 1.213, & x_6 = 1.222, & x_7 = 1.220, & x_8 = 1.229, \\ x_9 = 1.223, & x_{10} = 1.194, & x_{11} = 1.194, & x_{12} = 1.218, \\ x_{13} = 1.195, & x_{14} = 1.217, & x_{15} = 1.197, & x_{16} = 1.210, \\ x_{17} = 1.222, & x_{18} = 1.192, & x_{19} = 1.213, & x_{20} = 1.238. \end{array}$$

The tolerance of the machining shaft of a certain model being processed is 1.2 ± 0.05, that is, target $T = 1.2$ and $d = 0.05$. As noted above, the values of random variable $Y_i = (X_i - 1.2)/0.05$ and $(y_1, \ldots, y_i, \ldots, y_{20})$, the observed set of $(Y_1, \ldots, Y, \ldots, Y_{20})$, are shown as follows:

$$\begin{array}{llll} y_1 = 0.494, & y_2 = 0.276, & y_3 = 0.304, & y_4 = 0.315, \\ y_5 = 0.257, & y_6 = 0.447, & y_7 = 0.401, & y_8 = 0.571, \\ y_9 = 0.456, & y_{10} = -0.116, & y_{11} = -0.127, & y_{12} = 0.370, \\ y_{13} = -0.094, & y_{14} = 0.340, & y_{15} = -0.061, & y_{16} = 0.206, \\ y_{17} = 0.448, & y_{18} = -0.169, & y_{19} = 0.264, & y_{20} = 0.763. \end{array}$$

Then, δ_0^* and γ_0^*, the observed values of δ^* and γ^*, are denoted by:

$$\delta_0^* = \frac{1}{20}\sum_{i=1}^{20} y_i = 0.267 \text{ and } \gamma_0^* = \sqrt{\frac{1}{19}\sum_{i=1}^{20}(y_i - 0.261)^2} = 0.258.$$

Therefore,

$$E = \frac{Z_{0.0025}}{\sqrt{20}}\sqrt{\frac{19}{\chi^2_{0.9975;19}}} = \frac{2.807}{\sqrt{20}} \times \sqrt{\frac{19}{40.885}} = 0.428,$$

$$\begin{aligned} E_{\gamma_L} &= E \times \gamma_0^* = 0.428 \times 0.258 = 0.110, \\ \delta_L &= \delta_0^* - E_{\gamma_L} = 0.267 - 0.110 = 0.157, \\ \delta_R &= \delta_0^* + E_{\gamma_L} = 0.267 + 0.110 = 0.377. \end{aligned}$$

Obviously, $\delta_L = 0.157$ is bigger than zero. According to Equations (32) and (33), d_R and d_T are denoted as follows:

$$\begin{aligned} d_R = \; & C - 9\left\{(\delta_0^* - E \times \gamma_0^*)^2 + \left(\sqrt{\tfrac{n-1}{\chi^2_{0.9975;n-1}}}\gamma_0^*\right)^2\right\} \\ & = 0.97 - 9 \times \left\{(0.157)^2 + \left(\sqrt{\tfrac{19}{40.885}} \times 0.258\right)^2\right\} \;; \\ & = 0.97 - 9 \times (0.0246 + 0.0309) \\ & = 0.81 - 0.50 = 0.31 \end{aligned}$$

$$\begin{aligned} d_T = & 9\left\{(\delta_0^*)^2 + \left(\sqrt{\frac{n-1}{\chi^2_{0.5;n-1}}}\gamma_0^*\right)^2\right\} - 9\left\{(\delta_0^* - E\times\gamma_0^*)^2 + \left(\sqrt{\frac{n-1}{\chi^2_{0.9975;n-1}}}\gamma_0^*\right)^2\right\} \\ & = 9\times\left\{(0.267)^2 + \left(\sqrt{\frac{19}{18.338}}\times 0.258\right)^2\right\} - 9\left\{(0.157)^2 + \left(\sqrt{\frac{19}{40.885}}\times 0.258\right)^2\right\}. \\ & = 9\times(0.0713 + 0.0690) - 9\times(0.0246 + 0.0309) \\ & = 1.26 - 0.50 = 0.76 \end{aligned}$$

Therefore, the value of $d_R/2d_T$ is derived as:

$$d_R/2d_T = \frac{0.31}{2\times 0.76} = 0.20.$$

Let $\phi = 0.2$. According to the fuzzy testing rule (1), if $d_R/2d_T \leq \phi = 0.2$, then do not reject H_0 and conclude $C_{PP} > 0.81$. That means the process quality level has not reached 6 sigma, so an improvement must be made in the process.

Based on the above information, we have $\delta_0^* = 0.267$ and $\gamma_0^* = 0.258 = 1/3.876$. According to Chen and Lin [20] and Wu et al. [26], they mean that the process mean shift exceeds 1/4 of the tolerance, and the process only reaches 3.876-sigma quality level. Obviously, both the process accuracy and the process precision are insufficient, and they must be improved to boost the overall process quality level.

6. Conclusions, Research Limitations, and Future Research

The process incapability index, C_{PP}, which can reflect the expected loss and the yield in the process, is a good indicator to assess the process quality level. Because the process incapability index C_{PP} contains two unknown parameters, it is necessary to estimate the index with sample data. Given that enterprises consider costs and emphasize the timeliness of the quick response, the sample applied to sampling inspection is mostly small in size. A lot of research has indicated that a fuzzy testing method built on the confidence interval can be applied at this time since this method incorporates experts and experience gathered in the past. By means of this method, wrong judgements led by sampling error can be diminished, and the test accuracy can be boosted, as well [20,21,24,25]. Given this effect, we first deduced the process incapability index corresponding to the six-sigma quality level. Next, we defined the null hypothesis and the alternative hypothesis of the test based on engineers' requirements for the process quality level. Furthermore, we derived the $100\times(1-\alpha)\%$ lower confidence limit of the process incapability index and developed a confidence-interval-based fuzzy test of the index. Then, we put forward a fuzzy testing method based on the lower confidence limit to evaluate the quality level required by the process capability of the product. If the quality level fails to satisfy the requirement of the quality level, then an improvement must be made. Finally, an example was demonstrated to explain the application of the fuzzy testing method addressed in Section 4 so as to facilitate the application and promotion for relevant enterprises.

In the case study of Section 5, according to the equation $E[L(Y)] = k'(\delta^2 + \gamma^2)$, as long as the corresponding value of k' is calculated based on the company's relevant data, the point estimate of the expected loss can be obtained. However, the value of k' usually varies with different cases or companies. Moreover, it has been discovered from the fuzzy evaluation model proposed by this case study that both process accuracy and process precision are insufficient, so both of them must be improved to raise the overall process quality level. Like the value of k', the cost of enhancing accuracy and precision also varies with different enterprises, but they are not within the scope of this study. The future research can emphasize k', which is worth exploring, and construct the evaluation model of the improving costs or benefits.

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