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Bi-Objective Optimization for Interval Max-Plus Linear Systems

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Abstract: This paper investigates the interval-valued-multi-objective-optimization problem, whose objective function is a vector-valued max-plus interval function and the constraint function is a real-affine function. The strong and weak solvabilities of the interval-valued-optimization problem are introduced, and the solvability criteria are established. A necessary and sufficient condition for the strong solvability of the multi-objective-optimization problem is provided. In particular, for the bi-objective-optimization problem, a necessary and sufficient condition of the weak solvability is provided, and all the solvable sub-problems are found out. The interval optimal solution is obtained by constructing the set of all optimal solutions of the solvable sub-problems. The optimal load distribution is used to demonstrate how the presented results work in real-life examples.

Keywords: max-plus linear system; multi-objective optimization; interval-valued optimization; weak solvability; interval optimal solution

MSC: 15A80; 90C29; 93C73

1. Introduction

Multi-objective optimization is concerned with mathematical optimization problems involving more than one objective function to be optimized simultaneously. It is one of the most complex decision-making problems and has significant theoretical and applied potential in combination with a wide diversity of models. Many optimal solution methods have been proposed for linear- or nonlinear-optimization problems, such as the simplex method, the Karush–Kuhn–Tucker approach, heuristic algorithms (including the genetic algorithm, the simulated-annealing algorithm, particle-swarm optimization, etc.) and so on (see, e.g., Refs. [1–4]). In real-word applications, Zhang et al. [5], for example, established the Karush–Kuhn–Tucker-based-optimization algorithm for solving the torque-allocation problem, with the objective being to improve the stability performance of distributed-drive electric vehicles. Shafigh et al. [6] developed a linear-programming-embedded simulated-annealing algorithm for solving a comprehensive model for large-size problems in distributed-layout-based manufacturing systems, with the aim of minimizing the total cost of material handling, machine relocation, inventory holding, and internal-part production cost. Gangwar et al. [7] built a network-reconfiguration method for an unbalanced distribution system by using the repository-based constrained-nondominated-sorting genetic algorithm, with the aim being to minimize daily energy loss, energy not supplied and the cumulative-current-unbalance factor. The simplex method [3] perfectly solves linear-programming problems whose objective and constraint functions are both real linear functions. The Karush–Kuhn–Tucker approach [1] is effective for nonlinear optimization problems with a single objective. Heuristic algorithms [2] based on intuition and experience have a wide scope of application, while the deviation degree between the feasible solution and the optimal solution searched for by the heuristic algorithms cannot

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be estimated, in general. It is of great significance to develop an accurate method to solve specific types of multi-objective-nonlinear-optimization problems.

Max-plus algebra has a nice algebraic structure and is effectively used to model, analyze, control and optimize some nonlinear-time-evolution systems with synchronization but no concurrency (see, e.g., Refs. [8–11]). These nonlinear systems can be described by a max-plus linear-time-invariant model, which is called the max-plus linear system. Max-plus linear systems have wide applications in manufacturing, transportation, scheduling, robotics and high-throughput screening, as well as reinforcement learning and other fields (see, e.g., Refs. [12–17]). Many methods have been put forward to solve all kinds of optimization problems for max-plus linear systems. For example, Butkovič and MacCaig [18] studied the integer optimization of max-plus linear systems and found out the integer solutions. Xu et al. [19] investigated the optimistic optimization of max-plus linear systems, and they established efficient algorithms to find the approximation of globally optimal solutions for general nonlinear optimization. Gaubert et al. [20] studied the tropical-linear-fractional-programming problem, whose objective function is a max-plus rational function and a constraint condition that is a two-sided max-plus linear inequality, and they reduced such a problem to the auxiliary-mean-payoff-game problem. Goncalves et al. [21] provided efficient algorithms to solve the tropical-linear-fractional-programming problem, whose objective function is a max-plus function and a constraint condition that is a two-sided max-plus linear equation. Marotta et al. [22] presented a solution to the tropical-lexicographic-synchronization-optimization problem, whose objective function is a max-plus rational function and a constraint condition that is a two-sided max-plus linear equation. Tao et al. [23–26] studied the global-optimization problem of max-plus linear systems, whose objective function is a max-plus vector-valued function and a constraint function that is a real-affine function, and they provided the necessary and sufficient conditions for the existence and uniqueness of globally optimal solutions. Shu and Yang [27] solved the minimax-programming problem, in which the objective function is to minimize the maximum of all variables, while the constraint is a system of max-plus inequalities.

During the practical operation of a physical system, parameter perturbations are inevitable because of disturbances and errors in the estimation of processes. In practical applications, it is usually necessary to make an optimal decision in uncertainty environments. Necoara et al. [28] found a solution to a class of finite-horizon min-max controls for uncertain max-plus linear systems. Le Corronc et al. [29] synthesized an optimal controller to reduce the uncertainty at the output of interval max-plus linear systems. Myskova and Plavka [30–32] studied the robustness of interval max-plus linear systems, and they presented the necessary and sufficient conditions for interval matrices to be robust. Farahani et al. [33] constructed a solution for the optimization of stochastic max-min-plus scaling systems by using an approximation method based on moment-generating functions. Wang et al. [34] studied the optimal input design for uncertain max-plus linear systems, and they constructed the exact interval input to minimize the range of input that ensures the system can output at the desired point.

This paper studies the multi-objective-optimization problem for uncertain max-plus linear systems whose parameters are not exactly known but belong to an interval. Multi-objective optimization for interval max-plus systems is formulated as an interval-valued-optimization problem. The strong and weak solvabilities of the interval-valued-optimization problem are introduced based on the solvability of its sub-problems with deterministic parameters. The solvability criteria for the interval-valued-optimization problem are established. For one thing, it is pointed out that the problem is strongly solvable only if the interval objective function is degenerated to a max-plus function with deterministic coefficients, and then the strong solvability is reduced to the solvability of its unique sub-problem. A necessary and sufficient condition for the strong solvability of the multi-objective-optimization problem is established. For another, the weak solvability of the bi-objective-optimization problem is studied. A necessary and sufficient condition of weak solvability is provided and all the solvable sub-problems are found out. The interval optimal solution is obtained by constructing the set of all optimal solutions of the solvable sub-problems. To demonstrate the effectiveness of the proposed results in real-life examples, the bi-objective-optimization technique is applied in the load distribution of distributed systems, by which the minimum completion time can be advanced.

Comparing with the previous works, the main novelty and contribution of this paper are summarized as follows:

- The multi-objective-optimization problem is investigated. The global optimal solution and the global minimum are obtained.
- The hybrid-optimization problem is considered. More specifically, the constraint function is a real-affine function, while the objective function is a nonlinear function.
- The interval-valued-optimization problem is studied. The solvability criteria are established, and the interval optimal solution is constructed, to make the optimal decisions under uncertainty.

The remainder of this paper is organized as follows. Section 2 recalls some basic concepts and results from max-plus algebra. Section 3 establishes the multi-objective-optimization model of interval max-plus systems, and gives a necessary and sufficient condition for strong solvability. Section 4 studies the weak solvability of bi-objective optimization for interval max-plus systems, and finds the interval optimal solution. Section 5 presents an application example in optimal load distribution. Section 6 draws conclusions and highlights future works.

2. Preliminaries

This section introduces some notations, terminologies and properties from max-plus algebra, most of which can be found in Refs. [8–11] for more details.

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^+$ be the set of positive integers. For $n \in \mathbb{N}^+$, denote by $\mathbb{N}_n$ the set $\{1, 2, \ldots, n\}$. For $a, b \in \mathbb{R} \cup \{-\infty\}$, let

$$a \oplus b = \max\{a, b\} \text{ and } a \otimes b = a + b,$$

where $\max\{a, -\infty\} = a$ and $a + (-\infty) = -\infty$. The algebraic structure $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called *max-plus algebra* and is simply denoted by $\mathbb{R}_{\max}$, in which $-\infty$ and 0 are the zero and identity elements, denoted by ε and e, respectively. The symbol ϕ is used to represent the conventional $-$, i.e., for $a, b \in \mathbb{R}_{\max}$, $a \phi b = a - b$, which is valued in $\overline{\mathbb{R}}_{\max} (= \mathbb{R}_{\max} \cup \{+\infty\})$ not just in $\mathbb{R}_{\max}$. Note that by definition, $(+\infty) \otimes (-\infty) = -\infty$ and $(-\infty) \phi (-\infty) = +\infty$.

Let $\mathbb{R}^n_{\max}$ and $\mathbb{R}^{m\times n}_{\max}$ be the sets of n-dimensional vectors and $m \times n$ matrices with entries in $\mathbb{R}_{\max}$, respectively. To prevent confusion, matrices and vectors are represented by bold-type letters. The addition $\oplus$, multiplication $\otimes$ and scalar multiplication $\circ$ of max-plus matrices are defined as follows:

- For $\boldsymbol{A} = (a_{ij}), \boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{m\times n}_{\max}$, $(\boldsymbol{A} \oplus \boldsymbol{B})_{ij} = a_{ij} \oplus b_{ij}$;
- For $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times r}_{\max}$ and $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{r\times n}_{\max}$,

$$(\boldsymbol{A} \otimes \boldsymbol{B})_{ij} = \bigoplus_{k=1}^{r} a_{ik} \otimes b_{kj};$$

- For $d \in \mathbb{R}_{\max}$ and $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times n}_{\max}$, $(d \circ \boldsymbol{A})_{ij} = d \otimes a_{ij}$.

  In addition, for $d \in \mathbb{R}_{\max}$ and $\boldsymbol{x} = (x_j) \in \mathbb{R}^n_{\max}$, $(d \phi \boldsymbol{x})_j = d \phi x_j$.

  For $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times n}_{\max}$, let $\boldsymbol{A}^\top$ be the transposition of $\boldsymbol{A}$, and $\boldsymbol{A}_{i\cdot}$ be the ith row of $\boldsymbol{A}$, i.e.,

$$\boldsymbol{A}_{i\cdot} = (a_{i1}\ a_{i2}\ \ldots\ a_{in}).$$

For $a, b \in \mathbb{R}_{\max}$, $a \leqslant b$ if $a \oplus b = b$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_{\max}$, $\boldsymbol{x} \leqslant \boldsymbol{y}$ if $\boldsymbol{x} \oplus \boldsymbol{y} = \boldsymbol{y}$. For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m\times n}_{\max}$, $\boldsymbol{A} \leqslant \boldsymbol{B}$ if $\boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{B}$. If $\boldsymbol{A} \leqslant \boldsymbol{B}$ and $\boldsymbol{x} \leqslant \boldsymbol{y}$, then $\boldsymbol{A} \otimes \boldsymbol{x} \leqslant \boldsymbol{B} \otimes \boldsymbol{y}$.

For $\boldsymbol{A} \in \mathbb{R}^{m\times n}_{\max}$, the vector-valued function

$$\begin{aligned} F : \mathbb{R}^{n}_{\max} &\to \mathbb{R}^{m}_{\max} \\ \boldsymbol{x} &\mapsto \boldsymbol{A} \otimes \boldsymbol{x} \end{aligned}$$

is called a *max-plus function* of type (n, m). A *max-plus linear system* is a system that can be described by max-plus functions.

Given $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{m\times n}_{\max}$ and $\boldsymbol{b} = (b_i) \in \mathbb{R}^{m}_{\max}$, a *system of max-plus linear equations* with unknown $\boldsymbol{x}$ is represented by

$$\boldsymbol{A} \otimes \boldsymbol{x} = \boldsymbol{b}, \tag{1}$$

where $\boldsymbol{x} = (x_j) \in \overline{\mathbb{R}}^{n}_{\max}$. System (1) is said to be *solvable* if there exists $\tilde{\boldsymbol{x}} \in \overline{\mathbb{R}}^{n}_{\max}$, such that $\boldsymbol{A} \otimes \tilde{\boldsymbol{x}} = \boldsymbol{b}$ and $\tilde{\boldsymbol{x}}$ is called a *solution* of system (1). For $\tilde{\boldsymbol{x}} \in \overline{\mathbb{R}}^{n}_{\max}$, $\tilde{\boldsymbol{x}}$ is called a *subsolution* of system (1) if $\boldsymbol{A} \otimes \tilde{\boldsymbol{x}} \leqslant \boldsymbol{b}$. The greatest subsolution is constructed as below, to establish a criterion for the solvability of system (1).

Lemma 1 ([8])**.** *The greatest subsolution of system* (1)*, denoted by* $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b})$*, exists and is given by*

$$e \phi \boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = (e \phi \boldsymbol{b}) \otimes \boldsymbol{A}.$$

In a conventional framework, $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = (x^*_j(\boldsymbol{A}, \boldsymbol{b})) \in \overline{\mathbb{R}}^{n}_{\max}$ can be expressed as

$$x^*_j(\boldsymbol{A}, \boldsymbol{b}) = \min_{i\in\mathbb{N}_m} \{b_i - a_{ij}\},\ j \in \mathbb{N}_n. \tag{2}$$

The greatest subsolution $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b})$ naturally satisfies the following properties:

(i) $\boldsymbol{A} \otimes \boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) \leqslant \boldsymbol{b}$;

(ii) if $\tilde{\boldsymbol{x}}$ is a subsolution of system (1), then $\tilde{\boldsymbol{x}} \leqslant \boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b})$. In particular, if $\tilde{\boldsymbol{x}}$ is a solution of system (1), then $\tilde{\boldsymbol{x}} \leqslant \boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b})$.

System (1) is solvable if and only if the greatest subsolution is a solution, i.e., $\boldsymbol{A} \otimes \boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = \boldsymbol{b}$.

A (closed) *interval* in $\mathbb{R}_{\max}$ is a set of the form

$$\mathfrak{U} = [\underline{u}, \bar{u}] = \{u \in \mathbb{R}_{\max} \mid \underline{u} \leqslant u \leqslant \bar{u}\},$$

where $\underline{u}$, $\bar{u} \in \mathbb{R}_{\max}$ are the lower and upper bounds of interval $\mathfrak{U}$, respectively (see, e.g., [35]). Denote by $\mathcal{I}(\mathbb{R}_{\max})$ the set of closed intervals in $\mathbb{R}_{\max}$.

An *interval matrix* in $\mathbb{R}_{\max}$ is defined by

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \\ \vdots & \vdots & & \vdots \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \cdots & \mathcal{A}_{mn} \end{pmatrix} = \begin{pmatrix} [\underline{a}_{11}, \bar{a}_{11}] & [\underline{a}_{12}, \bar{a}_{12}] & \cdots & [\underline{a}_{1n}, \bar{a}_{1n}] \\ [\underline{a}_{21}, \bar{a}_{21}] & [\underline{a}_{22}, \bar{a}_{22}] & \cdots & [\underline{a}_{2n}, \bar{a}_{2n}] \\ \vdots & \vdots & & \vdots \\ [\underline{a}_{m1}, \bar{a}_{m1}] & [\underline{a}_{m2}, \bar{a}_{m2}] & \cdots & [\underline{a}_{mn}, \bar{a}_{mn}] \end{pmatrix}, \tag{3}$$

where $\mathcal{A}_{ij} = [\underline{a}_{ij}, \bar{a}_{ij}] \in \mathcal{I}(\mathbb{R}_{\max})$. Let $\underline{\boldsymbol{A}} = (\underline{a}_{ij}), \overline{\boldsymbol{A}} = (\bar{a}_{ij}) \in \mathbb{R}^{m\times n}_{\max}$. Then,

$$\boldsymbol{\mathcal{A}} = [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}] = \{\boldsymbol{A} \in \mathbb{R}^{m\times n}_{\max} \mid \underline{\boldsymbol{A}} \leqslant \boldsymbol{A} \leqslant \overline{\boldsymbol{A}}\}.$$

Denote by $\mathcal{I}(\mathbb{R}^{m\times n}_{\max})$ the set of $m \times n$ interval matrices in $\mathbb{R}_{\max}$. Specifically, $\mathcal{I}(\mathbb{R}^{1\times n}_{\max})$ and $\mathcal{I}(\mathbb{R}^{n\times 1}_{\max})$ are the sets of n-dimensional row and column interval vectors in $\mathbb{R}_{\max}$, respectively, which are simply denoted by $\mathcal{I}(\mathbb{R}^{n}_{\max})$.

For $\boldsymbol{\mathcal{A}} = [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}] \in \mathcal{I}(\mathbb{R}_{\max}^{m\times n})$, the interval-valued function

$$\boldsymbol{\mathcal{F}} : \mathbb{R}_{\max}^{n} \to \mathcal{I}(\mathbb{R}_{\max}^{m})$$
$$\boldsymbol{x} \mapsto \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x} = [\underline{\boldsymbol{A}} \otimes \boldsymbol{x}, \overline{\boldsymbol{A}} \otimes \boldsymbol{x}]$$

is called a *max-plus interval function* of type (n, m), which is a set of max-plus functions, i.e.,

$$\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x} = \{F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x} \mid \boldsymbol{A} \in \boldsymbol{\mathcal{A}}\}.$$

An *interval max-plus linear system* is a system that can be described by max-plus interval functions.

3. Multi-Objective Optimization of Interval Max-Plus Systems

This section establishes the multi-objective-optimization model for interval max-plus systems and considers the solvability of such an interval-valued-optimization problem.

The *multi-objective-optimization problem* for interval max-plus systems is formulated as

$$\min_{\boldsymbol{x}\in\mathcal{X}} \boldsymbol{\mathcal{F}}(\boldsymbol{x}), \tag{4}$$

where the decision variable is $\boldsymbol{x} = (x_j) \in \mathbb{R}^n$; the objective function is a max-plus interval function $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x}$, where $\boldsymbol{\mathcal{A}} \in \mathcal{I}(\mathbb{R}_{\max}^{m\times n})$ is given in (3); the constraint set is

$$\mathcal{X} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} k_j x_j = c,\ k_j > 0,\ c \in \mathbb{R} \right\},$$

which can be normalized as

$$\widetilde{\mathcal{X}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \sum_{j=1}^{n} \tilde{k}_j x_j = \tilde{c},\ \sum_{j=1}^{n} \tilde{k}_j = 1,\ \tilde{k}_j > 0,\ c \in \mathbb{R} \right\},$$

where $\tilde{k}_j = k_j / \sum_{j=1}^{n} k_j$ and $\tilde{c} = c / \sum_{j=1}^{n} k_j$. Without loss of generality, assume that $\sum_{j=1}^{n} k_j = 1$ in $\mathcal{X}$ in the discussion later in this paper.

For each $F(\boldsymbol{x}) \in \boldsymbol{\mathcal{F}}(\boldsymbol{x})$, the multi-objective-optimization problem

$$\min_{\boldsymbol{x}\in\mathcal{X}} F(\boldsymbol{x}) \tag{5}$$

is called a *sub-problem* of the interval-valued-optimization problem (4). The objective function of problem (5) is a max-plus function $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$, where $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$.

Definition 1 ([24])**.** *Problem* (5) *is said to be* solvable *if there exists* $\tilde{\boldsymbol{x}} \in \mathcal{X}$*, such that* $F(\tilde{\boldsymbol{x}}) \leqslant F(\boldsymbol{x})$ *for any* $\boldsymbol{x} \in \mathcal{X}$ *and* $\tilde{\boldsymbol{x}}$ *is called an optimal solution of problem* (5)*.*

Next, let us introduce the solvability of interval-valued-optimization problem (4) based on the solvability of its sub-problems.

Definition 2. *Problem* (4) *is said to be weakly solvable if there exists* $F(\boldsymbol{x}) \in \boldsymbol{\mathcal{F}}(\boldsymbol{x})$*, such that sub-problem* (5) *is solvable. Problem* (4) *is said to be strongly solvable if for any* $F(\boldsymbol{x}) \in \boldsymbol{\mathcal{F}}(\boldsymbol{x})$ *sub-problem* (5) *is solvable.*

In other words, interval-valued-optimization problem (4) is weakly solvable if it has at least one solvable sub-problem, and problem (4) is strongly solvable if each of its sub-problems is solvable. Before establishing the solvability criteria of problem (4), it is necessary to study the solvability of sub-problem (5).

Lemma 2 ([24]). *For sub-problem (5), let* $\boldsymbol{b} = (b_i) \in \mathbb{R}^m_{\max}$ *be defined by*

$$b_i = \sum_{j=1}^{n} k_j a_{ij} + c,\ i \in \mathbb{N}_m. \tag{6}$$

Then, $\boldsymbol{b}$ *is the greatest lower bound of* $F(\boldsymbol{x})$*, i.e.,*

(i) $F(\boldsymbol{x}) \geqslant \boldsymbol{b}$ *for any* $\boldsymbol{x} \in \mathcal{X}$*;*

(ii) *if* $\boldsymbol{y}$ *is a lower bound of* $F(\boldsymbol{x})$*, then* $\boldsymbol{y} \leqslant \boldsymbol{b}$*.*

Lemma 3 ([24,25]). *Sub-problem (5) is solvable if and only if*

$$\sum_{j=1}^{n} k_j x_j^*(\boldsymbol{A}, \boldsymbol{b}) = c, \tag{7}$$

where $\boldsymbol{b}$ *is the greatest lower bound given in (6). Moreover, if Equation (7) holds, then* $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b})$ *is the unique optimal solution of sub-problem (5).*

Next, let us give a necessary condition for strong solvability.

Theorem 1. *If problem (4) is strongly solvable, then* $\underline{\boldsymbol{A}} = \overline{\boldsymbol{A}}$*.*

Proof. Since problem (4) is strongly solvable, it follows that for any $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$, sub-problem (5) is solvable. Then, Equation (7) holds. It can be known from (2) that $x_j^*(\boldsymbol{A}, \boldsymbol{b}) \leqslant b_i \not\phi a_{ij}$ for any $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Suppose there exists $j_0 \in \mathbb{N}_n$, such that $x_{j_0}^*(\boldsymbol{A}, \boldsymbol{b}) < b_i \not\phi a_{ij_0}$. Then,

$$c = \sum_{j=1}^{n} k_j x_j^*(\boldsymbol{A}, \boldsymbol{b}) < \sum_{j=1}^{n} k_j (b_i - a_{ij}) = b_i - \sum_{j=1}^{n} k_j a_{ij} = \sum_{j=1}^{n} k_j a_{ij} + c - \sum_{j=1}^{n} k_j a_{ij} = c.$$

This contradiction implies that $x_j^*(\boldsymbol{A}, \boldsymbol{b}) = b_i \not\phi a_{ij}$ for any $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Let $p_j = x_1^*(\boldsymbol{A}, \boldsymbol{b}) \not\phi x_j^*(\boldsymbol{A}, \boldsymbol{b})$. Then, for any $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$,

$$a_{i1} \otimes p_j = a_{i1} \otimes x_1^*(\boldsymbol{A}, \boldsymbol{b}) \not\phi x_j^*(\boldsymbol{A}, \boldsymbol{b}) = a_{i1} \otimes (b_i \not\phi a_{i1}) \not\phi (b_i \not\phi a_{ij}) = a_{ij}. \tag{8}$$

That is, for any $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{\mathcal{A}}$,

$$a_{ij} \not\phi a_{i1} = a_{1j} \not\phi a_{11} = p_j,\ i \in \mathbb{N}_m,\ j \in \mathbb{N}_n. \tag{9}$$

Specifically, for $\underline{\boldsymbol{A}} \in \boldsymbol{\mathcal{A}}$, we have $\underline{a}_{ij} \not\phi \underline{a}_{i1} = \underline{a}_{1j} \not\phi \underline{a}_{11}$ for any $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. Suppose that there exist $i_0 \in \mathbb{N}_m$ and $j_0 \in \mathbb{N}_n$ such that $\underline{a}_{i_0 j_0} < \bar{a}_{i_0 j_0}$. Let $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{\mathcal{A}}$ be defined by

$$a_{ij} = \begin{cases} \bar{a}_{ij}, & i = i_0,\ j = j_0, \\ \underline{a}_{ij}, & \text{others}. \end{cases}$$

It follows from (9) that $a_{i_0 j_0} \not\phi a_{i_0 1} = a_{1 j_0} \not\phi a_{11}$, i.e., $\bar{a}_{i_0 j_0} \not\phi \underline{a}_{i_0 1} = \underline{a}_{1 j_0} \not\phi \underline{a}_{11}$. Hence,

$$\underline{a}_{1 j_0} \not\phi \underline{a}_{11} = \bar{a}_{i_0 j_0} \not\phi \underline{a}_{i_0 1} > \underline{a}_{i_0 j_0} \not\phi \underline{a}_{i_0 1} = \underline{a}_{1 j_0} \not\phi \underline{a}_{11}.$$

This contradiction implies that $\underline{\boldsymbol{A}} = \overline{\boldsymbol{A}}$. □

It can be seen from the theorem above that interval-valued-optimization problem (4) cannot be strongly solvable if $\underline{\boldsymbol{A}} \neq \overline{\boldsymbol{A}}$. In other words, problem (4) is strongly solvable only if the interval objective function is degenerated to a max-plus function with deterministic coefficients. Consequently, the strong solvability of problem (4) is reduced to the solvability of its unique sub-problem (5). Then, the following necessary and sufficient condition for the strong solvability can be obtained from Theorem 1 and Lemma 3.

Corollary 1. *Problem* (4) *is strongly solvable if and only if* $\underline{\boldsymbol{A}} = \overline{\boldsymbol{A}}$ *and Equation* (7) *holds for* $\boldsymbol{A} = \underline{\boldsymbol{A}}$ *(or* $\overline{\boldsymbol{A}}$*).*

4. Weak Solvability of Bi-Objective Optimization Problem

This section studies the weak solvability of bi-objective optimization problem for interval max-plus linear systems, that is, the weak solvability of problem (4) in the case of $m = 2$.

Consider the *bi-objective optimization problem*

$$\min_{x\in\mathcal{X}} \boldsymbol{\mathcal{F}}(\boldsymbol{x}), \tag{10}$$

where $\boldsymbol{x}$ and $\mathcal{X}$ are the same decision variable and constraint set as problem (4), respectively, and the objective function $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x}$ is a special case of problem (4) for $m = 2$, i.e.,

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \cdots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \cdots & \mathcal{A}_{2n} \end{pmatrix} = \begin{pmatrix} [\underline{a}_{11}, \bar{a}_{11}] & [\underline{a}_{12}, \bar{a}_{12}] & \cdots & [\underline{a}_{1n}, \bar{a}_{1n}] \\ [\underline{a}_{21}, \bar{a}_{21}] & [\underline{a}_{22}, \bar{a}_{22}] & \cdots & [\underline{a}_{2n}, \bar{a}_{2n}] \end{pmatrix}.$$

Let us first establish a weak solvability criterion for problem (10).

Theorem 2. *Problem* (10) *is weakly solvable if and only if*

$$\max_{j\in\mathbb{N}_n}\{\underline{a}_{2j} \phi \bar{a}_{1j}\} \leqslant \min_{j\in\mathbb{N}_n}\{\bar{a}_{2j} \phi \underline{a}_{1j}\}. \tag{11}$$

Proof. *Necessity.* Since interval problem (10) is weakly solvable, there exists $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{\mathcal{A}}$, such that sub-problem (5) is solvable. It can be known from the proof of Theorem 1 that Equation (8) holds. It follows that $a_{2j} \phi a_{1j} = (a_{21} \otimes p_j) \phi (a_{11} \otimes p_j) = a_{21} \phi a_{11}$. Let $d = a_{21} \phi a_{11}$. Then, $a_{2j} = a_{1j} \otimes d$. Hence, $\underline{a}_{2j} \phi \bar{a}_{1j} \leqslant a_{2j} \phi a_{1j} = d \leqslant \bar{a}_{2j} \phi \underline{a}_{1j}$, $\forall j \in \mathbb{N}_n$. This implies that $\max_{j\in\mathbb{N}_n}\{\underline{a}_{2j} \phi \bar{a}_{1j}\} \leqslant d \leqslant \min_{j\in\mathbb{N}_n}\{\bar{a}_{2j} \phi \underline{a}_{1j}\}$, i.e., Inequality (11) holds.

Sufficiency. Since Inequality (11) holds, there exists $d \in \mathbb{R}$, such that $\underline{a}_{2j} \phi \bar{a}_{1j} \leqslant d \leqslant \bar{a}_{2j} \phi \underline{a}_{1j}$ for any $j \in \mathbb{N}_n$. Let $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{2\times n}_{\max}$ be defined by

$$a_{1j} = \begin{cases} \underline{a}_{2j} \phi d, & \text{if } d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \underline{a}_{2j} \phi \underline{a}_{1j}], \\ \underline{a}_{1j}, & \text{if } d \in [\underline{a}_{2j} \phi \underline{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]; \end{cases} \tag{12}$$

$$a_{2j} = \begin{cases} \underline{a}_{2j}, & \text{if } d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \underline{a}_{2j} \phi \underline{a}_{1j}], \\ \underline{a}_{1j} \otimes d, & \text{if } d \in [\underline{a}_{2j} \phi \underline{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]. \end{cases} \tag{13}$$

For $j \in \mathbb{N}_n$, if $d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \underline{a}_{2j} \phi \underline{a}_{1j}]$, then

$$\underline{a}_{1j} = \underline{a}_{2j} \phi (\underline{a}_{2j} \phi \underline{a}_{1j}) \leqslant a_{1j} = \underline{a}_{2j} \phi d \leqslant \underline{a}_{2j} \phi (\underline{a}_{2j} \phi \bar{a}_{1j}) = \bar{a}_{1j},$$
$$\underline{a}_{2j} = a_{2j} \leqslant \bar{a}_{2j};$$

if $d \in [\underline{a}_{2j} \phi \underline{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]$, then

$$\underline{a}_{1j} = a_{1j} \leqslant \bar{a}_{1j},$$
$$\underline{a}_{2j} = \underline{a}_{1j} \otimes \underline{a}_{2j} \phi \underline{a}_{1j} \leqslant a_{2j} = \underline{a}_{1j} \otimes d \leqslant \underline{a}_{1j} \otimes \bar{a}_{2j} \phi \underline{a}_{1j} = \bar{a}_{2j}.$$

Hence, $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$. It can be seen from (12) and (13) that $a_{2j} = d \otimes a_{1j}$ for any $j \in \mathbb{N}_n$. Then,

$$b_2 = \sum_{j=1}^{n} k_j a_{2j} + c = \sum_{j=1}^{n} k_j (a_{1j} + d) + c = \sum_{j=1}^{n} k_j a_{1j} + c + d = b_1 + d.$$

It follows that $b_2 - a_{2j} = (b_1 + d) - (a_{1j} + d) = b_1 - a_{1j}$, and so

$$x_j^*(\boldsymbol{A}, \boldsymbol{b}) = \min_{i \in \mathbb{N}_2} \{b_i - a_{ij}\} = b_1 - a_{1j},\ j \in \mathbb{N}_n.$$

Hence,

$$\sum_{j=1}^{n} k_j x_j^*(\boldsymbol{A}, \boldsymbol{b}) = \sum_{j=1}^{n} k_j (b_1 - a_{1j}) = b_1 - \sum_{j=1}^{n} k_j a_{1j} = \sum_{j=1}^{n} k_j a_{1j} + c - \sum_{j=1}^{n} k_j a_{1j} = c.$$

From Lemma 3, the sub-problem with objective function $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$ is solvable. Hence, problem (10) is weakly solvable. □

Let us illustrate the theorem above with a numerical example.

Example 1. *Consider the interval-valued-optimization problem*

$$\min_{x \in \mathcal{X}} \boldsymbol{\mathcal{F}}(\boldsymbol{x}), \tag{14}$$

where the objective function is $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x}$,

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} [1,2] & [-2,3] & [-1,4] \\ [0,4] & [-3,0] & [0,5] \end{pmatrix},$$

and the constraint set is $\mathcal{X} = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,|\, 2x_1 + 3x_2 + x_3 = 6\right\}$, *which can be normalized as*

$$\mathcal{X} = \left\{\boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \frac{1}{3}x_1 + \frac{1}{2}x_2 + \frac{1}{6}x_3 = 1\right\}.$$

By a direct calculation,

$$\max_{j \in \mathbb{N}_3} \{\underline{a}_{2j} \phi \bar{a}_{1j}\} = \max\{-2, -6, -4\} = -2 < \min_{j \in \mathbb{N}_3} \{\bar{a}_{2j} \phi \underline{a}_{1j}\} = \min\{3, 2, 6\} = 2.$$

It then follows from Theorem 2 that problem (14) is weakly solvable.
Indeed, let $d = 0 \in [-2, 2]$. *According to (12) and (13), construct a matrix as below:*

$$\boldsymbol{A} = \begin{pmatrix} \underline{a}_{11} & \underline{a}_{12} & \underline{a}_{23} \\ \underline{a}_{11} & \underline{a}_{12} & \underline{a}_{23} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix} \in \boldsymbol{\mathcal{A}}.$$

Then, the sub-problem with objective function $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$ *is solvable, whose minimal value is* $\boldsymbol{b} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}^{\mathsf{T}}$, *and which is attained at the point* $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = \begin{pmatrix} -\frac{2}{3} & \frac{7}{3} & \frac{1}{3} \end{pmatrix}^{\mathsf{T}}$. *This implies that problem (14) is weakly solvable.*

Next, let us find all the solvable sub-problems of the interval-valued-optimization problem (10). For convenience of presentation, let

$$\underline{\alpha} = \max_{j \in \mathbb{N}_n} \{\underline{a}_{2j} \phi \bar{a}_{1j}\} \text{ and } \bar{\alpha} = \min_{j \in \mathbb{N}_n} \{\bar{a}_{2j} \phi \underline{a}_{1j}\}.$$

It can be seen from the proof of sufficiency of Theorem 2 that the solvable sub-problems have a common characteristic that $A_{1.}$ and $A_{2.}$ are proportional in the max-plus framework. The following corollary shows the existence of the matrices in $\boldsymbol{\mathcal{A}}$ whose rows are proportional.

Corollary 2. *For $d \in \mathbb{R}$, let $\mathcal{T}(d) = \{\boldsymbol{A} \in \boldsymbol{\mathcal{A}} \mid \boldsymbol{A}_{2.} = d \circ \boldsymbol{A}_{1.}\}$. Then, $\mathcal{T}(d) \neq \emptyset$ if and only if $d \in [\underline{\alpha}, \bar{\alpha}]$.*

Proof. *Necessity.* Since $\mathcal{T}(d) \neq \emptyset$, there exists $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$, such that $a_{2j} = d \otimes a_{1j}$ for any $j \in \mathbb{N}_n$. Then,

$$\underline{a}_{2j} \phi \bar{a}_{1j} \leqslant d = a_{2j} \phi a_{1j} \leqslant \bar{a}_{2j} \phi \underline{a}_{1j}, \ \forall j \in \mathbb{N}_n.$$

This implies that $d \in [\underline{\alpha}, \bar{\alpha}]$.

Sufficiency. For $d \in [\underline{\alpha}, \bar{\alpha}]$, let $\boldsymbol{A} = (a_{ij}) \in \mathbb{R}^{2\times n}_{\max}$ be defined by (12) and (13). It has been proved in Theorem 2 that $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$ and $\boldsymbol{A}_{2.} = d \circ \boldsymbol{A}_{1.}$. Hence, $\boldsymbol{A} \in \mathcal{T}(d)$, and so $\mathcal{T}(d) \neq \emptyset$. □

For $d \in [\underline{\alpha}, \bar{\alpha}]$, let

$$\mathcal{S}(d) = \left\{ \boldsymbol{A} \in \mathbb{R}^{2\times n}_{\max} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(d),\ \boldsymbol{A}_{2.} = d \circ \boldsymbol{A}_{1.} \right\},$$

where $\mathfrak{U}(d) = [\underline{\boldsymbol{u}}, \bar{\boldsymbol{u}}] = ([\underline{u}_j, \bar{u}_j]) \in \mathcal{I}(\mathbb{R}^{1\times n}_{\max})$ is defined by

$$\underline{u}_j = \begin{cases} \underline{a}_{2j} \phi d, & \text{if } d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \underline{a}_{2j} \phi \underline{a}_{1j}], \\ \underline{a}_{1j}, & \text{if } d \in [\underline{a}_{2j} \phi \underline{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]; \end{cases} \quad \bar{u}_j = \begin{cases} \bar{a}_{1j}, & \text{if } d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \bar{a}_{2j} \phi \bar{a}_{1j}], \\ \bar{a}_{2j} \phi d, & \text{if } d \in [\bar{a}_{2j} \phi \bar{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]. \end{cases} \tag{15}$$

Then, all the solvable sub-problems of problem (10) can be represented as follows.

Theorem 3. *If problem (10) is weakly solvable, then all the solvable sub-problems have the form*

$$\min_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}), \tag{16}$$

where $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$, $\boldsymbol{A} \in \mathcal{S}(d)$ and $d \in [\underline{\alpha}, \bar{\alpha}]$.

Proof. Since problem (10) is weakly solvable, it follows from Theorem 2 that $\underline{\alpha} \leqslant \bar{\alpha}$. Next, let us prove $\mathcal{S}(d) = \mathcal{T}(d)$ for $d \in [\underline{\alpha}, \bar{\alpha}]$, where $\mathcal{T}(d)$ is given in Corollary 2. On the one hand, let

$$\underline{\boldsymbol{A}}(d) = \begin{pmatrix} \underline{\boldsymbol{u}} \\ d \circ \underline{\boldsymbol{u}} \end{pmatrix} \text{and } \overline{\boldsymbol{A}}(d) = \begin{pmatrix} \bar{\boldsymbol{u}} \\ d \circ \bar{\boldsymbol{u}} \end{pmatrix}, \tag{17}$$

where $\underline{\boldsymbol{u}}, \bar{\boldsymbol{u}} \in \mathbb{R}^n_{\max}$ defined by (15) are the lower and upper bounds of $\mathfrak{U}(d)$, respectively. It has been proved in the sufficiency of Theorem 2 that $\underline{\boldsymbol{A}}(d) \in \boldsymbol{\mathcal{A}}$. Similarly, we can prove $\overline{\boldsymbol{A}}(d) = (a_{ij}) \in \boldsymbol{\mathcal{A}}$. Indeed, for $j \in \mathbb{N}_n$, if $d \in [\underline{a}_{2j} \phi \bar{a}_{1j}, \bar{a}_{2j} \phi \bar{a}_{1j}]$, then

$$\begin{aligned} \underline{a}_{1j} \leqslant a_{1j} = \bar{a}_{1j}, \\ \underline{a}_{2j} = \bar{a}_{1j} \otimes \underline{a}_{2j} \phi \bar{a}_{1j} \leqslant a_{2j} = \bar{a}_{1j} \otimes d \leqslant \bar{a}_{1j} \otimes \bar{a}_{2j} \phi \bar{a}_{1j} = \bar{a}_{2j}; \end{aligned}$$

if $d \in [\bar{a}_{2j} \phi \bar{a}_{1j}, \bar{a}_{2j} \phi \underline{a}_{1j}]$, then

$$\begin{aligned} \underline{a}_{1j} = \bar{a}_{2j} \phi (\bar{a}_{2j} \phi \underline{a}_{1j}) \leqslant a_{1j} = \bar{a}_{2j} \phi d \leqslant \bar{a}_{2j} \phi (\bar{a}_{2j} \phi \bar{a}_{1j}) = \bar{a}_{1j}, \\ \underline{a}_{2j} \leqslant a_{2j} = \bar{a}_{2j}. \end{aligned}$$

Hence, for any $\boldsymbol{A} \in \mathcal{S}(d)$, $\boldsymbol{A}_{2.} = d \circ \boldsymbol{A}_{1.}$ and $\underline{\boldsymbol{A}} \leqslant \underline{\boldsymbol{A}}(d) \leqslant \boldsymbol{A} \leqslant \overline{\boldsymbol{A}}(d) \leqslant \overline{\boldsymbol{A}}$, i.e., $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$. Therefore, $\boldsymbol{A} \in \mathcal{T}(d)$ and so $\mathcal{S}(d) \subseteq \mathcal{T}(d)$. On the other hand, for any $\boldsymbol{A} \in \mathcal{T}(d)$, let us prove that $\boldsymbol{A}_{1.} \in \mathfrak{U}(d)$. Suppose there exists $j_1 \in \mathbb{N}_n$ such that $a_{1j_1} < \underline{u}_{j_1}$. If $d \in [\underline{a}_{2j_1} \not\! \bar{a}_{1j_1}, \bar{a}_{2j_1} \not\! \bar{a}_{1j_1}]$, then

$$a_{2j_1} = a_{1j_1} \otimes d < \underline{u}_{j_1} \otimes d = \underline{a}_{2j} \not\! d \otimes d = \underline{a}_{2j_1};$$

if $d \in [\underline{a}_{2j_1} \not\! \underline{a}_{1j_1}, \bar{a}_{2j_1} \not\! \underline{a}_{1j_1}]$, then $a_{1j_1} < \underline{u}_{j_1} = \underline{a}_{1j_1}$, both of which are contradicted to $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$. Suppose that there exists $j_2 \in \mathbb{N}_n$, such that $a_{1j_2} > \bar{u}_{j_2}$. If $d \in [\underline{a}_{2j_2} \not\! \bar{a}_{1j_2}, \bar{a}_{2j_2} \not\! \bar{a}_{1j_2}]$, then $a_{1j_2} > \bar{u}_{j_2} = \bar{a}_{1j_2}$; if $d \in [\underline{a}_{2j_2} \not\! \underline{a}_{1j_2}, \bar{a}_{2j_2} \not\! \underline{a}_{1j_2}]$, then

$$a_{2j_2} = a_{1j_2} \otimes d > \bar{u}_{j_2} \otimes d = \bar{a}_{2j} \not\! d \otimes d = \bar{a}_{2j_2},$$

both of which are also contradicted to $\boldsymbol{A} \in \boldsymbol{\mathcal{A}}$. Hence, $a_{1j} \in [\underline{u}_j, \bar{u}_j]$ for any $j \in \mathbb{N}_n$, i.e., $\boldsymbol{A}_{1.} \in \mathfrak{U}(d)$. This implies that $\boldsymbol{A} \in \mathcal{S}(d)$ and, hence, $\mathcal{T}(d) \subseteq \mathcal{S}(d)$. Therefore, $\mathcal{T}(d) = \mathcal{S}(d)$. Hence, all the solvable sub-problems have the form (16). □

Definition 3. *The sub-problems with objective functions $F(\boldsymbol{x}) = \underline{\boldsymbol{A}}(d) \otimes \boldsymbol{x}$ and $F(\boldsymbol{x}) = \overline{\boldsymbol{A}}(d) \otimes \boldsymbol{x}$ are called the lower and upper* extreme solvable sub-problems *(relative to d) of problem* (10)*, respectively, where $\underline{\boldsymbol{A}}(d)$ and $\overline{\boldsymbol{A}}(d)$ are given in* (17)*.*

Let us illustrate the theorem above with the following example.

Example 2 (continued from Example 1)**.** *Find all solvable sub-problems of problem* (14)*. For $d \in [-2, 2]$, let $\mathfrak{U}(d) = [\underline{\boldsymbol{u}}, \bar{\boldsymbol{u}}] = ([\underline{u}_j, \bar{u}_j]) \in \mathcal{I}(\mathbb{R}^3_{\max})$ be defined by*

$$\underline{u}_1 = \begin{cases} e \not\! d, & \text{if } d \in [-2,-1], \\ 1, & \text{if } d \in [-1,3]; \end{cases} \quad \bar{u}_1 = \begin{cases} 2, & \text{if } d \in [-2,2], \\ 4 \not\! d, & \text{if } d \in [2,3]; \end{cases}$$
$$\underline{u}_2 = \begin{cases} -3 \not\! d, & \text{if } d \in [-6,-1], \\ -2, & \text{if } d \in [-1,2]; \end{cases} \quad \bar{u}_2 = \begin{cases} 3, & \text{if } d \in [-6,-3], \\ e \not\! d, & \text{if } d \in [-3,2]; \end{cases}$$
$$\underline{u}_3 = \begin{cases} e \not\! d, & \text{if } d \in [-4,1], \\ -1, & \text{if } d \in [1,6]; \end{cases} \quad \bar{u}_3 = \begin{cases} 4, & \text{if } d \in [-4,1], \\ 5 \not\! d, & \text{if } d \in [1,6]. \end{cases}$$

That is,

$$\mathfrak{U}(d) = \begin{cases} ([-d,2]\ [-3-d,-d]\ [-d,4]), & \text{if } d \in [-2,-1]; \\ ([1,2]\ [-2,-d]\ [-d,4]), & \text{if } d \in [-1,1]; \\ ([1,2]\ [-2,-d]\ [-1,5-d]), & \text{if } d \in [1,2]. \end{cases} \tag{18}$$

Hence, all the solvable sub-problems have the form

$$\min_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}),$$

where $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x},\ \boldsymbol{A} \in \mathcal{S}(d)$ and

$$\mathcal{S}(d) = \left\{ \boldsymbol{A} \in \mathbb{R}^{2\times n}_{\max} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(d),\ \boldsymbol{A}_{2.} = d \circ \boldsymbol{A}_{1.},\ \ d \in [-2,2] \right\}.$$

Specifically, for example, let $d = 1$. Then, $\mathfrak{U}(1) = ([1,2]\ [-2,-1]\ [-1,4])$. The solvable sub-problems (relative to 1) have the form

$$\min_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}),$$

where $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$, $\boldsymbol{A} \in \mathcal{S}(1)$ and $\mathcal{S}(1) = \{\boldsymbol{A} \in \mathbb{R}^{2\times n}_{\max} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(1),\ \boldsymbol{A}_{2.} = 1 \circ \boldsymbol{A}_{1.}\}$.

Finally, let us present the optimal solutions of problem (10).

Theorem 4. *The set of all optimal solutions of solvable sub-problems of problem* (10) *is*

$$\left\{ \boldsymbol{H}\boldsymbol{A}_{1.}^{\intercal} + \boldsymbol{c} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(d),\ d \in [\underline{\alpha}, \bar{\alpha}] \right\}, \tag{19}$$

where $\mathfrak{U}(d)$ *is defined by* (15)*, and*

$$\boldsymbol{H} = \begin{pmatrix} k_1 - 1 & k_2 & \cdots & k_n \\ k_1 & k_2 - 1 & \cdots & k_n \\ \vdots & \vdots & \ddots & \vdots \\ k_1 & k_2 & \cdots & k_n - 1 \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix}.$$

Proof. For $d \in [\underline{\alpha}, \bar{\alpha}]$ and $\boldsymbol{A} \in \mathcal{S}(d)$, we have $a_{2j} = d \otimes a_{1j}$ for any $j \in \mathbb{N}_n$. By (6),

$$b_1 = \sum_{j=1}^{n} k_j a_{1j} + c,$$
$$b_2 = \sum_{j=1}^{n} k_j a_{2j} + c = \sum_{j=1}^{n} k_j (d + a_{1j}) + c = \sum_{j=1}^{n} k_j a_{1j} + c + d = b_1 + d.$$

By (2), for $j_0 \in \mathbb{N}_n$,

$$\begin{aligned} x_{j_0}^*(\boldsymbol{A}, \boldsymbol{b}) &= \min\{b_1 - a_{1j_0}, b_2 - a_{2j_0}\} = \min\{b_1 - a_{1j_0}, (b_1 + d) - (a_{1j_0} + d)\} \\ &= b_1 - a_{1j_0} = \sum_{j=1}^{n} k_j a_{1j} + c - a_{1j_0} = \sum_{j \neq j_0} k_j a_{1j} + k_{j_0} a_{1j_0} - a_{1j_0} + c \\ &= \sum_{j \neq j_0} k_j a_{1j} + (k_{j_0} - 1) a_{1j_0} + c = \boldsymbol{H}_{j_0 *} \boldsymbol{A}_{1.}^{\intercal} + c. \end{aligned}$$

Then, $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = \boldsymbol{H}\boldsymbol{A}_{1.}^{\intercal} + \boldsymbol{c}$. According to Lemma 3 and Theorem 3, the set of all optimal solutions of solvable sub-problems of problem (10) can be represented by (19). □

Definition 4. *For each* $d \in [\underline{\alpha}, \bar{\alpha}]$*, the set of optimal solutions given in* (19) *is called an* interval optimal solution *of problem* (10)*.*

Let us find the interval optimal solution of problem (14) by using the theorem above.

Example 3 (continued from Example 2)**.** *According to Theorem 4, the set of all optimal solutions of solvable sub-problems of problem* (14) *is*

$$\left\{ \boldsymbol{H}\boldsymbol{A}_{1.}^{\intercal} + \boldsymbol{c} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(d),\ d \in [-2, 2] \right\},$$

where $\mathfrak{U}(d)$ *is given by* (18)*, and*

$$\boldsymbol{H} = \frac{1}{6} \begin{pmatrix} -4 & 3 & 1 \\ 2 & -3 & 1 \\ 2 & 3 & -5 \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{20}$$

Specifically, for example, let $d = 1$*. It has been shown in Example 2 that*

$$\mathcal{S}(1) = \left\{ \boldsymbol{A} \in \mathbb{R}_{\max}^{2 \times n} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(1),\ \boldsymbol{A}_{2.} = 1 \circ \boldsymbol{A}_{1.} \right\},$$

where $\mathfrak{U}(1) = ([1,2]\ [-2,-1]\ [-1,4])$. *The set of optimal solutions of the solvable sub-problems with an objective function obtained from* $\mathcal{S}(1)$ *is* $\{\boldsymbol{H}\boldsymbol{A}_{1.}^{\mathsf{T}} + \boldsymbol{c} \mid \boldsymbol{A}_{1.} \in \mathfrak{U}(1)\}$. *For*

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \in \mathcal{S}(1) \in \boldsymbol{\mathcal{A}},$$

the sub-problem with objective function $F(\boldsymbol{x}) = \boldsymbol{A} \otimes \boldsymbol{x}$ *is solvable. By a direct calculation, the minimal value of* $F(\boldsymbol{x})$ *is* $\boldsymbol{b} = \frac{1}{6}(7\ \ 13)^{\mathsf{T}}$, *which is attained at the point*

$$\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = \frac{1}{6}(1\ \ 13\ \ -5)^{\mathsf{T}} = \boldsymbol{H}\boldsymbol{A}_{1.}^{\mathsf{T}} + \boldsymbol{c},$$

where $\boldsymbol{H}$ *and* $\boldsymbol{c}$ *are given in* (20).

5. Application Example

Bi-objective-interval-valued-optimization problems appear on many occasions in real-life examples. This section takes the load-distribution problem as an example, to illustrate how the obtained results work in practical applications.

Consider the distributed system with the task precedence graph shown in Figure 1, in which an overall task is partitioned into 7 subtasks T_1, T_2, ..., T_7, the circles represent subtasks, the number inside each circle represents the corresponding task execution time, and the number associated with each link corresponds to the interprocessor communication time. It has been presented in Example 4 of Ref. [24] that the load-distribution problem can be described by the multi-objective-optimization problem

$$\min_{\boldsymbol{x} \in \mathcal{X}} F(\boldsymbol{x}), \tag{21}$$

where $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 0.6x_1 + 0.4x_2 = 2\}$, $F(\boldsymbol{x}) = \boldsymbol{H} \otimes \boldsymbol{x}$ and $\boldsymbol{H}^{\mathsf{T}} = \begin{pmatrix} 10 & 5 & 17 & 25 & 35 \\ 10 & 5 & 17 & 25 & 35 \end{pmatrix}$. The global minimum of problem (21) is $\boldsymbol{b} = (12\ 7\ 19\ 27\ 37)^{\mathsf{T}}$, which is attained at the point $\boldsymbol{x}^*(\boldsymbol{A}, \boldsymbol{b}) = (2\ \ 2)^{\mathsf{T}}$. This implies that if the execution times of tasks T_1 and T_2 are both 2 units, then the completion times of tasks T_3, T_4, T_5, T_6 and T_7 are 12, 7, 19, 27 and 37 units, respectively. Moreover, the overall completion time is 37 units.

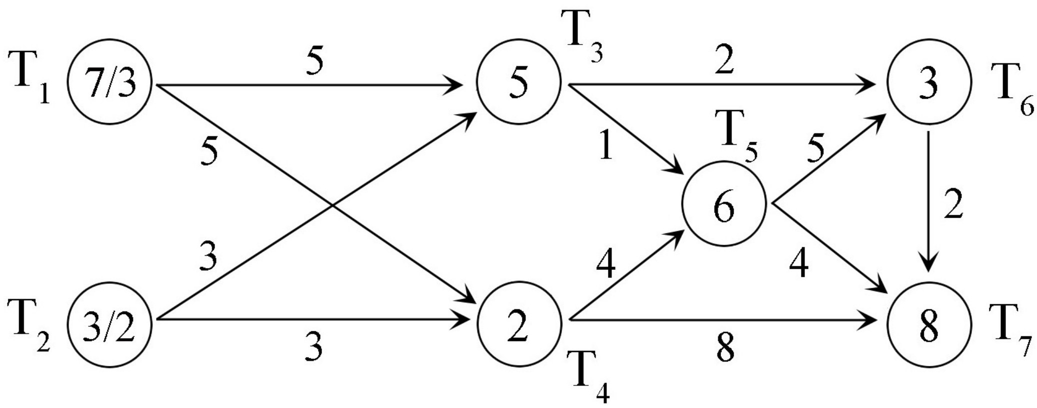

Figure 1. Task precedence graph of a distributed system [24].

Suppose that the entries of $\boldsymbol{H}^{\mathsf{T}}$ can be varied in the interval

$$\boldsymbol{\mathcal{H}}^{\mathsf{T}} = \begin{pmatrix} [9,11] & [2,6] & [16,18] & [22,27] & [33,38] \\ [8,11] & [4,7] & [15,18] & [23,28] & [32,36] \end{pmatrix} := \boldsymbol{\mathcal{A}}.$$

Next, let us minimize the completion time of the overall task by solving the interval-valued-optimization problem with coefficient matrix $\boldsymbol{\mathcal{A}}$. By a direct calculation,

$$\underline{\alpha} = \max_{j\in\mathbb{N}_5}\{\underline{a}_{2j} \phi \bar{a}_{1j}\} = \max\{-3, -2, -3, -4, -6\} = -2,$$
$$\bar{\alpha} = \min_{j\in\mathbb{N}_5}\{\bar{a}_{2j} \phi \underline{a}_{1j}\} = \min\{2, 5, 2, 6, 3\} = 2.$$

Since $\underline{\alpha} < \bar{\alpha}$, it follows from Theorem 2 that the interval-valued-optimization problem is solvable. This implies that the completion time of each subtask can be minimized simultaneously. Let $d = 2$. By (17), the coefficient matrix of the objective function of the lower extreme solvable sub-problem is

$$\underline{\boldsymbol{A}}(2) = \begin{pmatrix} \underline{\boldsymbol{u}} \\ 2 \circ \underline{\boldsymbol{u}} \end{pmatrix} = \begin{pmatrix} 9 & 2 & 16 & 22 & 33 \\ 11 & 4 & 18 & 24 & 35 \end{pmatrix} \in \boldsymbol{\mathcal{A}}.$$

If the coefficient matrix of the objective function of problem (21) is changed to $\underline{\boldsymbol{A}}(2)^{\top}$, then it follows from (6) that the global minimum is reduced to $(11.8\ 4.8\ 18.8\ 24.8\ 35.8)^{\top}$. This implies that the completion time of each subtask is advanced through adjusting the value of parameters within the allowable range $\boldsymbol{\mathcal{A}}$. Moreover, the overall completion time is reduced from 37 to 35.8.

6. Conclusions

This paper investigated multi-objective optimization for interval max-plus linear systems, which was formulated as an interval-valued-optimization problem. The strong and weak solvabilities were studied based on the solvability of sub-problems. It was found that the solvability of sub-problems is determined by the proportional relation of rows of the coefficient matrix of the objective function. Such a characteristic is a key to establish the solvability criteria for the interval-valued-optimization problem. A necessary and sufficient condition for the strong solvability of the multi-objective-optimization problem was established. For the bi-objective-optimization problem, a necessary and sufficient condition of the weak solvability was provided, and all the solvable sub-problems were found out. The interval optimal solution was obtained by constructing the set of all optimal solutions of the solvable sub-problems. The bi-objective-optimization technique was then used in load distribution, to advance the minimum completion time of the distributed system.

The global-optimization problem studied in Ref. [24] is a specific sub-problem of the interval-valued-multi-objective-optimization problem introduced. The interval model will have extensive applications in engineering practice. The solvability of the interval-valued-optimization problem with more than two objectives deserves further research.

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