mathematics

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Classifying Seven-Valent Symmetric Graphs of Order $8pq$

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Abstract: A graph is symmetric if its automorphism group is transitive on the arcs of the graph. Guo et al. determined all of the connected seven-valent symmetric graphs of order $8p$ for each prime p. We shall generalize this result by determining all of the connected seven-valent symmetric graphs of order $8pq$ with p and q to be distinct primes. As a result, we show that for each such graph of Γ, it is isomorphic to one of seven graphs.

Keywords: normal quotient; symmetric graph; automorphism group

MSC: 05C25

1. Introduction

We assume that the graphs in this paper are finite, simple, connected and undirected. For undefined terminologies of groups and graphs, we refer the reader to [1,2].

Let Γ be a graph. We denote $V\Gamma$, $E\Gamma$, $A\Gamma$ and $\mathrm{Aut}\Gamma$ as a vertex set, edge set, arc set and full automorphism group of the graph Γ, respectively. We define that the graph Γ is *vertex-transitive* if $\mathrm{Aut}\Gamma$ is transitive on the vertex set $V\Gamma$ of Γ, and Γ is an *arc-transitive* graph if $\mathrm{Aut}\Gamma$ is transitive on the arc set $A\Gamma$ of Γ. An *arc-transitive* graph is also called a *symmetric* graph.

Let G be a group, and let S be a subset of G such that $S = S^{-1} := \{s^{-1} | s \in S\}$. The Cayley graph $Cay(G, S)$ is defined to have a vertex set G and edge set $\{\{g, sg\} \| g \in G, s \in S\}$. Now, we denote the following Cayley graphs of dihedral groups by $\mathcal{CD}^k_{2pq}$.

Set $\mathcal{CD}^k_{2pq} = Cay(G, \{b, ab, a^{k+1}b, \ldots, a^{k^5+k^4+\cdots+k+1}b\})$, where $G = \langle a, b | a^{pq} = b^2 = 1, a^b = a^{-1}\rangle \cong D_{2pq}$, and k is a solution of the equation $x^6 + x^5 + \cdots + x + 1 \equiv 0 (\bmod\ pq)$.

There are many graph parameters to characterize the reliability and vulnerability of an interconnection network, such as spectral characterization, main eigenvalues, distance characteristic polynomials, and arc-transitivity. Among these parameters, the spectral characterizations, main eigenvalues, and distance characteristic polynomials are the better ones to measure the stability of a network; see [3–7], for example. For arc-transitivity, see [8], as an example. In this paper, we study the arc-transitivity of graphs.

Let p and q be distinct primes. By [9–11], symmetric graphs of orders p, $2p$, and $3p$ have been classified. Furthermore, Praeger et al. determined symmetric graphs of order pq in [12,13].

Recently, the classification of symmetric graphs with certain valency and with a restricted order has attracted much attention. For example, all cubic symmetric graphs of an order up to 768 have been determined by Conder and Dobcsa ń yi [14]. Tetravalent s-transitive graphs of order $6p$, $6p^2$, $8p$, $8p^2$, $10p$ or $10p^2$ were classified in [15–17]. More recently, a large number of papers on seven-valent symmetric graphs have been published. The classification of seven-valent symmetric graphs of order $8p$, $12p$, $16p$, $24p$ or $2pq$ were presented in [18–22]. We shall generalize these results by determining all connected seven-valent symmetric graphs of the order $8pq$.

In this paper, the main result we obtain is the following theorem.

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Citation: Jiang, Y.; Ling, B.; Yang, J.; Zhao, Y. Classifying Seven-Valent Symmetric Graphs of Order $8pq$. *Mathematics* **2024**, *12*, 787. https://doi.org/10.3390/math12060787

Academic Editosr: Irina Cristea and Alessandro Linzi

Received: 3 February 2024
Revised: 2 March 2024
Accepted: 5 March 2024
Published: 7 March 2024

Theorem 1. *Let $p < q$ be primes and let Γ be a seven-valent symmetric graph of the order $8pq$. Then, Γ is isomorphic to one of the graphs in Table 1.*

Table 1. seven-valent symmetric graphs of order $8pq$.

Γ	$\mathrm{Aut}\Gamma$	(p,q)
$\mathcal{C}_{48}$	$\mathrm{PGL}(2,7)\times \mathrm{D}_8$	(2,3)
$\mathcal{C}_{112}$	$(\mathbb{Z}_2^3\times \mathrm{D}_{14}){:}\mathrm{F}_{21}$	(2,7)
$\mathcal{C}_{120}$	S_7	(3,5)
$\mathcal{C}_{312}^i$	$\mathrm{PGL}(2,13)\times\mathbb{Z}_2$	$(3,13), i=1,2,3,4$
$\mathcal{C}_{312}^5$	$(\mathrm{PSL}(2,13)\times\mathbb{Z}_2){:}\mathbb{Z}_2$	(3,13)
$\mathcal{C}_{312}^6$	$\mathrm{PSL}(2,13){:}\mathrm{D}_8$	(3,13)
$\mathcal{C}_{(2^3,2q)}$	$(\mathbb{Z}_2^3\times \mathrm{D}_{2q}){:}\mathbb{Z}_7$	$(2, 7 \mid q-1)$

Some of the properties in Table 1 are obtained with the help of the Magma system [23]. The method of proving Theorem 1 is to reduce the automorphism groups of the graphs to some nonabelian simple groups. To make this method effective, we need to know the classification result of stabilizers of symmetric graphs. If the valency is a prime p, the method may still work. However, we need information about the stabilizers of prime-valent symmetric graphs and a more detailed discussion. Additionally, the term symmetric graph that is used in this paper has been also used for a different type of symmetry in other research works; see [24], for example. It studied the symmetry of graphs through characteristic polynomials, which is more interesting and detailed.

2. Preliminary Results

In this section, we will provide some necessary preliminary results to be used in later discussions.

For a graph Γ and its full automorphism group $\mathrm{Aut}\Gamma$, let G be a vertex-transitive subgroup of $\mathrm{Aut}\Gamma$ and let N be an intransitive normal subgroup of G on $V\Gamma$. We use V_N to denote the set of N-orbits in $V\Gamma$. The *normal quotient graph* Γ_N is a graph that satisfies the vertex set of V_N and two N-orbits B, and $C \in V_N$ are adjacent in Γ_N if and only if some vertex of B is adjacent in Γ to some vertex of C. The following Lemma ([25] Theorem 9) provides a basic method for studying our seven-valent symmetric graphs.

Lemma 1. *Let Γ be an G-arc-transitive graph of the prime valency p, where $p > 2$ and $G \leq \mathrm{Aut}\Gamma$, and let N be a normal subgroup of G and have at least three orbits on $V\Gamma$. Then, the following statements hold.*

(i) N is semi-regular on $V\Gamma$ and $G/N \leq \mathrm{Aut}\Gamma_N$, and Γ is a normal cover of Γ_N;
(ii) Γ is (G,s)-transitive if and only if Γ_N is $(G/N,s)$-transitive, where $1 \leq s \leq 5$ or $s = 7$.

By ([26] Theorem 3.4) and ([27] Theorem 1.1), we have the following lemma, which describes the vertex stabilizers of symmetric seven-valent graphs.

Lemma 2. *Let Γ be a seven-valent (G,s)-transitive graph, where $G \leq \mathrm{Aut}\Gamma$ and $s \geq 1$ are integers. Let $\alpha \in V\Gamma$. Then, $s \leq 3$ and one of the following holds, where F_{14}, F_{21} and F_{42} denote the Frobenius group of order 14, 21 and 42, respectively.*

(i) If G_α is soluble, then $|G_\alpha| \mid 2^2\cdot 3^2\cdot 7$. Further, the couple (s, G_α) lie in the following table.

s	1	2	3
G_α	$\mathbb{Z}_7, \mathrm{F}_{14}, \mathrm{F}_{21}, \mathrm{F}_{14}\times\mathbb{Z}_2, \mathrm{F}_{21}\times\mathbb{Z}_3$	$\mathrm{F}_{42}, \mathrm{F}_{42}\times\mathbb{Z}_2, \mathrm{F}_{42}\times\mathbb{Z}_3$	$\mathrm{F}_{42}\times\mathbb{Z}_6$

(ii) If G_α is insoluble, then $|G_\alpha| \mid 2^{24}\cdot 3^4\cdot 5^2\cdot 7$. Further, the couple (s, G_α) lie in the following table.

s	2	3
G_α	$\mathrm{PSL}(3,2), \mathrm{ASL}(3,2)$, $\mathrm{ASL}(3,2) \times \mathbb{Z}_2$, $\mathrm{A}_7, \mathrm{S}_7$	$\mathrm{PSL}(3,2) \times \mathrm{S}_4, \mathrm{A}_7 \times \mathrm{A}_6$, $\mathrm{S}_7 \times \mathrm{S}_6, (\mathrm{A}_7 \times \mathrm{A}_6) : \mathbb{Z}_2$, $\mathbb{Z}_2^6 : (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2)), [2^{20}] :$ $(\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$
$\lvert G_\alpha \rvert$	$2^3 \cdot 3 \cdot 7, 2^6 \cdot 3 \cdot 7, 2^7 \cdot 3 \cdot 7$, $2^3 \cdot 3^2 \cdot 5 \cdot 7, 2^4 \cdot 3^2 \cdot 5 \cdot 7$	$2^6 \cdot 3^2 \cdot 7, 2^6 \cdot 3^4 \cdot 5^2 \cdot 7, 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, $2^7 \cdot 3^4 \cdot 5^2 \cdot 7, 2^{10} \cdot 3^2 \cdot 7, 2^{24} \cdot 3^2 \cdot 7$

To construct seven-valent symmetric graphs, we need to introduce the Sabidussi coset graph. Let G be a finite group, and H is a core-free subgroup of G. Suppose D is a union of some double cosets of H in G, such that $D^{-1} = D$. The Sabidussi *coset graph* $\mathrm{Cos}(G, H, D)$ of G with respect to H and D is defined to have a vertex set $V\Gamma = [G : H]$ (the set of right cosets of H in G), and the edge set $E\Gamma = \{\{Hg, Hdg\} | g \in G, d \in D\}$ [28,29].

Proposition 1 ([30] Proposition 2.9)**.** *Let Γ be a graph and let G be a vertex-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Then, Γ is isomorphic to a Sabidussi coset graph $\mathrm{Cos}(G, H, D)$, where $H = G_\alpha$ is the stabilizer of $\alpha \in V\Gamma$ in G and D consists of all elements of G with a map of α to one of its neighbors. Further,*

(i) Γ is connected if and only if D generates the group G;
(ii) Γ is G-arc-transitive if and only if D is a single double coset. In particular, if $g \in G$ interchanges α and one of its neighbors, then $g^2 \in H$ and $D = HgH$;
(iii) The valency of the graph Γ is equal to $|D|/|H| = |H : H \cap H^g|$.

In the following lemmas, we provide classification information of seven-valent symmetric graphs of order $8p$ and $2pq$, where p and q are two distinct primes. By [19], we obtain the classification of seven-valent symmetric graphs of order $8p$.

Lemma 3. *Let Γ be a seven-valent symmetric graph of order $8p$. Then $\Gamma \cong K_{8,8} - 8K_2$ or $\mathcal{C}_{24}$.*

By [22], we can describe seven-valent symmetric graphs of order $2pq$.

Lemma 4. *Let $3 \le p < q$ be primes and let Γ be a seven-valent symmetric graph of order $2pq$. Then, the following statements hold:*

(i) $\Gamma \cong \mathcal{CD}_{2pq}^k$, where k is a solution of the equation $x^6 + x^5 + \cdots + x + 1 \equiv 0 (\mathrm{mod}\ pq)$, and $\mathrm{Aut}\Gamma \cong D_{2pq} : \mathbb{Z}_7$, where $p \mid q - 1$.
(ii) Γ lies in Table 2.

Table 2. Seven-valent symmetric graphs of order $2pq$.

Γ	$\mathrm{Aut}\Gamma$	(p, q)
$\mathcal{C}_{78}^1$	$\mathrm{PGL}(2,13)$	(3, 13)
$\mathcal{C}_{78}^2$	$\mathrm{PSL}(2,13)$	(3, 13)
$\mathcal{C}_{310}$	$\mathrm{PSL}(5,2).\mathbb{Z}_2$	(5, 31)
$\mathcal{C}_{30}$	S_8	(3, 5)

Next, we need some information about nonabelian simple groups. The first one has information about maximal subgroups of $\mathrm{PSL}(2, t)$ and $\mathrm{PGL}(2, t)$, where t is an odd prime; refer to ([31] Section 239) and ([32] Theorem 2) .

Lemma 5. *Let $G = \mathrm{PSL}(2, t)$ or $\mathrm{PGL}(2, t)$, where $t \ge 5$ is a prime, and let M be a maximal subgroup of G.*

(i) If $G = \mathrm{PSL}(2, t)$, then $M \in \{\mathrm{D}_{t-1}, \mathrm{D}_{t+1}, \mathrm{Z}_2 : \mathrm{Z}_{(t-1)/2}, \mathrm{A}_4, \mathrm{S}_4, \mathrm{A}_5\}$;
(ii) If $G = \mathrm{PGL}(2, t)$, then $M \in \{\mathrm{D}_{2(t-1)}, \mathrm{D}_{2(t+1)}, \mathrm{Z}_2 : \mathrm{Z}_{t-1}, \mathrm{S}_4, \mathrm{PSL}(2, t)\}$.

The next proposition is about nonabelian simple groups of order that are divisible by at most seven primes. By [2] (pp. 134–136), we have the following proposition.

Proposition 2. *Let T be a nonabelian simple group, such that $28pq \mid |T|$ and $|T| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$, where $5 \leq p < q$ are primes. Then, T is one of the groups in Table 3.*

Table 3. Simple group T with order dividing $2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

T	$\|T\|$	(p,q)	T	$\|T\|$	(p,q)
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$(5,11)$	$PSL(3,8)$	$2^9 \cdot 3^2 \cdot 7^2 \cdot 73$	$(7,73)$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(11,23)$	$PSL(3,16)$	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	$(13,17)$
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$(11,23)$	$PSL(2,5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	$(5,31)$
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	$(11,19)$	$PSL(2,7^2)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	$(5,7)$
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	$(5,11)$	$PSL(4,4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17 \cdot 17$	$(5,17)$
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$(5,11)$	$PSL(5,2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	$(5,31)$
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	$(5,13)$	$PSL(6,2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	$(7,31)$
$PSp(4,8)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	$(7,13)$	${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	$(7,13)$
$PSL(2,2^6)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	$(5,13)$	${}^2D_4(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	$(5,17)$
$PSL(2,2^9)$	$2^9 \cdot 3^2 \cdot 7 \cdot 19 \cdot 73$	$(19,73)$	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	$(5,13)$
$PSL(2,q)$	$\frac{q(q+1)(q-1)}{2}$				

Proof. Suppose T is a sporadic simple group, by [2] (pp.135–136), $T = M_{22}, M_{23}, M_{24}, J_1$, or HS. Suppose $T = A_n$ is an alternating group. Then, $T = A_{11}$ is the limitation of $|T|$.

Let X be one type of the Lie group, and let $t = r^f$ be a prime power. Now, suppose that $T = X(t)$ is a simple group of the Lie type, as T contains at most four 3-factors, three 5-factors, and two 7-factors [2] (p.135), and $T = PSL(2,q)$, $PSL(2,5^3)$ or $PSL(2,7^2)$.

Similarly, if $r = 2$, then $T = Sz(8)$, $PSp(4,8)$, $PSL(2,2^6)$, $PSL(2,2^9)$, $PSL(3,8)$, $PSL(3,16)$, $PSL(4,4)$, $PSL(5,2)$, $PSL(6,2)$, ${}^3D_4(2)$, ${}^2D_4(2)$ or $G_4(2)$. □

3. The Proof of Theorem 1

We will prove Theorem 1 through a series of lemmas in this section. To prove Theorem 1, we need information on seven-valent symmetric graphs of order $4pq$. Therefore, we first prove the following lemma.

Lemma 6. *Let $p < q$ be primes and let Γ be a seven-valent symmetric graph of order $4pq$. Then, $\Gamma \cong \mathcal{C}_{24}$, $\mathcal{C}_{60}$, $\mathcal{SG}^i_{156}$ or $\mathcal{CG}^j_{156}$, where $i = 1,2,3,4,5$ and $j = 1,2,3,4$.*

Proof. Let Γ be a seven-valent symmetric graph of the order $4pq$, where $p < q$ are primes. Let $A = \mathrm{Aut}\Gamma$. In Lemma 2, $|A| \mid 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$ is $|A_\alpha| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$, where $\alpha \in V\Gamma$. If $p = 2$, then Γ has the order $8q$; in Lemma 3, we have $q = 3$ and $\Gamma \cong \mathcal{C}_{24}$. If $p = 3$, then Γ has the order $12q$, and in [18,33], we have $q = 5$ or 13 and $\Gamma \cong \mathcal{C}_{60}$, $\mathcal{SG}^i_{156}$ or $\mathcal{CG}^j_{156}$, where $i = 1,2,3,4,5$ and $j = 1,2,3,4$. Therefore, we only need to prove that there is no seven-valent symmetric graph of order $4pq$ for $5 \leq p < q$, and the Lemma 6 is proved.

Now, we assume $5 \leq p < q$. By ([33] Theorem 1.1), we have $A \cong PSL(2,r) \times \mathbb{Z}_2$, $PGL(2,r) \times \mathbb{Z}_2$, $PSL(2,r)$ or $PGL(2,r)$, where $r \equiv \pm 1 (\mathrm{mod}\ 7)$ is a prime. If $A \cong PSL(2,r) \times \mathbb{Z}_2$ or $PGL(2,r) \times \mathbb{Z}_2$, then A has a normal subgroup $N \cong \mathbb{Z}_2$. It follows that Γ_N is a seven-valent symmetric graph of order $2pq$ and $A/N \leq \mathrm{Aut}\Gamma_N$. Since A/N is isomorphic to $PSL(2,r)$ or $PGL(2,r)$ for $5 \leq p < q$, there exists no such graph in Lemma 4. Hence, A is not isomorphic to $PSL(2,r) \times \mathbb{Z}_2$ or $PGL(2,r) \times \mathbb{Z}_2$.

If $A \cong PSL(2,r)$ or $PGL(2,r)$, then A has a normal subgroup $N \cong PSL(2,r)$. Assume that N has t orbits on the vertex set of Γ, $t \geq 3$. Then, N is semi-regular on $V\Gamma$ in Lemma 1 and thus $|N|$ divides $4pq$, contradicting with $N \cong PSL(2,r)$ and $5 \leq p < q$. Hence, $N_\alpha \neq 1$, N has, at most, two orbits on $V\Gamma$ and $2pq \mid |N : N_\alpha|$. Note that Γ is connected, $N \trianglelefteq A$, and $N_\alpha \neq 1$. Then, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \trianglelefteq A_\alpha^{\Gamma(\alpha)}$. This implies that $7 \mid |N_\alpha|$; thus, we have that $14pq \mid |N|$. And, $|N| \mid 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$ is $|N| \mid |A|$. Since $|A : N| \leq 2$, we have

$|A_\alpha : N_\alpha| \leq 2$. If A_α is insoluble, then N_α is also insoluble as $|A_\alpha : N_\alpha| \leq 2$. In Lemma 5, $N_\alpha = A_5$ (the alternating group on $\{1, 2, 3, 4, 5\}$), which contradicts with $7 \mid |N_\alpha|$. Therefore, A_α is soluble. It follows that $|A_\alpha| \mid 252$ in Lemma 2; thus, $|N_\alpha|$ divides 252. This implies that $|N| \mid 1008 \cdot p \cdot q$.

We claim that $r = q$, since $|V\Gamma| = |A|/|A_\alpha| = 4pq$ and $|A_\alpha| \mid 252$. Then, we have $4pq = \frac{r(r-1)(r+1)}{2|A_\alpha|}$ or $\frac{r(r-1)(r+1)}{|A_\alpha|}$. Since $r \equiv \pm 1 (\mathrm{mod}\ 7)$ is a prime and $|A_\alpha| \mid 252$, we have $r = p$ or q. Assume that $r = p$. Then, $4q = \frac{(r-1)(r+1)}{2|A_\alpha|}$ or $\frac{(r-1)(r+1)}{|A_\alpha|}$. This implies that $q = r + 1$ as $q > p$, which is impossible because $r + 1$ is not a prime. Thus, $r = q$ and $|N| = \frac{q(q-1)(q+1)}{2}$. Note that $(\frac{q+1}{2}, \frac{q-1}{2}) = 1$. Assume that $p \mid \frac{q-1}{2}$. Then, $q + 1 \mid 1008$. And then, we have $q = 7, 11, 13, 17, 23, 41, 47, 71, 83, 167, 251$ or 503. Assume that $p \mid \frac{q+1}{2}$. Then, $q - 1 \mid 1008$. And then, we have $q = 7, 13, 17, 19, 29, 37, 43, 73, 113, 127, 337$ or 1009. Note that $14pq \mid |N|$, $|N| \mid 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$ and $5 \leq p < q$. Therefore, N is one of the groups in the following table:

N	Order	N	Order
$\mathrm{PSL}(2,29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\mathrm{PSL}(2,41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$
$\mathrm{PSL}(2,43)$	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$	$\mathrm{PSL}(2,71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$
$\mathrm{PSL}(2,83)$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	$\mathrm{PSL}(2,113)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$
$\mathrm{PSL}(2,167)$	$2^3 \cdot 3 \cdot 7 \cdot 83 \cdot 167$	$\mathrm{PSL}(2,251)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 251$
$\mathrm{PSL}(2,337)$	$2^4 \cdot 3 \cdot 7 \cdot 13^2 \cdot 337$	$\mathrm{PSL}(2,503)$	$2^3 \cdot 3^2 \cdot 7 \cdot 251 \cdot 503$
$\mathrm{PSL}(2,1009)$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 101 \cdot 1009$		

Assume that $q = 29, 71, 113, 251$ or 1009. Note that $|N : N_\alpha| = 2pq$ or $4pq$. N has no subgroup of index $2pq$ or $4pq$ in Lemma 5, which is a contradiction.

Assume that $q = 337$. Then, $N = \mathrm{PSL}(2, 337)$, contradicting with $|N| \mid 2^{26} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

Assume that $q = 41$. Then, $N = \mathrm{PSL}(2, 41)$ and $(p, q) = (5, 41)$. Since N has no subgroup of index $2pq$ in Lemma 5, we have that N is transitive on $V\Gamma$, and thus $|N_\alpha| = 42$. Hence, $N_\alpha = F_{42}$ in Lemma 2. In Proposition 1, $\Gamma = \mathrm{Cos}(N, N_\alpha, N_\alpha g N_\alpha)$, where g is a 2-element in N such that $g^2 \in N_\alpha$ and $\langle N_\alpha, g\rangle = N$. In Magma [23], there is no such $g \in N$, which is a contradiction.

Finally, assume that $q = 43$. Then, $N = \mathrm{PSL}(2, 43)$ and $(p, q) = (11, 43)$. If N has two orbits on $V\Gamma$, then $A = \mathrm{PGL}(2, 43)$ and $A_\alpha = F_{42}$ in Lemma 2. This is impossible, as $\mathrm{PGL}(2, 41)$ has no subgroup isomorphic to F_{42}. Therefore, N is transitive on $V\Gamma$ and in Lemma 2, $N_\alpha = F_{21}$. In Lemma 5, $\mathrm{PSL}(2, 41)$ has no subgroup isomorphic to F_{21}, which is a contradiction. Similarly, $q \neq 83, 167$ or 503. This completes the proof. □

Now, let Γ be a seven-valent symmetric graph of the order $8pq$, where $p < q$ are primes. Let $\mathrm{A} := \mathrm{Aut}\Gamma$. Take $\alpha \in V\Gamma$. In Lemma 2, $|\mathrm{A}_\alpha| \mid 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$, and hence $|\mathrm{A}| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

If $p = 2$, then Γ has the order $16q$; by [20], we have $q = 3, 7$ or $7 \mid q - 1$, and Γ is isomorphic to $\mathcal{C}_{48}, \mathcal{C}_{112}$ or $\mathcal{C}_{(2^3,2q)}$. If $p = 3$, then Γ has the order $24q$; in [21], we have $q = 5$ or 13, and Γ is isomorphic to $\mathcal{C}_{120}, \mathcal{C}^i_{312}$ with $i = 1, 2, 3, 4$, $\mathcal{C}^5_{312}$ or $\mathcal{C}^6_{312}$. Therefore, we only need to prove that there is no seven-valent symmetric graph of the order $8pq$ for $5 \leq p < q$, and the Theorem 1 is proved. For the remainder of this paper, we let $5 \leq p < q$.

In the next lemma, we deal with the case where there is a soluble minimal normal subgroup of A.

Lemma 7. *Assume that A has a soluble minimal normal subgroup. Then, there exists no seven-valent symmetric graph of order $8pq$ for $5 \leq p < q$.*

Proof. Assuming N is a soluble minimal normal subgroup of the full automorphism group A. Then, N is an elementary abelian group. Since $|V\Gamma| = 8pq$, we have $N \cong \mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_p$ or $\mathbb{Z}_q$. It is easy to prove that N has more than two orbits on $V\Gamma$; if not, we have $4pq \mid |N|$, a contradiction. Therefore, in Lemma 1, $|N_\alpha| = 1$, and the quotient graph Γ_N of Γ relative to N is a seven-valent symmetric graph, with A/N as an arc-transitive subgroup of the automorphism of Γ_N.

If $N \cong \mathbb{Z}_2^3$, then Γ_N is a seven-valent symmetric graph of the order pq (pq is an odd number), which is a contradiction, as symmetric graphs of the odd order odd valent do not exist. If $N \cong \mathbb{Z}_2$, then Γ_N is a seven-valent symmetric graph of the order $4pq$. In Lemma 6, we note that $5 \leq p < q$, Γ_N does not exist, which is a contradiction. If $N \cong \mathbb{Z}_p$, then Γ_N is a seven-valent symmetric graph of the order $8q$. Γ_N does not exist in Lemma 3, which is a contradiction. Similarly, we obtain that $N \ncong \mathbb{Z}_q$.

If $N \cong \mathbb{Z}_2^2$, then Γ_N is a seven-valent symmetric graph of the order $2pq$. In Lemma 4, $\Gamma_N \cong \mathcal{C}_{310}$ or $\mathcal{CD}_{2pq}^k$, where k is a solution of the equation $x^6 + x^5 + \cdots + x + 1 \equiv 0(\mathrm{mod}\ pq)$ and $p \mid q - 1$.

Let $\Gamma_N \cong \mathcal{C}_{310}$. Then, $A/N \leq \mathrm{Aut}\mathcal{C}_{310} = \mathrm{PSL}(5,2).\mathbb{Z}_2$. Furthermore, A/N is arc-transitive on $V\Gamma_N$. By Magma [23], $\mathrm{Aut}\Gamma_N$ has a minimal arc-transitive subgroup, which is isomorphic to $\mathrm{PSL}(5,2)$. Thus, $\mathrm{PSL}(5,2) \leq A/N \leq \mathrm{PSL}(5,2).\mathbb{Z}_2$. Since the Schur Multiplier of $\mathrm{PSL}(5,2)$ is trivial, $A = \mathbb{Z}_2^2 \times \mathrm{PSL}(5,2)$ or $(\mathbb{Z}_2^2 \times \mathrm{PSL}(5,2)).\mathbb{Z}_2$. For the former case, in Proposition 1, $\Gamma = \mathrm{Cos}(A, A_\alpha, A_\alpha g A_\alpha)$, where g is a 2-element in A such that $g^2 \in A_\alpha$ and $\langle A_\alpha, g\rangle = A$. By Magma [23], there is no such $g \in A$, which is a contradiction. For the latter case, A/N has a normal subgroup, $M \cong \mathrm{PSL}(5,2)$. It is obvious that M has at most two orbits on $V\Gamma$. Since M has no subgroup of order 16128, M is transitive on $V\Gamma$, implying that $|M_\alpha| = 8064$; this is impossible in Lemma 2.

Let $\Gamma_N \cong \mathcal{CD}_{2pq}^k$, where k is a solution of the equation $x^6 + x^5 + \cdots + x + 1 \equiv 0(\mathrm{mod}\ pq)$. Note that A/N is an arc-transitive subgroup of $\mathrm{Aut}(\Gamma_N) = D_{2pq} : \mathbb{Z}_7$. Hence, $2pq \cdot 7 \mid |A/N|$. This implies that $A/N = D_{2pq} : \mathbb{Z}_7$. Let H be a normal subgroup of the order pq of D_{2pq} and Q be a Sylow q-subgroup of H. Then, in the Sylow Theorem, Q char H and thus $Q \trianglelefteq D_{2pq}$ is $H \trianglelefteq D_{2pq}$. Note that Q is also a Sylow q-subgroup of D_{2pq}. Then, Q char D_{2pq} and thus $Q \trianglelefteq A/N$ is $D_{2pq} \trianglelefteq A/N$. Then, $5 \leq p < q$ and $p \mid q-1$. Then, $q \geq 11$. Hence, Q is also a Sylow q-subgroup of A/N. Let $Q = G/N$. Then, $G/N \cong \mathbb{Z}_q$ and $|G| = 2^2 \cdot q$. In the Sylow Theorem, the Sylow q-subgroup of G is normal, at say L. Then, $L \cong \mathbb{Z}_q$, and thus $G = \mathbb{Z}_2^2 \times \mathbb{Z}_q = N \times L$. Hence, $L \trianglelefteq A$ is $G \trianglelefteq A$. Then, the normal quotient graph Γ_L of Γ relative to L is a seven-valent symmetric graph of order $8p$. In Lemma 3, there exists no graph for this case, which is a contradiction.

Thus, we complete the proof of Lemma 7. □

Now we move on to the case where there is no soluble minimal normal subgroup of A. Then, we have the following lemma.

Lemma 8. *Assume that A has no soluble minimal normal subgroup. Then, there exists no seven-valent symmetric graph of order $8pq$ for $5 \leq p < q$.*

Proof. Let N be an insoluble minimal normal subgroup of A, and let $C = C_A(N)$ be the centralizer of N in A. Then, N is isomorphic to T^d, where $d \geq 1$ and T are non-abelian simple groups. Assume that N has t orbits on the vertex set of Γ. If $t \geq 3$, then $N_\alpha = 1$ by Lemma 1 and thus $|N| = |T|^d \mid 8pq$, since N is insoluble. Then, $|N| = 4pq$ or $8pq$. Thus, N has two orbits or an orbit on $V\Gamma$, which is a contradiction. Hence, N has at most two orbits on $V\Gamma$, and it follows that $4pq \mid |N|$.

If $N_\alpha = 1$, then $|N| = 4pq$ or $8pq$, since $q \mid |N|$ and $q^2 \nmid |N|$. Then, $N = T$. Note that $5 \leq p < q$ [34]; no such simple group exists, and this is a contradiction. Hence, $N_\alpha \neq 1$. Since Γ is connected to $N \trianglelefteq A$ and $N_\alpha \neq 1$, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \trianglelefteq A_\alpha^{\Gamma(\alpha)}$. It follows that 7 divides $|N_\alpha|$. Then, we have that $28pq \mid |N|$.

Now, we claim that $d = 1$. Otherwise, $d \geq 2$, and thus $7^2 \mid |N|$. We have $d = 2$ as $|N| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$. So $p = 7$ or $q = 7$. If $p = 7$, then $q > 7$ and $q^2 \mid |T|^2$, which contradicts with $|N| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$. If $q = 7$, then $p = 5$. This implies that $|T| \mid 2^{13} \cdot 3^2 \cdot 5 \cdot 7$. Note that $35 \mid |T|$. By checking the nonabelian simple group of an order less than $2^{13} \cdot 3^2 \cdot 5 \cdot 7$, we have that $T = A_7$, A_8 or $\mathrm{PSL}(3,4)$, and $N = {A_7}^2$, ${A_8}^2$ or $\mathrm{PSL}(3,4)^2$ as $d = 2$. On the other side of the coin, $C \trianglelefteq A$, $C \cap N = 1$ and thus $\langle C, N \rangle = C \times N$. Because $|C \times N| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$ and $|N| = |T|^2 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2$ or $2^{12} \cdot 3^4 \cdot 5^2 \cdot 7^2$, C is a $\{2, p\}$-group, and hence soluble, where $p = 5$. So, $C = 1$ as A contains no soluble minimal normal subgroup. This implies $A = A/C \leq \mathrm{Aut}(N) \cong \mathrm{Aut}(T) wr \mathbb{Z}_2$. By Magma [23], no such graph exists, which is a contradiction. Therefore, we have $d = 1$, and $N = T \trianglelefteq A$ is a nonabelian simple group.

We next prove that $C = 1$. If $C \neq 1$, then C is insoluble, as $C \trianglelefteq A$ and A contain no soluble minimal normal subgroup. In the same argument as for the case N, we have 7 divides $|C_\alpha|$. Because $\langle C, N \rangle = C \times N$ and C, $N \trianglelefteq A$, we have $C_\alpha \times N_\alpha \leq A_\alpha$. Note that 7 divides $|N_\alpha|$; this concludes that $7^2 \mid |A_\alpha|$, which is a contradiction with Lemma 2. Therefore, we have $C = 1$, and thus $A \leq \mathrm{Aut}(T)$ is almost simple. It follows that $T = \mathrm{soc}(A)$ is a nonabelian simple group and satisfies the following condition.

Condition(*): $|T|$ lies in Table 3 such that $28pq \mid |T|$ and $|T| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

Assume first that $T \cong \mathrm{M}_{22}$, M_{23}, J_1, A_{11}, $\mathrm{PSL}(2, 2^9)$, $\mathrm{PSL}(3, 16)$, $\mathrm{PSL}(2, 5^3)$, $\mathrm{PSL}(2, 7^2)$, $\mathrm{PSL}(4, 4)$, $\mathrm{PSL}(6, 2)$, $\mathrm{PSp}(4, 8)$, HS, ${}^2\mathrm{D}_4(2)$, ${}^3\mathrm{D}_4(2)$, or $\mathrm{G}_2(4)$. Note that $|T : T_\alpha| = 4pq$ or $8pq$. T has no subgroup of index $4pq$ or $8pq$ by Atlas [35], which is a contradiction.

Assume that $T \cong \mathrm{M}_{24}$. Since T has no subgroup of index $4pq$, we show that T is transitive on $V\Gamma$, and thus $|T_\alpha| = 120{,}960$. In Proposition 1, $\Gamma = \mathrm{Cos}(T, T_\alpha, T_\alpha g T_\alpha)$, where g is a 2-element in T such that $g^2 \in T_\alpha$ and $\langle T_\alpha, g \rangle = T$. In Magma [23], there is no such $g \in T$, which is a contradiction. Similarly, T is not isomorphic to $\mathrm{Sz}(8)$, $\mathrm{PSL}(2, 2^6)$ or $\mathrm{PSL}(5, 2)$.

Assume that $T \cong \mathrm{PSL}(3, 8)$. If T has two orbits on $V\Gamma$, then Γ is bipartite and $|T_\alpha| = 2^7 \cdot 3^2 \cdot 7$. Recall that A is almost simple. Thus, $\mathrm{A} \leq \mathrm{Aut}(T)$. Since $\mathrm{Aut}(T) = \mathrm{PSL}(3, 8).\mathbb{Z}_6$, we have $A \cong \mathrm{PSL}(3, 8).\mathbb{Z}_2$, $\mathrm{PSL}(3, 8).\mathbb{Z}_3$ or $\mathrm{PSL}(3, 8).\mathbb{Z}_6$, and thus $|A_\alpha| = 2^7 \cdot 3^2 \cdot 7$, $2^6 \cdot 3^3 \cdot 7$ or $2^7 \cdot 3^3 \cdot 7$, which is impossible according to Lemma 2. Thus, T is transitive on $V\Gamma$. In Proposition 1, $\Gamma = \mathrm{Cos}(T, T_\alpha, T_\alpha g T_\alpha)$, where g is a 2-element in T such that $g^2 \in T_\alpha$ and $\langle T_\alpha, g \rangle = T$. By Magma [23], there is no such $g \in T$, which is a contradiction.

Finally, assume that $T \cong \mathrm{PSL}(2, q)$. Then, $T \leq A \leq \mathrm{Aut}(T) = \mathrm{PGL}(2, q)$ ($\mathrm{PGL}(2, q) = \mathrm{PSL}(2, q).\mathbb{Z}_2$) and $|A : T| \leq 2$. If A_α is insoluble, then T_α is also insoluble as $|A_\alpha : T_\alpha| \leq 2$. $T_\alpha = A_5$ in Lemma 5, contradicting with 7, divides $|T_\alpha|$. Therefore, A_α is soluble, and $|A_\alpha|$ divides by 252 in Lemma 2, and so $|T_\alpha|$ divides 252. This implies that $|T| \mid 2016 \cdot p \cdot q$. Note that $|T| = \dfrac{q(q-1)(q+1)}{2}$ and $(\dfrac{q+1}{2}, \dfrac{q-1}{2}) = 1$. If $p \mid \dfrac{q-1}{2}$, then $q + 1 \mid 2016$. It follows that $q = 7, 11, 13, 17, 23, 31, 41, 47, 71, 83, 167, 223, 251$ or 503. If $p \mid \dfrac{q+1}{2}$, then $q - 1 \mid 2016$. It follows that $q = 7, 13, 17, 19, 29, 37, 43, 73, 97, 113, 127, 337, 673, 1009$ or 2017. Note that T meets the condition (*) and $5 \leq p < q$. Therefore, T is one of the groups in the following table:

T	Order	T	Order
$\mathrm{PSL}(2,29)$	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$	$\mathrm{PSL}(2,41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$
$\mathrm{PSL}(2,43)$	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$	$\mathrm{PSL}(2,71)$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$
$\mathrm{PSL}(2,83)$	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$	$\mathrm{PSL}(2,97)$	$2^5 \cdot 3 \cdot 7^2 \cdot 97$
$\mathrm{PSL}(2,113)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 113$	$\mathrm{PSL}(2,167)$	$2^3 \cdot 3 \cdot 7 \cdot 83 \cdot 167$
$\mathrm{PSL}(2,223)$	$2^5 \cdot 3 \cdot 7 \cdot 37 \cdot 223$	$\mathrm{PSL}(2,251)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 251$
$\mathrm{PSL}(2,337)$	$2^4 \cdot 3 \cdot 7 \cdot 13^2 \cdot 337$	$\mathrm{PSL}(2,503)$	$2^3 \cdot 3^2 \cdot 7 \cdot 251 \cdot 503$
$\mathrm{PSL}(2,673)$	$2^5 \cdot 3 \cdot 7 \cdot 337 \cdot 673$	$\mathrm{PSL}(2,1009)$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 101 \cdot 1009$
$\mathrm{PSL}(2,2017)$	$2^5 \cdot 3^2 \cdot 7 \cdot 1009 \cdot 2017$		

Assume that $q = 29, 71, 97, 113, 223, 251, 337$ or 1009. Note that $|T : T_\alpha| = 4pq$ or $8pq$. T has no subgroup of index $4pq$ or $8pq$ in Lemma 5, which is a contradiction.

Assume that $q = 337$. Then, $T = \mathrm{PSL}(2, 337)$, which contradicts with $|T| \mid 2^{27} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

Assume that $q = 43$. Then, $T = \mathrm{PSL}(2, 43)$ and $(p, q) = (11, 43)$, since T has no subgroup of index $8pq$. Then, T is not transitive to $V\Gamma$. If T has two orbits on $V\Gamma$, then $|T_\alpha| = 21$. As A is almost simple, $A = \mathrm{PGL}(2, 43)$, and $A_\alpha = F_{21}$ in Lemma 2. In Proposition 1, $\Gamma = \mathsf{Cos}(A, A_\alpha, A_\alpha g A_\alpha)$, where g is a 2-element in A such that $g^2 \in A_\alpha$ and $\langle A_\alpha, g\rangle = A$. In Magma [23], there is no such $g \in A$, which is a contradiction.

Finally, assume that $q = 41$. Then, $T = \mathrm{PSL}(2, 41)$ and $(p, q) = (5, 11)$. If T has two orbits on $V\Gamma$, then $|T_\alpha| = 42$. As A is almost simple, $A = \mathrm{PGL}(2, 41)$, and $A_\alpha = F_{42}$ in Lemma 2. This is impossible, as $\mathrm{PGL}(2, 41)$ has no subgroup isomorphic to F_{42}. Therefore, T is transitive to $V\Gamma$ and in Lemma 2, $T_\alpha = F_{21}$. In Lemma 5, $\mathrm{PSL}(2, 41)$ has no subgroup isomorphic to F_{21}, which is a contradiction. Similarly, $q \neq 167, 503, 673$ or 2017.

Thus, we complete the proof of Lemma 8. □

By combining Lemma 6, 7 and 8, we have completed the proof of Theorem 1.

4. Conclusions

Through the classification of seven-valent symmetric graphs of the order $8pq$, we obtain many highly symmetric graphs in Table 1. These graphs can be applied to the design of the interconnection network. With induction, we may further classify seven-valent symmetric graphs of the order $8n$, where n is an odd square-free integer. We can even classify p-valent symmetric graphs of the order $2^k n$, where k is a positive integer and n is an odd square-free integer.

Author Contributions: Formal analysis, J.Y. and Y.Z.; Writing—original draft, Y.J.; Writing—review & editing, B.L. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the National Natural Science Foundation of China (12061089, 11861076, 11701503, 11761079), and the Natural Science Foundation of Yunnan Province (202201AT070022, 2019FB139).

Data Availability Statement: The data will be made available by the authors on request.

Conflicts of Interest: The authors declare no potential conflicts of interest.

Nomenclature

$G, H, \ldots$	*Groups*
$a, b, \ldots$	*Elements of groups*
a^b	$b^{-1}ab$
D_n	*Dihedral group of order n*
S_n, A_n	*Symmetric, alternating groups of degree n*
$\mathbb{Z}$	*Sets of integers*
$\mathbb{Z}_n$	$\mathbb{Z}/n\mathbb{Z}$
M_{22}, M_{23}, M_{24}	*Mathieu groups*
$\mathrm{ASL(n, R)}$	*Affine group over R*
$\mathrm{Sz(2^n)}$	*Suzuki group*
$\mathrm{SL(n, R)}$	*Linear groups over R*
J_1	*Janko group*
HS	*Higman, Sims group*
$\mathrm{PSp}(4, 8)$	*Symplectic group*
$^2\mathrm{D}_4(2)$	*Orthogonal group*
$^3\mathrm{D}_4(2)$	*Triality twisted group*
$\mathrm{G}_2(4)$	*Chevalley group*
$\mathrm{PGL}(n, R), \mathrm{PSL}(n, R)$	*Projective general linear and projective special linear groups*
Γ	*Graph*
$V\Gamma, E\Gamma, A\Gamma$	*Vertex set, edge set, arc set of* Γ

Γ_N	*Quotient graph*
α	*Element of graph*
$\mathsf{Aut}(\Gamma), \mathsf{Aut}(\Gamma_N)$	*Automorphism group of* Γ *and* Γ_N
$\mathcal{C}_n$	*Symmetric graph of order n*
G_α	*Stabilizer of* α *in G*
$G \times H, G^n$	*Direct product, direct power*
$GwrH$	*Wreath product*
$G.H$	*An extension of G by H*
$\|G\|$	*Cardinality of the group G*
G/N	*Quotient group*
F_n	*Frobenius group of order n*
$H \cong G$	*H is isomorphic with G*
$\langle N_\alpha, g \rangle$	*Group generated by* N_α *and g*
$G \trianglelefteq A$	*G is a normal subgroup of A*
$\mathsf{Aut}(T)$	*Automorphism group of T*
$soc(A)$	*Socle of G*
$C_A(N)$	*Centralizer of N in G*
$\|A : N\|$	*Index of the subgroup N in A*

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