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Exploring Thermoelastic Effects in Damped Bresse Systems with Distributed Delay

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Abstract: In this work, we consider the one-dimensional thermoelastic Bresse system by addressing the aspects of nonlinear damping and distributed delay term acting on the first and the second equations. We prove a stability result without the common assumption regarding wave speeds under Neumann boundary conditions. We discover a new relationship between the decay rate of the solution and the growth of ϖ at infinity. Our results were achieved using the multiplier method and the perturbed modified energy, named Lyapunov functions together with some properties of convex functions.

Keywords: Lyapunov functions; distributed delay; Bresse system; general decay; thermoelasticity; second sound; relaxation function; partial differential equations

MSC: 35B40; 93D15; 93D20

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1. Introduction and Relevance of Subject

Originally, the Bresse system consists of three wave equations where the main variables describe the longitudinal, vertical and shear angle displacements, which can be represented in [1] as

$$\begin{cases} \rho_1 \varphi_{tt} = Q_x + lN + F_1 \\ \rho_2 \psi_{tt} = M_x - Q + F_2 \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases} \tag{1}$$

where

$$N = k_0(w_x - l\varphi), Q = k(\varphi_x + lw + \psi), M = b\psi_x. \tag{2}$$

We use N, Q and M to denote the axial force, the shear force and the bending moment. By w, φ and ψ, we are denoting the longitudinal, vertical and shear angle displacements. Here, $\rho_1 = \rho A = \rho I, k_0 = EA, k = k'GA$ and $l = R^{-1}$. We use ρ for density, E for the modulus of elasticity, G for the shear modulus, K for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature, and we assume that all these quantities are positives. Also, by F_i we are denoting external forces. The Bresse system (1) is more general than the well-known Timoshenko system where the longitudinal displacement w is not considered $l = 0$. There are a number of publications concerning the stabilization of Timoshenko system with a different kind of damping; in this regard, we note the next references (see [2–7]).

System (1) is an un-damped system and its associated energy remains constant when the time t evolves. To stabilize system (1), many damping terms have been considered by several authors (see [8–14]).

By considering damping terms as infinite memories acting in the three equations, the system (1) has been recently studied in [11]

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \int_0^\infty g_1(t-s)\varphi_{xx}(s)ds = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \int_0^\infty g_2(t-s)\psi_{xx}(s)ds = 0 \\ \rho_1 w_{tt} Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \int_0^\infty g_3(t-s)w_{xx}(s)ds = 0, \end{cases}$$

where $(x,t) \in]0, L[\times \mathbf{R}_+, g_i : \mathbf{R}_+ \to \mathbf{R}_+, i = 1,2,3$ are given functions. The authors proved, under suitable conditions on the initial data and the memories g_i, that the system is well-posed and its energy converges to zero when time goes to infinity, and they provided a connection between the decay rate of energy and the growth of g_i at infinity, whereas in our system (3), the nonlinear damping dominates and makes the energy decay following its rate with some assumptions on distributed delay, as seen in (40) since $\mu_3(s)\vartheta(s)$ is the coefficient of nonlinear damping. The proof is based on the semi-groups theory for the well-posedness, and the energy method with the approach introduced in [15] for the stability.

In [8], the authors considered the Bresse system in a bounded domain with delay terms in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt} - Gh(\varphi_x + lw + \psi)_x - Ehl(w_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x, t-\tau_1) = 0 \\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + lw + \psi) + \widehat{\mu_1}\psi_t + \widehat{\mu_2}\psi_t(x, t-\tau_2) = 0 \\ \rho_1 w_{tt} Eh(w_x - l\varphi)_x + lGh(\varphi_x + lw + \psi) + \widehat{\widehat{\mu_1}}\psi_t + \widehat{\widehat{\mu_2}}\psi_t(x, t-\tau_3) = 0 \end{cases}$$

where $\tau_i > 0 (i = 1,2,3)$ are a time delay, $\mu_1, \mu_2, \widehat{\mu_1}, \widehat{\mu_2}, \widehat{\widehat{\mu_1}}, \widehat{\widehat{\mu_2}}$ are a positive real numbers. This system is subjected to the Dirichlet boundary conditions and to the initial conditions which belong to a suitable Sobolev space. First, the authors proved the global existence of its solutions in Sobolev spaces by means of the semi-group theory under a condition between the weight of the delay terms in the feedback and the weight of the terms without delay. Furthermore, the authors studied the asymptotic behavior of solutions using multiplies methods.

Motivated by the works mentioned above, for $x \in (0,1), t \in (0,\infty)$, we investigate the following Bresse system:

$$\begin{cases} \rho_1\varphi_{tt} = k(\varphi_x + lw + \psi)_x + k_0 l(w_x - l\varphi) - \mu_1\varphi_t \\ \qquad - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\varphi_t(x, t-s) \\ \rho_2\psi_{tt} = \beta\psi_{xx} - k(\varphi_x + lw + \psi) \\ \qquad - \int_0^t \varpi(t-s)[\nu(x)\psi_x(s)]_x ds - \mu_3(t)\theta(x)f(\psi_t) - \gamma\varrho_x \\ \rho_1 w_{tt} = k_0(w_x - l\varphi)_x - kl(\varphi_x + lw + \psi) \\ \rho_3\varrho_t = -kq_x - \gamma\psi_{tx} \\ \rho_4 q_t = -\delta q - k\varrho_x, \end{cases} \tag{3}$$

with boundary conditions

$$\begin{array}{l} \varphi(x=0,t) = \varphi(x=1,t) = 0 \\ \psi(x=0,t) = \psi(x=1,t) = 0 \\ w(x=0,t) = w(x=1,t) = 0 \\ w_x(x=0,t) = w_x(x=1,t) = 0 \\ \varphi_x(x=0,t) = \varphi_x(x=1,t) = 0, \quad t \geq 0, \end{array} \tag{4}$$

$$\begin{array}{l} \varrho(x=0,t) = \varrho(x=1,t) = 0 \\ q(x=1,t) = q(x=0,t) = 0, \quad t \geq 0, \end{array}$$

and the initial data

$$\begin{array}{l}\varphi(x,t=0)=\varphi_0(x),\varphi_t(x,t=0)=\varphi_1(x),\\ \psi(x,t=0)=\psi_0(x),\psi_t(x,t=0)=\psi_1(x),\\ w(x,t=0)=w_0(x),w_t(x,t=0)=w_1(x),\\ \varrho(x,t=0)=\varrho_0(x),q(x,t=0)=q_0(x).\end{array} \tag{5}$$

and

$$\psi_t(x,-t)=f_0(x,t).$$

Here ν,θ,f are specific functions, and $\rho_1,\rho_2,\rho_3,\rho_4,\delta,\beta,\gamma,k,l,k_0$ are constitutive constants, while τ_1,τ_2 are two real numbers with $0\le\tau_1\le\tau_2$, μ_1 is positive constant, μ_2 is an L^∞function, μ_3 is a bounded function and ϖ is the relaxation function satisfying:

Hypothesis 1 (H1). *$\varpi:\mathbf{R}_+\to\mathbf{R}_+$ is a C^1 function satisfying*

$$\varpi(0)>0,\beta-\|\nu(x)\|_\infty\int_0^\infty\varpi(s)ds=r>0. \tag{6}$$

Hypothesis 2 (H2). *There exists a positive nonincreasing differentiable function $\vartheta:\mathbf{R}_+\to\mathbf{R}_+$ satisfying*

$$\varpi'(t)\le-\vartheta(t)\varpi(t),t\ge0. \tag{7}$$

Hypothesis 3 (H3). *$\mu_2:[\tau_1,\tau_2]\to\mathbf{R}$ is a bounded function satisfying*

$$\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\le\mu_1. \tag{8}$$

Hypothesis 4 (H4). *The functions ν,θ satisfy*

$$\begin{array}{l}\nu\in C^1([0,1]),\\ \nu=0, or\ \ \nu(0)+\nu(1)>0,\\ \inf\limits_{x\in[0,1]}\{\nu(x)+\theta(x)\}>0.\end{array}$$

Hypothesis 5 (H5). *$f:\mathbf{R}\to\mathbf{R}$ is a continuous and non-decreasing function such that there exist positive constants k_1,k_2 and l_1 and $h:\mathbf{R}_+\to\mathbf{R}_+$ is a convex, continuous and increasing function of class $C^1(\mathbf{R}_+)\cap C^2(]0,\infty[)$ satisfying: $h(0)=0$, and $h''=0$ on $[0,l_1]$ or $h'(0)=0$ and $h''>0$ on $[0,l_1]$ such that*

$$\begin{array}{l}h(s^2+f^2(s))\le f(s)s, for\ \ |s|\le l_1,\\ k_1s^2\le f(s)s\le k_2s^2, for\ \ |s|>l_1.\end{array}$$

Remark 1. *Since ϖ is positive and $\varpi(0)>0$; then, for any $t_0>0$, we have*

$$\varpi_1=\int_0^\infty\varpi(s)ds>\int_0^t\varpi(s)ds\ge\int_0^{t_0}\varpi(s)ds=\varpi_0>0.$$

Using the fact that $\nu(0)>0$ and a is continuous, then there exist $\varepsilon>0$ such that $\inf\limits_{x\in[0,\varepsilon]}\nu(x)\ge0$. Let us denote

$$d=min\{\varepsilon,\inf_{x\in[0,\varepsilon]}\{\nu(x)+\theta(x)\}\}>0,$$

and let $\alpha \in C^1([0,1])$ be such that $0 \leq \alpha \leq a$ and

$$\begin{cases} \alpha(x) = 0, if \;\; \nu(x) \leq \frac{d}{4} \\ \\ \alpha(x) = \nu(x), if \;\; \nu(x) \geq \frac{d}{4}. \end{cases}$$

Under suitable assumptions, we show that, even in the presence of the thermoelastic, we can establish a general energy decay of the solution for (3). We prove our result by using the energy method together with some properties of convex functions. The advantage to propose system (3) is to discover the interaction between the distributed delay term (located in the first equation) and nonlinear damping in its general case (located in the second equation) with the presence of linear memory and their influence on the stability of the system. We found a good interaction between them by outlining minimal conditions to stabilize the system. We consider that these two terms are considered as damping and each term has a special way to stabilize the system.

Lemma 1 ([16])**.** *The function α is not identically zero and satisfies*

$$\inf_{x\in[0,1]} \{\alpha(x) + \theta(x)\} \geq \frac{d}{2}.$$

Let us denote by h^* the conjugate function in the sense of Young of the convex function h as in [17]

$$h^*(p) = \sup_{t\in\mathbf{R}_+} (pt - h(t)).$$

Assume that $h'' > 0$; then, for $p \geq 0$ a given number, h^* is the Legendre transform of h, which is given as in [18] by

$$h^*(p) = p[h']^{-1}(p) - h([h']^{-1}(p)), \tag{9}$$

and which satisfies the following inequality (Young's inequality):

$$px \leq h(x) + h^*(p), \; \forall p, x \geq 0.$$

The relation (9) and the fact that $h(0) = 0$ and $(h')^{-1}$, h are increasing functions yield

$$h^*(p) \leq p[h']^{-1}(p), \forall p \geq 0.$$

Now, for ε_0, we define the functions J and K by

$$J(t) = \begin{cases} t, & if \;\; h'' = 0 \;\; on \;\; [0, l_1] \\ \\ th'(\varepsilon_0 t), & if \;\; h'(0) = 0 \;\; and \;\; h'' > 0 \;\; on \;\; [0, l_1], \end{cases} \tag{10}$$

and

$$K(t) = \int_t^1 \frac{1}{J(s)} ds. \tag{11}$$

The following notations will be used:

$$(\varpi \diamond u)(t) = \int_0^1 \alpha(x) \int_0^t \varpi(s)(u(t) - u(s)) ds dx, \forall u \in L^2(0,1)$$

$$(\varpi o u)(t) = \int_0^1 \nu(x) \int_0^t \varpi(t-s)(u(t) - u(s))^2 ds dx, \forall u \in L^2(0,1),$$

there exists a positive constant c so that

$$(\varpi \diamond u)^2 \leq c\varpi o u_x, \forall u \in H_0^1(0,1). \tag{12}$$

We organized our paper as follows: In Section 2, we prepare some Lemmas and present some appropriate functional to state the main Theorem 1. Notably, the research establishes a stability result in Section 3 without the conventional assumptions on wave speeds, particularly under Neumann boundary conditions.

2. Main Result

In this section, we prove our decay result for the energy of the systems (3)–(5) using the multiplier technique. To achieve our goal, as in [19], we use the following new variable:

$$z(x,\tau,s,t) = \varphi_t(x, t - s\tau),$$

and then we obtain

$$\begin{cases} sz_t(x,\tau,s,t) + z_\tau(x,\tau,s,t) = 0 \\ z(x,0,s,t) = \varphi_t(x,t). \end{cases}$$

consequently, the problem is equivalent to

$$\begin{cases} \rho_1\varphi_{tt} = k(\varphi_x + lw + \psi)_x + k_0 l(w_x - l\varphi) - \mu_1\varphi_t \\ \qquad - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds \\ \rho_2\psi_{tt} = \beta\psi_{xx} - k(\varphi_x + lw + \psi) \\ \qquad - \int_0^t \varpi(t-s)[\nu(x)\psi_x(s)]_x ds - \mu_3(t)\theta(x) f(\psi_t) - \gamma\varrho_x \\ \rho_1 w_{tt} = k_0(w_x - l\varphi)_x - kl(\varphi_x + lw + \psi) \\ \rho_3\varrho_t = -kq_x - \gamma\psi_{tx} \\ \rho_4 q_t = -\delta q - k\varrho_x \\ sz_t(x,\tau,s,t) + z_\tau(x,\tau,s,t) = 0, \end{cases} \tag{13}$$

where

$$(\tau, s) \in (0,1) \times (\tau_1, \tau_2).$$

We need a several Lemmas.

Lemma 2. *The energy functional $\mathcal{E}$ is defined by*

$$\begin{aligned} \mathcal{E}(t) \quad = \quad & \frac{1}{2}\int_0^1 [\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + (\beta - \nu(x)\int_0^t \varpi(s)ds)\psi_x^2 + \rho_3\varrho^2 \\ & + \rho_4 q^2 + k(\varphi_x + lw + \psi)^2 + k_0(w_x - l\varphi)^2]dx + \frac{1}{2}\varpi o\psi_x \\ & + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\tau,s,t) ds d\tau dx, \end{aligned} \tag{14}$$

satisfies

$$\begin{aligned} \mathcal{E}'(t) \quad = \quad & -\delta\int_0^1 q^2 dx + \frac{1}{2}\varpi' o\psi_x - \frac{1}{2}\varpi(t)\int_0^1 \nu(x)\psi_x^2 dx \\ & - \mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t) dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right)\int_0^1 \varphi_t^2 dx, \end{aligned} \tag{15}$$

and

$$\mathcal{E}'(t) \leq -\delta\int_0^1 q^2 dx + \frac{1}{2}\varpi' o\psi_x - \eta_0\int_0^1 \varphi_t^2 dx - \mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t) dx \leq 0, \tag{16}$$

where $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \geq 0.$

Proof. Multiplying (3)$_1$ by φ_t, (3)$_2$ by ψ_t, (3)$_3$ by w_t, (3)$_4$ by ϱ and (3)$_5$ by q, with integration by parts over $(0,1)$ and using (4), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + \beta\psi_x^2 + \rho_3\varrho^2 + \rho_4 q^2 + k(\varphi_x + lw + \psi)^2$$
$$+k_0(w_x - l\varphi)^2\Big]dx$$
$$+\frac{1}{2}\varpi o\psi_x + \delta\int_0^1 q^2dx + \mu_1\int_0^1 \varphi_t^2 dx + \int_0^1 \varphi_t\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,s,t)dsdx$$
$$-\int_0^1 \psi_{xt}\int_0^t \varpi(t-s)\nu(x)\psi_x(s)dsdx + \mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t)dx = 0. \tag{17}$$

The last term in the LHS of (17) is estimated as follows:

$$-\int_0^1 \psi_{xt}\int_0^t \varpi(t-s)\nu(x)\psi_x(s)dsdx$$
$$= \frac{1}{2}\frac{d}{dt}\varpi o\psi_x - \frac{1}{2}\frac{d}{dt}\int_0^t \varpi(s)ds\int_0^1 \nu(x)\psi_x^2$$
$$-\frac{1}{2}\varpi' o\psi_x + \frac{1}{2}\varpi(t)\int_0^1 \nu(x)\psi_x^2 dx. \tag{18}$$

Now, multiplying the last equation in (13) by $z|\mu_2(s)|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx\\
=\;& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|zz_\tau(x,\tau,s,t)dsd\tau dx\\
=\;& -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\tau}z^2(x,\tau,s,t)dsd\tau dx\\
=\;& \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|(z^2(x,0,s,t) - z^2(x,1,s,t))dsdx\\
=\;& \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds)\int_0^1 \varphi_t^2 dx\\
& -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)dsdx.
\end{aligned}$$

From (14), (17), (18) and (19), we obtain (15).
The energy functional satisfies

$$\begin{aligned}
\mathcal{E}'(t) \;\leq\;& -\delta\int_0^1 q^2dx + \frac{1}{2}\varpi' o\psi_x - (\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|)\int_0^1 \varphi_t^2 dx\\
& -\mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t)dx,
\end{aligned}$$

then, by (8), there exists a positive constant η_0 such that

$$\mathcal{E}'(t) \leq -\delta\int_0^1 q^2dx + \frac{1}{2}\varpi' o\psi_x - \eta_0\int_0^1 \varphi_t^2 dx - \mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t)dx,$$

then, we obtain $\mathcal{E}$ as a non-increasing function. □

Lemma 3. *The functional*

$$
\begin{aligned}
F_1(t) &= -\rho_2 \int_0^1 \alpha(x)\psi_t \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\
&\quad + \frac{\gamma\rho_4}{k}\int_0^1 \alpha(x) q \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx,
\end{aligned}
$$

satisfies

$$
\begin{aligned}
F_1'(t) &\leq -\frac{\rho_2\varpi_0}{2}\int_0^1 \alpha(x)\psi_t^2 dx + 3c\varepsilon_1 \int_0^1 \psi_x^2 dx + k^2\varepsilon_2 \int_0^1 (\varphi_x + lw + \psi)^2 dx \\
&\quad + c\int_0^1 \theta(x) f^2(\psi_t)dx + c\int_0^1 q dx + c(1+\varepsilon_1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2})\varpi o \psi_x \\
&\quad + c\varpi' o \psi_x dsdx.
\end{aligned} \tag{19}
$$

Proof. We set

$$F_1(t) = I_1(t) + I_2(t),$$

where

$$
\begin{aligned}
I_1(t) &= -\rho_2 \int_0^1 \alpha(x)\psi_t \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx, \\
I_2(t) &= \frac{\gamma\rho_4}{k}\int_0^1 \alpha(x) q \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx.
\end{aligned}
$$

Differentiation of I_1 gives

$$
\begin{aligned}
I_1'(t) &= -\rho_2 \int_0^1 \alpha(x)\psi_{tt} \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\
&\quad - \rho_2 \int_0^1 \alpha(x)\psi_t \int_0^t \varpi'(t-s)(\psi(t)-\psi(s))dsdx \\
&\quad - \rho_2 \int_0^1 \alpha(x)\psi_t^2 \int_0^t \varpi(s)dsdx.
\end{aligned} \tag{20}
$$

Using (13), we obtain

$$
\begin{aligned}
&- \rho_2 \int_0^1 \alpha(x)\psi_{tt} \int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\
&= -\beta \int_0^1 \alpha(x)\psi_x \int_0^t \varpi(t-s)(\psi_x(t)-\psi_x(s))dsdx \\
&+ k\int_0^1 \alpha(x)(\varphi_x + lw + \psi)\int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\
&- \int_0^1 \alpha(x)\nu(x)\Big(\int_0^t \varpi(t-s)\psi_x(s)ds\Big)\Big(\int_0^t \varpi(t-s)(\psi_x(t)-\psi_x(s))ds\Big)dx \\
&+ \mu_3(t)\int_0^1 \alpha(x)\theta(x) f(\psi_t)\Big(\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds\Big)dx \\
&+ \gamma\int_0^1 \alpha(x)\varrho_x\Big(\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds\Big)dx \\
&+ \int_0^1 \alpha'(x)\Big(\beta\psi_x - \nu(x)\int_0^t \varpi(s)\psi_x(s)ds\Big)\Big(\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds\Big)dx.
\end{aligned} \tag{21}
$$

Next, by using (12), we have for any $\delta_1 > 0$

$$-\rho_2 \int_0^1 \alpha(x)\psi_t \int_0^t \varpi'(t-s)(\psi(t)-\psi(s))dsdx \leq c\delta_1 \int_0^1 \alpha(x)\psi_t^2 dx - \frac{c}{\delta_1}\varpi' o \psi_x. \tag{22}$$

Also

$$\begin{aligned} I_2'(t) &= \frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q_t\int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\ &+\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\int_0^t \varpi'(t-s)(\psi(t)-\psi(s))dsdx \\ &\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\psi_t(\int_0^t \varpi(s)ds)dx. \end{aligned}$$

Using (13), we obtain

$$\begin{aligned} I_2'(t) &= -\frac{\gamma\delta}{k}\int_0^1 \alpha(x)q\int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\ &-\gamma\int_0^1 \alpha(x)\varrho_x(\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds)dx \\ &-\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\int_0^t \varpi'(t-s)(\psi(t)-\psi(s))dsdx \\ &\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\psi_t(\int_0^t \varpi(s)ds)dx. \end{aligned} \tag{23}$$

Similarly to (22), we treat the terms in the RHS of (21) as follows:

$$\beta\int_0^1 \alpha(x)\psi_x\int_0^t \varpi(t-s)(\psi_x(t)-\psi_x(s))dsdx \le c\varepsilon_1\int_0^1 \psi_x^2dx + \frac{c}{\varepsilon_1}\varpi o\psi_x. \tag{24}$$

Also,

$$\begin{aligned} &k\int_0^1 \alpha(x)(\varphi_x+lw+\psi)\int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \\ &\le c\varepsilon_2\int_0^1 (\varphi_x+lw+\psi)^2dx + \frac{c}{\varepsilon_2}\varpi o\psi_x. \end{aligned} \tag{25}$$

By the same method used in [18], we have these estimates

$$\begin{aligned} &-\int_0^1 \alpha(x)\nu(x)\Big(\int_0^t \varpi(s)\psi_x(s)ds\Big)\Big(\int_0^t \varpi(t-s)(\psi_x(t)-\psi_x(s))ds\Big)dx \\ &\le c\varepsilon_1\int_0^1 \psi_x^2dx + c(\varepsilon_1+\frac{1}{\varepsilon_1})\varpi o\psi_x, \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\mu_3(t)\int_0^1 \alpha(x)\theta(x)f(\psi_t)\Big(\int_0^t \varpi(t-s)(\psi_x(t)-\psi_x(s))ds\Big)dx \\ &\le c\int_0^1 \theta(x)f^2(\psi_t)dx + c\varpi o\psi_x. \end{aligned} \tag{27}$$

Finally,

$$\begin{aligned} &\int_0^1 \alpha'(x)\Big(\beta\psi_x - \nu(x)\int_0^t \varpi(s)\psi_x(s)ds\Big)\Big(\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds\Big)dx \\ &\le c\varepsilon_1\int_0^1 \psi_x^2dx + c(\varepsilon_1+\frac{1}{\varepsilon_1})\varpi o\psi_x. \end{aligned} \tag{28}$$

As in (22), we easily find that

$$\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\int_0^t \varpi'(t-s)(\psi(t)-\psi(s))dsdx \le c\int_0^1 q^2dx - c\varpi' o\psi_x. \tag{29}$$

Also, we estimate the first term in the RHS of (23) as follows:

$$-\frac{\gamma\delta}{k}\int_0^1 \alpha(x)q\int_0^t \varpi(t-s)(\psi(t)-\psi(s))dsdx \leq c\int_0^1 q^2dx - c\varpi o\psi_x, \tag{30}$$

and

$$\frac{\gamma\rho_4}{k}\int_0^1 \alpha(x)q\psi_t\int_0^t \varpi(s)dsdx \leq c\delta_2\int_0^1 \psi_t^2dx + \frac{c}{\delta_2}\int_0^1 q^2dx. \tag{31}$$

By combining the estimates (20)–(31), we complete the proof. □

Lemma 4. *Assume that (H1) hold. Then, the functional*

$$F_2(t) = \rho_2\int_0^1 \psi\psi_t dx,$$

satisfies

$$\begin{aligned} F_2'(t) \quad &\leq \quad -\frac{r}{2}\int_0^1 \psi_x^2dx + \rho_2\int_0^1 \psi_t^2dx + \frac{2ck^2}{r}\int_0^1(\varphi_x + lw + \psi)^2dx \\ &\quad +c\int_0^1 \theta(x)f^2(\psi_t)dx + c\int_0^1 \varrho^2dx + c\varpi o\psi_x. \end{aligned} \tag{32}$$

Proof. With direct computations using integration by parts and Young's inequality, we have

$$\begin{aligned} F_2'(t) \quad &= \quad \rho_2\int_0^1 \psi_t^2dx - \beta\int_0^1 \psi_x^2dx - k\int_0^1 \psi(\varphi_x + lw + \psi)dx \\ &\quad +\int_0^1 \psi_x\int_0^t \varpi(t-s)\nu(x)\psi_x(s)dsdx - \gamma\int_0^1 \psi\varrho_x dx \\ &\quad -\mu_3(t)\int_0^1 \theta(x)\psi f(\psi_t)dx. \end{aligned} \tag{33}$$

Using $(H1)$, Young's, Cauchy–Schwarz and Poincare's inequalities, we obtain (32).
□

Lemma 5. *The functional*

$$F_3(t) = \rho_3\rho_4\int_0^1 \varrho\int_0^x q(y)dydx,$$

satisfies

$$F_3'(t) \quad \leq \quad -\frac{k\rho_3}{2}\int_0^1 \varrho^2dx + \varepsilon_3\int_0^1 \psi_t^2dx + c(1+\frac{1}{\varepsilon_3})\int_0^1 q^2dx. \tag{34}$$

Proof. Direct computations give

$$\begin{aligned} F_3'(t) \quad &= \quad -k\rho_3\int_0^1(\varphi_x + lw + \psi)\varrho dx + \rho_3k_0l\int_0^1(w_x - l\varphi)\int_0^x \varrho(y)dydx \\ &\quad +\gamma\rho_3\int_0^1 \varrho^2dx + \rho_1\int_0^1 q\varphi_t dx - \rho_1\gamma\int_0^1 \varphi_t^2dx + \rho_1 l\int_0^1 \varphi_t\varrho_x dx. \end{aligned}$$

by using Young's inequality, we obtain (34). □

In the following Lemma, we suppose that

$$k = k_0. \tag{35}$$

Lemma 6. *The functional*

$$F_4(t) = -\rho_1 \int_0^1 \varphi_t(w_x - l\varphi)dx - \rho_1 \int_0^1 w_t(\varphi_x + lw + \psi)dx.$$

satisfies

$$\begin{aligned} F_4'(t) \leq & -\frac{k_0 l}{2} \int_0^1 (w_x - l\varphi)^2 dx - \frac{\rho_1 l}{2} \int_0^1 w_t^2 dx + c \int_0^1 \varphi_t^2 dx + c \int_0^1 \psi_t^2 dx \\ & + kl \int_0^1 (\varphi_x + lw + \psi)^2 dx + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx, \end{aligned} \tag{36}$$

Proof. By differentiating F_4, then using (3), integration by parts and (4), we obtain

$$\begin{aligned} F_4'(t) = & -k \int_0^1 (\varphi_x + lw + \psi)_x (w_x - l\varphi) dx - k_0 l \int_0^1 (w_x - l\varphi)^2 dx \\ & + \mu_1 \int_0^1 \varphi_t (w_x - l\varphi) dx - \int_0^1 (w_x - l\varphi) \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t) ds dx \\ & - \rho_1 \int_0^1 \varphi_t (w_{tx} - l\varphi_t) dx - k_0 \int_0^1 (w_x - l\varphi)_x (\varphi_x + lw + \psi) dx \\ & + kl \int_0^1 (\varphi_x + lw + \psi)^2 dx - \rho_1 \int_0^1 w_t (\varphi_{xt} + lw_t + \psi_t) dx. \end{aligned}$$

Using Young's, Cauchy–Schwarz and Poincare's inequalities, bearing in mind (35) yields (36). □

In the following Lemma, we suppose that

$$-\gamma - \frac{k\beta\rho_3}{\rho_1\gamma}\chi + \frac{k^2\beta}{\rho_4\gamma}\chi = 0, \tag{37}$$

where $\chi = (\frac{\rho_1}{k} - \frac{\rho_2}{\beta})$ and (35) holds.

Lemma 7. *The functional*

$$\begin{aligned} F_5(t) = & \rho_2 \int_0^1 \psi_t(\varphi_x + lw + \psi) dx + \frac{\beta\rho_1}{k} \int_0^1 \psi_x \varphi_t dx + \frac{\beta\rho_3}{\gamma}\chi \int_0^1 \varrho\varphi_t dx \\ & - \frac{k\beta}{\gamma}\chi \int_0^1 q(\varphi_x + lw + \psi) dx - \frac{\beta\rho_2 l^2}{k_0} \int_0^1 \psi\psi_t dx + \frac{\beta\rho_1 l}{k_0} \int_0^1 \psi w_t dx \\ & - \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t \varpi(t-s)\nu(x)\psi_x(s) ds dx \\ & + \frac{l\rho_1}{k} \int_0^t \varpi(s) ds \int_0^1 \nu(x)\psi_x \int_0^x w_t(y) dy d, \end{aligned}$$

satisfies

$$\begin{aligned} F_5'(t) \leq & -\frac{k}{2} \int_0^1 (\varphi_x + lw + \psi)^2 dx + 5\varepsilon_4 \int_0^1 w_t^2 dx + 3c\varepsilon_6 \int_0^1 (w_x - l\varphi)^2 dx \\ & + 8c\varepsilon_5 \int_0^1 \psi_x^2 dx + \Big(\frac{\beta^2 l^2}{k_0} + \frac{3l^4\varpi_0 c}{2k}\Big) \int_0^1 \psi_x^2 dx \\ & + c(1 + \frac{1}{\varepsilon_4}) \int_0^1 \psi_t^2 dx + c(1 + \frac{1}{\varepsilon_5}) \int_0^1 \varphi_t^2 dx + c(1 + \frac{1}{\varepsilon_4}) \int_0^1 q^2 dx \\ & + c(1 + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6}) \int_0^1 \varrho^2 dx + c(1 + \frac{1}{\varepsilon_5}) \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t) ds dx \\ & + c(1 + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5}) \varpi o \psi_x - c\varpi' o \psi_x. \end{aligned} \tag{38}$$

Proof. Direct computations give

$$
\begin{aligned}
F_5'(t) &= \rho_2 \int_0^1 \psi_{tt}(\varphi_x + lw + \psi)dx + \rho_2 \int_0^1 \psi_t(\varphi_{tx} + lw_t + \psi_t)dx \\
&+ \frac{\beta\rho_1}{k} \int_0^1 \psi_{tx}\varphi_t dx + \frac{\beta\rho_1}{k} \int_0^1 \psi_x \varphi_{tt} dx + \frac{\beta\rho_3}{\gamma}\chi \int_0^1 \varrho_t \varphi_t dx \\
&+ \frac{\beta\rho_3}{\gamma}\chi \int_0^1 \varrho \varphi_{tt} dx - \frac{k\beta}{\gamma}\chi \int_0^1 q_t(\varphi_x + lw + \psi)dx \\
&- \frac{k\beta}{\gamma}\chi \int_0^1 q(\varphi_{tx} + lw_t + \psi_t)dx - \frac{\beta\rho_2 l^2}{k_0} \int_0^1 \psi\psi_{tt} dx \\
&- \frac{\beta\rho_2 l^2}{k_0} \int_0^1 \psi_t^2 dx + \frac{\beta\rho_1 l}{k_0} \int_0^1 \psi_t w_t dx \\
&+ \frac{\beta\rho_1 l}{k_0} \int_0^1 \psi w_{tt} dx - \frac{\rho_1}{k} \int_0^1 \varphi_{tt} \int_0^t \varpi(t-s)\nu(x)\psi_x(s)dsdx \\
&- \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^t \varpi'(t-s)\nu(x)\psi_x(s)dsdx \\
&+ \frac{l\rho_1}{k} \int_0^t \varpi(s)ds \int_0^1 \nu(x)\psi_{tx} \int_0^x w_t(y)dydx \\
&+ \frac{l\rho_1}{k} \int_0^t \varpi(s)ds \int_0^1 \nu(x)\psi_x \int_0^x w_{tt}(y)dydx.
\end{aligned}
$$

Using Young, Cauchy–Schwarz and Poincare's inequalities, bearing in mind (35) and (37) yields (38). □

Lemma 8. *The functional*

$$
F_6(t) = \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\tau}|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx,
$$

satisfies

$$
\begin{aligned}
F_6'(t) &\leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx + \mu_1 \int_0^1 \varphi_t^2 dx \\
&\quad -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)dsdx
\end{aligned} \tag{39}
$$

where η_1 is a positive constant.

Proof. By differentiating F_6, with respect to t and using the last equation in (13), we have

$$
\begin{aligned}
F_6'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\tau}|\mu_2(s)|zz_\tau(x,\tau,s,t)dsd\tau dx \\
&= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\tau}|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx \\
&\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|[e^{-s}z^2(x,1,s,t) - z^2(x,0,s,t)]dsdx.
\end{aligned}
$$

Using the fact that $z(x,0,s,t) = \varphi_t(x,t)$, and $e^{-s} \leq e^{-s\tau} \leq 1$, for all $0 < \tau < 1$, we obtain

$$
\begin{aligned}
F_6'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx \\
&\quad - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \int_0^1 \varphi_t^2 dx.
\end{aligned}
$$

Due to the fact that $-e^{-s}$ is a increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$.

Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling $(H3)$, we obtain (39). □

We are now ready to prove the following result.

Theorem 1. *Assume (H1)–(H5), and there exist positive constants λ_1 and λ_2 such that the energy functional given by (14) satisfies*

$$\mathcal{E}(t) \leq \lambda_2 K^{-1}\Big(\lambda_1 \int_0^t \mu_3(s)\vartheta(s)ds\Big), \forall t \geq 0, \tag{40}$$

where $\vartheta = 1$ if $\nu \equiv 0$. K is defined in (11).

3. Proof of Main Result

We define a Lyapunov functional

$$\begin{aligned} \mathcal{L}(t) &= N\mathcal{E}(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) \\ &+ N_5F_5(t) + N_6F_6(t), \end{aligned} \tag{41}$$

where N, N_1, N_2, N_3, N_5 and N_6 are positive constants to be selected later.

Proof. (Of Theorem 1) By differentiating (41) and using (16), (19), (32), (34), (36), (38), (39), we have

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left[\frac{\rho_1\varpi_0}{2}\right]\int_0^1 (\alpha(x)+\theta(x))\psi_t^2dx + \frac{\rho_1\varpi_0}{2}\int_0^1 \theta(x)\psi_t^2dx \\ & -\left[\frac{rN_2}{2} - 2c\varepsilon_1N_1 + 8c\varepsilon_5N_5 - N_5(\frac{\beta^2l^2}{k_0} + \frac{3l^4\varpi_0c}{2k})\right]\int_0^1 \psi_x^2dx \\ & +\left[\rho_2N_2 + \varepsilon_3N_3 + c(1+\frac{1}{\varepsilon_4})N_5 + c\right]\int_0^1 \psi_t^2dx \\ & -\left[N\eta_0 - cN_5(1+\frac{1}{\varepsilon_5}) - \mu_1N_6 - c\right]\int_0^1 \varphi_t^2dx \\ & -\left[\frac{\rho_1 l}{2} - 5\varepsilon_4N_5\right]\int_0^1 w_t^2dx \\ & -\left[N\delta - cN_1 - cN_3(1+\frac{1}{\varepsilon_3}) - cN_5(1+\frac{1}{\varepsilon_4})\right]\int_0^1 q^2dx \\ & -\left[N_3\frac{k\rho_3}{2} - cN_2 - cN_5(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6})\right]\int_0^1 \varrho^2dx \\ & +[cN_2 + cN_1(1+\varepsilon_2+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}) + cN_5(1+\frac{1}{\varepsilon_4}+\frac{1}{\varepsilon_5})]\varpi o\psi_x \\ & +[\frac{N}{2} - cN_1 - cN_5]\varpi' o\psi_x \\ & -\left[N_6\eta_1 - cN_5(1+\frac{1}{\varepsilon_5}) - c\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)dsdx \\ & -[N_6\eta_1]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx \\ & -\left[\frac{k}{2}N_5 - k^2\varepsilon_2N_1 - \frac{2ck^2}{r}N_2\right]\int_0^1 (\varphi_x + lw + \psi)^2dx \\ & -\left[\frac{k_0 l}{2} - 3c\varepsilon_6N_5\right]\int_0^1 (w_x - l\varphi)^2dx \end{aligned}$$

$$-[cN_1 + cN_2 + cN_5] \int_0^1 \theta(x) f^2(\psi_t) dx$$
$$-N\mu_3(t) \int_0^1 \theta(x) \psi_t f(\psi_t) dx. \quad (42)$$

By setting

$$\varepsilon_4 = \frac{l\rho_1}{20N_5}, \varepsilon_6 = \frac{k_0 l}{12cN_5}, \varepsilon_5 = \frac{1}{8cN_5}, N_2 = \frac{r}{8kc} N_5,$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{\rho_1 \varpi_0}{2}\right] \int_0^1 (\alpha(x) + \theta(x)) \psi_t^2 dx + \frac{\rho_1 \varpi_0}{2} \int_0^1 \theta(x) \psi_t^2 dx \\
& -\left[\frac{rN_2}{2}(1 - 4l^2 m) - 2c\varepsilon_1 N_1\right] \int_0^1 \psi_x^2 dx \\
& +[\rho_2 N_2 + \varepsilon_3 N_3 + c(1 + N_5) N_5 + c] \int_0^1 \psi_t^2 dx \\
& -[N\eta_0 - cN_5(1 + N_5) - \mu_1 N_6 - c] \int_0^1 \varphi_t^2 dx \\
& -\left[\frac{\rho_1 l}{4}\right] \int_0^1 w_t^2 dx - \left[N\delta - cN_1 - cN_3(1 + \frac{1}{\varepsilon_3}) - cN_5(1 + N_5)\right] \int_0^1 q^2 dx \\
& -\left[N_3 \frac{k\rho_3}{2} - cN_2 - cN_5(1 + N_5)\right] \int_0^1 \varrho^2 dx \\
& +[cN_2 + cN_1(1 + \varepsilon_2 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}) + cN_5(1 + N_5)] \varpi o \psi_x \\
& +[\frac{N}{2} - cN_1 - cN_5] \varpi' o \psi_x \\
& -[N_6 \eta_1 - cN_5(1 + N_5) - c] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx \\
& -[N_6 \eta_1] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x, \tau, s, t) ds d\tau dx \\
& -\left[\frac{k}{4} N_5 - k^2 \varepsilon_2 N_1 - c\right] \int_0^1 (\varphi_x + lw + \psi)^2 dx \\
& -\left[\frac{k_0 l}{4}\right] \int_0^1 (w_x - l\varphi)^2 dx \\
& -[cN_1 + cN_2 + cN_5] \int_0^1 \theta(x) f^2(\psi_t) dx \\
& -N\mu_3(t) \int_0^1 \theta(x) \psi_t f(\psi_t) dx. \quad (43)
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. Let us take $\varepsilon_1 = \varepsilon_2 = \frac{1}{N_1}$ and we choose N_5 large enough such that

$$\alpha_1 = \frac{kN_5}{4} - \varepsilon_2 N_1 - c > 0,$$

then, we choose l small enough and we fix N_2 such that

$$\alpha_2 = \frac{r}{2}(1 - 4l^2 m) N_2 - 2\varepsilon_2 N_1 - c > 0,$$

where

$$m = (\frac{\beta^2 c}{r^2} + \frac{3l^2\varpi_0 c}{2r^2}),$$

then, we take $\varepsilon_3 = \frac{1}{N_3}$ and we choose N_1 large enough such that

$$\alpha_3 = \frac{d\rho_1}{4}N_1 - \rho_2 N_2 - \varepsilon_3 N_3 - cN_5(1+N_5) > 0,$$

then, we choose N_3 large enough such that

$$\alpha_4 = \frac{\rho_3 k}{2}N_3 - cN_1(1+N_1) - cN_2 - cN_5(1+N_5) > 0,$$

and N_6 large enough such that

$$\alpha_5 = \eta_1 N_6 - cN_5(1+N_5) > 0.$$

Thus, we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \quad \leq \quad & -\alpha_2 \int_0^1 \psi_x^2 dx - \alpha_3 \int_0^1 \psi_t^2 dx - [N\eta_0 - c]\int_0^1 \varphi_t^2 dx - \alpha_7 \int_0^1 w_t^2 dx \\
& -\alpha_5 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,1,s,t)dx - \alpha_1 \int_0^1 (\varphi_x + lw + \psi)^2 dx \\
& -\alpha_8 \int_0^1 (w_x - l\varphi)^2 dx - [\delta N - c]\int_0^1 q^2 dx - \alpha_4 \int_0^1 \varrho^2 dx \\
& +[\frac{N}{2} - c]\varpi' o\psi_x + c\varpi o\psi_x - \alpha_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx \\
& +c\int_0^1 \theta(x)f^2(\psi_t)dx - N\mu_3(t)\int_0^1 \theta(x)\psi_t f(\psi_t)dx,
\end{aligned} \tag{44}$$

where $\alpha_6 = \eta_1 N_6, \alpha_7 = \frac{\rho_1 l}{4}, \alpha_8 = \frac{k_0 l}{4}$.

On the other hand, if we let

$$\mathcal{T}(t) = N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t) + F_4(t) + N_5 F_5(t) + N_6 F_6(t),$$

then

$$\begin{aligned}
|\mathcal{T}(t)| \quad \leq \quad & \rho_2 N_1 \int_0^1 |\alpha(x)\psi_t \int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds|dx \\
& +\frac{\gamma\rho_4}{k}N_1 \int_0^1 |\alpha(x)q\int_0^t \varpi(t-s)(\psi(t)-\psi(s))ds|dx + N_2\int_0^1 |\psi\psi_t|dx \\
& +\rho_3\rho_4 N_3 \int_0^1 \left|\varrho \int_0^x q(y)dy\right|dx + \rho_1 \int_0^1 |\varphi_t(w_x - l\varphi)|dx \\
& +\rho_1 \int_0^1 |w_t(\varphi_x + lw + \psi)|dx + \rho_2 N_5 \int_0^1 |\psi_t(\varphi_x + lw + \psi)|dx \\
& +\frac{\beta\rho_1}{k}N_5 \int_0^1 |\psi_x\varphi_t|dx + \frac{\beta\rho_3}{\gamma}\chi N_5 \int_0^1 |\varrho\varphi_t|dx \\
& +\frac{\beta k}{\gamma}\chi N_5 \int_0^1 |q(\varphi_x + lw + \psi)|dx + \frac{\beta\rho_2 l^2}{k_0}N_5 \int_0^1 |\psi\psi_t|dx \\
& +\frac{\beta\rho_1 l}{k_0}N_5 \int_0^1 |\psi w_t|dx + \frac{\rho_1}{k}N_5 \int_0^1 \left|\nu(x)\varphi_t \int_0^t \varpi(t-s)\psi_x(s)ds\right|dx \\
& +\frac{l\rho_1}{k}\Big(\int_0^t \varpi(s)ds\Big)N_5 \int_0^1 \left|\nu(x)\psi_x \int_0^x w_t(y)dy\right|dx + N_6|F_6(t)|.
\end{aligned}$$

Exploiting Young, Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned} |\mathcal{T}(t)| &\leq c\int_0^1 \Big(\varphi_t^2+\psi_t^2+\psi_x^2+w_t^2+(\varphi_x+lw+\psi)^2+(w_x-l\varphi)^2+\varrho^2+q^2\Big)dx \\ &\quad +c\varpi o\psi_x + c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\tau,s,t)dsd\tau dx. \\ &\leq c\mathcal{E}(t). \end{aligned}$$

Consequently, we obtain

$$|\mathcal{T}(t)| = |\mathcal{L}(t) - N\mathcal{E}(t)| \leq c\mathcal{E}(t),$$

that is

$$(N-c)\mathcal{E}(t) \leq \mathcal{L}(t) \leq (N+c)\mathcal{E}(t). \tag{45}$$

Now, by choosing N large enough such that

$$N-c>0, \frac{N}{2}-c>0, N\delta - c>0, N\eta_0 - c>0,$$

and exploiting (14), estimates (44) and (45), respectively, give

$$c_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq c_2\mathcal{E}(t), \forall t \geq 0, \tag{46}$$

for some $c_1, c_2 > 0$.

We have

$$\mathcal{L}(t) \sim \mathcal{E}(t),$$

and

$$\mathcal{L}'(t) \leq -c\mathcal{E}(t) + c\varpi o\psi_x + c\int_0^1 \theta(x)(\psi_t^2 + f^2(\psi_t))dx, \ \forall t \geq t_0. \tag{47}$$

Let us define the following sets:

$$\begin{aligned} \Sigma_\psi &= \{x \in (0,1) : |\psi_t(x,t)| > l_1\}, \\ \varrho_\psi &= (0,1) \setminus \Sigma_\psi. \end{aligned}$$

We estimate the last term in the RHS of (47).

First, note that

$$\begin{aligned} \int_0^1 \theta(x)(\psi_t^2+f^2(\psi_t))dx &= \int_{\Sigma_\psi} \theta(x)(\psi_t^2+f^2(\psi_t))dx \\ &\quad + \int_{\varrho_\psi} \theta(x)(\psi_t^2+f^2(\psi_t))dx. \end{aligned}$$

Using assumption $(H5)$ and (16), we easily show that

$$\begin{aligned} \mu_3(t)\int_{\Sigma_\psi} \theta(x)(\psi_t^2+f^2(\psi_t))dx &\leq (k_1^{-1}+k_2)\int_{\Sigma_\psi} \mu_3(t)\theta(x)\psi_t f(\psi_t)dx \\ &\leq (k_1^{-1}+k_2)\int_0^1 \mu_3(t)\theta(x)\psi_t f(\psi_t)dx \\ &\leq -c\mathcal{E}'(t). \end{aligned} \tag{48}$$

If $h''=0$ on $[0,l_1]$:

This implies that there exist k_1', k_2' such that $k_1's^2 \leq f(s)s \leq k_2's^2$ for all $s \in \mathbf{R}_+$ and then (48) is also satisfied for $|\psi_t(x,t)| \leq l_1$, then on all $(0,1)$. From (47) and (48) and the fact that $\mu_3' \leq 0$, we arrive at

$$(\mu_3(t)\mathcal{L}+c\mathcal{E}(t))' \leq -\mu_3(t)J(\mathcal{E}(t))+c\varpi o\psi_x, \forall t \geq t_0, \tag{49}$$

where J is defined in (10).

If $h'(0)=0$ **and** $h''>0$ **on** $(0,l_1]$**:**

Since h is convex and increasing, h^{-1} is concave and increasing, and by using (H5), the reversed Jensen's inequality for concave function (see [20]), and (16), we obtain

$$\begin{aligned}
\mu_3(t)\int_{\varrho_\psi}\theta(x)(\psi_t^2+f^2(\psi_t))dx &\leq \mu_3(t)\int_{\varrho_\psi}\theta(x)h^{-1}(\psi_t f(\psi_t))dx \\
&\leq \mu_3(t)\int_{\varrho_\psi}h^{-1}(\theta(x)\psi_t f(\psi_t))dx \\
&\leq \mu_3(t)|\varrho_\psi|h^{-1}(\int_{\varrho_\psi}h^{-1}\frac{1}{|\varrho_\psi|}(\theta(x)\psi_t f(\psi_t))dx) \\
&\leq c\mu_3(t)h^{-1}(\int_{\varrho_\psi}h^{-1}(\theta(x)\psi_t f(\psi_t))dx) \\
&\leq c\mu_3(t)h^{-1}(\int_0^1 h^{-1}(\theta(x)\psi_t f(\psi_t))dx) \\
&\leq c\mu_3(t)h^{-1}(-c\mathcal{E}'(t)).
\end{aligned} \tag{50}$$

Therefore, from (47), (48) and (50), we find that

$$\mu_3\mathcal{L}'(t) \leq -c\mu_3\mathcal{E}(t)+c\mu_3 h^{-1}(-c\mathcal{E}'(t))-c\mathcal{E}'(t)+c\varpi o\psi_x, \forall t \geq t_0.$$

By using Young's inequality (49) and the fact that

$$h^*(p) \leq p[h']^{-1}(p), \mathcal{E}' \leq 0, h''>0 \quad and \quad \mu_3' \leq 0,$$

we obtain for $\varepsilon_0>0$ small enough and $c_0>0$ large enough,

$$\begin{aligned}
&[h'(\varepsilon_0\mathcal{E}(t))[\mu_3(t)\mathcal{L}(t)+c\mathcal{E}(t)]+c_0\mathcal{E}(t)]' \\
=\; & \varepsilon_0\mathcal{E}'(t)h''(\varepsilon_0\mathcal{E}(t))[\mu_3\mathcal{L}(t)+c\mathcal{E}(t)] \\
& +h'(\varepsilon_0\mathcal{E}(t))[\mu_3\mathcal{L}'(t)+\mu_3'\mathcal{L}(t)+c\mathcal{E}'(t)]+c_0\mathcal{E}'(t) \\
\leq\; & -c\mu_3 h'(\varepsilon_0\mathcal{E}(t))\mathcal{E}(t)+c\mu_3 h'(\varepsilon_0\mathcal{E}(t))h^{-1}(-c\mathcal{E}'(t)) \\
& +c_0\mathcal{E}'(t)+ch'(\varepsilon_0\mathcal{E}(t))\varpi o\psi_x \\
\leq\; & -c\mu_3 h'(\varepsilon_0\mathcal{E}(t))\mathcal{E}(t)+c\mu_3 h^*(h'(\varepsilon_0\mathcal{E}(t))-c\mathcal{E}'(t) \\
& +c_0\mathcal{E}'(t)+ch'(\varepsilon_0\mathcal{E}(t))\varpi o\psi_x \\
\leq\; & -c\mu_3 h'(\varepsilon_0\mathcal{E}(t))\mathcal{E}(t)+c\varepsilon_0\mu_3 h'(\varepsilon_0\mathcal{E}(t)\mathcal{E}(t)+c\varpi o\psi_x \\
\leq\; & -c\mu_3 h'(\varepsilon_0\mathcal{E}(t))\mathcal{E}(t)+c\varpi o\psi_x=-c\mu_3 J(\mathcal{E}(t))+c\varpi o\psi_x.
\end{aligned} \tag{51}$$

Now, let us define the following functional:

$$\mathcal{D}(t) = \begin{cases} \mu_3(t)\mathcal{L}(t) - c\mathcal{E}(t) & if \quad h'' = 0, \ on \ [0, l_1] \\ h'(\varepsilon_0\mathcal{E}(t))[\mu_3(t)\mathcal{L}(t) - c\mathcal{E}(t)] + c_0\mathcal{E}'(t), & if h'(0) = 0 \ and \ h'' > 0 \ on \ [0, l_1]. \end{cases}$$

Using (46), we have

$$\mathcal{D}(t) \sim \mathcal{E}(t),$$

and exploiting (49) and (51), we easily deduce that

$$\mathcal{D}'(t) \leq -c\mu_3(t)J(\mathcal{E}(t)) + c\varpi o\psi_x, \forall t \geq t_0. \tag{52}$$

By using (16) and (H2), we obtain

$$\begin{aligned} (\vartheta(t)\mathcal{D}(t))' &= \vartheta'(t)\mathcal{D}(t) + \vartheta(t)\mathcal{D}'(t) \\ &\leq -c\mu_3(t)\vartheta(t)J(\mathcal{E}(t)) + c\vartheta(t)\varpi o\psi_x \\ &\leq -c\mu_3(t)\vartheta(t)J(\mathcal{E}(t)) + c(\vartheta(t)\varpi)o\psi_x \\ &\leq -c\mu_3(t)\vartheta(t)J(\mathcal{E}(t)) - c\varpi' o\psi_x \\ &\leq -c\mu_3(t)\vartheta(t)J(\mathcal{E}(t)) - c\mathcal{E}'(t). \end{aligned}$$

Next, let

$$\mathcal{H} = \varepsilon(\vartheta(t)\mathcal{D}(t) + c\mathcal{E}(t)),$$

where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\vartheta(t)\mathcal{D}(t) + c\mathcal{E}(t) \leq \frac{1}{\bar{\varepsilon}}\mathcal{E}(t), \forall t \geq 0.$$

We also have

$$\mathcal{H} \sim \mathcal{E}, \tag{53}$$

and for $\forall t \geq t_0$

$$\mathcal{H}'(t) \leq -c\varepsilon\mu_3(t)\vartheta(t)J(\mathcal{H}). \tag{54}$$

Noting that $K' = -\frac{1}{J}$ (see (11)), we obtain from (54)

$$\mathcal{H}'(t)K'(\mathcal{H}(t)) \geq c\varepsilon\mu_3(t)\vartheta(t), \forall t \geq t_0.$$

A simple integration over (t_0, t) then yields

$$K(\mathcal{H}(t)) \geq K(\mathcal{H}(t_0)) + c\varepsilon\int_0^t \mu_3(s)\vartheta(s)ds - c\varepsilon\int_0^{t_0} \mu_3(s)\vartheta(s)ds.$$

On the other hand, since $\lim\limits_{t\to 0^+} K(t) = +\infty$ and

$$0 \leq \mathcal{H}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}\mathcal{E}(t_0) \leq \frac{\varepsilon}{\bar{\varepsilon}}\mathcal{E}(0).$$

We obtain for ε small enough

$$K(\mathcal{H}(t_0)) - c\varepsilon \int_0^{t_0} \mu_3(s)\vartheta(s)ds > 0.$$

Then, thanks to the fact that K^{-1} is decreasing, we infer

$$\begin{aligned} \mathcal{H}(t) &\leq K^{-1}(K(\mathcal{H}(t_0)) + c\varepsilon \int_0^t \mu_3(s)\vartheta(s)ds - c\varepsilon \int_0^{t_0} \mu_3(s)\vartheta(s)ds) \\ &\leq K^{-1}(c\varepsilon \int_0^t \mu_3(s)\vartheta(s)ds). \end{aligned}$$

From this end inequality and (53), we easily obtain (40). Then, the proof is completed. □

4. Conclusions

Our system concerns a Bresse system along with structural damping, distributed delay and in the presence of both temperatures of second sound type effects introduced in (3). As a main novelty, a general decay result for the solution with few constraints regarding the speeds of wave propagation is obtained. This new result is considered, as far as we know, as an extension of previous results in the literature for such type of system. We mention here that the nonlinear damping in our system creates a good interaction between the distributed delay and the other damping terms of system (3). This type of damping gives more information and qualitative properties on the solution and its impact on stability is also very important, as it is shown in the requirement of Theorem 1. Of course, the other terms (both temperatures and strong damping effects) act as balances in the stability of the system.

Open problem: Open problem: It will be very interesting to analysis the stability of the same system in more complicated cases from the mathematical point of view, but it will be very useful for the application point of view. Namely, one can consider the system

$$\begin{cases} \rho_1\varphi_{tt} = k(\varphi_x + lw + \psi)_x + k_0 l(w_x - l\varphi) - \mu_1\varphi_t \\ \rho_2\psi_{tt} = \beta\psi_{xx} - k(\varphi_x + lw + \psi) - \mu_2(t)\theta(x)f(\psi_t) - \gamma\varrho_x \\ \rho_1 w_{tt} = k_0(w_x - l\varphi)_x - kl(\varphi_x + lw + \psi) \\ \rho_3\varrho_t = c\int_0^\infty \varpi(s)\varrho_{xx}(t-s)ds - \gamma\psi_{tx}, \end{cases} \tag{55}$$

with a good choice of boundary and initial conditions. The thermoplastic effect is taken in viscoelasticity, and the most interesting question will be asked as follows: Can the system be stabilized? What kind of stability can be found? What is the role of $\mu_2(t)\theta(x)f(\psi_t)$?

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References

1. Bresse, J.A.C. *Cours de Mecanique Appliqueee*; Mallet Bachelier: Paris, France, 1859.
2. Feng, B.; Xin-Guang, Y. Long-time dynamics for a nonlinear Timoshenko system with delay. *Appl. Anal.* **2017**, *96*, 606–625. [CrossRef]
3. Houasni, M.; Zitouni, S.; Amiar, R. General decay for a viscoelastic damped Timoshenko system of second sound with distributed delay. *Math. Emgine. Sci. Aero.* **2019**, *10*, 323–340.
4. Hao, J.; Wang, F. Energy decay in a Timoshenko-type system for thermoelasticity of type III with distributed delay and past history. *Elect. J. Differ. Equ.* **2018**, *208*, 1–27.

5. Keddi, A.; Apalara, T.A.; Messaoudi, S.A. Exponential and Polynomial Decay in a Thermoelastic-Bresse System with Second Sound. *Appl. Math. Opti.* **2018**, *77*, 315–341. [CrossRef]
6. Timoshenko, S.P. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philos. Mag.* **1921**, *6*, 744–746. [CrossRef]
7. Dimplekumar, C.; Austin, S.; Brad, L. On Applications of Generalized Functions in the Discontinuous Beam Bending Differential Equations. *Appl. Math.* **2016**, *7*, 1943–1970.
8. Benaissa, A.; Miloudi, M.; Mokhtari, M. Global existence and energy decay of solutions to a Bresse system with delay terms. *Comment. Math. Univ. Carolin.* **2015**, *56*, 169–186. [CrossRef]
9. Bouzettouta, L.; Zitouni, S.; Zennir, K.; Guesmia, A. Stability of Bresse system with internal distributed delay. *J. Math. Comput. Sci.* **2017**, *7*, 92–118.
10. Elhindi, M.; Zennir, K.; Ouchenane, D.; Choucha, A.; El Arwadi, T. Bresse-Timoshenko type systems with thermodiffusion effects: Well-possedness, stability and numerical results. *Rend. Circ. Mat. Palermo II. Ser* **2023**, *72*, 169–194. [CrossRef]
11. Guesmia, A.; Kafini, M. Bresse system with infinite memories. *Math. Meth. Appl. Sci.* **2015**, *38*, 2389–2402. [CrossRef]
12. Zitouni, S.; Bouzattoota, L.; Zennir, K.; Ouchenane, D. Exponential decay of thermo-elastic Bresse system with distributed delay term. *Hacettepe. J. Math. Stat.* **2018**, *47*, 1216–1230. [CrossRef]
13. Dimplekumar, C.; Ravikumar, K.; Ramkumar, K.; Dhanalakshmi, K.; John A.D. Trajectory controllability of neutral stochastic integrodifferential equations with mixed fractional Brownian motion. *J. Cont. Desi.* **2023**, 1–15. . [CrossRef]
14. Durga, N.; Djemai, M.; Chalishajar, D. Solvability and trajectory controllability of impulsive stochastic MHD equations with Rosenblatt process. *Chaos Soliton Fract.* **2023**, *175*, 114013. [CrossRef]
15. Guesmia, A. Asymptot stability of abstract dissipative system with infinite memory. *J. Math. Anal. Appl.* **2011**, *382*, 748–760. [CrossRef]
16. Cavalcanti, M.M.; Oquendo, H.P. Frictional versus viscoelastic damping in a semilinear wave equation. *SIAM J. Control Optim.* **2003**, *42*, 1310–1324. [CrossRef]
17. Arnold, V.I. *Mathematical Methods of Classical Mechanics*; Springer: New York, NY, USA, 1989.
18. Liu, W.J.; Zuazua, E. Decay rates for dissipative wave equations. *Ric. Mat.* **1999**, *XLVIII*, 61–75.
19. Nicaise, A.S.; Pignotti, C. Stabilization of the wave equation with boundary or internal distributed delay. *Diff. Int. Equ.* **2008**, *21*, 935–958. [CrossRef]
20. Rudin, W. *Real and Complex Analysis*, 2nd ed.; McGraw-Hill: New York, NY, USA, 1974.