

Article

5th-Order Multivariate Edgeworth Expansions for Parametric Estimates

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Abstract: The only cases where exact distributions of estimates are known is for samples from exponential families, and then only for special functions of the parameters. So statistical inference was traditionally based on the asymptotic normality of estimates. To improve on this we need the *Edgeworth expansion* for the distribution of the standardised estimate. This is an expansion in $n^{-1/2}$ about the normal distribution, where n is typically the sample size. The first few terms of this expansion were originally given for the special case of a sample mean. In earlier work we derived it for *any* standard estimate, hugely expanding its application. We define an estimate \hat{w} of an unknown vector w in R^p , as a *standard estimate*, if $E \hat{w} \rightarrow w$ as $n \rightarrow \infty$, and for $r \geq 1$ the r th-order cumulants of \hat{w} have magnitude n^{1-r} and can be expanded in n^{-1} . Here we present a significant extension. We give the expansion of the distribution of *any smooth function* of \hat{w} , say $t(\hat{w})$ in R^q , giving its distribution to $n^{-5/2}$. We do this by showing that $t(\hat{w})$, is a standard estimate of $t(w)$. This provides far more accurate approximations for the distribution of $t(\hat{w})$ than its asymptotic normality.

Keywords: Edgeworth expansions; parametric inference; standard estimates; chain rules for cumulant coefficients

MSC: 60B12; 60B20; 60E05; 62E20; 62F12; 62G86; 62H10



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1. Introduction and Summary

Suppose that \hat{w} is a *standard* or *Type A* estimate of an unknown w in R^p with respect to a given parameter n . That is, $E \hat{w} \rightarrow w$ as $n \rightarrow \infty$ and for $r \geq 1$, its r th-order cumulants have magnitude n^{1-r} and can be expanded as

$$\bar{k}^{1-r} = \kappa(\hat{w}^{i_1}, \dots, \hat{w}^{i_r}) = \sum_{e=r-1}^{\infty} n^{-e} \bar{k}_e^{1-r} \text{ for } 1 \leq i_1, \dots, i_r \leq p, \quad (1)$$

where the *cumulant coefficients* $\bar{k}_e^{1-r} = k_e^{j_1 \dots j_r}$ do not depend on n , or at least are bounded as $n \rightarrow \infty$. So $\bar{k}_0^1 = w^{i_1}$. For example, (1) holds for \hat{w} a function of a sample mean. We show that if $t(\hat{w})$ is a smooth function of a standard estimate \hat{w} , then it is a standard estimate of $t(w)$. We establish this for unbiased \hat{w} in Theorem 2, and for biased \hat{w} in Theorem 3. More generally, we define \hat{w} as a *Type B* estimate if $E \hat{w} \rightarrow w$ as $n \rightarrow \infty$, and for $r \geq 1$,

$$\bar{k}^{1-r} = \sum_{d=2r-2}^{\infty} n^{-d/2} \bar{b}_d^{1-r} \text{ for } 1 \leq i_1, \dots, i_r \leq p, \bar{b}_d^{1-r} = b_d^{i_1 \dots i_r}.$$

For example, this type arises when considering one-sided confidence regions. If $t(\hat{w})$ is a smooth function of a Type B estimate, then it is a Type B estimate of $t(w)$. So for a Type A estimate, \bar{b}_d^{1-r} is \bar{k}_e^{1-r} for $d = 2e$ and 0 for d odd. n is typically the sample size or the minimum sample size if there is more than one sample.

Sections 3 and 4 show that a smooth function of \hat{w} , say $t(\hat{w})$, is a standard estimate of $t = t(w)$. These sections provide the cumulant coefficients of $t(\hat{w})$ in terms of those

of \hat{w} and the derivatives of $t(w)$. Section 3 does this for \hat{w} unbiased and Section 4 for \hat{w} biased. So they can be thought of as *chain rules* for obtaining the cumulant coefficients for $t(\hat{w})$ from those of \hat{w} . We use the notation $Y_n = O(n^{-\gamma})$ to mean that $n^\gamma Y_n$ is bounded as $n \rightarrow \infty$. We provide the cumulant coefficients required for Edgeworth expansions of \hat{t} to $O(n^{-5/2})$. Cumulant coefficients up to $O(n^{-1})$ were given in [1]. Cumulant coefficients up to $O(n^{-r/2})$ use the r th derivatives of $t(w)$. Section 5 specialises to univariate $t(w)$ with examples. Theorem 3 and Corollary 4 rectify $\bar{a}_2^{12} = K_2^{j_1 j_2}$ and a_{22} on pages 67 and 59 of [2]. Section 2 extends the *shorthand* bar notation above and gives the foundation theorem.

We now summarise the expressions for Edgeworth expansions of \hat{w} for standard and Type B estimates in terms of the cumulant coefficients \bar{k}_e^{1-r} and \bar{b}_d^{1-r} given in [3–5]:

$$Prob.(Y_{nw} \leq x) = \sum_{r=0}^{\infty} n^{-r/2} P_r(x), \quad p_{Y_{nw}}(x) = \sum_{r=0}^{\infty} n^{-r/2} p_r(x), \tag{2}$$

$$\text{where } Y_{nw} = n^{1/2}(\hat{w} - w - b_1 n^{-1/2}), \quad (b_1)_i = b_1^i, \quad P_0(x) = \Phi_V(x), \tag{3}$$

$$P_r(x) = \tilde{B}_r(e(-\partial/\partial x)) \Phi_V(x) \text{ for } r \geq 1, \tag{4}$$

$$e_j(t) = \sum_{r=1}^{j+2} \bar{b}_{r+j}^{1\dots r} t_{i_1} \dots t_{i_r} / r!, \quad \bar{b}_{r+j}^{1\dots r} = b_{r+j}^{i_1 \dots i_r}, \tag{5}$$

$\Phi_V(x)$ is the multivariate normal distribution with zero mean and covariance $V = (\bar{b}_2^{12})$, $\tilde{B}_r(e)$ is the complete ordinary Bell polynomial of [6]:

$$\begin{aligned} \tilde{B}_1(e) &= e_1, \quad \tilde{B}_2(e) = e_2 + e_1^2, \quad \tilde{B}_3(e) = e_3 + 2e_1 e_2 + e_1^3, \\ \tilde{B}_4(e) &= e_4 + 2e_1 e_3 + e_2^2 + 3e_1^2 e_2 + e_1^4. \end{aligned}$$

This equation provides the 5th-order Edgeworth expansion for the distribution of Y_{nw} , extending it up to $O(n^{-5/2})$. It is important to note that (5) utilises the *tensor summation convention* of implicitly summing i_1, \dots, i_r over their range $1, \dots, p$. For example,

$$\text{for } \partial_i = \partial/\partial x_i \text{ and } \bar{\partial}_k = \partial_{i_k},$$

$$\begin{aligned} P_1(x) &= e_1(-\partial/\partial x) \Phi_V(x) = \sum_{r=1}^3 \bar{b}_{r+1}^{1\dots r} (-\bar{\partial}_1) \dots (-\bar{\partial}_r) \Phi_V(x) / r! \\ &= \bar{k}_1^1 (-\bar{\partial}_1) \Phi_V(x) + \bar{k}_2^{1-3} (-\bar{\partial}_1)(-\bar{\partial}_2)(-\bar{\partial}_3) \Phi_V(x) / 6 \end{aligned}$$

for a standard estimate. For a standard estimate, $b_1 = 0$ in (3) and the cumulant coefficients needed for $P_r(x), p_r(x)$ of (2) are $\bar{k}_0^1 = w^{i_1}$,

$$\text{for } r = 0 : \bar{k}_1^{12}; \quad \text{for } r = 1 : \bar{k}_1^1, \bar{k}_2^{1-3}; \quad \text{for } r = 2 : \bar{k}_2^{12}, \bar{k}_3^{1-4}; \tag{6}$$

$$\text{for } r = 3 : \bar{k}_2^1, \bar{k}_3^{1-3}, \bar{k}_4^{1-5}; \quad \text{for } r = 4 : \bar{k}_3^{12}, \bar{k}_4^{1-4}, \bar{k}_5^{1-6}. \tag{7}$$

Therefore, to derive the 5th-order Edgeworth expansion for the distribution of $n^{1/2}(t(\hat{w}) - t(w))$ for \hat{w} a standard estimate, we simply substitute the coefficients in (6) and (7) in the expression for $P_r(x)$, $r \leq 4$, with those corresponding to $t(\hat{w})$ as provided in Sections 3–5.

Equation (9) of [3] provides $P_r(x)$ for the more general case where $P_0(x)$ is the distribution function of Y in R^p which depends on n but is asymptotic to $\Phi_V(x)$ and has a Type B expansion. One can choose $P_0(x)$ so that the number of terms in each $P_r(x)$ greatly reduces: see Withers and Nadarajah (2012d) [7,8]. When \hat{w} is lattice, further terms need to be added: see for example Chapter 5 of [9], [10], and for the density of Y_{nw} , p211 of [11], Section 5 of [12], and Section 6 of [13]. Corollary 1 of [3] gives the *tilted* Edgeworth expansion for $t(\hat{w})$, sometimes called the *saddlepoint approximation*, or the *small sample* expansion as it is a series in n^{-1} not just $n^{-1/2}$. It is very useful for the tails of the distribution where Edgeworth expansions perform poorly. Cumulant coefficients are also needed for bias

reduction, Bayesian inference, confidence regions and power. See [7,8,14–18]. for examples. For a historical overview of Edgeworth expansions, refer to Section 7.

In summary, this paper gives high-order expansions for the distribution of a wide range of estimates, by determining the cumulant coefficients required for any smooth function of a standard estimate. This approach offers unprecedented accuracy for these distributions and eliminates the necessity for simulation methods.

2. Foundations

Considering $w = (w^1, \dots, w^p)$ in R^p and an estimate \hat{w} , assume that $E \hat{w} \rightarrow w$ as $n \rightarrow \infty$ and that for $r \geq 1$, its r th-order cumulants have magnitude n^{1-r} . Given i_1, \dots, i_r in $1, 2, \dots, p$, we write these cumulants in shorthand as

$$\bar{k}^{1-r} = k^{i_1 \dots i_r} = \kappa(\hat{w}^{i_1}, \dots, \hat{w}^{i_r}) = O(n^{1-r}) \text{ as } n \rightarrow \infty. \tag{8}$$

For example, if $\hat{w} = \bar{X}$ is the mean of a random sample of size n , then (8) holds since $\bar{k}^{1-r} = n^{1-r} \kappa(X^{i_1}, \dots, X^{i_r})$ where X^i is the i th component of X . According to Theorem 1, Equation (8) is valid if \hat{w} is a smooth function of one or more sample means. Let $t : R^p \rightarrow R^q$ be a smooth function in a neighbourhood of w with j th component $t^j = t^j(w)$, $j = 1, \dots, q$ and finite partial derivatives

$$\bar{t}_{sr\dots}^k = t_{i_s i_r \dots}^{j_k} = \partial_{i_s} \partial_{i_r} \dots t^{j_k}(w), \quad \bar{t}_{s-r}^k = t_{i_s \dots i_r}^{j_k} \text{ for } s \leq r$$

where $\partial_i = \partial/\partial w^i$. Superscripts i are reserved for the cumulants of \hat{w} and subscripts for partial derivatives of $t(w)$. Superscripts j are reserved for the components of $t(w)$ and for the joint cumulants of $\hat{t} = t(\hat{w})$. This bar shorthand allows us to shorten expressions by suppressing the i s and j s. We write the cumulants of $\hat{t} = t(\hat{w})$ as

$$\bar{K}^{1-r} = K^{j_1 \dots j_r} = \kappa(\hat{t}^{j_1}, \dots, \hat{t}^{j_r}) \text{ where } \hat{t} = t(\hat{w}), \hat{t}^j = t^j(\hat{w}). \tag{9}$$

For example, $\bar{k}^{12} = k^{i_1 i_2}$ and $\bar{K}^{12} = K^{j_1 j_2}$ imply that the covariance of \hat{w} is represented by (\bar{k}^{12}) , and the covariance of \hat{t} is represented by (\bar{K}^{12}) , both of which scale as $O(n^{-1})$. Next, we demonstrate that

$$\bar{K}^{12} = {}_1\bar{K}^{12} + O(n^{-2}) \text{ where } {}_1\bar{K}^{12} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}^{12}.$$

In other words, ${}_1K^{j_1 j_2} = t_{i_1}^{j_1} t_{i_2}^{j_2} k^{i_1 i_2}$, employing the tensor sum convention. The rest of this section and all proofs can be skipped on a first reading. Theorem 1 provides the cumulants of $\hat{t} = t(\hat{w})$ when \hat{w} is unbiased.

We use the notation $\sum^N f^{j_1 j_2 \dots}$ to denote summing over all N permutations of j_1, j_2, \dots resulting distinct terms.

Theorem 1. Suppose $E \hat{w} = w$ and Equation (8) holds. Then for $r \geq 1$ and $1 \leq j_1, \dots, j_r \leq q$, \bar{K}^{1-r} of (9) satisfies

$$\bar{K}^{1-r} = \sum_{e=r-1}^{\infty} {}_e\bar{K}^{1-r} \text{ where } {}_e\bar{K}^{1-r} = {}_eK^{j_1 \dots j_r} = O(n^{-e}) \text{ as } n \rightarrow \infty, \tag{10}$$

and the leading ${}_e\bar{K}^{1-r}$ are as follows.

$${}_0\bar{K}^1 = \bar{t}^1, \text{ that is, } {}_0K^1 = t^1.$$

$${}_1\bar{K}^1 = \bar{t}_{12}^1 \bar{k}^{12} / 2, \text{ that is, } {}_1K^1 = t_{i_1 i_2}^{j_1} k^{i_1 i_2} / 2 = \sum_{i_1, i_2=1}^p t_{i_1 i_2}^{j_1} k^{i_1 i_2} / 2.$$

$${}_2\bar{K}^1 = \bar{t}_{1-3}^1 \bar{k}^{1-3} / 6 + \bar{t}_{1-4}^1 \bar{k}^{12} \bar{k}^{34} / 8,$$

that is,

$$\begin{aligned}
 {}_2K^j &= t_{i_1 i_2 i_3}^{j_1} k^{i_1 i_2 i_3} / 6 + t_{i_1 i_4}^{j_1} \bar{k}^{i_1 i_2} \bar{k}^{i_3 i_4} / 8. \\
 {}_3\bar{K}^1 &= \bar{t}_{1-4}^1 \bar{k}^{1-4} / 24 + \bar{t}_{1-5}^1 \bar{k}^{1-3} \bar{k}^{45} / 12 + \bar{t}_{1-6}^1 \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} / 48, \\
 {}_4\bar{K}^1 &= \bar{t}_{1-5}^1 \bar{k}^{1-5} / 120 + \bar{t}_{1-6}^1 (\bar{k}^{1-4} \bar{k}^{56} / 48 + \bar{k}^{1-3} \bar{k}^{4-6} / 72) + \bar{t}_{1-7}^1 \bar{k}^{1-3} \bar{k}^{45} \bar{k}^{67} / 48 \\
 &\quad + \bar{t}_{1-8}^1 \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} \bar{k}^{78} / 384. \\
 {}_1\bar{K}^{12} &= \bar{t}_1^1 \bar{t}_2^2 \bar{k}^{12}, \quad {}_2\bar{K}^{12} = T_{1-3}^{12} \bar{k}^{1-3} / 2 + T_{1-4}^{12} \bar{k}^{12} \bar{k}^{34} / 2
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-3}^{12} &= \sum_{i=1}^2 \bar{t}_{12}^1 \bar{t}_3^2, \quad T_{1-4}^{12} = \sum_{i=1}^2 \bar{t}_{1-3}^1 \bar{t}_4^2 + \bar{t}_{13}^1 \bar{t}_{24}^2, \quad \sum_{i=1}^2 \bar{t}_{a-b}^1 \bar{t}_{c-d}^2 = \bar{t}_{a-b}^1 \bar{t}_{c-d}^2 + \bar{t}_{a-b}^2 \bar{t}_{c-d}^1. \\
 {}_3\bar{K}^{12} &= U_{1-4}^{12} \bar{k}^{1-4} + T_{1-5}^{12} \bar{k}^{1-3} \bar{k}^{45} + T_{1-6}^{12} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} / 4
 \end{aligned}$$

where

$$\begin{aligned}
 U_{1-4}^{12} &= \sum_{i=1}^2 \bar{t}_{1-3}^1 \bar{t}_4^2 / 6 + \bar{t}_{12}^1 \bar{t}_{34}^2 / 4, \\
 T_{1-5}^{12} &= \sum_{i=1}^2 (\bar{t}_{1-4}^1 \bar{t}_5^2 / 6 + \bar{t}_{1245}^1 \bar{t}_3^2 / 4 + \bar{t}_{124}^1 \bar{t}_{35}^2 / 2 + \bar{t}_{145}^1 \bar{t}_{23}^2 / 4), \\
 T_{1-6}^{12} &= \sum_{i=1}^2 \bar{t}_{1-5}^1 \bar{t}_6^2 / 2 + \sum_{i=1}^2 \bar{t}_{1235}^1 \bar{t}_{46}^2 + \bar{t}_{1-3}^1 \bar{t}_{4-6}^2 + 2\bar{t}_{135}^1 \bar{t}_{246}^2 / 3. \\
 {}_2\bar{K}^{1-3} &= \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}^{1-3} + T_{1-4}^{1-3} \bar{k}^{12} \bar{k}^{34}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-4}^{1-3} &= \sum_{i=1}^3 \bar{t}_{13}^1 \bar{t}_2^2 \bar{t}_4^3. \\
 {}_3\bar{K}^{1-3} &= T_{1-4}^{1-3} \bar{k}^{1-4} / 2 + T_{1-5}^{1-3} \bar{k}^{1-3} \bar{k}^{45} + T_{1-6}^{1-3} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-5}^{1-3} &= \sum_{i=1}^6 \bar{t}_{124}^1 \bar{t}_3^2 \bar{t}_5^3 / 2 + \sum_{i=1}^3 \bar{t}_{145}^1 \bar{t}_2^2 \bar{t}_3^3 / 2 + \sum_{i=1}^6 \bar{t}_{12}^1 \bar{t}_{34}^2 \bar{t}_5^3 / 2 + \sum_{i=1}^3 \bar{t}_{14}^1 \bar{t}_{25}^2 \bar{t}_3^3, \\
 T_{1-6}^{1-3} &= \sum_{i=1}^3 \bar{t}_{1235}^1 \bar{t}_4^2 \bar{t}_6^3 / 2 + \sum_{i=1}^6 \bar{t}_{1-3}^1 \bar{t}_{45}^2 \bar{t}_6^3 + \sum_{i=1}^6 \bar{t}_{135}^1 \bar{t}_{24}^2 \bar{t}_6^3 / 2 + \bar{t}_{13}^1 \bar{t}_{25}^2 \bar{t}_{46}^3. \\
 {}_3\bar{K}^{1-4} &= \bar{t}_1^1 \dots \bar{t}_4^4 \bar{k}^{1-4} + T_{1-5}^{1-4} \bar{k}^{1-3} \bar{k}^{45} + T_{1-6}^{1-4} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-5}^{1-4} &= \sum_{i=1}^{12} \bar{t}_{14}^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_5^4, \quad T_{1-6}^{1-4} = \sum_{i=1}^4 \bar{t}_{135}^1 \bar{t}_2^2 \bar{t}_4^3 \bar{t}_6^4 + \sum_{i=1}^{12} \bar{t}_{13}^1 \bar{t}_{25}^2 \bar{t}_4^3 \bar{t}_6^4. \\
 {}_4\bar{K}^{1-4} &= U_{1-5}^{1-4} \bar{k}^{1-5} / 2 + U_{1-6}^{1-4} \bar{k}^{1-4} \bar{k}^{56} + V_{1-6}^{1-4} \bar{k}^{1-3} \bar{k}^{4-6} + T_{1-7}^{1-4} \bar{k}^{1-3} \bar{k}^{45} \bar{k}^{67} \\
 &\quad + T_{1-8}^{1-4} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} \bar{k}^{78}
 \end{aligned}$$

where

$$\begin{aligned}
 U_{1-5}^{1-4} &= \sum_{i_1, i_2, i_3, i_4, i_5}^4 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4, \\
 U_{1-6}^{1-4} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^6 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4, \\
 V_{1-6}^{1-4} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^6 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4, \\
 T_{1-7}^{1-4} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4, \\
 T_{1-8}^{1-4} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^4 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{24} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{12} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 / 2 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^3 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4. \\
 4\bar{K}^{1-5} &= \bar{t}_1^1 \dots \bar{t}_5^5 \bar{k}^{1-5} + T_{1-6}^{1-5} \bar{k}^{1-4} \bar{k}^{56} + U_{1-6}^{1-5} \bar{k}^{1-3} \bar{k}^{4-6} + T_{1-7}^{1-5} \bar{k}^{1-3} \bar{k}^{45} \bar{k}^{67} \\
 &+ T_{1-8}^{1-5} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} \bar{k}^{78}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-6}^{1-5} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{20} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5, \quad U_{1-6}^{1-5} = \sum_{i_1, i_2, i_3, i_4, i_5, i_6}^{15} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5, \\
 T_{1-7}^{1-5} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{30} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5, \\
 T_{1-8}^{1-5} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^5 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 / 5 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5. \\
 5\bar{K}^{1-6} &= \bar{t}_1^1 \dots \bar{t}_6^6 \bar{k}^{1-6} + T_{1-7}^{1-6} \bar{k}^{1-5} \bar{k}^{67} + U_{1-7}^{1-6} \bar{k}^{1-4} \bar{k}^{5-7} + T_{1-8}^{1-6} \bar{k}^{1-4} \bar{k}^{56} \bar{k}^{78} \\
 &+ U_{1-8}^{1-6} \bar{k}^{1-3} \bar{k}^{4-6} \bar{k}^{78} + T_{1-9}^{1-6} \bar{k}^{1-3} \bar{k}^{45} \bar{k}^{67} \bar{k}^{89} + T_{1-10}^{1-6} \bar{k}^{12} \bar{k}^{34} \bar{k}^{56} \bar{k}^{78} \bar{k}^{9,10}
 \end{aligned}$$

where

$$\begin{aligned}
 T_{1-7}^{1-6} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{30} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6, \quad U_{1-7}^{1-6} = \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6, \\
 T_{1-8}^{1-6} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{180} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{120} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6, \\
 U_{1-8}^{1-6} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{90} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8}^{90} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6, \\
 T_{1-9}^{1-6} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{60} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{180} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{120} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{720} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6, \\
 T_{1-10}^{1-6} &= \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^6 \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^{120} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^{90} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 \\
 &+ \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6 + \sum_{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9, i_{10}}^{360} \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{t}_4^4 \bar{t}_5^5 \bar{t}_6^6.
 \end{aligned}$$

Note 1. For reference regarding N in Σ^N , refer to page 48 of [19]. It is important to note that the notation Σ^N in terms like T_{1-s}^{1-r} only applies for $N < r!$ in the context where they are used.

For example, writing $(abc) = \bar{t}_{13}^a \bar{t}_{24}^b \bar{t}_4^c$ and recalling that Σ^N only permutes superscripts but leaves subscripts alone, we have

$$T_{1-4}^{1-3} = \sum_{(123)}^N = (123) + (213) + (321) \tag{11}$$

with $N = 3$ not $3!$ since

$$\sum_{(123)}^{3!} = (123) + (132) + (213) + (231) + (321) + (312) = \sum_{k=1}^6 S_k$$

say, when multiplied by $\bar{k}^{12} \bar{k}^{34}$, as in ${}_2\bar{K}^{1-3}$, gives $\sum_{k=1}^6 S'_k$ say, where for $k = 1, 2, 3$, $S'_{2k} = S'_{2k-1}$. For example, $T_{1-4}^{1-3} \bar{k}^{12} \bar{k}^{34}$ in ${}_2\bar{K}^{1-3}$ above is shorthand for $\sum^3 \bar{t}_{13}^1 \bar{t}_{24}^2 \bar{t}_4^3 \bar{k}^{12} \bar{k}^{34}$. For,

$$S'_2 = \bar{t}_4^2 \bar{k}^{43} \bar{t}_{31}^1 \bar{k}^{12} \bar{t}_2^3 = \bar{t}_1^2 \bar{k}^{12} \bar{t}_{13}^1 \bar{k}^{34} \bar{t}_4^3 = S'_1 \Rightarrow T_{1-4}^{1-3} \bar{k}^{12} \bar{k}^{34} = S'_1 + S'_3 + S'_5.$$

Proof. This result can be derived by substituting $\bar{A}_{1-r}^j = A_{i_1 \dots i_r}^j$ by $\bar{t}_{1-r}^j / r! = t_{i_1 \dots i_r}^j / r!$ according to [19]. □

Likewise, one can readily derive ${}_4\bar{K}^{12}$, ${}_4\bar{K}^{1-3}$ from pages 51–53 of [19]. The tensor form $2\bar{K}_1^{12} = \bar{t}_{12}^1 \bar{k}_1^{12}$ can be conceptualised as a *molecule* or *molecular structure* of 2 atoms, \bar{t}_{12}^1 and \bar{k}_1^{12} , connected by the *double bond* 1, 2, represented as i_1, i_2 . ${}_2\bar{K}^1$ is a linear combination of $\bar{t}_{1-3}^1 \bar{k}^{1-3}$, 2 atoms linked by the triple bond 1,2,3, and secondly $\bar{k}^{12} \bar{t}_{1-4}^1 \bar{k}^{34}$. The last expression has the structure of CO₂, with 2 identical atoms each linked by a double bond to a central atom. Just as such bonds are depicted in chemistry to illustrate the structure of a molecule, they can be very useful here to illustrate the difference in structure of similar mathematical expressions. S'_1 of Note 1 is a linear molecular form with the 4 single bonds 1,2,3,4 and 4 distinct atoms, \bar{t}_1^2 , \bar{t}_1^1 , \bar{t}_{12}^1 , and \bar{k}^{12} . Other expressions have more complex structures. Doubling the last term in ${}_2\bar{K}^{12}$ yields $T_{1-4}^{12} \bar{k}^{12} \bar{k}^{34} = S^{12} + S^{21} + S$ where $S^{12} = \bar{k}^{12} \bar{t}_{1-3}^1 \bar{k}^{34} \bar{t}_4^2$ exhibits a linear structure with a double bond between 1 and 2, followed by two single bonds, 3 and 4. Additionally, $S = \bar{t}_{31}^1 \bar{k}^{12} \bar{t}_{24}^2 \bar{k}^{43}$ forms a square or rectangle with four single bonds 1,2,4,3 arranged along successive edges of the square. These pictorial forms are a very useful way to distinguish similar expressions in $\Sigma^N f^{i_1 i_2 \dots}$.

Section 6 provides the ‘more complicated’ terms referred to (but not given) on p49 of [19] when \hat{w} is biased. It can be used for an alternative Proof of Theorem 3 below. From Theorem 1, Edgeworth expansions can be obtained for the distribution and density of the standardised form of $t(\hat{w})$,

$$Y_{nt} = n^{1/2}(\hat{t} - t) = n^{1/2}(t(\hat{w}) - t(w)), \tag{12}$$

of the form

$$Prob.(Y_{nt} \leq x) = \sum_{r=0}^{\infty} P_{rn}(x), \quad p_{Y_{nt}}(x) = \sum_{r=0}^{\infty} p_{rn}(x), \tag{13}$$

where $P_{rn}(x), p_{rn}(x)$ are $O(n^{-r/2})$. The ${}_e\bar{K}^{1-r}$ of Theorem 1 needed for $P_{rn}(x), p_{rn}(x)$ are as follows.

- For $P_{0n}(x), p_{0n}(x)$: ${}_0\bar{K}^1 = \bar{t}_1^1, {}_1\bar{K}^{12}$. For $P_{1n}(x), p_{1n}(x)$: ${}_1\bar{K}^1, {}_2\bar{K}^{1-3}$.
- For $P_{2n}(x), p_{2n}(x)$: ${}_2\bar{K}^{12}, {}_3\bar{K}^{1-4}$. For $P_{3n}(x), p_{3n}(x)$: ${}_2\bar{K}^1, {}_3\bar{K}^{1-3}, {}_4\bar{K}^{1-5}$.
- For $P_{4n}(x), p_{4n}(x)$: ${}_3\bar{K}^{1-2}, {}_4\bar{K}^{1-4}, {}_5\bar{K}^{1-6}$.

3. Cumulant Coefficients for $t(\hat{w})$ when $E \hat{w} = w$

We now show that for $r \geq 1$ and $1 \leq j_1, \dots, j_r \leq q$, the cumulant coefficient \bar{K}^{1-r} from Equation (10) can be expanded as

$$\bar{K}^{1-r} = K^{j_1 \dots j_r} = \kappa(\hat{t}^{j_1}, \dots, \hat{t}^{j_r}) = \sum_{e=r-1}^{\infty} n^{-e} \bar{K}_e^{1-r} \tag{14}$$

Substituting $\{\bar{k}^{1-r}\}$ with $\{\bar{K}^{1-r}\}$ on the right-hand side of (4), denoted as RHS (4), provides the Edgeworth expansion for Y_{nt} as in Equation (12). If π is a product of cumulants as in Equation (1), let $(\pi)_e$ denote the coefficient of n^{-e} in the expansion of π . For example,, $(\bar{k}^{1-r})_e = \bar{k}_e^{1-r}$,

$$\begin{aligned} (\bar{k}^{12} \bar{k}^{34})_3 &= \bar{k}_1^{12} \bar{k}_2^{34} + \bar{k}_2^{12} \bar{k}_1^{34}, \quad (\bar{k}^{12} \bar{k}^{34})_4 = \bar{k}_1^{12} \bar{k}_3^{34} + \bar{k}_2^{12} \bar{k}_2^{34} + \bar{k}_3^{12} \bar{k}_1^{34}, \\ (\bar{k}^{1-3} \bar{k}^{45})_4 &= \bar{k}_2^{1-3} \bar{k}_2^{45} + \bar{k}_3^{1-3} \bar{k}_1^{45}, \\ (\bar{k}^{12} \bar{k}^{34} \bar{k}^{56})_4 &= \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_2^{56} + \bar{k}_1^{12} \bar{k}_2^{34} \bar{k}_1^{56} + \bar{k}_2^{12} \bar{k}_1^{34} \bar{k}_1^{56}. \end{aligned} \tag{15}$$

Now, let us provide the elements of the expansion (14) when $E \hat{w} = w$.

Theorem 2. Assume that \hat{w} is an unbiased estimate of w satisfying Equation (1) and $t(w)$ has finite derivatives. In this case, Equation (14) holds with bounded cumulant coefficients

$$\begin{aligned} \bar{K}_e^{1-r} &= K_e^{j_1 \dots j_r} = \sum_{k=r-1}^e k \bar{K}_e^{1-r} : \\ \bar{K}_{r-1}^{1-r} &= r_{-1} \bar{K}_{r-1}^{1-r}, \\ \bar{K}_r^{1-r} &= r_{-1} \bar{K}_r^{1-r} + r \bar{K}_r^{1-r}, \end{aligned} \tag{16}$$

and so forth. The leading coefficients needed for $P_r(x), p_r(x)$ of (4) for the distribution of Y_{nt} of (12) are given in the T, U, V notation of Theorem 1 as follows.

$$\begin{aligned} \bar{K}_0^1 &= \bar{t}^1, \text{ that is, } K_0^1 = t^1 = t^1(w). \quad {}_0\bar{K}_e^1 = 0 \text{ for } e \geq 1. \\ \text{For } P_0(x) : \bar{K}_1^{12} &= {}_1\bar{K}_1^{12} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}_1^{12}, \text{ that is, } K_1^{j_1 j_2} = t_{i_1}^{j_1} t_{i_2}^{j_2} k_1^{i_1 i_2}. \\ \text{For } P_1(x) : \bar{K}_1^1 &= {}_1\bar{K}_1^1 = \bar{t}_{12}^1 \bar{k}_1^{12} / 2, \text{ that is, } K_1^1 = t_{i_1 i_2}^1 k_1^{i_1 i_2} / 2, \\ \bar{K}_2^{1-3} &= {}_2\bar{K}_2^{1-3} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_2^{1-3} + T_{1-4}^{1-3} \bar{k}_1^{12} \bar{k}_1^{34}. \\ \text{For } P_2(x) : \bar{K}_2^{12} &= {}_1\bar{K}_2^{12} + {}_2\bar{K}_2^{12} \text{ for } {}_1\bar{K}_2^{12} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}_2^{12}, \\ {}_2\bar{K}_2^{12} &= T_{1-3}^{12} \bar{k}_2^{1-3} / 2 + T_{1-4}^{12} \bar{k}_1^{12} \bar{k}_1^{34} / 2, \\ \bar{K}_3^{1-4} &= {}_3\bar{K}_3^{1-4} = (\bar{t}_1^1 \dots \bar{t}_4^4) \bar{k}_3^{1-4} + T_{1-5}^{1-4} \bar{k}_2^{1-3} \bar{k}_1^{45} + T_{1-6}^{1-4} \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56}. \\ \text{For } P_3(x) : \bar{K}_2^1 &= {}_1\bar{K}_2^1 + {}_2\bar{K}_2^1 \text{ for } {}_1\bar{K}_2^1 = \bar{t}_{12}^1 \bar{k}_2^{12} / 2, \quad {}_2\bar{K}_2^1 = \bar{t}_{1-3}^1 \bar{k}_2^{1-3} / 6 \\ &+ \bar{t}_{1-4}^1 \bar{k}_1^{12} \bar{k}_1^{34} / 8, \text{ that is, } K_2^1 = t_{i_1 i_2}^1 k_2^{i_1 i_2} / 2 + t_{i_1 i_2 i_3}^1 k_2^{i_1 i_2 i_3} / 6 + t_{i_1 i_2 i_3 i_4}^1 k_1^{i_1 i_2} k_1^{i_3 i_4} / 8, \\ \bar{K}_3^{1-3} &= \sum_{k=2}^3 k \bar{K}_3^{1-3} \text{ for } {}_2\bar{K}_3^{1-3} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_3^{1-3} + T_{1-4}^{1-3} (\bar{k}^{12} \bar{k}^{34})_3, \\ {}_3\bar{K}_3^{1-3} &= T_{1-4}^{1-3} \bar{k}_3^{1-4} / 2 + T_{1-5}^{1-3} \bar{k}_2^{1-3} \bar{k}_1^{45} + T_{1-6}^{1-3} \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56}, \\ \bar{K}_4^{1-5} &= {}_4\bar{K}_4^{1-5} = \bar{t}_1^1 \dots \bar{t}_5^5 \bar{k}_4^{1-5} + T_{1-6}^{1-5} \bar{k}_3^{1-4} \bar{k}_1^{56} + U_{1-6}^{1-5} \bar{k}_2^{1-3} \bar{k}_2^{4-6} \\ &+ T_{1-7}^{1-5} \bar{k}_2^{1-3} \bar{k}_1^{45} \bar{k}_1^{67} + T_{1-8}^{1-5} \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56} \bar{k}_1^{78}. \\ \text{For } P_4(x) : \bar{K}_3^{12} &= \sum_{k=1}^3 k \bar{K}_3^{12} \text{ for } {}_1\bar{K}_3^{12} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}_3^{12}, \end{aligned}$$

$$\begin{aligned}
 {}_2\bar{K}_3^{12} &= T_{1-3}^{12} \bar{k}_3^{1-3}/2 + T_{1-4}^{12} (\bar{k}^{12}\bar{k}^{34})_3/2, \\
 {}_3\bar{K}_3^{12} &= U_{1-4}^{12} \bar{k}_3^{1-4} + T_{1-5}^{12} \bar{k}_2^{1-3}\bar{k}_1^{45} + T_{1-6}^{12} \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}/4, \\
 \bar{K}_4^{1-4} &= \sum_{k=3}^4 {}_k\bar{K}_4^{1-4} \text{ for } {}_3\bar{K}_4^{1-4} = (\bar{t}_1^1 \cdots \bar{t}_4^4) \bar{k}_4^{1-4} + T_{1-5}^{1-4} (\bar{k}^{1-3}\bar{k}^{45})_4 \\
 &+ T_{1-6}^{1-4} (\bar{k}^{12}\bar{k}^{34}\bar{k}^{56})_4, {}_4\bar{K}_4^{1-4} = U_{1-5}^{1-4} \bar{k}_4^{1-5}/2 + U_{1-6}^{1-4} \bar{k}_3^{1-4}\bar{k}_1^{56} \\
 &+ V_{1-6}^{1-4} \bar{k}_2^{1-3}\bar{k}_2^{4-6} + T_{1-7}^{1-4} \bar{k}_2^{1-3}\bar{k}_1^{45}\bar{k}_1^{67} + T_{1-8}^{1-4} \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}\bar{k}_1^{78}, \\
 \bar{K}_5^{1-6} &= {}_5\bar{K}_5^{1-6} = \bar{t}_1^1 \cdots \bar{t}_6^6 \bar{k}_5^{1-6} + T_{1-7}^{1-6} \bar{k}_4^{1-5}\bar{k}_1^{67} + U_{1-7}^{1-6} \bar{k}_3^{1-4}\bar{k}_2^{5-7} + T_{1-8}^{1-6} \bar{k}_3^{1-4}\bar{k}_1^{56}\bar{k}_1^{78} \\
 &+ U_{1-8}^{1-6} \bar{k}_2^{1-3}\bar{k}_2^{4-6}\bar{k}_2^{78} + T_{1-9}^{1-6} \bar{k}_2^{1-3}\bar{k}_1^{45}\bar{k}_1^{67}\bar{k}_1^{89} + T_{1-10}^{1-6} \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}\bar{k}_1^{78}\bar{k}_1^{9,10}.
 \end{aligned}$$

Also, for $\bar{K}_0^1, \bar{K}_1^1, \bar{K}_2^1$ above, $E t^{j_1}(\hat{w}) = \sum_{e=0}^4 n^{-e} \bar{K}_e^1 + O(n^{-5})$ where

$$\begin{aligned}
 \bar{K}_3^1 &= \sum_{k=1}^3 {}_k\bar{K}_3^1 \text{ for } {}_1\bar{K}_3^1 = \bar{t}_{12}^1 \bar{k}_3^{12}/2, {}_2\bar{K}_3^1 = \bar{t}_{1-3}^1 \bar{k}_3^{1-3}/6 + \bar{t}_{1-4}^1 \bar{k}_1^{12}\bar{k}_2^{34}/4, \\
 {}_3\bar{K}_3^1 &= \bar{t}_{1-4}^1 \bar{k}_3^{1-4}/24 + \bar{t}_{1-5}^1 \bar{k}_2^{1-3}\bar{k}_1^{45}/12 + \bar{t}_{1-6}^1 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}/48, \\
 \bar{K}_4^1 &= \sum_{k=1}^4 {}_k\bar{K}_4^1 \text{ for } {}_1\bar{K}_4^1 = \bar{t}_{12}^1 \bar{k}_4^{12}/2, \\
 {}_2\bar{K}_4^1 &= \bar{t}_{1-3}^1 \bar{k}_4^{1-3}/6 + \bar{t}_{1-4}^1 (2\bar{k}_1^{12}\bar{k}_3^{34} + \bar{k}_2^{12}\bar{k}_2^{34})/8, \\
 {}_3\bar{K}_4^1 &= \bar{t}_{1-4}^1 \bar{k}_4^{1-4}/24 + \bar{t}_{1-5}^1 (\bar{k}_2^{1-3}\bar{k}_2^{45} + \bar{k}_3^{1-3}\bar{k}_1^{45})/12 + \bar{t}_{1-6}^1 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_2^{56}/16, \\
 {}_4\bar{K}_4^1 &= \bar{t}_{1-5}^1 \bar{k}_4^{1-5}/120 + \bar{t}_{1-6}^1 (\bar{k}_3^{1-4}\bar{k}_1^{56}/48 + \bar{k}_2^{1-3}\bar{k}_2^{4-6}/72) + \bar{t}_{1-7}^1 \bar{k}_2^{1-3}\bar{k}_1^{45}\bar{k}_1^{67}/48 \\
 &+ \bar{t}_{1-8}^1 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}\bar{k}_1^{78}/384.
 \end{aligned}$$

Proof. Substituting (1) into ${}_k\bar{K}^{1-r}$ of Theorem 1 gives ${}_k\bar{K}^{1-r} = \sum_{e=k}^\infty {}_k\bar{K}_e^{1-r} n^{-e}$ say. So by (10), (14) and (16) hold. ${}_k\bar{K}_e^{1-r} = {}_kK_e^{j_1 \cdots j_r}$ is ${}_k\tilde{V}_e^{j_1 \cdots j_r}$ of [2]. □

Note 2. (11) made explicit the 3 terms needed in T_{1-4}^{1-3} for $P_1(x)$ of Theorem 2. Similarly $P_2(x)$ needs the 12 terms

$$\begin{aligned}
 T_{1-5}^{1-4} &= \sum_{12} (1234) = (1234) + (1243) + (2413) + (2431) + (3124) + (3142) \\
 &+ (3241) + (3412) + (4123) + (4132) + (4231) + (4321)
 \end{aligned}$$

where $(abcd) = \bar{t}_{14}^a \bar{t}_2^b \bar{t}_3^c \bar{t}_5^d$. It also needs the 4 + 12 terms $T_{1-6}^{1-4} = A + B$ where

$$\begin{aligned}
 A &= \sum_{4} (1234) = (1234) + (2134) + (3124) + (4123) \text{ for } (abcd) = \bar{t}_{135}^a \bar{t}_2^b \bar{t}_4^c \bar{t}_6^d, \\
 B &= \sum_{12} (1234) = (1234) + (1423) + (1432) + (1324) + (2134) + (2314) + (2413) \\
 &+ (3124) + (3214) + (3412) + (4213) + (4312) \text{ for } (abcd) = \bar{t}_{13}^a \bar{t}_{25}^b \bar{t}_4^c \bar{t}_6^d.
 \end{aligned}$$

4. Cumulant Coefficients for $t(\hat{w})$ when $E \hat{w} \neq w$

We proceed by removing the assumption of \hat{w} being unbiased. We utilise \bar{K}_e^{1-r} from Theorem 2, and the shorthand $\bar{f}_{.m} = \partial_{i_m} f$ where again $\partial_i = \partial/\partial w^i$. A significant distinction arises compared to Theorem 2: in that case, \bar{k}_e^{1-r} was treated as an algebraic expression. However, now we must consider each of them as a function of w . Thus, we assume that the distribution of \hat{w} is determined by w . This assumption is necessary to derive higher order confidence intervals for $t(w)$ when $q = 1$: see [20]. It is demonstrated that for Y_{nt} from Equation (12), $P_2(x), p_2(x)$ require the first derivatives $\bar{k}_{1,i}^{12} = \partial_i \bar{k}_1^{12}$ where $\partial_i = \partial/\partial w^i$, $P_3(x), p_3(x)$ need the 1st derivatives $\bar{k}_{2,4}^{1-3}$, and so on. The derivatives of \bar{K}_e^{1-r} are computed using Leibniz’s rule for the derivatives of a product. For example,

$$\begin{aligned} \bar{K}_{1,3}^{12} &= (\bar{t}_1^1 \bar{t}_2^2 \bar{k}_1^{12})_{.3} = \left(\sum_{12} \bar{t}_1^1 \bar{t}_{2,3}^2\right) \bar{k}_1^{12} + \bar{t}_1^1 \bar{t}_2^2 \bar{k}_{1,3}^{12} \text{ for } \sum_{12} \bar{t}_1^1 \bar{t}_{2,3}^2 = \bar{t}_{1,3}^1 \bar{t}_2^2 + \bar{t}_1^1 \bar{t}_{2,3}^2, \\ \bar{K}_{1,3}^1 &= (\bar{t}_{1,3}^1 \bar{k}_1^{12})_{.3} / 2 = \bar{t}_{1-3}^1 \bar{k}_1^{12} / 2 + \bar{t}_{1,2}^1 \bar{k}_{1,3}^{12} / 2, \\ \bar{K}_{1,34}^{12} &= \sum_{12} [(\bar{t}_{1,4}^1 \bar{t}_{2,3}^2 + \bar{t}_1^1 \bar{t}_{2-4}^2) \bar{k}_1^{12} + \bar{t}_1^1 \bar{t}_{2,3}^2 \bar{k}_{1,4}^{12} + \bar{t}_{1,4}^1 \bar{t}_2^2 \bar{k}_{1,3}^{12}] + \bar{t}_1^1 \bar{t}_2^2 \bar{k}_{1,34}^{12}, \\ (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_2^{1-3})_{.4} &= (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3)_{.4} \bar{k}_2^{1-3} + \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_{2,4}^{1-3}, (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3)_{.4} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_{3,4}^3 + \bar{t}_1^1 \bar{t}_{2,4}^2 \bar{t}_3^3 + \bar{t}_{1,4}^1 \bar{t}_2^2 \bar{t}_3^3, \\ (T_{1-4}^{1-3} \bar{k}_1^{12} \bar{k}_1^{34})_{.5} &= T_{1-4,5}^{1-3} \bar{k}_1^{12} \bar{k}_1^{34} + T_{1-4}^{1-3} (\bar{k}_1^{12} \bar{k}_1^{34})_{.5}, \\ T_{1-4,5}^{1-3} &= \sum_3 (\bar{t}_{1,3,5}^1 \bar{t}_2^2 \bar{t}_4^3 + \bar{t}_{1,3}^1 \bar{t}_{2,5}^2 \bar{t}_4^3 + \bar{t}_{1,3}^1 \bar{t}_2^2 \bar{t}_{4,5}^3), (\bar{k}_1^{12} \bar{k}_1^{34})_{.5} = \bar{k}_{1,5}^{12} \bar{k}_1^{34} + \bar{k}_1^{12} \bar{k}_{1,5}^{34}. \end{aligned}$$

Theorem 3. Let \hat{w} in R^p be a biased standard estimate of w satisfying (1) where \bar{k}_e^{1-r} depend on w . Then $\hat{t} = t(\hat{w})$ in R^q is a standard estimate of $t(w)$:

$$\kappa(\hat{t}^{j_1}, \dots, \hat{t}^{j_r}) = \sum_{e=r-1}^{\infty} n^{-e} \bar{a}_e^{1-r} \text{ for } r \geq 1, 1 \leq j_1, \dots, j_r \leq q, \tag{17}$$

$$\text{where } \bar{a}_e^{1-r} = \bar{K}_e^{1-r} + \bar{D}_e^{1-r}, \bar{D}_{r-1}^{1-r} = 0, \tag{18}$$

for \bar{K}_e^{1-r} of Theorem 2, and the other $\bar{D}_e^{1-r} = D_e^{j_1 \dots j_r}$ needed for $P_r(x), p_r(x)$ of (4) for Y_{nt} of (12) are as follows.

$$\begin{aligned} \text{For } P_0(x) : \bar{D}_1^{12} &= 0 \Rightarrow \bar{a}_1^{12} = \bar{K}_1^{12} = K_1^{j_1 j_2} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}_1^{12}. \\ \text{For } P_1(x) : \bar{D}_1^1 &= \bar{t}_1^1 \bar{k}_1^1 \Rightarrow \bar{a}_1^1 = \bar{K}_1^1 + \bar{D}_1^1 = \bar{t}_1^1 \bar{k}_1^1 + \bar{t}_{1,2}^1 \bar{k}_1^{12} / 2, \\ \text{For } P_2(x) : \bar{D}_2^{12} &= \bar{K}_{1,3}^{12} \bar{k}_1^3 = [(\bar{t}_{1,3}^1 \bar{t}_2^2 + \bar{t}_1^1 \bar{t}_{2,3}^2) \bar{k}_1^{12} + \bar{t}_1^1 \bar{t}_2^2 \bar{k}_{1,3}^{12}] \bar{k}_1^3 \\ &\Rightarrow \bar{a}_2^{12} = \bar{t}_1^1 \bar{t}_2^2 \bar{k}_2^{12} + T_{1-3}^{12} \bar{k}_2^{1-3} / 2 + T_{1-4}^{12} \bar{k}_1^{12} \bar{k}_1^{34} / 2 \\ &+ [(\bar{t}_{1,3}^1 \bar{t}_2^2 + \bar{t}_1^1 \bar{t}_{2,3}^2) \bar{k}_1^{12} + \bar{t}_1^1 \bar{t}_2^2 \bar{k}_{1,3}^{12}] \bar{k}_1^3. \\ \text{For } P_3(x) : \bar{D}_2^1 &= \bar{K}_{1,1}^1 + \bar{K}_{0,2}^1, \bar{K}_{1,1}^1 = \bar{K}_{1,3}^1 \bar{k}_1^3, \bar{K}_{0,2}^1 = \bar{t}_1^1 \bar{k}_2^1 + \bar{t}_{1,2}^1 \bar{k}_1^1 \bar{k}_1^2 / 2 \Rightarrow \\ \bar{a}_2^1 &= \bar{t}_1^1 \bar{k}_2^1 + \bar{t}_{1,2}^1 (\bar{k}_2^{12} + \bar{k}_1^1 \bar{k}_2^1 + \bar{k}_{1,3}^{12} \bar{k}_1^3) / 2 + \bar{t}_{1-3}^1 (\bar{k}_2^{1-3} / 6 + \bar{k}_1^1 \bar{k}_2^3 / 2) + \bar{t}_{1-4}^1 \bar{k}_1^1 \bar{k}_1^{12} \bar{k}_1^{34} / 8, \\ \bar{D}_3^{1-3} &= \bar{K}_{2,4}^{1-3} \bar{k}_1^4 = (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_2^{1-3})_{.4} \bar{k}_1^4 + (T_{1-4}^{1-3} \bar{k}_1^{12} \bar{k}_1^{34})_{.5} \bar{k}_1^5. \\ \text{For } P_4(x) : \bar{D}_3^{12} &= \bar{K}_{2,1}^{12} + \bar{K}_{1,2}^{12}, \bar{K}_{2,1}^{12} = \bar{K}_{2,3}^{12} \bar{k}_1^3, \bar{K}_{1,2}^{12} = \bar{K}_{1,3,4}^{12} \bar{k}_2^3 / 2 + \bar{K}_{1,34}^{12} \bar{k}_1^3 \bar{k}_1^4, \\ \bar{D}_4^{1-4} &= \bar{K}_{3,5}^{1-4} \bar{k}_1^5, \bar{D}_5^{1-6} = 0. \end{aligned}$$

For $E^{t^h}(\hat{w})$ to $O(n^{-5})$ we also need $\bar{D}_j = \bar{D}_j^1, j = 3, 4$, given by

$$\begin{aligned} \bar{D}_3 &= \bar{K}_{2,1} + \bar{K}_{1,2} + \bar{K}_{0,3}, \bar{K}_{2,1} = \bar{K}_{2,1} \bar{k}_1^1, \\ \bar{K}_{2,1} &= (\bar{t}_{1-3} \bar{k}_2^{23} + \bar{t}_{2,3} \bar{k}_{2,1}^{23}) / 2 + (\bar{t}_{1-4} \bar{k}_2^{2-4} + \bar{t}_{2-4} \bar{k}_{2,1}^{2-4}) / 6 + \bar{t}_{1-5} \bar{k}_1^{23} \bar{k}_1^{45} / 8 + \bar{t}_{2-5} \bar{k}_1^{23} \bar{k}_{1,1}^{45} / 4, \\ \bar{K}_{1,2} &= \bar{K}_{1,1} \bar{k}_2^1 + \bar{K}_{1,1,2} \bar{k}_1^1 \bar{k}_2^1 / 2, \\ 2\bar{K}_{1,1} &= \bar{t}_{1-3} \bar{k}_1^{23} + \bar{t}_{2,3} \bar{k}_{1,1}^{23}, 2\bar{K}_{1,1,2} = \bar{t}_{1-4} \bar{k}_1^{34} + \sum_{12} \bar{t}_{2-4} \bar{k}_{1,1}^{34} + \bar{t}_{3,4} \bar{k}_{1,1,2}^{34}, \\ \bar{K}_{0,3} &= \bar{t}_1 \bar{k}_3^1 + \bar{t}_{1,2} \bar{k}_1^1 \bar{k}_2^1 + \bar{t}_{1-3} \bar{k}_1^1 \bar{k}_1^2 \bar{k}_1^3 / 6, \\ \bar{D}_4 &= \bar{K}_{3,1} + \bar{K}_{2,2} + \bar{K}_{1,3} + \bar{K}_{0,4}, \bar{K}_{3,1} = \bar{K}_{3,1} \bar{k}_1^1, \\ \bar{K}_{3,1} &= \bar{t}_{1-3} \bar{k}_3^{23} / 2 + \bar{t}_{2,3} \bar{k}_{3,1}^{23} / 2 + \bar{t}_{1-4} \bar{k}_3^{2-4} / 6 + \bar{t}_{2-4} \bar{k}_{3,1}^{2-4} / 6 + \bar{t}_{1-5} \bar{k}_1^{23} \bar{k}_2^{45} / 4 \\ &+ \bar{t}_{2-5} \bar{k}_1^{23} \bar{k}_{2,1}^{45} / 2 + (\bar{t}_{1-5} \bar{k}_3^{2-5} + \bar{t}_{2-5} \bar{k}_{3,1}^{2-5}) / 24 + (\bar{t}_{1-6} \bar{k}_2^{2-4} \bar{k}_1^{56} + \bar{t}_{2-6} \bar{k}_{2,1}^{2-4} \bar{k}_1^{56} \\ &+ \bar{t}_{2-6} \bar{k}_2^{2-4} \bar{k}_{1,1}^{56}) / 12 + \bar{k}_1^{23} \bar{k}_1^{45} (\bar{t}_{1-7} \bar{k}_1^{67} / 48 + \bar{t}_{2-7} \bar{k}_{1,1}^{67} / 16), \end{aligned}$$

$$\begin{aligned} \bar{K}_{2,2} &= \bar{K}_{2,1}\bar{k}_2^1 + \bar{K}_{2,12}\bar{k}_1^1\bar{k}_1^2/2, \\ \bar{K}_{2,1} &= \bar{t}_{1-3}\bar{k}_2^{23}/2 + \bar{t}_{23}\bar{k}_{2,1}^{23}/2 + \bar{t}_{1-4}\bar{k}_2^{2-4}/6 + \bar{t}_{2-4}\bar{k}_{2,1}^{2-4}/6 + \bar{t}_{1-5}\bar{k}_1^{23}\bar{k}_1^{45}/8 + \bar{t}_{2-5}\bar{k}_1^{23}\bar{k}_{1,1}^{45}/4, \\ 2\bar{K}_{2,12} &= \bar{t}_{1-4}\bar{k}_2^{34} + \sum_{12}^2 \bar{t}_{2-4}\bar{k}_{2,1}^{34} + \bar{t}_{34}\bar{k}_{2,12}^{34} + (\bar{t}_{1-5}\bar{k}_2^{3-5} + \sum_{12}^2 \bar{t}_{13-5}\bar{k}_{2,2}^{3-5} + \bar{t}_{3-5}\bar{k}_{2,12}^{3-5})/3 \\ &+ \bar{t}_{1-6}\bar{k}_1^{34}\bar{k}_1^{56}/4 + \sum_{12}^2 \bar{t}_{13-6}\bar{k}_1^{34}\bar{k}_{1,2}^{56}/2 + \bar{t}_{3-6}(\bar{k}_{1,2}^{34}\bar{k}_{1,1}^{56} + \bar{k}_1^{34}\bar{k}_{1,12}^{56})/2, \\ \bar{K}_{1,3} &= \bar{K}_{1,1}\bar{k}_3^1 + \bar{K}_{1,12}\bar{k}_1^1\bar{k}_2^2 + \bar{K}_{1,123}\bar{k}_1^1\bar{k}_1^2\bar{k}_1^3/6, \quad 2\bar{K}_{1,1} = \bar{t}_{1-3}\bar{k}_1^{23} + \bar{t}_{23}\bar{k}_{1,1}^{23}, \quad 2\bar{K}_{1,12} = \bar{t}_{1-4}\bar{k}_1^{34} + \sum_{12}^2 \bar{t}_{134}\bar{k}_{1,2}^{34} + \bar{t}_{34}\bar{k}_{1,12}^{34}, \\ 2\bar{K}_{1,123} &= \bar{t}_{1-5}\bar{k}_1^{45} + \sum_{1-3}^3 (\bar{t}_{1345}\bar{k}_{1,2}^{45} + \bar{t}_{3-5}\bar{k}_{1,12}^{45}) + \bar{t}_{45}\bar{k}_{1,1-3}^{45}, \\ \bar{K}_{0,4} &= \bar{t}_1\bar{k}_4^1 + \bar{t}_{12}(\bar{k}_1^1\bar{k}_3^2 + \bar{k}_2^1\bar{k}_2^2/2) + \bar{t}_{1-3}\bar{k}_1^1\bar{k}_1^2\bar{k}_2^3/2 + \bar{t}_{1-4}\bar{k}_1^1\bar{k}_1^2\bar{k}_1^3\bar{k}_1^4/24. \end{aligned}$$

Proof. $\bar{K}^{1-r}(w) = \bar{K}^{1-r}$ and $\bar{K}_e^{1-r}(w) = \bar{K}_e^{1-r}$ are functions of w . By (14)

$$\bar{K}^{1-r}(w_n) = \sum_{e=r-1}^{\infty} n^{-e} \bar{K}_e^{1-r}(w_n) \text{ for } w_n = E \hat{w} = w + d_n,$$

where by (1), d_n has i_1 th component $\bar{d}_n^{i_1} = d_n^{i_1} = \sum_{e=1}^{\infty} n^{-e} \bar{k}_e^1$. Consider the Taylor series expansion

$$\bar{K}_k^{1-r}(w + d_n) = \bar{K}_k^{1-r} + \bar{K}_{k,1}^{1-r} \bar{d}_n^1 + \bar{K}_{k,12}^{1-r} \bar{d}_n^1 \bar{d}_n^2 / 2! + \dots = \sum_{e=0}^{\infty} \bar{K}_{k,e}^{1-r} n^{-e} \text{ say.}$$

Substituting into (14) gives (17) with

$$\bar{a}_c^{1-r} = \sum_{k+e=c} \bar{K}_{k,e}^{1-r} = \sum_{e=0}^{c-r+1} \bar{K}_{c-e,e}^{1-r}. \tag{19}$$

Also $\bar{K}_{k,0}^{1-r} = \bar{K}_k^{1-r}$ so that (18) holds with

$$\begin{aligned} \bar{D}_c^{1-r} &= \sum_{e=1}^{c-r+1} \bar{K}_{c-e,e}^{1-r} : \\ \bar{D}_r^{1-r} &= \bar{K}_{r-1,1}^{1-r}, \quad \bar{D}_{r+1}^{1-r} = \sum_{e=1}^2 \bar{K}_{r+1-e,e}^{1-r} \dots \end{aligned} \tag{20}$$

□

An alternative proof can be obtained using Section 6. This corrects $C_e = \bar{a}_e^1$ given in Appendix B of [21]. Ref. [2] uses $K_k^{j_1 \dots j_r} = \bar{K}_k^{1-r}$ for \bar{a}_k^{1-r} but the expression for K_2^{ab} on p67, lines 2–3 omitted the term $A_i^a A_j^b k_{1,k}^{ij} k_1^k$. That is, the last term in \bar{a}_2^{12} of Theorem 3 was omitted. Similarly the results on p67 for $r = 3, 4$ are only true when the \hat{w} is unbiased or the cumulant coefficients of \hat{w} do not depend on w , as they omit the derivatives of \bar{k}_e^{1-r} . The examples given there are not affected as \hat{w} is unbiased. Nor are the nonparametric examples of [22] and [23] affected, as the empirical distribution is an unbiased estimate of a distribution. Likewise \hat{w} is unbiased for the examples of [20]. M-estimates are biased but the results of [16] are not affected as only $K_1^{j_1 j_2}, K_1^{j_1}, K_2^{j_1 j_2 j_3}$ are given. No changes are needed for [3,4,17,24]. Applications to non-parametric and parametric confidence intervals were given in [22] and [20,23] and to ellipsoidal confidence regions and power in [4] and [25]. For nonparametric problems, $F(x)$ and its empirical distribution $F_n(x)$ play the role of w and \hat{w} ; since it is unbiased, no corrections are needed. For $q = 1$, $a_{ri} = \bar{a}_i^{1-r}$ were given

for parametric and non-parametric problems in [22] and [2,23] and expressions for the classic Edgeworth expansion of Y_{nt} in terms of a_{ri} were given in [14]. For $q \geq 1$, \bar{a}_i^{1-r} for parametric problems were given in [2], and can be obtained easily from a_{ri} given when $q = 1$ for 1-sample and multi-sample non-parametric problems in [22] and [23] and for semi-parametric problems in [16,24]. All these results can be extended to samples with independent non-identically distributed residuals, as done in [26] Section 6 and [17]. The extension to *matrix* \hat{w} just needs a slight change in notation. For example, in [17], \hat{w} can be viewed as a function of the mean of n independent *complex random matrices*, although n is actually the number of transmitters or receivers. Extensions to dependent random variables are also possible: see [27].

5. Cumulant Coefficients for Univariate $t(\hat{w})$

Now suppose that $q = 1$. Let ${}_k K_{re}$ be the coefficient of n^{-e} in ${}_k K^{1-r}$. We write \bar{K}_e^{1-r} as K_{re} . For $E \hat{w} = w$, (14), (16) and (20) become

$$K^r = \kappa_r(\hat{t}) = \sum_{e=r-1}^{\infty} n^{-e} K_{re}, r \geq 1; K_{re} = \sum_{k=r-1}^e {}_k K_{re} : \tag{21}$$

$$K_{r,r-1} = {}_{r-1} K_{r,r-1}, K_{rr} = \sum_{k=r-1}^r {}_k K_{rr}, K_{r,r+1} = \sum_{k=r-1}^{r+1} {}_k K_{r,r+1}, \dots$$

For $E \hat{w} \neq w$, (17)–(19) become

$$K^r = \kappa_r(\hat{t}) = \sum_{e=r-1}^{\infty} n^{-e} a_{re}, r \geq 1; a_{re} = K_{re} + D_{re}, D_{rc} = \sum_{e=1}^{c-r+1} K_{r,c-e,e} :$$

$$D_{r,r-1} = 0, D_{rr} = K_{r,r-1,1}, D_{r,r+1} = \sum_{e=1}^2 K_{r,r+1-e,e}, \dots$$

Here, we give the cumulant coefficients K_{re} needed for the Edgeworth expansion of Y_{nt} of (12) for $P_r(x)$, $r \leq 4$. We do this when $E \hat{w} = w$ in Corollary 1 and when $E \hat{w} \neq w$ in Corollaries 3 and 4. To show more clearly the expressions we need in molecular form, we introduce the following *ions*, (expressions with unpaired suffixes),

$$s^{i_1} = \bar{s}^1 = \bar{k}_1^{12} \bar{t}_2, \bar{u}_1 = \bar{t}_{12} \bar{s}^2 = \bar{t}_{12} \bar{k}_1^{23} \bar{t}_3, \bar{X}^{34} = \bar{k}_1^{31} \bar{t}_{12} \bar{k}_1^{24}, \bar{z}_{12} = \bar{t}_{1-3} \bar{s}^3,$$

$$\bar{v}^1 = \bar{k}_1^{12} \bar{u}_2 = \bar{X}^{14} \bar{t}_4, \bar{x}_1 = \bar{t}_{12} \bar{v}^2, \bar{S}^1 = \bar{k}_2^{12} \bar{t}_2, \bar{y}^1 = \bar{k}_2^{1-3} \bar{t}_2 \bar{t}_3, \bar{Y}_1 = \bar{t}_{12} \bar{y}^2. \tag{22}$$

where a suffix does not have a match then summation does not occur. For example, the RHS of $\bar{s}^1 = \bar{k}_1^{12} \bar{t}_2$ sums over i_2 but not i_1 . Let $v, c_{01}, c_{02}, c_{21}, c_{22}, c_{23}, c_{11}, \dots, c_{1,10}, c_{31}, \dots, c_{3,11}$ be the 27 functions of ω given on p4234–4235 of [20], labelled there as $I_2(\binom{2}{0}), I_1(\binom{1}{0}), \dots, I_{301}(\binom{222}{000})$. By Corollaries 1 and 3 below, those needed for $P_r(x)$, $r \leq 2$, of (4), that is, for the Edgeworth expansion of Y_{nt} of (12) to $O(n^{-3/2})$, are the following molecules.

For $P_0(x)$: $v = K_{21} = \bar{t}_1 \bar{k}_1^{12} \bar{t}_2$.

For $P_1(x)$, K_{11} : $c_{02} = \bar{t}_{12} \bar{k}_1^{12}$; for D_{11} : $c_{01} = \bar{t}_1 \bar{k}_1^1$;

for K_{32} : $c_{21} = \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{k}_2^{1-3} = \bar{t}_1 \bar{y}^1, c_{23} = \bar{s}^1 \bar{t}_{12} \bar{s}^2 = \bar{s}^1 \bar{u}_1$.

For $P_2(x)$, K_{22} : $c_{11} = \bar{t}_1 \bar{k}_2^{12} \bar{t}_2 = \bar{t}_1 \bar{S}^1, c_{15} = \bar{t}_1 \bar{k}_2^{1-3} \bar{t}_{23}, c_{19} = \bar{t}_{12} \bar{X}^{12}$,

$$c_{1,10} = \bar{s}^1 \bar{t}_{1-3} \bar{k}_1^{23} = \bar{z}_{23} \bar{k}_1^{23};$$

for D_{22} : $c_{12} = \bar{k}_1^1 \bar{k}_{1,1}^{23} \bar{t}_2 \bar{t}_3, c_{16} = \bar{k}_1^1 \bar{u}_1 = \bar{k}_1^1 \bar{t}_{12} \bar{k}_1^{23} \bar{t}_3$;

for K_{43} : $c_{31} = \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4 \bar{k}_3^{1-4}, c_{36} = \bar{y}^3 \bar{u}_3, c_{3,10} = \bar{u}_1 \bar{k}_1^{12} \bar{u}_2, c_{3,11} = \bar{s}^1 \bar{s}^2 \bar{s}^3 \bar{t}_{1-3}$.

Each molecule can be written as a shape. For example, c_{19} is a rectangle. We now give the molecules L_j, L_{ij} needed for the Edgeworth expansion to $O(n^{-5/2})$, that is, for $P_r(x)$ for $r = 3, 4$. Note that $P_r(x)$ needs the derivatives of $t(w)$ up to order $r + 1$.

$$\begin{aligned}
 &\text{For } P_3(x), K_{12} : L_1 = \bar{t}_{12}\bar{k}_2^{12}, L_2 = \bar{t}_{1-3}\bar{k}_2^{1-3}, L_3 = \bar{t}_{1-4}\bar{k}_1^{12}\bar{k}_1^{34}, \\
 &\text{for } K_{33} : L_4 = \bar{t}_1\bar{t}_2\bar{t}_3\bar{k}_3^{1-3}, L_5 = \bar{u}_1\bar{S}^1, L_6 = \bar{t}_{13}\bar{t}_2\bar{t}_4\bar{k}_3^{1-4}, L_{71} = \bar{z}_{12}\bar{k}_2^{1-3}\bar{t}_3, \\
 &L_{72} = \bar{y}^1\bar{t}_{145}\bar{k}_1^{45}, L_{73} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{u}_3, L_{74} = \bar{t}_{14}\bar{k}_1^{45}\bar{t}_{52}\bar{k}_2^{1-3}\bar{t}_3, \\
 &L_{81} = \bar{k}_1^{12}\bar{t}_{1-4}\bar{s}^3\bar{s}^4, L_{82} = \bar{k}_1^{12}\bar{t}_{1-3}\bar{v}^3, L_{83} = \bar{X}^{34}\bar{z}_{34}, \\
 &L_{84} = \bar{X}^{14}\bar{t}_{45}\bar{k}_1^{56}\bar{t}_{61}, \text{ a hexagon,} \\
 &\text{for } K_{54} : L_9 = \bar{t}_1 \cdots \bar{t}_5\bar{k}_4^{1-5}, L_{10} = \bar{u}_1\bar{t}_2\bar{t}_3\bar{t}_4\bar{k}_3^{1-4}, L_{11} = \bar{y}^1\bar{Y}_1 = \bar{y}^1\bar{t}_{12}\bar{y}^2, \\
 &L_{121} = \bar{y}^1\bar{t}_{1-3}\bar{s}^2\bar{s}^3, L_{122} = \bar{t}_1\bar{k}_2^{1-3}\bar{u}_2\bar{u}_3, L_{123} = \bar{Y}_2\bar{v}^2, \\
 &L_{131} = \bar{s}^1 \cdots \bar{s}^4\bar{t}_{1-4}, L_{132} = \bar{s}^1\bar{s}^2\bar{t}_{1-3}\bar{v}^3, L_{133} = \bar{v}^1\bar{t}_{12}\bar{v}^2 = \bar{v}^1\bar{x}_1. \\
 &\text{For } P_4(x), K_{23} : L_{14} = \bar{t}_1\bar{t}_2\bar{k}_3^{12}, L_{15} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{t}_3, L_{161} = \bar{S}^1\bar{t}_{1-3}\bar{k}_1^{23}, \\
 &L_{162} = \bar{z}_{12}\bar{k}_2^{12}, L_{171} = \bar{X}^{24}\bar{t}_{24}. L_{181} = \bar{t}_{1-3}\bar{k}_3^{1-4}\bar{t}_4, L_{182} = \bar{t}_{12}\bar{k}_3^{1-4}\bar{t}_{34}, L_{191} = \bar{k}_2^{1-3}\bar{t}_{1-4}\bar{s}^4, \\
 &L_{192} = \bar{k}_1^{12}\bar{t}_{1-4}\bar{k}_2^{3-5}\bar{t}_5, L_{193} = \bar{t}_{1-3}\bar{k}_2^{2-4}\bar{t}_{45}\bar{k}_1^{51}, L_{194} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{t}_{3-5}\bar{k}_1^{45}, \\
 &L_{201} = \bar{k}_1^{12}\bar{k}_1^{34}\bar{t}_{1-5}\bar{s}^5, L_{202} = \bar{k}_1^{12}\bar{t}_{1-4}\bar{X}^{34}, L_{203} = \bar{k}_1^{12}\bar{t}_{1-3}\bar{k}_1^{34}\bar{t}_{4-6}\bar{k}_1^{56}, \\
 &L_{204} = \bar{t}_{135}(\bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56})\bar{t}_{246}; \\
 &\text{for } K_{44} : L_{21} = \bar{t}_1 \cdots \bar{t}_4\bar{k}_4^{1-4}, L_{221} = \bar{Y}_2\bar{k}_2^{23}\bar{t}_3, L_{222} = \bar{u}_1\bar{t}_2\bar{t}_3\bar{k}_3^{1-3}, \\
 &L_{231} = \bar{s}^1\bar{z}_{12}\bar{S}^2. L_{241} = \bar{x}_2\bar{S}^2, L_{242} = \bar{u}_1\bar{k}_2^{12}\bar{u}_2. \\
 &L_{25} = \bar{t}_{12}\bar{k}_4^{1-5}\bar{t}_3\bar{t}_4\bar{t}_5, L_{261} = \bar{t}_1\bar{t}_2\bar{k}_3^{1-4}\bar{t}_{3-5}\bar{s}^5, L_{262} = \bar{t}_1\bar{t}_2\bar{t}_3\bar{k}_3^{1-4}\bar{t}_{4-6}\bar{k}_1^{56}, \\
 &L_{263} = \bar{t}_{12}\bar{k}_3^{1-4}\bar{t}_3\bar{u}_4, L_{264} = \bar{t}_1\bar{t}_2\bar{k}_3^{1-4}(\bar{t}_{35}\bar{t}_{46})\bar{k}_1^{56}, \\
 &L_{271} = \bar{t}_1\bar{k}_2^{1-3}\bar{t}_{2-4}\bar{y}^4, L_{272} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{Y}_3, L_{273} = \bar{t}_1\bar{k}_2^{1-3}(\bar{t}_{24}\bar{t}_{35})\bar{k}_3^{4-6}\bar{t}_6, \\
 &L_{281} = \bar{t}_1\bar{k}_2^{1-3}\bar{t}_{2-5}\bar{s}^4\bar{s}^5, L_{282} = \bar{s}^1\bar{k}_1^{23}\bar{t}_{1-4}\bar{y}^4, L_{283} = \bar{u}_1\bar{k}_2^{1-3}\bar{z}_{23}, \\
 &L_{284} = \bar{t}_1\bar{k}_2^{1-3}\bar{t}_{2-4}\bar{v}^4, L_{285} = \bar{Y}_2\bar{k}_1^{23}\bar{t}_{3-5}\bar{k}_1^{45}, L_{286} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{z}_{34}\bar{s}^4, \\
 &L_{287} = \bar{t}_1\bar{k}_2^{1-3}\bar{t}_{24}\bar{k}_1^{45}\bar{z}_{53}, L_{288} = \bar{y}^1\bar{t}_{1-3}\bar{X}^{23}, \\
 &L_{289} = \bar{t}_{12}\bar{k}_2^{1-3}\bar{x}_3, L_{2810} = \bar{u}_1\bar{k}_2^{1-3}(\bar{t}_{24}\bar{t}_{35})\bar{k}_1^{45}, \\
 &L_{2811} = \bar{t}_1\bar{k}_2^{1-3}(\bar{t}_{24}\bar{k}_1^{45}\bar{t}_{36}\bar{k}_1^{67})\bar{t}_{57}, L_{291} = \bar{k}_1^{12}\bar{t}_{1-5}\bar{s}^3\bar{s}^4\bar{s}^5, L_{292} = \bar{k}_1^{12}\bar{t}_{1-4}\bar{v}^3\bar{s}^4, \\
 &L_{293} = \bar{X}^{34}\bar{t}_{3-6}\bar{s}^5\bar{s}^6, L_{294} = \bar{k}_1^{12}\bar{t}_{1-3}\bar{k}_1^{34}\bar{t}_{4-6}\bar{s}^5\bar{s}^6, L_{295} = \bar{k}_1^{12}(\bar{z}_{13}\bar{z}_{24})\bar{k}_1^{34}, \\
 &L_{296} = \bar{k}_1^{12}\bar{t}_{1-3}\bar{k}_1^{34}\bar{x}_4, L_{297} = \bar{k}_1^{12}\bar{t}_{135}\bar{v}^5\bar{t}_{24}\bar{k}_1^{34}, \\
 &L_{298} = \bar{X}^{14}\bar{t}_{45}\bar{k}_1^{56}\bar{z}_{61}, L_{299} = \bar{X}^{14}\bar{t}_{45}\bar{X}^{58}; \\
 &\text{for } K_{65} : L_{30} = \bar{t}_1 \cdots \bar{t}_6\bar{k}_5^{1-6}, L_{31} = \bar{u}_1\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_5\bar{k}_4^{1-5}, L_{32} = \bar{t}_1\bar{t}_2\bar{t}_3\bar{k}_4^{1-5}\bar{t}_{45}, \\
 &L_{331} = \bar{t}_1\bar{t}_2\bar{t}_3\bar{k}_3^{1-4}\bar{z}_{45}\bar{s}^5, L_{332} = \bar{u}_1\bar{u}_2\bar{k}_3^{1-4}\bar{t}_3\bar{t}_4, L_{333} = \bar{t}_1\bar{t}_2\bar{t}_3\bar{k}_3^{1-4}\bar{x}_4 \\
 &L_{341} = \bar{t}_{1-3}\bar{y}^1\bar{s}^2\bar{y}^3, L_{342} = \bar{Y}_2\bar{k}_2^{2-4}\bar{t}_3\bar{u}_4, L_{343} = \bar{Y}_1\bar{k}_1^{12}\bar{Y}_2, L_{351} = \bar{y}^3\bar{t}_{3-6}\bar{s}^4\bar{s}^5\bar{s}^6, \\
 &L_{352} = \bar{t}_1\bar{u}_2\bar{k}_2^{1-3}\bar{z}_{34}\bar{s}^4, L_{353} = \bar{y}^1\bar{t}_{12}\bar{v}^2, L_{354} = \bar{Y}_4\bar{k}_1^{45}\bar{t}_{5-7}\bar{s}^6\bar{s}^7, \\
 &L_{355} = \bar{u}_1\bar{u}_2\bar{u}_3\bar{k}_2^{1-3}, L_{356} = \bar{t}_1\bar{u}_2\bar{k}_2^{1-3}\bar{x}_3, L_{357} = \bar{t}_{3-5}\bar{y}^3\bar{v}^4\bar{s}^5. \\
 &L_{361} = \bar{s}^1 \cdots \bar{s}^5\bar{t}_{1-5}, L_{362} = \bar{s}^1\bar{s}^2\bar{s}^3\bar{t}_{1-4}\bar{v}^4, L_{363} = \bar{s}^1\bar{z}_{13}\bar{k}_1^{34}\bar{t}_{4-6}\bar{s}^5\bar{s}^6, \\
 &L_{364} = \bar{v}^1\bar{z}_{12}\bar{v}^2, L_{365} = \bar{s}^1\bar{z}_{13}\bar{k}_1^{34}\bar{x}_4, L_{366} = \bar{x}_1\bar{k}_1^{12}\bar{x}_2.
 \end{aligned}$$

These c_{rs} and L_j do not use derivatives of \bar{k}_e^{1-r} , the cumulant coefficients of \hat{w} .

Corollary 1. Suppose that \hat{w} is an unbiased standard estimate of w in R^p with respect to n , and that $q = 1$. Then the cumulants of $\hat{t} = t(\hat{w})$ can be expanded as (21) with bounded cumulant

coefficients K_{re} . The leading coefficients needed for $P_r(x)$ of (4) for the distribution of Y_{nt} of (12) are as follows.

$$\begin{aligned}
 K_{10} &= \bar{t} = t(w). \text{ For } P_0(x) : K_{21} = v = \bar{t}_1 \bar{k}_1^{12} \bar{t}_2. \\
 \text{For } P_1(x) : K_{11} &= c_{02}/2, K_{32} = c_{21} + 3c_{23}. \\
 \text{For } P_2(x) : K_{22} &= \sum_{k=1}^2 kK_{22}, {}_1K_{22} = c_{11}, {}_2K_{22} = c_{15} + c_{19}/2 + c_{1,10}, \\
 K_{43} &= c_{31} + 12c_{36} + 12c_{3,10} + 4c_{3,11}. \\
 \text{For } P_3(x) : K_{12} &= \sum_{k=1}^2 kK_{12}, {}_1K_{12} = L_1/2, {}_2K_{12} = L_2/6 + L_3/8; \\
 K_{33} &= \sum_{k=2}^3 kK_{33}, {}_2K_{33} = L_4 + 6L_5, {}_3K_{33} = 3L_6/2 + 3L_7 + L_8 \text{ where} \tag{23}
 \end{aligned}$$

$$L_7 = L_{71} + 3L_{72}/2 + 3 \sum_{k=3}^4 L_{7k}, L_8 = 3L_{81}/2 + 6L_{82} + 3L_{83} + 3L_{84}, \tag{24}$$

$$\begin{aligned}
 K_{54} &= L_9 + 20L_{10} + 15L_{11} + 30L_{12} + L_{13} \text{ where} \\
 L_{12} &= L_{121} + 2L_{122} + 2L_{123}, L_{13} = L_{131} + 60(L_{132} + L_{133}). \tag{25}
 \end{aligned}$$

$$\text{For } P_4(x) : K_{23} = \sum_{k=1}^3 kK_{23}, {}_1K_{23} = L_{14}, {}_2K_{23} = L_{15} + \sum_{k=1}^2 L_{16k} + L_{171},$$

$${}_3K_{23} = L_{181}/3 + L_{182}/4 + \sum_{k=19}^{20} L_k \text{ where } L_{19} = L_{191}/3 + \sum_{k=2}^4 L_{19k},$$

$$L_{20} = L_{201}/4 + L_{202}/2 + L_{203}/4 + L_{204}/6;$$

$$K_{44} = \sum_{k=3}^4 kK_{44}, {}_3K_{44} = L_{21} + 12 \sum_{k=1}^2 L_{22k} + 12L_{231} + 24L_{241} + 12L_{242}.$$

$${}_4K_{44} = 2L_{25} + 2L_{26} + 6 \sum_{k=1}^3 L_{27k} + L_{28} + L_{29} \text{ where}$$

$$L_{26} = 3L_{261} + L_{262} + 6L_{263} + 3L_{264},$$

$$L_{28} = \sum_{k=1}^{11} c_k L_{28k}, c_1 = 6, c_3 = 24, c_8 = 4, c_k = 12 \text{ otherwise,}$$

$$L_{29} = \sum_{k=1}^9 h_k L_{29k}, h_1 = 2, h_2 = h_6 = 12, h_7 = 24, h_9 = 3, h_k = 6 \text{ otherwise;}$$

$$K_{65} = L_{30} + 30L_{31} + 60vL_{32} + 60L_{33} + 90L_{34} + 60L_{35} + 6L_{36} \text{ for}$$

$$L_{33} = L_{331} + 3L_{332} + 2L_{333}, L_{34} = L_{341} + 4L_{342} + L_{343},$$

$$L_{35} = \sum_{k=1}^7 d_k L_{35k}, d_1 = 1, d_2 = d_3 = d_7 = 6, d_4 = 3, d_5 = 2, d_6 = 12,$$

$$L_{36} = \sum_{k=1}^6 e_k L_{36k}, e_1 = 1, e_2 = 20, e_3 = 15, e_7 = 6, e_4 = e_5 = e_6 = 60.$$

$$\text{Also, } K_{13} = \bar{t}_{12} \bar{k}_3^{12} / 2 + \bar{t}_{1-3} \bar{k}_3^{1-3} / 6 + \bar{t}_{1-4} (\bar{k}^{12} \bar{k}^{34})_3 / 8 + \bar{t}_{1-4} \bar{k}_3^{1-4} / 24$$

$$+ \bar{t}_{1-5} \bar{k}_2^{1-3} \bar{k}_1^{45} / 12 + \bar{t}_{1-6} \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56} / 48 \text{ where } (ab)_3 = a_1 b_2 + a_2 b_1,$$

$$K_{14} = \bar{t}_{12} \bar{k}_4^{12} / 2 + \bar{t}_{1-3} \bar{k}_4^{1-3} / 6 + \bar{t}_{1-4} [(\bar{k}^{12} \bar{k}^{34})_4 / 8 + \bar{k}_4^{1-4} / 24] + \bar{t}_{1-5} [\bar{k}_4^{1-5} / 120$$

$$+ (\bar{k}^{1-3} \bar{k}^{45})_4 / 12] + \bar{t}_{1-6} [(\bar{k}^{12} \bar{k}^{34} \bar{k}^{56})_4 / 48 + \bar{k}_3^{1-4} \bar{k}_1^{56} / 48 + \bar{k}_2^{1-3} \bar{k}_2^{4-6} / 72]$$

$$+ \bar{t}_{1-7} \bar{k}_2^{1-3} \bar{k}_1^{45} \bar{k}_1^{67} / 48 + \bar{t}_{1-8} \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56} \bar{k}_1^{78} / 384$$

$$\text{where the 1st } (ab)_4 = a_1 b_3 + a_2 b_2 + a_3 b_1,$$

$$\text{the 2nd } (ab)_4 = a_2 b_2 + a_3 b_1,$$

$$\text{and } (abc)_4 = a_1 b_1 c_2 + a_1 b_2 c_1 + a_2 b_1 c_1.$$

Proof. Since $q = 1$, Σ^N becomes N . We write $T_{1-s}^{1-r}, U_{1-s}^{1-r}, V_{1-s}^{1-r}$ as $T_{1-s}^r, U_{1-s}^r, V_{1-s}^r$. By Theorem 2 we need the following.

$$\begin{aligned}
 T_{1-4}^3/3 &= \bar{t}_{13}\bar{t}_2\bar{t}_4, T_{1-3}^2/2 = \bar{t}_{12}\bar{t}_3, T_{1-4}^2/2 = \bar{t}_{1-3}\bar{t}_4 + \bar{t}_{13}\bar{t}_{24}, \\
 T_{1-5}^4/12 &= \bar{t}_{14}\bar{t}_2\bar{t}_3\bar{t}_5, T_{1-6}^4/4 = \bar{t}_{135}\bar{t}_2\bar{t}_4\bar{t}_6 + 3\bar{t}_{13}\bar{t}_{25}\bar{t}_4\bar{t}_6, \\
 T_{1-5}^3/3 &= \bar{t}_{124}\bar{t}_3\bar{t}_5 + 3\bar{t}_{145}\bar{t}_2\bar{t}_3/2 + 3\bar{t}_{12}\bar{t}_{34}\bar{t}_5 + 3\bar{t}_{14}\bar{t}_{25}\bar{t}_3, \\
 T_{1-6}^3 &= 3\bar{t}_{1235}\bar{t}_4\bar{t}_6/2 + 6\bar{t}_{1-3}\bar{t}_{45}\bar{t}_6 + 3\bar{t}_{135}\bar{t}_{24}\bar{t}_6 + \bar{t}_{13}\bar{t}_{25}\bar{t}_{46}, \\
 T_{1-6}^5/20 &= \bar{t}_{15}\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_6, U_{1-6}^5/15 = \bar{t}_{14}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_6, \\
 T_{1-7}^5/30 &= \bar{t}_{146}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_7 + 2\bar{t}_{14}\bar{t}_{26}\bar{t}_3\bar{t}_5\bar{t}_7 + 2\bar{t}_{14}\bar{t}_{56}\bar{t}_2\bar{t}_3\bar{t}_7, \\
 T_{1-8}^5 &= \bar{t}_{1357}\bar{t}_2\bar{t}_4\bar{t}_6\bar{t}_8 + 60\bar{t}_{135}\bar{t}_{27}\bar{t}_4\bar{t}_6\bar{t}_8 + 60\bar{t}_{13}\bar{t}_{25}\bar{t}_{47}\bar{t}_6\bar{t}_8. U_{1-4}^2 = \bar{t}_{1-3}\bar{t}_4/3 + \bar{t}_{12}\bar{t}_{34}/4, \\
 T_{1-5}^2 &= \bar{t}_{1-4}\bar{t}_5/3 + \bar{t}_{1245}\bar{t}_3/2 + \bar{t}_{124}\bar{t}_{35} + \bar{t}_{145}\bar{t}_{23}/2, \\
 T_{1-6}^2 &= \bar{t}_{1-5}\bar{t}_6 + 2\bar{t}_{1235}\bar{t}_{46} + \bar{t}_{1-3}\bar{t}_{4-6} + 2\bar{t}_{135}\bar{t}_{246}/3, \\
 T_{1-5}^4/12 &= \bar{t}_{14}\bar{t}_2\bar{t}_3\bar{t}_5, U_{1-5}^4/4 = \bar{t}_{12}\bar{t}_3\bar{t}_4\bar{t}_5, \\
 U_{1-6}^4/2 &= 3\bar{t}_{125}\bar{t}_3\bar{t}_4\bar{t}_6 + 2\bar{t}_{156}\bar{t}_2\bar{t}_3\bar{t}_4 + 6\bar{t}_{12}\bar{t}_{35}\bar{t}_4\bar{t}_6 + 3\bar{t}_{15}\bar{t}_{26}\bar{t}_3\bar{t}_4, \\
 V_{1-6}^4/6 &= \bar{t}_{124}\bar{t}_3\bar{t}_5\bar{t}_6 + \bar{t}_{12}\bar{t}_{34}\bar{t}_5\bar{t}_6 + \bar{t}_{14}\bar{t}_{25}\bar{t}_3\bar{t}_6, \\
 T_{1-7}^4 &= 6\bar{t}_{1246}\bar{t}_3\bar{t}_5\bar{t}_7 + 3\bar{t}_{1456}\bar{t}_2\bar{t}_3\bar{t}_7 + 24\bar{t}_{124}\bar{t}_{36}\bar{t}_5\bar{t}_7 + 12\bar{t}_{124}\bar{t}_{56}\bar{t}_3\bar{t}_7 \\
 &+ 12\bar{t}_{145}\bar{t}_{26}\bar{t}_3\bar{t}_7 + 12\bar{t}_{146}\bar{t}_{23}\bar{t}_5\bar{t}_7 + 24\bar{t}_{146}\bar{t}_{25}\bar{t}_3\bar{t}_7 + 4\bar{t}_{146}\bar{t}_{57}\bar{t}_2\bar{t}_3 \\
 &+ 12\bar{t}_{456}\bar{t}_{17}\bar{t}_2\bar{t}_3 + 12\bar{t}_{12}\bar{t}_{34}\bar{t}_5\bar{t}_6\bar{t}_7 + 12\bar{t}_{14}\bar{t}_{25}\bar{t}_{36}\bar{t}_7 + 12\bar{t}_{14}\bar{t}_{26}\bar{t}_5\bar{t}_7\bar{t}_3, \\
 T_{1-8}^4 &= 2\bar{t}_{12357}\bar{t}_4\bar{t}_6\bar{t}_8 + 12\bar{t}_{1235}\bar{t}_{47}\bar{t}_6\bar{t}_8 + 6\bar{t}_{1357}\bar{t}_{24}\bar{t}_6\bar{t}_8 \\
 &+ 6\bar{t}_{123}\bar{t}_{457}\bar{t}_6\bar{t}_8 + 6\bar{t}_{135}\bar{t}_{247}\bar{t}_6\bar{t}_8 + 12\bar{t}_{123}\bar{t}_{45}\bar{t}_{67}\bar{t}_8 + 24\bar{t}_{135}\bar{t}_{24}\bar{t}_{67}\bar{t}_8 \\
 &+ 6\bar{t}_{135}\bar{t}_{27}\bar{t}_{48}\bar{t}_6 + 3\bar{t}_{13}\bar{t}_{25}\bar{t}_{47}\bar{t}_6\bar{t}_8, T_{1-7}^6/30 = \bar{t}_{16}\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_5\bar{t}_7, U_{1-7}^6/60 = \bar{t}_{15}\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_6\bar{t}_7, \\
 T_{1-8}^6/60 &= \bar{t}_{157}\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_6\bar{t}_8 + 3\bar{t}_{15}\bar{t}_{27}\bar{t}_3\bar{t}_4\bar{t}_6\bar{t}_8 + 2\bar{t}_{15}\bar{t}_{67}\bar{t}_2\bar{t}_3\bar{t}_4\bar{t}_8, \\
 U_{1-8}^6/90 &= \bar{t}_{147}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_6\bar{t}_8 + 4\bar{t}_{14}\bar{t}_{27}\bar{t}_3\bar{t}_5\bar{t}_6\bar{t}_8 + \bar{t}_{17}\bar{t}_{48}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_6, \\
 T_{1-9}^6/60 &= \bar{t}_{1468}\bar{t}_2\bar{t}_3\bar{t}_5\bar{t}_7\bar{t}_9 + 6\bar{t}_{146}\bar{t}_{28}\bar{t}_3\bar{t}_5\bar{t}_7\bar{t}_9 + 6\bar{t}_{146}\bar{t}_{58}\bar{t}_2\bar{t}_3\bar{t}_7\bar{t}_9 \\
 &+ 3\bar{t}_{468}\bar{t}_{15}\bar{t}_2\bar{t}_3\bar{t}_7\bar{t}_9 + 2\bar{t}_{14}\bar{t}_{26}\bar{t}_{38}\bar{t}_5\bar{t}_7\bar{t}_9 + 12\bar{t}_{14}\bar{t}_{26}\bar{t}_{58}\bar{t}_3\bar{t}_7\bar{t}_9 + 6\bar{t}_{14}\bar{t}_{56}\bar{t}_{78}\bar{t}_2\bar{t}_3\bar{t}_9, \\
 T_{1-10}^6/6 &= \bar{t}_{13579}\bar{t}_2\bar{t}_4\bar{t}_6\bar{t}_8\bar{t}_{10} + 20\bar{t}_{1357}\bar{t}_{29}\bar{t}_4\bar{t}_6\bar{t}_8\bar{t}_{10} \\
 &+ 15\bar{t}_{135}\bar{t}_{279}\bar{t}_4\bar{t}_6\bar{t}_8\bar{t}_{10} + 60\bar{t}_{135}\bar{t}_{27}\bar{t}_{49}\bar{t}_6\bar{t}_8\bar{t}_{10} + 60\bar{t}_{135}\bar{t}_{27}\bar{t}_{89}\bar{t}_4\bar{t}_6\bar{t}_{10} + 60\bar{t}_{13}\bar{t}_{25}\bar{t}_{47}\bar{t}_{69}\bar{t}_8\bar{t}_{10}.
 \end{aligned}$$

For $P_1(x)$: $T_{1-4}^3 \bar{k}_1^{12}\bar{k}_1^{34} = 3c_{23}$.

For $P_2(x)$: $T_{1-3}^2 \bar{k}_2^{1-3}/2 = c_{15}$, $T_{1-4}^2 \bar{k}_1^{12}\bar{k}_1^{34}/2 = c_{19}/2 + c_{1,10}$;

$K_{43} = c_{31} + g_1 + g_2$ for $g_1 = T_{1-5}^4 \bar{k}_2^{1-3}\bar{k}_1^{45} = 12c_{36}$,

$g_2 = T_{1-6}^4 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56} = 4c_{3,11} + 12c_{3,10}$.

For $P_3(x)$: ${}_2K_{33} = L_4 + 3L'_5$, $L'_5 = T_{1-4}^3 (\bar{k}^{12}\bar{k}^{34})_3 = \bar{t}_{13}\bar{t}_2\bar{t}_4 (\bar{k}^{12}\bar{k}^{34})_3 = 2L_5$,

$L_6 = T_{1-4}^3/\bar{k}_3^{1-4}$, $L_7 = T_{1-5}^3/\bar{k}_2^{1-3}\bar{k}_1^{45}$,

$L_{71} = \bar{s}^4\bar{t}_{412} \bar{k}_2^{1-3}\bar{t}_3$, $L_8 = T_{1-6}^3 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}$,

$L_{81} = \bar{t}_{1235}\bar{t}_4\bar{t}_6 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}$, $L_{82} = \bar{t}_{1-3}\bar{t}_{45}\bar{t}_6 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}$, $L_{83} = \bar{t}_{135}\bar{t}_{24}\bar{t}_6 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}$,

$L_{84} = \bar{t}_{13}\bar{t}_{25}\bar{t}_{46} \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}$.

K_{54} is given by (25) with $L_{10} = T_{1-6}^5/\bar{k}_3^{1-4}\bar{k}_1^{56}$, $L_{11} = U_{1-6}^5/15 \bar{k}_2^{1-3}\bar{k}_2^{4-6}$,

$L_{12} = T_{1-7}^5/\bar{k}_2^{1-3}\bar{k}_1^{45}\bar{k}_1^{67}$, $L_{121} = \bar{t}_2\bar{t}_3\bar{k}_2^{1-3}\bar{t}_{146}\bar{s}^4\bar{s}^6$, $L_{13} = T_{1-8}^5 \bar{k}_1^{12}\bar{k}_1^{34}\bar{k}_1^{56}\bar{k}_1^{78}$.

For $P_4(x)$: ${}_2K_{23} = L_{15} + L_{16} + L_{17}/2$, $L_{15} = T_{1-3}^2/\bar{k}_3^{1-3}$, $L_{16} = \sum_{k=1}^2 L_{16k}$,

$$\begin{aligned}
 L_{162} &= \bar{s}^2 \bar{t}_{2-4} \bar{k}_2^{34}, L_{17} = T_{1-4}^2 (\bar{k}^{12} \bar{k}^{34})_3 = \sum_{k=1}^2 L_{17k}, L_{172} = L_{171}; \\
 3K_{23} &= \sum_{k=18}^{20} L_k : L_{18} = U_{1-4}^2 \bar{k}_3^{1-4} = L_{181}/3 + L_{182}/4, \\
 L_{19} &= T_{1-5}^2 \bar{k}_2^{1-3} \bar{k}_1^{45} = \sum_{k=1}^4 L_{19k}/g_k \text{ for } g_1 = 3, g_2 = g_3 = g_4 = 1, \\
 L_{192} &= \bar{t}_3 \bar{k}_2^{1-3} \bar{t}_{1245} \bar{k}_1^{45}, L_{193} = \bar{t}_{412} \bar{k}_2^{1-3} \bar{t}_{35} \bar{k}_1^{54}, L_{194} = \bar{k}_1^{45} \bar{t}_{451} \bar{k}_2^{1-3} \bar{t}_{23}, \\
 L_{20} &= T_{1-6}^2 \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56}/4 = L_{201}/4 + L_{202}/2 + L_{203}/4 + L_{204}/6, \\
 L_{202} &= \bar{k}_1^{12} \bar{t}_{1235} (\bar{k}_1^{34} \bar{k}_1^{56}) \bar{t}_{46}. \\
 3K_{44} &= L_{21} + 12L_{22} + 4L, L = T_{1-6}^4 (\bar{k}^{12} \bar{k}^{34} \bar{k}^{56})_4 = L_{23} + 3L_{24}, \\
 L_{22} &= T_{1-5}^4 /12 (\bar{k}^{1-3} \bar{k}^{45})_4 = (L_{221} + L_{222}) \text{ by (15)}, \\
 L_{23} &= \bar{t}_{135} \bar{t}_2 \bar{t}_4 \bar{t}_6 (\bar{k}^{12} \bar{k}^{34} \bar{k}^{56})_4 = \sum_{k=1}^3 L_{23k} \text{ by (15)}, L_{23k} \equiv L_{231}. \text{ By (15)}, \\
 L_{24} &= \bar{t}_{13} \bar{t}_{25} \bar{t}_4 \bar{t}_6 (\bar{k}^{12} \bar{k}^{34} \bar{k}^{56})_4 = 2L_{241} + L_{242}; \\
 4K_{44} &= 2L_{25} + 2L_{26} + 6L_{27} + L_{28} + L_{29}, L_{25} = \bar{k}_4^{1-5} U_{1-5}^4 /4, \\
 L_{26} &= U_{1-6}^4 /2 \bar{k}_3^{1-4} \bar{k}_1^{56}, L_{27} = V_{1-6}^4 /6 \bar{k}_2^{1-3} \bar{k}_2^{4-6} = \sum_{k=1}^3 L_{27k}, \\
 L_{28} &= T_{1-7}^4 \bar{k}_2^{1-3} \bar{k}_1^{45} \bar{k}_1^{67}, L_{29} = T_{1-8}^4 \bar{t}_{12} \bar{k}_1^{34} \bar{k}_1^{56} \bar{k}_1^{78}; \\
 \text{for } K_{65} : L_{31} &= T_{1-7}^6 \bar{k}_4^{1-5} \bar{k}_1^{67} /30, L_{32} = U_{1-7}^6 \bar{k}_4^{1-5} \bar{k}_1^{67} /60v, \\
 L_{33} &= T_{1-8}^6 \bar{k}_3^{1-4} \bar{k}_1^{56} \bar{k}_1^{78} /60, L_{34} = U_{1-8}^6 \bar{k}_2^{1-3} \bar{k}_2^{4-6} \bar{k}_1^{78} /90, \\
 L_{35} &= T_{1-9}^6 \bar{k}_2^{1-3} \bar{k}_1^{45} \bar{k}_1^{67} \bar{k}_1^{89} /60, L_{36} = T_{1-10}^6 \bar{k}_1^{12} \bar{k}_1^{34} \bar{k}_1^{56} \bar{k}_1^{78} \bar{k}_1^{9,10} /6.
 \end{aligned}$$

□

Example 1. Suppose that $E \hat{w} = w$ and $t(w)$ is linear in w . Then $K_{1e} = 0$ for $e \geq 1$. For $r \leq 4$, the K_{ij} needed for $P_r(x)$ of (4) for the distribution of Y_{nt} of (12) are as follows.

$$\begin{aligned}
 K_{10} &= \bar{t} = t(w). \text{ For } P_0(x) : K_{21} = v. \text{ For } P_1(x) : K_{32} = c_{21}. \\
 \text{For } P_2(x) : K_{22} &= {}_1K_{22} = c_{11}, K_{43} = c_{31}. \\
 \text{For } P_3(x) : K_{33} &= {}_2K_{33} = L_4, K_{54} = L_9. \\
 \text{For } P_4(x) : K_{23} &= {}_1K_{23} = L_{14}, K_{44} = {}_3K_{44} = L_{21}, K_{65} = L_{30}.
 \end{aligned}$$

For, $\bar{u}_1, \bar{v}^1, \bar{x}_1, \bar{z}_{12}$ are 0, as are most c_{ij}, L_k and ${}_2K_{22}, {}_3K_{33}, {}_2K_{23}, {}_3K_{23}, {}_4K_{44}$. So for $r = 0, \dots, 4$, for $P_r(x)$ we only need to calculate these 3 c_{ij} and 5 L_j .

Let G_γ be a gamma random variable with known mean γ . Its r th cumulant is $(r - 1)! \gamma$. For a standard exponential random variable $\gamma = 1$.

Example 2. Linear combinations of scale parameters. Suppose that $E \hat{w} = w$ and $t(w)$ is linear, the components of \hat{w} are independent, and for $1 \leq i \leq p$, \hat{w}^i / w^i has a distribution with known r th cumulant $n^{1-r} \kappa_{ri}$. Then, $K_{re} = 0$ for $e \neq r - 1$ and

$$K_{r,r-1} = \sum_{i=1}^p t_{.i}^r \kappa_{ri} (w^i)^r.$$

For example, if \hat{w}^i / w^i is a gamma random variable with mean γ , then $\kappa_{ri} = (r - 1)! \gamma$.

For $s \leq r$ and any function $f^{i_s \dots i_r}$, set $\sum_{s-r} \bar{f}^{s-r} = \sum_{i_s, \dots, i_r} f^{i_s \dots i_r}$ summed over their range. In Example 3 their range is 1, 2; for example in L_{123} , $\sum_1 t_{.2i_1} \bar{v}^1 = \sum_{i_1=1}^k t_{.2i_1} \bar{v}^1$. In Example 4 their range is 1, \dots , k ; for example $\bar{u}_1 = \sum_2 \bar{t}_{12} \bar{s}^2 = \sum_{i_2=1}^k \bar{t}_{12} \bar{s}^2$.

Example 3. Suppose that $\hat{\mu} \sim \mathcal{N}_1(\mu, V/n)$ and $\hat{V}/V = \chi^2_f/f = G_\gamma/\gamma$ are independent, where $\gamma = f/2$ has magnitude n . Set $v = \gamma/n$. Then $\kappa_r(\hat{\mu}) = \mu \delta_{r1} + Vn^{-1} \delta_{r2}$, $\kappa_r(\hat{V}) = k_r n^{1-r}$ for $k_r = (r-1)!v^{1-r}V^r$, and cross-cumulants of \hat{w} are zero. Take $p = 2$, $w_1 = \mu$, $w_2 = V$. Then by Corollary 1, K_{re} are given in terms of

$$s^1 = t_{.1}V, s^2 = t_{.2}k_2, u_1 = t_{.11}t_{.1}V + t_{.12}t_{.2}k_2, u_2 = t_{.12}t_{.1}V + t_{.22}t_{.2}k_2, \\ v^1 = Vu_1, v^2 = k_2u_2,$$

as follows.

For $P_0(x)$: $K_{21} = v = t_{.1}^2V + t_{.2}^2k_2/2$.

For $P_1(x)$: $c_{02} = t_{.11} + t_{.22}k_2, c_{21} = t_{.2}^3k_3, c_{23} = \sum_{i=1}^2 t_{.ii}(s^i)^2$.

For $P_2(x)$: $c_{11} = 0, c_{15} = t_{.22}t_{.2}k_3, c_{19} = (t_{.11}V)^1 + (t_{.22}k_2)^2 + 2t_{.12}Vk_2,$
 $c_{31} = t_{.2}^4k_4, c_{36} = t_{.2}^2k_3u_2, c_{3,10} = u_1^2V + u_2^2k_2,$

$c_{3,11} = \sum_{i=1}^2 (s^i)^3 t_{.iii} + 3 \sum_{12} (s^1)^2 s^2 t_{.112}$ where $\sum_{12} f_{12} = f_{12} + f_{21}$.

For $P_3(x)$: $L_1 = L_4 = L_5 = 0, L_2 = t_{.222}k_3,$

$L_3 = t_{.1111}V^2 + 2t_{.1122}Vk_2 + t_{.2222}k_2^2,$

$L_6 = t_{.22}t_{.2}^2k_4, L_{71} = z_{22}k_3t_2, L_{72} = y^2(t_{.211}V + t_{.222}k_3), L_{73} = t_{.22}k_3u_2,$

$L_{74} = (Vt_{.12}^2 + k_2t_{.22}^2)k_3t_{.2}, L_{81} = (Vt_{.1i_2i_3i_4} + k_2t_{.2i_2i_3i_4})\bar{s}^3\bar{s}^4,$

$L_{82} = \sum_1 (Vt_{.11i_1} + k_2t_{.22i_1})v^{i_1}, L_{83} = V^2t_{.11}z_{11} + 2Vk_2t_{.12}z_{12} + k_2^2t_{.22}z_{22},$

$L_{84} = \sum_{1-3} \bar{k}_1^{11} \bar{k}_1^{22} \bar{k}_1^{33} \bar{t}_{12} \bar{t}_{23} \bar{t}_{31},$

$L_9 = t_{.2}^5k_5, L_{10} = u_2t_{.2}^3k_4, L_{11} = (y^2)^2t_{.22}, L_{121} = y^2 \sum_{12} t_{.2i_1i_2} \bar{s}^1 \bar{s}^2,$

$L_{122} = t_{.2}k_3u_2^2, L_{123} = y^2 \sum_1 t_{.2i_1} \bar{v}^1, L_{13} = \sum_{1-3} \bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{k}_1^{33} \bar{u}_3.$

Similarly, one can write down the L_s needed for $P_4(x)$.

Example 4. Suppose that we have the summary statistics from k samples of size n_i from normal populations with means and variances $\mu_i, V_i, 1 \leq i \leq k$. Take $p = 2k, w_i = \mu_i, w_{i+k} = V_i, 1 \leq i \leq k$. So we have p independent statistics, $\hat{\mu}_i \sim \mathcal{N}(\mu_i, V_i/n_i)$ and $\hat{V}_i \sim V_i \chi^2_{f_i}/f_i = V_i G_{\gamma_i}/\gamma_i, 1 \leq i \leq k$ where $\gamma_i = f_i/2$ has magnitude n , the total sample size. Set

$$v_i = \gamma_i/n, \lambda_i = n_i/n, \tau_i = V_i/\lambda_i.$$

Then $\kappa_r(\hat{\mu}_i) = \mu_i \delta_{r1} + V_i(\lambda_i n)^{-1} \delta_{r2}, \kappa_r(\hat{V}_i) = k_{ri}(\lambda_i n)^{1-r}$ for $k_{ri} = (r-1)!v_i^{1-r}$, and cross-cumulants of \hat{w} are zero. Suppose that $t(w)$ only depends on μ_1, \dots, μ_k , as in Example 3.3 of [2]. (The notation there is slightly different). Then

$$\bar{s}^1 = \bar{t}_1 \bar{\tau}_1, \bar{u}_1 = \sum_2 \bar{t}_{12} \bar{s}^2, \bar{v}^1 = \bar{\tau}_1 \bar{u}_1, \bar{z}_{12} = \sum_3 \bar{t}_{1-3} \bar{s}^3,$$

and by Corollary 1, the coefficients needed are as follows.

$$\begin{aligned}
 \text{For } P_0(x) : K_{21} &= v = \bar{t}_1 \bar{s}^1 = \sum_1 \bar{t}_1^2 \bar{t}_1, \\
 \text{For } P_1(x) : c_{02} &= \sum_1 \bar{t}_{11} \bar{t}_1, c_{21} = 0, c_{23} = \bar{s}^1 \bar{t}_{12} \bar{s}^2 = \sum_{12} \bar{t}_1 \bar{t}_1 \bar{t}_{12} \bar{t}_2, \\
 \text{For } P_2(x), K_{22} : c_{11} &= c_{15} = 0, c_{19} = \sum_{12} \bar{t}_{12}^2 \bar{t}_1 \bar{t}_2, c_{1,10} = \sum_{12} \bar{t}_1 \bar{t}_1 \bar{t}_{122} \bar{t}_2; \\
 \text{For } K_{43} : c_{31} &= c_{36} = 0, c_{3,10} = \sum_1 \bar{u}_1^2 \bar{t}_1, c_{3,11} = \sum_{1-3} \bar{s}^1 \bar{s}^2 \bar{s}^3 \bar{t}_{1-3}. \\
 \text{For } P_3(x), K_{12} &= L_3/6 = \sum_{12} \bar{t}_{1122} \bar{t}_1 \bar{t}_2 / 6 \text{ as } L_1 = L_2 = 0; \\
 \text{for } K_{33} : L_4 &= L_5 = L_6 = L_7 = 0, L_{81} = \sum_{1-3} \bar{t}_1 \bar{t}_{1123} \bar{s}^2 \bar{s}^3, L_{82} = \sum_{12} \bar{t}_1 \bar{t}_{122} \bar{v}^2, \\
 L_{83} &= \sum_{12} \bar{t}_{12} \bar{t}_1 \bar{t}_2 \bar{z}_{12} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{12} \bar{t}_{123} \bar{t}_3, L_{83} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{12} \bar{t}_{23} \bar{t}_{31}; \\
 \text{for } K_{54} : L_k &= 0 \text{ for } 10 \leq k \leq 14. \\
 \text{For } P_4(x), K_{23} : L_{i,k} &= 0 \text{ for } i = 15, 16, 18, 19, L_{171} = \sum_{12} \bar{t}_{12}^2 \bar{t}_1 \bar{t}_2, \\
 L_{201} &= \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_{11223} \bar{s}^5, L_{202} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{1123} \bar{t}_{23}, L_{203} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{112} \bar{t}_{233}, \\
 L_{204} &= \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{123}^2; \\
 K_{44} &= L_{29} \text{ as } L_k = 0 \text{ for } k = 21, (22, k), (23, 1), (24, k), 25, (26, k), (27, k), (28, k), \\
 L_{291} &= \sum_{1-4} \bar{t}_1 \bar{s}^2 \bar{s}^3 \bar{s}^4 \bar{t}_{11234}, L_{292} = \sum_{1-3} \bar{t}_1 \bar{t}_{1123} \bar{v}^2 \bar{s}^3, L_{293} = \sum_{1-4} \bar{t}_1 \bar{t}_2 \bar{s}^3 \bar{s}^4 \bar{t}_{1234}, \\
 L_{294} &= \sum_{1-4} \bar{t}_1 \bar{t}_2 \bar{t}_{112} \bar{t}_{234} \bar{s}^3 \bar{s}^4, L_{295} = \sum_{12} \bar{t}_1 \bar{z}_{12}^2 \bar{t}_2, L_{296} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{112} \bar{t}_{23}, \\
 L_{297} &= \sum_{12} \bar{t}_1 \bar{x}_{12} \bar{t}_{12} \text{ for } \bar{x}_{12} = \sum_3 \bar{t}_{123} \bar{v}^3, L_{298} = \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{12} \bar{t}_{23} \bar{z}_{31}, \\
 L_{299} &= \sum_{1-4} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4 \bar{t}_{12} \bar{t}_{23} \bar{t}_{34} \bar{t}_{41}; \\
 K_{65} &= 6L_{36}, \text{ as the other components of } K_{65} \text{ are 0. Also,} \\
 K_{13} &= \sum_{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_{112233} / 48, K_{14} = \sum_{1-4} \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4 \bar{t}_{11223344} / 384.
 \end{aligned}$$

Corollary 2. Set $Y_n = v^{-1/2} Y_{nt} = (n/v)^{1/2} (t(\hat{w}) - t(w))$. Then

$$\text{Prob.}(Y_n \leq x) = \sum_{r=0}^{\infty} n^{-r/2} P_{rv}(x)$$

where $P_{rv}(x)$ is $P_r(x)$ of Corollary 1 with K_{re} replaced by $K_{re}/v^{r/2}$.

Proof. This is straightforward. \square

Looking at $K_{re}, c_{ab}, v, \bar{s}^2, \bar{u}_3$ as functions of w , we denote their partial derivatives with respect to $\bar{w}_3 = w_r i_3$, say, by $\bar{K}_{re,3}, \bar{c}_{ab,3}, \bar{v}_{.3}, \bar{s}_{.3}^2, \bar{u}_{3,3}$ and similarly for higher derivatives. We shall give the ones we need in Lemma 1. When constructing confidence regions, one needs to assume that the distribution of \hat{w} is determined by w . So far, we have not assumed this. For \hat{w} biased, we need

Corollary 3. Let \hat{w} in R^p be a biased standard estimate of w satisfying (1) where \bar{k}_e^{1-r} may depend on w . Then for $q = 1, \hat{t} = t(\hat{w})$ is a standard estimate of $t = t(w)$ in R :

$$\kappa_r(\hat{t}) = \sum_{e=r-1}^{\infty} n^{-e} a_{re} \text{ for } r \geq 1, \text{ where } a_{re} = K_{re} + D_{re}, D_{r,r-1} = 0, \tag{26}$$

and the other D_{re} needed for $P_j(x)$ of (4) for the distribution of Y_{nt} of (12) are as follows.

For $P_0(x)$: $D_{21} = 0 \Rightarrow a_{21} = K_{21} = v = \bar{t}_1 \bar{k}_1^{12} \bar{t}_2$.
 For $P_1(x)$: $D_{11} = c_{01} \Rightarrow a_{11} = c_{01} + c_{02}/2, a_{32} = K_{32} = c_{21} + 3c_{23}$,
 For $P_2(x)$: $D_{22} = \bar{v}_{.3} \bar{k}_1^3 = c_{12} + 2c_{16}, a_{43} = K_{43}$.
 For $P_3(x)$: $D_{12} = K_{11,1} + K_{10,2}, K_{11,1} = \bar{K}_{11,3} \bar{k}_1^3, K_{10,2} = \bar{t}_1 \bar{k}_2^1 + \bar{k}_1^1 \bar{t}_1 \bar{k}_1^2 / 2$,
 $D_{33} = \bar{K}_{32,4} \bar{k}_1^4, a_{54} = K_{54}$.
 For $P_4(x)$: $D_{23} = K_{22,1} + K_{21,2}, K_{22,1} = \bar{K}_{22,3} \bar{k}_1^3, K_{21,2} = \bar{v}_{.3} \bar{k}_2^3 / 2 + \bar{v}_{.34} \bar{k}_1^3 \bar{k}_1^4$,
 $D_{44} = \bar{K}_{43,5} \bar{k}_1^5, a_{65} = K_{65}$.

Proof. This follows from Theorem 3. $D_{rc} = \sum_{e=1}^{c-r+1} K_{r,c-r,e}$ where K_{rke} is the coefficient of n^{-e} in the expansion of $K_{rk}(E \hat{w})$ about K_{rk} . \square

For $r \leq s$, and any \bar{X}_{r-s} , let $\sum_{r-s}^N \bar{X}_{r-s}$ sums over all N permutations of i_r, \dots, i_s giving distinct terms. For example,

$$\sum_{3-5}^3 \bar{t}_{13} \bar{t}_{245} = \bar{t}_{13} \bar{t}_{245} + \bar{t}_{14} \bar{t}_{235} + \bar{t}_{15} \bar{t}_{234}.$$

The derivatives of $v = K_{21}$ and K_{re} needed for Corollary 3 are given by

Lemma 1.

$$\bar{v}_{.3} = 2\bar{u}_3 + \bar{T}_3, \bar{v}_{.34} = \sum_{k=0}^2 \bar{v}_{34k}, \text{ where } \bar{T}_3 = \bar{t}_1 \bar{t}_2 \bar{k}_{1,3}^{12}, \tag{27}$$

$$\bar{v}_{340} = 2\bar{z}_{34} + 2\bar{t}_{31} \bar{k}_1^{12} \bar{t}_{24}, \bar{v}_{341} = 2\bar{t}_1 (\bar{k}_{1,3}^{12} \bar{t}_{24} + \bar{k}_{1,4}^{12} \bar{t}_{23}), \bar{v}_{342} = \bar{t}_1 \bar{t}_2 \bar{k}_{1,3,4}^{12}.$$

$$\bar{v}_{.3-5} = \sum_{k=0}^3 \bar{v}_{3-5k} \text{ for } \bar{v}_{3-50} = 2\bar{k}_1^{62} \sum_{3-5}^3 \bar{t}_{63} \bar{t}_{245} + \bar{s}^6 \bar{t}_{3-6}, \tag{28}$$

$$\bar{v}_{3-51} = 2 \sum_{3-5}^3 (\bar{t}_1 \bar{k}_{1,3}^{12} \bar{t}_{245} + \bar{t}_{31} \bar{k}_{1,4}^{12} \bar{t}_{25}), \bar{v}_{3-52} = 2\bar{t}_1 \sum_{3-5}^3 \bar{t}_{23} \bar{k}_{1,4,5}^{12}, \bar{v}_{3-53} = \bar{t}_1 \bar{t}_2 \bar{k}_{1,3-5}^{12},$$

$$2\bar{K}_{11,3} = c_{02,3} = \bar{k}_1^{12} \bar{t}_{1-3} + \bar{t}_{12} \bar{k}_{1,3}^{12}, \bar{K}_{32,4} = c_{21,4} + 3c_{23,4}$$

$$\text{for } c_{21,4} = 3\bar{Y}_4 + \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{k}_{2,4}^{1-3}, c_{23,4} = \bar{s}^1 \bar{s}^2 \bar{t}_{124} + 2b_4, b_4 = \bar{s}_{.4}^1 \bar{u}_1 = \bar{x}_4 + \bar{u}_1 \bar{k}_{1,4}^{12} \bar{t}_2.$$

$$\bar{K}_{22,3} = c_{11,3} + c_{15,3} + c_{19,3}/2 + c_{1,10,3} \text{ for } c_{11,3} = 2\bar{t}_{31} \bar{S}^1 + \bar{t}_1 \bar{t}_2 \bar{k}_{2,3}^{12},$$

$$c_{15,4} = \sum_{k=1}^3 c_{15k4}, c_{1514} = \bar{t}_{14} \bar{k}_2^{1-3} \bar{t}_{23}, c_{1524} = \bar{t}_1 \bar{k}_2^{1-3} \bar{t}_{2-4}, c_{1534} = \bar{t}_1 \bar{k}_{2,4}^{1-3} \bar{t}_{23},$$

$$c_{19,5} = 2c_{1915} + 2c_{1925}, c_{1915} = \bar{t}_{34} \bar{k}_1^{31} \bar{k}_1^{42} \bar{t}_{125}, c_{1925} = \bar{k}_{1,5}^{23} \bar{t}_{34} \bar{k}_1^{41} \bar{t}_{12},$$

$$c_{1,10,4} = \sum_{k=1}^3 c_{1,10k4}, c_{1,1014} = \bar{k}_1^{23} \bar{t}_{1-3} \bar{s}_{.4}^1 = \bar{k}_1^{23} \bar{t}_{1-3} (\bar{k}_{1,5}^{15} \bar{t}_{54} + \bar{t}_5 \bar{k}_{1,4}^{15}),$$

$$c_{1,1024} = \bar{s}^1 \bar{k}_1^{23} \bar{t}_{1-4}, c_{1,1034} = \bar{s}^1 \bar{t}_{1-3} \bar{k}_{1,4}^{23},$$

$$\bar{K}_{43,5} = c_{31,5} + 12c_{36,5} + 12c_{3,10,5} + 4c_{3,11,5} \text{ for } c_{31,5} = 4\bar{t}_{51} \bar{k}_3^{1-4} \bar{t}_2 \bar{t}_3 \bar{t}_4$$

$$+ \bar{t}_1 \dots \bar{t}_4 \bar{k}_{3,5}^{1-4}, c_{36,5} = 2\bar{t}_{51} \bar{k}_2^{1-3} \bar{t}_2 \bar{u}_3 + \bar{t}_1 \bar{t}_2 \bar{u}_3 \bar{k}_{2,5}^{1-3} + \bar{y}^3 \bar{t}_{3-5} \bar{s}^4 + \bar{Y}_4 (\bar{k}_{1,5}^{41} \bar{t}_1 + \bar{k}_1^{41} \bar{t}_{15}).$$

$$c_{3,10,3} = \bar{u}_1 \bar{k}_{1,3}^{12} \bar{u}_2 + 2\bar{v}^1 \bar{s}^2 \bar{t}_{1-3} + 2\bar{t}_1 \bar{x}_2 \bar{k}_{1,3}^{12} + 2\bar{x}_1 \bar{k}_1^{12} \bar{t}_{23},$$

$$c_{3,11,5} = 3\bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{s}_{.5}^3 = 3\bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{t}_4 \bar{k}_{1,5}^{34} + 3\bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{k}_1^{34} \bar{t}_{45}. \tag{29}$$

Proof. For example, substitute $\bar{u}_{3,5} = \bar{t}_{3-5} s^4 + \bar{t}_{34} \bar{k}_{1,5}^{41} \bar{t}_1 + \bar{t}_{34} \bar{k}_1^{41} \bar{t}_{15}$ into $c_{36,5} = 2\bar{t}_1 \bar{t}_{25} \bar{k}_2^{1-3} \bar{u}_3 + \bar{t}_1 \bar{t}_2 (\bar{k}_{2,5}^{1-3} \bar{u}_3 + \bar{k}_2^{1-3} \bar{u}_{3,5})$. \square

So now we can write D_{re} needed for Corollary 3 in molecular form:

Corollary 4. Assume that the conditions of Corollary 3 hold. Then D_{re} and $K_{re,j}$ given there satisfy

$$\begin{aligned}
 D_{22} &= c_{12} + 2c_{16} \Rightarrow a_{22} = c_{11} + c_{15} + c_{19}/2 + c_{1,10} + c_{12} + 2c_{16}. \\
 \text{For } D_{12}, K_{11,1} &= (\bar{t}_{1-3} \bar{k}_1^{12} + \bar{t}_{12} \bar{k}_{1,3}^{12}) \bar{k}_1^3 / 2. \\
 D_{33} &= (3\bar{y}^3 \bar{t}_{34} + \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{k}_{2,4}^{1-3} + 3\bar{s}^1 \bar{s}^2 \bar{t}_{124} + 6\bar{v}^1 \bar{t}_{14} + 6\bar{u}_1 \bar{k}_{1,4}^{12} \bar{t}_2) \bar{k}_1^4. \\
 \text{For } D_{23}, K_{22,1} &= (2\bar{t}_{31} \bar{s}^1 + \bar{t}_1 \bar{t}_2 \bar{k}_{2,3}^{12}) \bar{k}_1^3 + (\bar{t}_{14} \bar{k}_2^{1-3} \bar{t}_{23} + \bar{t}_1 \bar{k}_2^{1-3} \bar{t}_{2-4} + \bar{t}_1 \bar{k}_{2,4}^{1-3} \bar{t}_{23}) \bar{k}_1^4 \\
 &+ (\bar{t}_{125} \bar{x}^{12} + \bar{k}_{1,5}^{23} \bar{t}_{34} \bar{k}_1^{41} \bar{t}_{12}) \bar{k}_1^5 + [\bar{k}_1^{23} \bar{t}_{1-3} (\bar{k}_1^{15} \bar{t}_{54} + \bar{t}_5 \bar{k}_{1,4}^{15}) + \bar{s}^1 \bar{k}_1^{23} \bar{t}_{1-4} + \bar{s}^1 \bar{t}_{1-3} \bar{k}_{1,4}^{23}] \bar{k}_1^4, \\
 K_{21,2} &= (2\bar{u}_3 + \bar{t}_1 \bar{t}_2 \bar{k}_{1,3}^{12}) \bar{k}_2^3 / 2 + [4\bar{t}_1 \bar{k}_{1,3}^{12} \bar{t}_{24} + \bar{s}^1 \bar{t}_{134} + \bar{t}_{31} \bar{k}_1^{12} \bar{t}_{24} \\
 &+ \bar{t}_1 \bar{t}_2 \bar{k}_{1,34}^{12}] \bar{k}_1^3 \bar{k}_1^4. \\
 D_{44} &= [4\bar{t}_{51} \bar{k}_3^{1-4} \bar{t}_2 \bar{t}_3 \bar{t}_4 + \bar{t}_1 \cdots \bar{t}_4 \bar{k}_{3,5}^{1-4} + 24\bar{t}_{51} \bar{k}_2^{1-3} \bar{t}_2 \bar{u}_3 + 12\bar{t}_1 \bar{t}_2 \bar{u}_3 \bar{k}_{2,5}^{1-3} \\
 &+ 12\bar{y}^3 (\bar{t}_{3-5} s^4 + \bar{t}_{34} \bar{k}_{1,5}^{41} \bar{t}_1 + \bar{t}_{34} \bar{k}_1^{41} \bar{t}_{15})] \bar{k}_1^5 + 12[2\bar{v}^2 \bar{t}_{2-4} \bar{s}^4 + 2\bar{t}_1 \bar{x}_2 \bar{k}_{1,3}^{12} \\
 &+ 2\bar{x}_1 \bar{k}_1^{12} \bar{t}_{23} + 12\bar{u}_1 \bar{k}_{1,3}^{12} \bar{u}_2] \bar{k}_1^3 + 12(\bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{t}_4 \bar{k}_{1,5}^{34} + \bar{s}^1 \bar{s}^2 \bar{t}_{1-3} \bar{k}_1^{34} \bar{t}_{45}) \bar{k}_1^5.
 \end{aligned}$$

Proof. a_{1e} were given for $i \leq 4$ by Theorem 3. Corollaries 5.3, 5.4 agree with $a_{11}, a_{32}, a_{22}, a_{43}$ given for $P_r(x), r \leq 2$ on p59 of [2] except that D_{22} in a_{22} was overlooked. \square

Fisher and Cornish (1960) [28] showed the accuracy available using a few terms for the quantile expansions for the chi-square (or gamma), Student’s t , and F distributions. Similar results can be given for the accuracy of their Edgeworth expansions in approximating their distributions.

6. An Extension to Theorem 1

Here we remove the condition in Theorem 1 that $E \hat{w} = w$ and give the extra terms referred to but not given on p49 of [19]. We use ${}_e \bar{k}^{1-r}$ of Theorem 1, and the shorthand $\bar{f}_{.m} = \partial_{i_m} \bar{f}$ where $\partial_i = \partial / \partial w_i$. Suppose that for $r \geq 1$, the r th-order cumulants of (8) can be expanded as

$$\bar{k}^{1-r} = \kappa(\hat{w}^{i_1}, \dots, \hat{w}^{i_r}) = \sum_{e=r-1}^{\infty} {}_e \bar{k}^{1-r} \text{ for } 1 \leq i_1, \dots, i_r \leq p, \tag{30}$$

where ${}_e \bar{k}^{1-r} = O(n^{-e})$, and that $\hat{w} \rightarrow w$ as $n \rightarrow \infty$, so that ${}_0 \bar{k}^1 = \bar{w}^1$.

There is a key difference with Theorem 1: there, ${}_e \bar{k}^{1-r}$ was treated as an algebraic expression. But now we must view each of them as a function of w . So, we assume that the distribution of \hat{w} is determined by w .

The derivatives of ${}_e \bar{k}^{1-r}$ of Theorem 1 are given by Leibniz’s rule for the derivatives of a product:

$$\begin{aligned}
 {}_1 \bar{K}_{.3}^{12} &= (\bar{t}_1^1 \bar{t}_2^2 \bar{k}^{12})_{.3} = (\bar{t}_{13}^1 \bar{t}_2^2 + \bar{t}_1^1 \bar{t}_{23}^2) \bar{k}^{12} + \bar{t}_1^1 \bar{t}_2^2 \bar{k}_{.3}^{12}, \\
 {}_1 \bar{K}_{.3}^1 &= (\bar{t}_1^1 \bar{k}^{12})_{.3} / 2 = \bar{t}_{1-3}^1 \bar{k}^{12} / 2 + \bar{t}_{12}^1 \bar{k}_{.3}^{12} / 2, \\
 (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}^{1-3})_{.4} &= (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3)_{.4} \bar{k}^{1-3} + \bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 \bar{k}_{.4}^{1-3}, (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3)_{.4} = \bar{t}_1^1 \bar{t}_2^2 \bar{t}_{34}^3 + \bar{t}_1^1 \bar{t}_{24}^2 \bar{t}_3^3 + \bar{t}_{14}^1 \bar{t}_2^2 \bar{t}_3^3, \\
 (T_{1-4}^{1-3} \bar{k}^{12} \bar{k}^{34})_{.5} &= (T_{1-4}^{1-3})_{.5} \bar{k}^{12} \bar{k}_1^{34} + T_{1-4}^{1-3} (\bar{k}^{12} \bar{k}^{34})_{.5}, \\
 (T_{1-4}^{1-3})_{.5} &= \sum_{i=1}^3 (\bar{t}_{135}^1 \bar{t}_2^2 \bar{t}_4^3 + \bar{t}_{13}^1 \bar{t}_{25}^2 \bar{t}_4^3 + \bar{t}_{13}^1 \bar{t}_2^2 \bar{t}_{45}^3), (\bar{k}^{12} \bar{k}^{34})_{.5} = \bar{k}_{.5}^{12} \bar{k}^{34} + \bar{k}^{12} \bar{k}_{.5}^{34}.
 \end{aligned}$$

Theorem 4. Let \hat{w} in R^p be a biased standard estimate of w satisfying (30). Then $\hat{t} = t(\hat{w})$ in R^p is a standard estimate of $t = t(w)$:

$$\kappa(\hat{t}^{j_1}, \dots, \hat{t}^{j_r}) = \sum_{e=r-1}^{\infty} e\bar{a}^{1-r} \text{ for } r \geq 1, 1 \leq j_1, \dots, j_r \leq q, \tag{31}$$

$$\text{where } e\bar{a}^{1-r} = e\bar{K}^{1-r} + e\bar{D}^{1-r}, \quad r-1\bar{D}^{1-r} = 0, \tag{32}$$

and the other $e\bar{D}^{1-r} = eD^{i_1 \dots j_r}$ needed for $P_m(x)$ of (13) for the distribution of Y_{nt} of (12) are as follows.

For $P_0(x)$: $1\bar{D}^{12} = 0 \Rightarrow 1\bar{a}^{12} = 1\bar{K}^{12} = 1K^{j_1 j_2} = \bar{t}_1^1 \bar{t}_2^2 1\bar{k}^{12}$,
 For $P_1(x)$: $1\bar{D}^1 = 1\bar{t}^1 \bar{k}^1 \Rightarrow 1\bar{a}^1 = \bar{t}_1^1 1\bar{k}^1 + \bar{t}_{12}^1 1\bar{k}^{12}/2$,
 For $P_2(x)$: $2\bar{D}^{12} = 1\bar{K}_{.3}^{12} 1\bar{k}^3 \Rightarrow \bar{a}_2^{12} = \bar{t}_1^1 \bar{t}_2^2 2\bar{k}^{12} + T_{1-3}^{12} 2\bar{k}^{1-3}/2$
 $+ T_{1-4}^{12} 1\bar{k}^{12} 1\bar{k}^{34}/2 + (\bar{t}_{13}^1 \bar{t}_2^2 + \bar{t}_1^1 \bar{t}_{23}^2) 1\bar{k}^{12} + \bar{t}_1^1 \bar{t}_2^2 1\bar{k}_{.3}^{12} 1\bar{k}^3$.
 For $P_3(x)$: $2\bar{D}^1 = 1\bar{K}_1^1 + 0\bar{K}_2^1, 1\bar{K}_1^1 = (1\bar{K}^1)_{.3} 1\bar{k}^3$,
 $0\bar{K}_2^1 = \bar{t}_1^1 2\bar{k}^1 + \bar{t}_{12}^1 1\bar{k}^1 1\bar{k}^2/2 \Rightarrow$
 $2\bar{a}^1 = \bar{t}_1^1 2\bar{k}^1 + \bar{t}_{12}^1 (2\bar{k}^{12} + 1\bar{k}^1 1\bar{k}^2 + 1\bar{K}_{.3}^{12} 1\bar{k}^3)/2$
 $+ \bar{t}_{1-3}^1 (2\bar{k}^{1-3}/6 + 1\bar{k}^1 1\bar{k}^{23}/2) + \bar{t}_{1-4}^1 1\bar{k}^{12} 1\bar{k}^{34}/8$,
 $3\bar{D}^{1-3} = 2\bar{K}_{.4}^{1-3} \bar{k}_1^4 = (\bar{t}_1^1 \bar{t}_2^2 \bar{t}_3^3 2\bar{k}^{1-3})_{.4} 1\bar{k}^4 + (T_{1-4}^{1-3} 1\bar{k}^{12} 1\bar{k}^{34})_{.5} 1\bar{k}^5$.
 For $P_4(x)$: $3\bar{D}^{12} = 2\bar{K}_1^{12} + 1\bar{K}_2^{12}, 2\bar{K}_1^{12} = 2\bar{K}_{.3}^{12} 1\bar{k}^3$,
 $1\bar{K}_2^{12} = 1\bar{K}_{.3}^{12} 2\bar{k}^3/2 + 1\bar{K}_{.34}^{12} 1\bar{k}^3 1\bar{k}^4, 4\bar{D}^{1-4} = 3\bar{K}_{.5}^{1-4} 1\bar{k}^5, 5\bar{D}^{1-5} = 0$.

Proof. $\bar{K}^{1-r}(w) = \bar{K}^{1-r}$ and $e\bar{K}^{1-r}(w) = e\bar{K}^{1-r}$ are functions of w . By (10),

$$\bar{K}^{1-r}(w_n) = \sum_{e=r-1}^{\infty} e\bar{K}^{1-r}(w_n) \text{ for } w_n = E \hat{w} = w + d_n, \tag{33}$$

where by (30), d_n has i_1 th component $\bar{d}_n^1 = \bar{d}_n^{i_1} = \sum_{e=1}^{\infty} e\bar{k}^1$. Consider the Taylor series expansion

$$e\bar{K}^{1-r}(w + d_n) = e\bar{K}^{1-r} + e\bar{K}_{.1}^{1-r} \bar{d}_n^1 + e\bar{K}_{.12}^{1-r} \bar{d}_n^1 \bar{d}_n^2/2! + \dots = \sum_{e=0}^{\infty} e\bar{K}_e^{1-r}$$

say. Substituting into (10) gives (31) with

$$c\bar{a}^{1-r} = \sum_{k=e=c} e\bar{K}_e^{1-r} = \sum_{e=0}^{c-r+1} c-e\bar{K}_e^{1-r}$$

Also $e\bar{K}_0^{1-r} = e\bar{K}^{1-r}$ so that (32) holds with

$$c\bar{D}^{1-r} = \sum_{e=1}^{c-r+1} c-e\bar{K}_e^{1-r} : r\bar{D}^{1-r} = r-1\bar{K}_1^{1-r}, \quad r+1\bar{D}^{1-r} = \sum_{e=1}^2 r+1-e\bar{K}_e^{1-r}, \dots$$

□

The Edgeworth expansion (13) holds if $\{e\bar{K}^{1-r}\}$ are replaced by $\{e\bar{a}^{1-r}\}$.

7. Discussion

Approximations to the distributions of estimates is of vital importance in statistical inference. Asymptotic normality uses just the first term of the Edgeworth expansion. That approximation can be greatly improved with further terms. When the estimate is a sample mean, basic results were given by [29] and [30] with major advances by [31–33], Corollary

20.4 of [9], and many others. For an application to the jackknife, see [34]. See [35] for some historical references. For an application to the bootstrap, see [26]. For an application to transport, see [36]. For an application to medical research, see [27]. For an application to econometrics, see [37]. For an extension to order stats for a finite population, see [38]. For a first-order application to inference on networks, see [39]. For more historical references and a recent application to option and derivative pricing, see [40].

Extensions to stationary sequences were given by [41,42]. For a derivation of the Edgeworth expansion for a sample mean from the Gram–Charlier expansion, see [5,43] for the univariate and vector cases. These showed for the first time that the coefficients in these expansions were Bell polynomials in the cumulants.

The first extension from a sample mean for univariate estimates was by [28,44]. They assumed that the r th cumulant of the estimate was $\kappa_r(\hat{w}) = n^{1-r}k_r$ where k_r is a constant. However, in applications they assumed that \hat{w} was a Type A estimate, and collected terms. It was not until [14] that explicit results were given a univariate Type A estimate. Major advances were made in [3]. This gave explicit results for the terms in the Edgeworth expansion of a Type A or B estimate using Bell polynomials, as outlined in Section 1. It also allowed for expansions about *asymptotically* normal random variables. The advantage of this approach in greatly reducing the number of terms in each $P_r(x)$ was illustrated in [7].

For univariate estimates, Cornish and Fisher (1937) [44] also showed how to invert the Edgeworth expansion to obtain an expansion for the distribution quantiles. This was extended to Type A estimates in [14]. For extensions to transformations of multivariate estimates, like $t(\hat{w}) = (\hat{w} - w)'V^{-1}(\hat{w} - w)$, see [4,45,46]. An application to the amplitude and phase of the mean of a complex sample is given in [47].

Turning now to smooth functions of a Type A estimate, the first univariate results were given by [2] for parametric problems and [22] for nonparametric problems. These built on a deep result of [19]. This is why if \hat{w} is a Type A (or B) estimate of w , then a smooth function of \hat{w} , say $t(\hat{w})$, is a Type A (or B) estimate of $t(w)$.

The extension from a vector to a matrix estimate is just a matter of relabelling: a single sum becomes a double sum. The first examples of this we know of are in [17,24]. The extension to a *complex* scalar or vector or matrix w was given in these same papers. The first of these three papers applied it to the multi-tone problem in electrical engineering, and the other papers to channel capacity problems where \hat{w} is a weighted mean of complex matrix random variables, and n is no longer a sample size, but the number of transmitters or receivers.

A different type of extension can be obtained by identifying a sample mean $\hat{w} = \bar{X}$ from a distribution $F(x)$ with its empirical distribution $F_n(x)$, and $t(w)$ with $T(F)$, a smooth functional of $F(x)$, such as the bivariate correlation. $T(F_n)$ is a Type A estimate of $T(F)$, and its cumulant coefficient can be read off those of $t(\hat{w})$. In this way one obtains the Edgeworth expansion for $n^{1/2}(T(F_n) - T(F))$. See [15] and its references for one or more weighted samples. An extension to samples from a linear process was given by [18].

A caveat on the use of an Edgeworth expansion is that including more terms makes it more inaccurate in the tails. This is where the *tilted* expansions, also known as *saddlepoint*, or *small sample* expansions, become essential. Results for the density of Y_{nw} for a sample mean, were given in Section 5 of [12] and Section 6 of [13]. For a discussion and more references on tilting, mainly for a sample mean, see [11]. Ref. [3] shows how the cumulant coefficients given in this paper can be used to obtain the tilted expansion for the distribution and density of *any* Type A estimate.

8. Conclusions

Let \hat{w} represent a Type A estimate of an unknown parameter w belonging to R^p . Its cumulant coefficients, as defined by (1), serve as the foundational elements for the Edgeworth expansion (2) as a series $n^{-1/2}$, where n is typically the sample size.

The necessary coefficients for the r th term, $P_r(x)$, are provided in (6) and (7). Consider a smooth function $t(\hat{w})$ mapping to \mathbb{R}^q , which in turn is regarded as a Type A estimate of

$t(w)$ in \mathbb{R}^q . Consider a smooth function $t(\hat{w})$ mapping to \mathbb{R}^q , which in turn is regarded as a Type A estimate of $t(w)$ mapping to \mathbb{R}^q . This paper presents the cumulant coefficients for $t(\hat{w})$ in terms of those of \hat{w} and the derivatives of \hat{w} . Substituting these coefficients into (2) yields the Edgeworth expansion of $n^{1/2}(t(\hat{w}) - t(w))$ up to $n^{-5/2}$.

The tilted Edgeworth expansion for \hat{w} , crucial for tail accuracy, was previously delineated in [3] in terms of its cumulant coefficients. By incorporating those of $t(\hat{w})$ as presented here, we derive the tilted Edgeworth expansion for $t(\hat{w})$.

In many practical statistical estimation problems, simulations serve as a favored method for approximating distributions. However, their limitation lies in their inability to comprehensively represent the entire parametric landscape.

We have showcased some applications in electrical engineering. For instance, ref. [17] offered numerical comparisons of the initial three approximations to channel capacity for multiple arrays with multiple frequencies and delay spread. Given $p = 1$, this permitted an expansion for the percentile. There exist myriad other potential applications across electrical engineering and allied fields.

Lastly, we outline potential future research avenues. Chain rules applied to $t(\hat{w})$ can yield the cumulant coefficients of its Studentised form, paving the way for expansions in the coverage probability of confidence regions and enhancements in their accuracy. These coefficients find applications in bias reduction, Bayesian inference, confidence regions, and power analysis. While the Edgeworth expansion can sometimes yield negative values in distribution tails, tilted expansions circumvent this issue. Alternatively, selecting y in \mathbb{R}^p such that $\nabla_{nx} = Prob.(Y_{nw} \leq x + n^{-1/2}y) - \Phi_V(x)$ is $O(n^{-1})$ offers another approach. For $p > 1$ the diversity of y choices allows for potential reductions to $\nabla_{nx} = O(n^{-3/2})$ or smaller. Introducing $n^{-1/2}y$ replacements such as $y(n) = n^{-1/2}y_0 + n^{-1}y_1$ further expands the range of options.

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Appendix A. Some Comments on the References

Here, we give some comments and corrections to some of our papers.

Withers (1982) [2]: To the expression for a_{22} on p59 add $c_{12} + 2c_{16}$ where

$$c_{12} = I_2 \binom{12}{01} = t_i t_j k_{1,k}^{ij} k_1^k, \quad c_{16} = I_{11} \binom{12}{00} = t_i k_1^{ij} t_{jk} k_1^k.$$

This correction does not effect applications in which \hat{w} is unbiased, as in [2,23].

In the expression on p60 for $(a_{22})_2$, $I_{31} \binom{23}{22}$ should be $c_{36} = I_{31} \binom{23}{00}$.

On p61, 4 lines before Table 1, replace $(n/2)^{r-1}$ by $(n/2)^{1-r}$.

On p67, add to K_2^{ab} , $t_i^a t_{jb} k_{1,k}^{ij} k_1^k$. For $r = 3, 4$ see Section 4.

On p68 in (A3), replace $\binom{r+2}{r}$ by $\binom{r+2}{2}$. Changing to the simpler notation of [15], denote the expressions for $I_2 \binom{2}{0}, \dots, I_{301} \binom{222}{000}$ and $I_3 \binom{22}{01}, \dots, I_{31} \binom{222}{001}$ given on pages 58–59 by $\sigma^2 = V = V(w) = t_i k_1^{ij} t_j$, c_{01}, c_{02}, c_{21} , and $c_{23}, c_{11}, c_{15}, c_{19}, c_{1,10}, c_{31}, c_{36}, c_{3,10}, c_{3,11}, c_{22}, c_{12}, c_{14}, c_{17}, c_{32}, c_{34}, c_{35}, c_{38}, c_{39}$. So, the expressions on pages 59–60 become

$$\begin{aligned}
 a_{11} &= c_{01} + c_{02}/2, \quad a_{32} = c_{21} + 3c_{23}, \\
 a_{22} &= c_{11} + c_{15} + c_{19}/2 + c_{1,10} + c_{12} + 2c_{16} \text{ by Corollary 3,} \\
 a_{43} &= c_{31} + 12c_{36} + 12c_{3,10} + 4c_{3,11}, \\
 (a_{11})_1 &= \sigma^{-1}(c_{01} + c_{02}/2) - \sigma^{-3}(c_{22}/2 + c_{23}), \quad (a_{32})_2 = \sigma^{-3}(c_{21} - 3c_{22} - 3c_{23}), \\
 (a_{22})_2 &= \sum_{i=1}^3 V^{-i} A_i, \quad (a_{43})_4 = \sum_{i=2}^3 V^{-i} B_i \text{ for} \\
 A_1 &= c_{11} - c_{14}/2 + c_{15} - 2c_{17} - c_{19}/2, \quad A_2 = -(c_{01} + c_{02}/2)(c_{22} + 2c_{23}) \\
 &\quad - c_{32} - c_{34} + c_{35} - 2c_{36} - 4c_{38} + 2c_{39} - 2c_{3,10} - 2c_{3,11}, \\
 A_3 &= 7(c_{22} + 2c_{23})^2/4, \quad B_2 = c_{31} - 6c_{32} - 6c_{34} + 3c_{35} - 24c_{38} - 12c_{3,10} - 8c_{3,11}, \\
 B_3 &= 6(c_{22} + 2c_{23})(-c_{21} + 3c_{22} + 3c_{23}).
 \end{aligned}$$

We now illustrate how the results on p.60 were obtained. Let c'_{rs} denote c_{rs} when $t(\hat{w})$ is replaced by its Studentised form $t_{(0)}(\hat{w})$. Then

$$(a_{22})_2 = c'_{11} + c'_{15} + c'_{19}/2 + c'_{1,10} + c'_{12} + 2c'_{16}, \tag{A1}$$

The first few derivatives of $t_{(0)}(\hat{w})$ at w , and of $V(w)$, are

$$\begin{aligned}
 t_{0,i} &= V^{-1/2}t_i, \quad t_{0,ij} = V^{-1/2}t_{ij} - V^{-3/2}(t_iV_j + t_jV_i)/2, \\
 t_{0,ijk} &= V^{-1/2}t_{ijk} - V^{-3/2} \sum_{ijk} (t_{ij}V_k + V_{ij}t_k)/2 + 3V^{-5/2} \sum_{ijk} t_iV_jV_k/4, \\
 V_i &= 2t_{ia}k_1^{ab}t_b + k_{1,i}^{ab}t_a t_b, \text{ and} \\
 V_{ij} &= 2t_{ija}k_1^{ab}t_b + 2t_{ia}k_1^{ab}t_{jb} + 2 \sum_{ij} t_{ia}k_{1,j}^{ab}t_b + k_{1,ij}^{ab}t_a t_b, \text{ where}
 \end{aligned}$$

$$\sum_{ij}^2 a_i b_j = a_i b_j + a_j b_i, \quad \sum_{ijk}^3 a_{ij} b_k = a_{ij} b_k + a_{ik} b_j + a_{jk} b_i \text{ for } a_{ij} = a_{ji}.$$

$$\begin{aligned}
 \text{So } c'_{11} &= V^{-1}c_{11}, \quad c'_{15} = V^{-1}c_{15} - V^{-2}M_1, \\
 c'_{19} &= V^{-1}c_{19} - 2V^{-2}M_4 + V^{-3}(VM_2 + M_3^2)/2, \\
 c'_{1,10} &= V^{-1}c_{1,10} - V^{-2}(M_3c_{02} + 2M_4 + VM_5 + 2M_6)/2 + 3V^{-3}(VM_2 + 2M_3^2)/4, \\
 c'_{12} &= V^{-1}c_{12}, \quad c'_{16} = V^{-1}c_{16} - V^{-2}(VM_7 + c_{01}M_3)/2, \text{ where} \\
 M_1 &= t_i t_j k_2^{ijk} V_k = c_{32} + 2c_{36}, \quad M_2 = V_i k_1^{ij} V_j = c_{35} + 4c_{39} + 4c_{3,10}, \\
 M_3 &= t_i k_1^{ij} V_j = c_{22} + 2c_{23} = \tilde{c}_{22} \text{ say, } \quad M_4 = t_i k_1^{ij} t_{jk} k^{kl} V_l = c_{39} + 2c_{3,10}, \\
 M_5 &= k_1^{ij} V_{ij} = 2c_{1,10} + 2c_{19} + c_{14} + 4c_{17}, \\
 M_6 &= t_i k_1^{ij} V_{jk} k^{kl} t_l = 2c_{3,10} + 2c_{3,11} + c_{34} + 4c_{38}, \\
 M_7 &= k_1^i V_i = c_{12} + 2c_{16} \Rightarrow \\
 c'_{19} &= V^{-1}c_{19} + 2V^{-2}(4c_{35} - c_{3,10}) + V^{-3}\tilde{c}_{22}^2/2, \\
 c'_{1,10} &= \sum_{i=1}^3 V^{-i} b_i \text{ for } b_1 = -c_{14}/2 - 2c_{17} - c_{19}, \quad b_3 = 3\tilde{c}_{22}^2/2, \\
 b_2 &= -\tilde{c}_{22}c_{02}/2 - c_{34} + 3c_{35}/4 - 4c_{38} + 2c_{39} - c_{3,10} - 2c_{3,11}.
 \end{aligned}$$

Substitution into (A1) yields $(a_{22})_2$. The other $a'_{ri} = (a_{ri})_r$ given on p.60 are obtained similarly.

Withers (1984) [14]:

p393: In the 5-line expression for $f_4(x, L)$, replace $(x^3 - x)/30$ by $(x^3 - 3x)/30$, and $+2473)/7776$ by $+2473x)/7776$.

p394: In (3.4), replace $(x^2 - 1)/2$, by $(x^2 - 1)/6$,

p394: In line 2 of Section 4, 'of Section 2' should be 'of Section 3'.

The following corrigendum for a printer's error appeared in Withers, C.S. *J. R. Stat. Soc. B*, **1986**, 48, p258:

The expression $-l_3^4(252x^5 - 1688x^3 + 1511x)/7776$ should be added to the last line on p393.

That also gives $g_5(y)$ and $g_6(y)$ for the last line on p393.

Withers (1987) [21]:

p2371: (2.4): $\sum_{i=1}^{\infty} n^{-i} C_i$ need not converge. We only require an asymptotic expansion. The same is true for (3.2) p2375.

p2371, 3rd to last paragraph: Replace 'Appendix C, which also' with 'Appendix D. Appendix C'

p2372, Example 2.2, line 2: Replace $t^{2j}(\theta)$ with $t^{(2j)}(\theta)$, the $2j$ derivative of $t(\theta)$. In line 3 and in Example 3.1, $(y)_j = y(y-1) \cdots (y-j+1)$.

p2377 line 2: Replace (1.2) with (2.4)

p2377 line 3: Replace $|N|$ with N

p2377 line 9: Replace 'Section 2' with 'Section 3' Since $E_1 = 0$, C_1 of Example 3.2 p2376 is unaffected. p2378: These expression for C_j are correct if $\hat{\theta}$ is unbiased. In that case, the terms on p2378 with a 1 in the top line are 0 so that C_j has only m_j terms where $m_1 = 1, m_2 = 3, m_3 = 6, m_4 = 12$. However, if $\hat{\theta}$ is biased, then these expression for C_j did not allow for contributions from replacing θ by $E \hat{\theta}$ in the cumulant coefficients $k_j^{a_1 \cdots a_r}$ of (3.2). These are corrected in Withers, C.S. and Nadarajah, S. (Submitted), Bias-reduced estimates for parametric problems.

p2379 Appendix D: Add at start: For $p = 1$ see (3.4) of Withers, C.S. and Nadarajah, S., *Journal of Multivariate Analysis*, **2013**, 118, 138–147.

Withers (1988) [23]:

p729: In the 10th line from the bottom, replace "their range $1 \cdots p$ " with "their range $1 \cdots k$ "

p732 line 9: $T_{x_i x_j \dots}^{a_i a_j \dots}$ should be $T_{x_1 x_2 \dots}^{a_1 a_2 \dots}$.

p734: In the expression for h_1 in the 5th to last line, replace He_2 with $He_2/6$.

p737: In line 11, "Sections 1 and 2 of Withers (1983a)" should read "Sections 1 and 2 of Withers (1983b)".

p741: In the 4th equation from the bottom, at the end of the line, replace $[1^j]T_2^j$ with $[1^j]_1 T_2^j$,

Withers and Nadarajah (2008) [15]:

p743 para 2, line 4: Replace 'about zero.' with 'about zero when G puts mass 1 at x '.

p754, p756: Replace w_{ina} with w_{ina} . Different samples can have different weights.

p754, 2nd-to-last line: The first term on RHS, $c_{11}/2$, should be c_{11} .

p755, line 6: There is a typesetting error in the first of the 2 lines for a_{220} . Replace the first line with

$$a_{220} = \sigma^{-2}(c_{11} - c_{12} - c_{14}/2 + c_{15} - 2c_{17} - c_{19}/2) - \sigma^{-4}[(c_{01} + c_{02}/2)(c_{22} + 2c_{23}) + c_{32}$$

p756: The 3rd and 4th lines after (8), should be

$$[1^r]_a = \int T_F(x)^r dF_a(x),$$

$$[1, 12, 2^r]_{a_1 a_2} = \int \int T_F(x_1) T_F(x_1 x_2)^{a_1 a_2} T_F(x_2)^r dF_{a_1}(x_1) dF_{a_2}(x_2),$$

Withers and Nadarajah (2009) [43]:

p272. Line 3: Convergence of $S(t)$ is not needed, since B_{rk} is a finite sum. κ_r on LHS(1.1) should be k_r .

p273 last paragraph: Also, $f(x)/\phi(x)$ is only meaningful if X is dimension-free.
 p275. (2.8) is correct, but since $B_1 = B_2 = 0$, (2.8) can also be written as

$$\int f^2/\phi = 1 + \sum_{k=3}^{\infty} B_k^2/k!.$$

In the 5th line of Section 3, insert after $B_{rk}(\alpha)$, ‘at $\alpha_1 = \alpha_2 = 0$ ’.
 The first line of (3.1) should read

$$B_r = \sum(B_{rk}e^{r-2k} : 1 \leq k \leq r/3)$$

(3.2) can be written $K_s = s - 2[s/3]$ where $[x]$ is the integral part of x .

p276: In the expression for B_{10} , $B_{10,18}$ should be $B_{10,1}$.

p277: In the 2nd-to-last line, b_{64} should be B_{61} .

p278: In the expression for b_4 , the first term should be doubled. In the expression for b_5 , b_{82} should be B_{82} .

Withers and Nadarajah (2010a) [16]:

p3: In the 5th- and 6th-to-last lines, replace $= n^{-2}K_{ab} + O(n^{-2})$ with $= n^{-1}K_{ab} + O(n^{-2})$

p5: 2 lines above Theorem 2.2, replace “third moments” with “third central moments”
 p7, lines 2–3: delete “and its Studentised version”

p7, lines 3–4: delete “or $n^{-1/2}(\beta_a - \beta_a)\hat{f}_{aa}^{-1/2}$ ”

p7, line 7–10: delete from “So, a one-sided” to “by $O(n^{-1/2})$ ”;

p9: Move “Set $\phi' = \dots m = p + 1$. on the last 2 lines of p9 and 1st line of p10 to just before “Set” on p9 line 9.

p10, lines 14–15: replace “ g_{ij} where” by “ g_{ij} .” and move the rest of the sentence, “ $g_{ij} = \dots g_{N,ij}$...” to the line after (6.1) p9, preceded by the word “Set”

Withers and Nadarajah (2010b) [3]:

p1129, line 7: replace $b_k^{i_1 \dots i_r}(Y_n)$ by $b_k^{i_1 \dots i_r}(Y)$

To the 9th-to-last line we can add

$$\tilde{P}_3(t, B) = e_3(t) + e_1(t)e_2(t) + e_1(t)^3/6,$$

From p1130 line 6 to the end of Section 5: Replace s with p , the dimension of θ .
 p1130 line 7 is clearer, we replace line 8 with

$$\text{for } p_{(n)}^{(k)}(y) = \partial_{k_1} \dots \partial_{k_p} p_{(n)}(y), \partial_k = \partial/\partial y_k,$$

p1130 line 9: Replace $H_{v+k}(y)$ with $H_{v+k}(y, V)$

p1130, 5th- and 6th-to-last lines: for example $K_j^{i_1 \dots i_r} = k_j^{(i_1 \dots i_r)}(t) = \partial_{i_1} \dots \partial_{i_r} k_j(t)$ where $\partial_i = \partial/\partial t_i$.

p1132: A note on Corollary 3.2. For the duality of $I(x)$ and $k_0(t)$ see p176 of McCullagh, P., *Tensor methods in statistics*. Chapman and Hall, London, 1987.

p1133: In line 14, replace $H_{v+\lambda}(\theta, V_t)$ with $H_{v+\lambda}(0, V_t)$

Withers and Nadarajah (2014a) [5]:

p81: In (2.14), replace $J_r(x)$ and $J'_r(x)$ with $J_r(x)/r!$ and $J'_r(x)/r!$.

p81: The 2nd line after (2.15) should read

$$J'_r(x) = \int_{V\bar{y} \leq V\bar{x}} \bar{H}_r(\bar{y}, \bar{V}) \phi_{\bar{V}}(\bar{y}) d\bar{y}$$

The next line is correct:

$$J'_r(x) = \int_{V\bar{y} \leq V\bar{x}} (-\partial_{\bar{y}})^r \phi_{\bar{V}}(\bar{y}) d\bar{y}.$$

p82: In (2.20), replace $J_r(y_1, y_2)$ with $J_r(y_1, y_2)/r!$.

p85: In Withers, C.S. and Nadarajah, S. (2009), replace ‘via’ with ‘in terms of’.
Withers and Nadarajah (2014b) [7]:

p676. Multiply RHS of (1.13) by $n^{1/2}$. That is, replace it by

$$Y_{JK\theta} = (n/s_{2K\theta})^{1/2}(\hat{\theta} - s_{1J\theta}), J \geq 0, K \geq 1.$$

p699: In the editing of the original paper of 64 pages down to 21 pages, some details had to be removed. Here, are some more details for Theorem 1.2 after (1.24)

$$\begin{aligned} \nabla_1 &= 0 \text{ if } J \geq 1, \nabla_2 = \bar{A}_{43}H_2 \text{ if } K \geq 2, \\ \nabla_3 &= \bar{A}_{33}H_2 + \bar{A}_{54}H_4 \text{ if } J \geq 2, \nabla_4 = \bar{A}_{44}H_3 + \bar{A}_{65}H_5 \text{ if } K \geq 3, \\ \nabla_5 &= \bar{A}_{34}H_2 + \bar{A}_{55}H_4 + \bar{A}_{76}H_6 \text{ if } J \geq 3, \\ \nabla_6 &= \bar{A}_{45}H_3 + \bar{A}_{66}H_5 + \bar{A}_{87}H_7 \text{ if } K \geq 4. \\ \nabla_{re} &= 0 \text{ if } r \leq 3 \text{ for } e = h, f, g, \nabla_{4e} = [4^2]_0 e(4^2), \\ \nabla_{5e} &= [45]_0 e(45) + [34]_1 e(34), \\ \nabla_{6e} &= \sum\{[\pi]_0 e(\pi) : \pi = 5^2, 46, 4^3\} + \sum\{[\pi]_1 e(\pi) : \pi = 4^2, 35\} + [3^2]_2 e(3^2), \end{aligned}$$

where the $e(\pi), [\pi]_i$ needed for $e_r(x), 1 \leq r \leq 6$, are as follows.

$$\begin{aligned} h(ij \dots) &= H_{i+j+\dots-1} \text{ so that } h(1^i 2^i \dots) = H_{1i_1+2i_2+\dots-1}. \\ \text{For } r = 4 : [4^2]_0 &= \bar{A}_{43}^2/2!, h(4^2) = H_7, \\ f(4^2) &= H_7 - H_1H_3^2, g(4^2) = H_7 - 2H_3H_4 + H_1H_3^2. \\ \text{For } r = 5 : [45]_0 &= \bar{A}_{43}\bar{A}_{54}, h(45) = H_8, \\ f(45) &= H_8 - H_1H_3H_4, g(45) = H_8 - H_3H_5 - H_4^2 + H_1H_3H_4, \\ [34]_1 &= \bar{A}_{33}\bar{A}_{43}, h(34) = H_6, f(34) = H_6 - H_1H_2H_3, \\ g(34) &= H_6 - H_2H_4 - H_3^2 + H_1H_2H_3. \\ \text{For } r = 6 : [5^2]_0 &= \bar{A}_{54}^2/2!, h(5^2) = H_9, \\ f(5^2) &= H_9 - H_1H_4^2, g(5^2) = H_9 - 2H_4H_5 + H_1H_4^2, \\ [46]_0 &= \bar{A}_{43}\bar{A}_{65}, h(46) = H_9, \\ f(46) &= H_9 - H_1H_3H_5, g(46) = H_9 - H_3H_6 - H_4H_5 + H_1H_3H_5, \\ [4^3]_0 &= \bar{A}_{43}^3/3!, h(4^3) = H_{11}, f(4^3) = H_{11} - 3H_1H_3H_7 - H_2H_3^3 + 3H_1^2H_3^3, \\ g(4^3) &= H_{11} - 3H_3H_8 - 3H_4H_7 + 3H_1H_3H_7 + 3H_3^2H_5 + 6H_3H_4^2 \\ &\quad - 9H_1H_3^2H_4 - H_2H_3^3 + 3H_1^2H_3^3, \\ [4^2]_1 &= \bar{A}_{43}\bar{A}_{44}, \\ [35]_1 &= \bar{A}_{33}\bar{A}_{54}, f(35) = H_7 - H_1H_2H_4, g(35) = H_7 - H_3H_4 - H_2H_5, \\ [3^2]_2 &= \bar{A}_{33}^2/2!, f(3^2) = H_5 - H_1H_2^2, g(3^2) = H_5 - 2H_2H_3 + H_1H_2^2. \end{aligned}$$

p702 Section 2: In the 3rd equation of Theorem 1, $\ln \Gamma(\mu)$ should be $\ln \Gamma(m)$. p704: Disregard Table 3.

Withers and Nadarajah (2015) [8]:

In (22) and the formulas for d_2, \dots, d_6 that follow, replace d_r by

$$\bar{d}_r = rd_r = c_r(x)/(r-1)!.$$

As stated this gives $c_r = c_r(x) = r!d_r$. For example,

$$\begin{aligned}c_2 &= a_1, \quad c_3 = 2a_1^2 + a_2, \quad c_4 = 3!a_1^3 + 7a_1a_2 + a_3, \\c_5 &= 4!a_1^4 + 46a_1^2a_2 + 11a_1a_3 + 7a_2^2 + a_4, \\c_6 &= 5!a_1^5 + 326a_1^3a_2 + 101a_1^2a_3 + 127a_1a_2^2 + 16a_1a_4 + 25a_2a_3 + a_5.\end{aligned}$$

In the first reference, [1], replace J.J. Alfredo with J.A. Jimenez.

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