

Article

On Soft ω_δ -Open Sets and Some Decomposition Theorems

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Abstract: In this paper, we present a novel family of soft sets named “soft ω_δ -open sets”. We find that this class constitutes a soft topology that lies strictly between the soft topologies of soft δ -open sets and soft ω^0 -open sets. Also, we introduce certain sufficient conditions for the equivalence between this new soft topology and several existing soft topologies. Moreover, we verify several relationships that contain soft covering properties, such as soft compactness and soft Lindelofness, which are related to this new soft topology. Furthermore, in terms of the soft interior operator in certain soft topologies, we define four classes of soft sets. Via them, we obtain new decomposition theorems for soft δ -openness and soft θ -openness, and we characterize the soft topological spaces that have the soft “semi-regularization property”. In addition, via soft ω_δ -open sets, we introduce and investigate a new class of soft functions named “soft ω_δ -continuous functions”. Finally, we look into the connections between the newly proposed soft concepts and their counterparts in classical topological spaces.

Keywords: soft δ -open sets; soft θ -open sets; soft ω^0 -open sets; super-continuity; soft generated soft topological spaces

MSC: 54A10; 54A40; 54D1



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1. Introduction and Preliminaries

In today’s complex world, accurate modeling and management of many types of uncertainty are essential to tackle difficult issues in different fields, including environmental science, economics, engineering, social sciences, and medicine. While well-known techniques like probability theory, fuzzy sets [1], and rough sets [2] help handle ambiguity and uncertainty, they are not without limitations. These mathematical methods all share the same flaw, which is insufficient parameterization capabilities. In 1999, Molodtsov [3] introduced soft set theory as a solution to the shortcomings of earlier uncertainty-handling techniques. After that, the interpretation of soft sets for modeling uncertainty has been conducted; advancements in this area are described in [4,5]. Equipped with soft sets, parameter sets offer a defined framework that is naturally adaptable, facilitating the modeling of unclear data. Soft set theory and related fields have advanced greatly as a result very soon. As may be observed in [6–12], this has led to several applications of soft sets in real-world fields.

Numerous mathematicians have used soft set theory to introduce various mathematical structures, including soft group theory [13], soft ring theory [14], soft convex structures [15], and soft ideals [16]. These papers highlight the use of soft set theory in handling challenging mathematical problems.

Shabir and Naz [17] created soft topology first, and since then, a lot of researchers have focused on extending the topological concepts to include the field of soft topology. For instance, soft metric spaces [18–20], soft connected spaces [21], soft covering properties [22–24], and generalized soft open sets [25–29] are a few of the notions mentioned.

Recent papers [30–37] show that research in soft topology is currently ongoing and that there is still an opportunity for important contributions.

The generalizations of soft open sets play an effective role in the structure of soft topology by using them to redefine and investigate some soft topological concepts such as soft continuity, soft compactness, and soft separation axioms. This paper follows this area of research.

The arrangement of this article is as follows:

In Section 2, we define soft ω_δ -open sets. We study the features of sets and show how they relate to well-known other classes of soft sets, like soft δ -open sets and soft ω^0 -open sets. Furthermore, we investigate the links between this class of soft sets and its classical topology analogs. We also investigate several relationships that contain soft covering properties, such as soft compactness and soft Lindelofness.

In Section 3, we define four new classes of soft sets. We use them to provide novel decomposition theorems for soft δ -openness and soft θ -openness, as well as characterize semi-regularized soft topological spaces.

In Section 4, via soft ω_δ -open sets, we define soft ω_δ -continuous functions as a new class of soft functions and investigate some of their properties. We give several characterizations of it. Also, we investigate the links between this class of soft functions and its analogs in general topology. Moreover, we show that soft ω_δ -continuity is strictly weaker than soft ω^0 -continuity.

In Section 5, we give some findings and potential future studies.

Throughout this paper, we will use the concepts and terminology as they appear in [38,39].

Here, we recall some basic definitions and results that will be needed in this sequel.

Let M be an initial universe and Z be a set of parameters. A soft set over M relative to Z is a function $T : Z \rightarrow \mathcal{P}(M)$, where $\mathcal{P}(M)$ is the power set of M . The collection of all soft sets over M relative to Z is denoted by $SS(M, Z)$. Let $G \in SS(M, Z)$. If $G(a) = \emptyset$ for every $a \in Z$, then G is called the null soft set over M relative to Z and denoted by 0_Z . If $G(a) = M$ for all $a \in Z$, then G is called the absolute soft set over M relative to Z and denoted by 1_Z . If there exist $x \in M$ and $a \in Z$ such that $G(a) = \{x\}$ and $G(b) = \emptyset$ for all $b \in Z - \{a\}$, then G is called a soft point over M relative to Z and denoted by a_x . The collection of all soft points over M relative to Z is denoted by $SP(M, Z)$. If for some $a \in Z$ and $X \subseteq M$, $G(a) = X$ and $G(b) = \emptyset$ for all $b \in Z - \{a\}$, then G will be denoted by a_X . If for some $X \subseteq M$, $G(a) = X$ for all $a \in Z$, then G will be denoted by C_X . G is called a countable soft set over M relative to Z if $G(a)$ is countable for all $a \in Z$. The collection of all countable soft sets over M relative to Z will be denoted by $C(M, Z)$. If $G \in SS(M, Z)$ and $a_x \in SP(M, Z)$, then a_x is said to belong to G (notation: $a_x \in G$) if $x \in G(a)$.

Soft topological spaces were defined in [17] as follows: A triplet (M, \mathcal{Y}, Z) , where $\mathcal{Y} \subseteq SS(M, Z)$, is called a soft topological space if $0_Z, 1_Z \in \mathcal{Y}$, and \mathcal{Y} is closed under finite soft intersections and arbitrary soft unions.

Let (M, \mathcal{Y}, Z) be a soft topological space, and let $H \in SS(M, Z)$. Then the members of \mathcal{Y} are called soft open sets. The soft complements of the members of \mathcal{Y} are called soft closed sets in (M, \mathcal{Y}, Z) . The family of all soft closed sets in (M, \mathcal{Y}, Z) will be denoted by \mathcal{Y}^c . The soft interior and the soft closure of H in (M, \mathcal{Y}, Z) will be denoted by $Int_{\mathcal{Y}}(H)$ and $Cl_{\mathcal{Y}}(H)$, respectively. Let (M, λ) be a topological space, and let $U \subseteq M$. The interior and the closure of U in (M, λ) will be denoted by $Int_{\lambda}(U)$ and $Cl_{\lambda}(U)$, respectively.

Definition 1 ([40]). *Let (M, λ) be a topological space, and $V \subseteq M$. Then V is said to be a δ -open set in (M, λ) if for every $x \in V$, we find $D \in \lambda$ such that $x \in D \subseteq Int_{\lambda}(Cl_{\lambda}(D)) \subseteq V$. λ_{δ} denotes the family of all δ -open sets in (M, λ) .*

It is well known that (M, λ_{δ}) is a topological space with $\lambda_{\delta} \subseteq \lambda$.

Definition 2 ([41]). Let (M, λ) be a topological space, and $V \subseteq M$. Then V is said to be a ω_δ -open set in (M, λ) if for every $x \in V$, we find $D \in \lambda$ such that $x \in D$ and $D - \text{Int}_{\lambda_\delta}(V)$ is a countable set. λ_{ω_δ} denotes the family of all ω_δ -open sets in (M, λ) .

It is proved in [41] that $(M, \lambda_{\omega_\delta})$ is a topological space.

Definition 3 ([41]). A function $g : (M, \lambda) \rightarrow (N, \gamma)$ between the topological spaces (M, λ) and (N, γ) is called ω_δ -continuous if $g^{-1}(V) \in \lambda_{\omega_\delta}$ for every $V \in \gamma$.

Definition 4 ([39]). Let (M, \mathcal{Y}, Z) be a soft topological space and $K \in SS(M, Z)$. Then
 (a) K is a soft ω -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} K$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - K \in C(M, Z)$. \mathcal{Y}_ω will denote the family of all soft ω -open sets in (M, \mathcal{Y}, Z) .
 (b) K is a soft ω -closed set in (M, \mathcal{Y}, Z) if $1_Z - K \in \mathcal{Y}_\omega$.

It is proved in [39] that $(M, \mathcal{Y}_\omega, Z)$ is a soft topological space, $\mathcal{Y} \subseteq \mathcal{Y}_\omega$, and $\mathcal{Y} \neq \mathcal{Y}_\omega$ in general.

Definition 5. Let (M, \mathcal{Y}, Z) be a soft topological space and $H \in SS(M, Z)$. Then
 Ref. [42] (a) H is a soft θ -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G \subseteq \text{Cl}_\mathcal{Y}(G) \subseteq H$. The family of all soft θ -open sets in (M, \mathcal{Y}, Z) will be denoted by \mathcal{Y}_θ .
 Ref. [43] (b) H is a soft δ -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G \subseteq \text{Int}_\mathcal{Y}(\text{Cl}_\mathcal{Y}(G)) \subseteq H$.
 Ref. [39] (c) H is a soft ω -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - H \in C(M, Z)$.
 Ref. [44] (d) H is a soft ω^0 -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - \text{Int}_\mathcal{Y}(H) \in C(M, Z)$.
 Ref. [45] (e) H is a soft ω_θ -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - \text{Int}_{\mathcal{Y}_\theta}(H) \in C(M, Z)$.
 Ref. [46] (f) H is a soft regular-open set in (M, \mathcal{Y}, Z) if $H = \text{Int}_\mathcal{Y}(\text{Cl}_\mathcal{Y}(H))$.

\mathcal{Y}_δ (\mathcal{Y}_ω , \mathcal{Y}_{ω^0} , $\mathcal{Y}_{\omega_\theta}$, and $RO(\mathcal{Y})$) will denote the family of all soft δ -open (resp. ω -open, ω^0 -open, ω_θ -open, and regular open) sets in (M, \mathcal{Y}, Z) .

It is known that \mathcal{Y}_θ , \mathcal{Y}_δ , \mathcal{Y}_ω , \mathcal{Y}_{ω^0} , and $\mathcal{Y}_{\omega_\theta}$ are all soft topologies such that $\mathcal{Y}_\theta \subseteq \mathcal{Y}_\delta \subseteq \mathcal{Y} \subseteq \mathcal{Y}_{\omega^0} \subseteq \mathcal{Y}_\omega$ and $\mathcal{Y}_\theta \subseteq \mathcal{Y}_{\omega_\theta} \subseteq \mathcal{Y}_{\omega^0}$.

Definition 6. A soft topological space (M, \mathcal{Y}, Z) is called:
 Ref. [39] (a) Soft locally countable if it has a soft base $\mathcal{K} \subseteq C(M, Z)$.
 Ref. [39] (b) Soft anti-locally countable (soft A-L-C) if $\mathcal{Y} \cap C(M, Z) = \{0_Z\}$.
 Ref. [24] (c) Soft Lindelof if for every $\mathcal{H} \subseteq \mathcal{Y}$ such that $\tilde{\cup}_{H \in \mathcal{H}} H = 1_Z$, there is a countable subcollection $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $\tilde{\cup}_{H \in \mathcal{H}_1} H = 1_Z$.
 Ref. [47] (d) Soft nearly compact if for every $\mathcal{H} \subseteq RO(\mathcal{Y})$ such that $\tilde{\cup}_{H \in \mathcal{H}} H = 1_Z$, there is a finite subcollection $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $\tilde{\cup}_{H \in \mathcal{H}_1} H = 1_Z$.
 Ref. [47] (e) Soft nearly Lindelof if for every $\mathcal{H} \subseteq RO(\mathcal{Y})$ such that $\tilde{\cup}_{H \in \mathcal{H}} H = 1_Z$, there is a countable subcollection $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $\tilde{\cup}_{H \in \mathcal{H}_1} H = 1_Z$.
 Ref. [48] (f) Soft regular if for every $a_x \in SP(M, Z)$ and every $G \in \mathcal{Y}$ such that $a_x \tilde{\in} G$, there exists $H \in \mathcal{Y}$ such that $a_x \tilde{\in} H \subseteq \text{Cl}_\mathcal{Y}(H) \subseteq G$.
 Ref. [49] (g) Soft semi-regularization topology if $\mathcal{Y} = \mathcal{Y}_\delta$.

Definition 7 ([50]). A soft function $f_{q\theta} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is called soft ω^0 -continuous if $f_{q\theta}^{-1}(K) \in \mathcal{Y}_{\omega^0}$ for every $K \in \mathcal{X}$.

Theorem 1 ([17]). For any soft topological space (M, \mathcal{Y}, Z) and any $a \in Z$, the family

$$\{G(a) : G \in \mathcal{Y}\}$$

forms a topology on M . This topology is denoted by \mathcal{Y}_a .

Theorem 2 ([38]). For any family of topological spaces $\{(M, \beta_a) : a \in A\}$, the family

$$\{G \in SS(M, A) : G(a) \in \beta_a \text{ for all } a \in A\}$$

forms a soft topology on M relative to A . This soft topology is denoted by $\bigoplus_{a \in A} \beta_a$.

Theorem 3 ([38]). For any topological space (M, λ) and any set of parameters Z , the family $\{G \in SS(M, Z) : G(z) \in \lambda \text{ for all } z \in Z\}$ defines a soft topology on M relative to Z . $\tau(\lambda)$ denotes this soft topology.

2. Soft ω_δ -Open Sets

Definition 8. Let (M, \mathcal{Y}, Z) be a soft topological space and $K \in SS(M, Z)$. Then

- (a) K is a soft ω_δ -open set in (M, \mathcal{Y}, Z) if for any $z_m \tilde{\in} K$, we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - \text{Int}_{\mathcal{Y}_\delta}(K) \in C(M, Z)$. $\mathcal{Y}_{\omega_\delta}$ will denote the family of all soft ω_δ -open sets in (M, \mathcal{Y}, Z) .
- (b) K is a soft ω_δ -closed set in (M, \mathcal{Y}, Z) if $1_Z - K \in \mathcal{Y}_{\omega_\delta}$.

Theorem 4. Let (M, \mathcal{Y}, Z) be a soft topological space and $H \in SS(M, Z)$. Then $H \in \mathcal{Y}_{\omega_\delta}$ if and only if for each $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ and $R \in C(M, Z)$ such that $z_m \tilde{\in} G$ and $G - R \subseteq \text{Int}_{\mathcal{Y}_\delta}(H)$.

Proof. Necessity. Suppose that $H \in \mathcal{Y}_{\omega_\delta}$. Let $z_m \tilde{\in} H$. Then we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - \text{Int}_{\mathcal{Y}_\delta}(H) \in C(M, Z)$. Let $R = G - \text{Int}_{\mathcal{Y}_\delta}(H)$. Then $R \in C(M, Z)$ and $G - R = \text{Int}_{\mathcal{Y}_\delta}(H) \subseteq \text{Int}_{\mathcal{Y}_\delta}(H)$.

Sufficiency. Suppose that for each $z_m \tilde{\in} H$, we find $G \in \mathcal{Y}$ and $R \in C(M, Z)$ such that $z_m \tilde{\in} G$ and $G - R \subseteq \text{Int}_{\mathcal{Y}_\delta}(H)$. Let $z_m \tilde{\in} H$. Then, by assumption, we find $G \in \mathcal{Y}$ and $R \in C(M, Z)$ such that $z_m \tilde{\in} G$ and $G - R \subseteq \text{Int}_{\mathcal{Y}_\delta}(H)$. Since $G - R \subseteq \text{Int}_{\mathcal{Y}_\delta}(H)$, then $G - \text{Int}_{\mathcal{Y}_\delta}(H) \subseteq R \in C(M, Z)$, and thus, $G - \text{Int}_{\mathcal{Y}_\delta}(H) \in C(M, Z)$. Therefore, $H \in \mathcal{Y}_{\omega_\delta}$. \square

Theorem 5. For any soft topological space (M, \mathcal{Y}, Z) , $\mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta} \subseteq \mathcal{Y}_{\omega^0}$.

Proof. To see that $\mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta}$, let $G \in \mathcal{Y}_\delta$ and $z_m \tilde{\in} G$. Since $G \in \mathcal{Y}_\delta$, then $\text{Int}_{\mathcal{Y}_\delta}(G) = G$. Thus, we have $z_m \tilde{\in} G \in \mathcal{Y}$ such that $G - \text{Int}_{\mathcal{Y}_\delta}(G) = 0_Z \in C(M, Z)$, and hence $G \in \mathcal{Y}_{\omega_\delta}$.

To prove that $\mathcal{Y}_{\omega_\delta} \subseteq \mathcal{Y}_{\omega^0}$, let $G \in \mathcal{Y}_{\omega_\delta}$ and $z_m \tilde{\in} G$. Then we find $H \in \mathcal{Y}$ such that $z_m \tilde{\in} H$ and $H - \text{Int}_{\mathcal{Y}_\delta}(G) \in C(M, Z)$. Since $\text{Int}_{\mathcal{Y}_\delta}(G) \subseteq \text{Int}_{\mathcal{Y}}(G)$, then $H - \text{Int}_{\mathcal{Y}}(G) \subseteq H - \text{Int}_{\mathcal{Y}_\delta}(G)$, and so $H - \text{Int}_{\mathcal{Y}}(G) \in C(M, Z)$. Hence, $G \in \mathcal{Y}_{\omega^0}$. \square

Theorem 6. For any soft topological space (M, \mathcal{Y}, Z) , $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is a soft topological space.

Proof. Since by Proposition 4.2 of [43], $(M, \mathcal{Y}_\delta, A)$ is a soft topological space, then $0_Z, 1_Z \in \mathcal{Y}_\delta$. Thus, by Theorem 5, $0_Z, 1_Z \in \mathcal{Y}_{\omega_\delta}$.

Let $K, N \in \mathcal{Y}_{\omega_\delta}$ and $z_m \tilde{\in} K \tilde{\cap} N$. Then $z_m \tilde{\in} K \in \mathcal{Y}_{\omega_\delta}$ and $z_m \tilde{\in} N \in \mathcal{Y}_{\omega_\delta}$. So, we find $H, L \in \mathcal{Y}$ such that $z_m \tilde{\in} H \tilde{\cap} L \in \mathcal{Y}$ and $H - \text{Int}_{\mathcal{Y}_\delta}(K), L - \text{Int}_{\mathcal{Y}_\delta}(N) \in C(M, Z)$. Since $\text{Int}_{\mathcal{Y}_\delta}(K \tilde{\cap} N) = \text{Int}_{\mathcal{Y}_\delta}(K) \tilde{\cap} \text{Int}_{\mathcal{Y}_\delta}(N)$, then

$$\begin{aligned} (H \tilde{\cap} L) - (\text{Int}_{\mathcal{Y}_\delta}(K \tilde{\cap} N)) &= (H \tilde{\cap} L) - (\text{Int}_{\mathcal{Y}_\delta}(K) \tilde{\cap} \text{Int}_{\mathcal{Y}_\delta}(N)) \\ &= ((H \tilde{\cap} L) - \text{Int}_{\mathcal{Y}_\delta}(K)) \tilde{\cup} ((H \tilde{\cap} L) - \text{Int}_{\mathcal{Y}_\delta}(N)) \in C(M, Z). \end{aligned}$$

Hence, $K \tilde{\cap} N \in \mathcal{Y}_{\omega_\delta}$.

Let $\{G_\alpha : \alpha \in \Delta\} \subseteq \mathcal{Y}_{\omega_\delta}$ and $z_m \tilde{\in} \bigcup_{\alpha \in \Delta} G_\alpha$. Then there exists $\alpha_0 \in \Delta$ such that $z_m \tilde{\in} G_{\alpha_0}$. So, by Theorem 4, we find $H \in \mathcal{Y}$ and $R \in C(M, Z)$ such that $z_m \tilde{\in} H$ and $H - R \subseteq \text{Int}_{\mathcal{Y}_\delta}(G_{\alpha_0}) \subseteq \text{Int}_{\mathcal{Y}_\delta}(\bigcup_{\alpha \in \Delta} G_\alpha)$. Hence, $\bigcup_{\alpha \in \Delta} G_\alpha \in \mathcal{Y}_{\omega_\delta}$. \square

Theorem 7. If (M, \mathcal{Y}, Z) is soft locally countable, then $\mathcal{Y}_{\omega_\delta} = SS(M, Z)$.

Proof. Let (M, \mathcal{Y}, Z) be soft locally countable. Let $H \in SS(M, Z)$ and $z_m \tilde{\in} H$. Choose $K \in C(M, Z) \cap \mathcal{Y}$ such that $z_m \tilde{\in} K \subseteq H$. Thus, we have $K \in C(M, Z)$, $z_m \tilde{\in} K \in \mathcal{Y}$, and $K - Int_{\mathcal{Y}_\delta}(H) \in C(M, Z)$. Hence, $H \in \mathcal{Y}_{\omega_\delta}$. \square

Theorem 8. *If (M, \mathcal{Y}, Z) is a soft semi-regularization topology, then $\mathcal{Y}_{\omega_\delta} = \mathcal{Y}_{\omega_0}$.*

Proof. By Theorem 5, it is sufficient to see that $\mathcal{Y}_{\omega_0} \subseteq \mathcal{Y}_{\omega_\delta}$. Let $H \in \mathcal{Y}_{\omega_0}$ and $z_m \tilde{\in} H$. Then we find $G \in \mathcal{Y}$ such that $z_m \tilde{\in} G$ and $G - Int_{\mathcal{Y}}(H) \in C(M, Z)$. Since (M, \mathcal{Y}, Z) is a soft semi-regularization topology, then $\mathcal{Y}_\delta = \mathcal{Y}$, and so $Int_{\mathcal{Y}_\delta}(H) = Int_{\mathcal{Y}}(H)$. This shows that $H \in \mathcal{Y}_{\omega_\delta}$. \square

Theorem 9. *For any soft topological space (M, \mathcal{Y}, Z) , $\mathcal{Y}_{\omega_\theta} \subseteq \mathcal{Y}_{\omega_\delta}$.*

Proof. Let $G \in \mathcal{Y}_{\omega_\theta}$ and $z_m \tilde{\in} G$. Then we find $H \in \mathcal{Y}$ such that $z_m \tilde{\in} H$ and $H - Int_{\mathcal{Y}_\theta}(G) \in C(M, Z)$. Since $Int_{\mathcal{Y}_\theta}(G) \subseteq Int_{\mathcal{Y}_\delta}(G)$, then $H - Int_{\mathcal{Y}_\delta}(G) \subseteq H - Int_{\mathcal{Y}_\theta}(G)$, and so $H - Int_{\mathcal{Y}_\delta}(G) \in C(M, Z)$. Hence, $G \in \mathcal{Y}_{\omega_\delta}$. \square

Lemma 1. *Let (M, \mathcal{Y}, Z) be a soft topological space, and $K \in SS(M, Z)$. Then, for each $a \in Z$, $(Int_{\mathcal{Y}_\delta}(K))(a) \subseteq Int_{(\mathcal{Y}_a)_\delta}(K(a))$.*

Proof. Let $m \in (Int_{\mathcal{Y}_\delta}(K))(a)$. Then $a_m \tilde{\in} Int_{\mathcal{Y}_\delta}(K)$, and so, we find $G \in \mathcal{Y}_\delta$ such that $a_m \tilde{\in} G \subseteq K$. Thus, we have $m \in G(a) \subseteq K(a)$ and $G(a) \in (\mathcal{Y}_\delta)_a$. Since, by Theorem 30 of [51], $G(a) \in (\mathcal{Y}_a)_\delta$, then $m \in Int_{(\mathcal{Y}_a)_\delta}(K(a))$. \square

Theorem 10. *Let (M, \mathcal{Y}, Z) be a soft topological space. Then, for every $a \in Z$, $(\mathcal{Y}_{\omega_\delta})_a \subseteq (\mathcal{Y}_a)_{\omega_\delta}$.*

Proof. Let $a \in Z$. Let $V \in (\mathcal{Y}_{\omega_\delta})_a$ and $m \in V$. Then, there exists $K \in \mathcal{Y}_{\omega_\delta}$ such that $V = K(a)$. Thus, $a_m \tilde{\in} K \in \mathcal{Y}_{\omega_\delta}$, and by Theorem 4, we find $G \in \mathcal{Y}$ and $R \in C(M, Z)$ such that $a_m \tilde{\in} G$ and $G - R \subseteq Int_{\mathcal{Y}_\delta}(K)$. So, we have $m \in G(a) \in \mathcal{Y}_a$, $R(a)$ is a countable set, and $G(a) - R(a) = (G - R)(a) \subseteq (Int_{\mathcal{Y}_\delta}(K))(a)$. On the other hand, by Lemma 1, $(Int_{\mathcal{Y}_\delta}(K))(a) \subseteq Int_{(\mathcal{Y}_a)_\delta}(K(a))$. This shows that $V \in (\mathcal{Y}_a)_{\omega_\delta}$. \square

Corollary 1. *Let (M, \mathcal{Y}, Z) be a soft topological space, and $K \in \mathcal{Y}_{\omega_\delta}$. Then $K(a) \in (\mathcal{Y}_a)_{\omega_\delta}$ for all $a \in Z$.*

Proof. Let $s \in S$. Since $G \in \mathcal{Y}_{\omega_\delta}$, then $G(s) \in (\mathcal{Y}_{\omega_\delta})_s$. Thus, by Theorem 9, $G(s) \in (\mathcal{Y}_s)_{\omega_\delta}$. \square

Theorem 11. *Let $\{(M, \beta_z) : z \in Z\}$ be a collection of topological spaces. Then $(\oplus_{z \in Z} \beta_z)_{\omega_\delta} = \oplus_{z \in Z} (\beta_z)_{\omega_\delta}$.*

Proof. To show that $(\oplus_{z \in Z} \beta_z)_{\omega_\delta} \subseteq \oplus_{z \in Z} (\beta_z)_{\omega_\delta}$, let $H \in (\oplus_{z \in Z} \beta_z)_{\omega_\delta}$. Let $b \in Z$. We will show that $H(b) \in (\beta_b)_{\omega_\delta}$. Let $m \in H(b)$. Then $b_m \tilde{\in} H$. Since $H \in (\oplus_{z \in Z} \beta_z)_{\omega_\delta}$, we find $G \in \oplus_{z \in Z} \beta_z$ and $R \in C(M, Z)$ such that $b_m \tilde{\in} G$ and $G - R \subseteq Int_{(\oplus_{z \in Z} \beta_z)_\delta}(H)$. Now, by Theorem 31 of [51], $(\oplus_{z \in Z} \beta_z)_\delta = \oplus_{z \in Z} (\beta_z)_\delta$. Thus, $G - R \subseteq Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H)$ and so $G(b) - R(b) = (G - R)(b) \subseteq (Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H))(b)$. In contrast, by Lemma 4.9 of [52], $(Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H))(b) = Int_{(\beta_b)_\delta}(H(b))$. Therefore, we have $m \in G(b) \in \beta_b$, $R(b)$ is a countable set, and $G(b) - R(b) = Int_{(\beta_b)_\delta}(H(b))$. Hence, $H(b) \in (\beta_b)_{\omega_\delta}$.

To show that $\oplus_{z \in Z} (\beta_z)_{\omega_\delta} \subseteq (\oplus_{z \in Z} \beta_z)_{\omega_\delta}$, let $H \in \oplus_{z \in Z} (\beta_z)_{\omega_\delta}$. Let $b_m \tilde{\in} H$. Then $m \in H(b) \in (\beta_b)_{\omega_\delta}$. So, we find $V \in \beta_b$ such that $m \in V$ and $V - Int_{(\beta_b)_\delta}(H(b))$ is a countable set. By Lemma 4.9 of [52], $(Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H))(b) = Int_{(\beta_b)_\delta}(H(b))$ and so $(V - (Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H)))(b) = V - (Int_{\oplus_{z \in Z} (\beta_z)_\delta}(H))(b)$ is a countable set. There-

fore, we have $b_m \tilde{\in} b_V \in \bigoplus_{z \in Z} \beta_z$ and $b_V - \left(\text{Int}_{\bigoplus_{z \in Z} (\beta_z)_\delta} (H) \right) \in C(M, Z)$. This shows that $H \in \left(\bigoplus_{z \in Z} \beta_z \right)_{\omega_\delta}$. \square

Corollary 2. For any topological space (M, β) and any set of parameters Z , $(\tau(\beta))_{\omega_\delta} = \tau(\beta_{\omega_\delta})$.

Proof. Let $\beta_z = \beta$ for every $z \in Z$. Then $\tau(\beta) = \bigoplus_{z \in Z} \beta_z$. Thus, by Theorem 11,

$$\begin{aligned} (\tau(\beta))_{\omega_\delta} &= \left(\bigoplus_{z \in Z} \beta_z \right)_{\omega_\delta} \\ &= \bigoplus_{z \in Z} (\beta_z)_{\omega_\delta} \\ &= \tau(\beta_{\omega_\delta}). \end{aligned}$$

\square

The following examples show that equality cannot be used to replace either of the two soft inclusions in Theorem 5:

Example 1. Let $M = \mathbb{Q}$, $A = \mathbb{N}$, $\mathcal{Y} = \{0_A\} \cup \{K \in SS(M, A) : M - K(a) \text{ is a finite set for every } a \in A\}$. Since (M, \mathcal{Y}, A) is soft locally countable, then by Theorem 7, $\mathcal{Y}_{\omega_\delta} = SS(M, A)$. Therefore, $C_Z \in \mathcal{Y}_{\omega_\delta} - \mathcal{Y}_\delta$.

Example 2. Let $M = \mathbb{R}$, $Z = \{a, b, d\}$, and $\mathcal{Y} = \{0_Z, 1_Z, b_{(0, \infty)}\}$. Suppose that $\text{Int}_{\mathcal{Y}_\delta} (b_{(0, \infty)}) \neq 0_Z$. Then we find $m \in (0, \infty)$ such that $b_m \tilde{\in} \text{Int}_{\mathcal{Y}_\delta} (b_{(0, \infty)})$. So, we find $K \in \mathcal{Y}$ such that $b_m \tilde{\in} K \tilde{\subseteq} \text{Int}_{\mathcal{Y}} (Cl_{\mathcal{Y}}(K)) \tilde{\subseteq} b_{(0, \infty)}$. Thus, $K = b_{(0, \infty)}$, and so $\text{Int}_{\mathcal{Y}} (Cl_{\mathcal{Y}}(K)) = \text{Int}_{\mathcal{Y}} (1_Z) = 1_Z \tilde{\subseteq} b_{(0, \infty)}$. Hence, $\text{Int}_{\mathcal{Y}_\delta} (b_{(0, \infty)}) = 0_Z$. Suppose that $b_{(0, \infty)} \in \mathcal{Y}_{\omega_\delta}$, then we find $H \in \mathcal{Y}$ such that $b_1 \tilde{\in} H$ and $H - \text{Int}_{\mathcal{Y}_\delta} (b_{(0, \infty)}) = H \in C(M, Z)$. Since $H \in \mathcal{Y} - \{0_Z\}$, then $H \in \{1_Z, b_{(0, \infty)}\}$. But $\{1_Z, b_{(0, \infty)}\} \cap C(M, Z) = \emptyset$. Therefore, $b_{(0, \infty)} \notin \mathcal{Y}_{\omega_\delta}$. In contrast, by Theorem 5 of [44], $b_{(0, \infty)} \in \mathcal{Y}_{\omega^0}$.

Additionally, Example 2 demonstrates that \mathcal{Y} need not always be a subset of $\mathcal{Y}_{\omega_\delta}$. The inclusion in Theorem 9 need not be equality in general:

Example 3. Let $M = \mathbb{R}$, $Z = \mathbb{N}$, and $\mathcal{Y} = \{K \in SS(M, Z) : K(a) \in \{\emptyset, M, \mathbb{Q} \cap (1, 2), \mathbb{R} - \mathbb{Q}, (\mathbb{Q} \cap (1, 2)) \cup (\mathbb{R} - \mathbb{Q})\} \text{ for all } a \in Z\}$. Then $C_{\mathbb{R} - \mathbb{Q}} \in \mathcal{Y}_{\omega_\delta} - \mathcal{Y}_{\omega^0}$.

Theorem 12. Let (M, \mathcal{Y}, Z) be a soft topological space. If $C_V \in (\mathcal{Y} \cap \mathcal{Y}_{\omega_\delta}) - \{0_Z\}$, then $(\mathcal{Y}_{\omega_\delta})_V \subseteq (\mathcal{Y}_V)_{\omega_\delta}$.

Proof. Let $K \in (\mathcal{Y}_{\omega_\delta})_V$ and $z_m \tilde{\in} K$. Choose $T \in \mathcal{Y}_{\omega_\delta}$ such that $K = T \tilde{\cap} C_V$. Since $C_V \in \mathcal{Y}_{\omega_\delta}$, then $K \in \mathcal{Y}_{\omega_\delta}$. So, we find $D \in \mathcal{Y}$ and $E \in C(M, Z)$ such that $z_m \tilde{\in} D$ and $D - E \tilde{\subseteq} \text{Int}_{\mathcal{Y}_\delta} (K)$. So, we have $z_m \tilde{\in} D \tilde{\cap} C_V \in \mathcal{Y}_V$, $E \tilde{\cap} C_V \in C(V, A)$, and $(D \tilde{\cap} C_V) - (E \tilde{\cap} C_V) \tilde{\subseteq} (D - E) \tilde{\cap} C_V \tilde{\subseteq} \text{Int}_{\mathcal{Y}_\delta} (K) \tilde{\cap} C_V \tilde{\subseteq} \text{Int}_{(\mathcal{Y}_V)_\delta} (K)$. This shows that $K \in (\mathcal{Y}_V)_{\omega_\delta}$. \square

Corollary 3. Let (M, \mathcal{Y}, Z) be a soft topological space. If $C_V \in \mathcal{Y}_\delta - \{0_Z\}$, then $(\mathcal{Y}_{\omega_\delta})_V \subseteq (\mathcal{Y}_V)_{\omega_\delta}$.

Theorem 12 requires the condition “ $C_V \in \mathcal{Y} \cap \mathcal{Y}_{\omega_\delta}$ ”, as the following example demonstrates.

Example 4. Let $M = \mathbb{R}$, $V = \mathbb{R} - \mathbb{Q}$, $Z = \mathbb{N}$, λ be the usual topology on M , and $\mathcal{Y} = \{C_W : W \in \lambda\}$. Since $C_{(3,\infty)} \in \mathcal{Y}$, then by Theorem 5 of [44], $C_{(3,\infty)} \in \mathcal{Y}_{\omega^0}$. Since (M, \mathcal{Y}, Z) is soft regular and $C_{(3,\infty)} \in \mathcal{Y}$, then by Theorem 8, $C_{(3,\infty)} \in \mathcal{Y}_{\omega_\delta}$. Thus, $C_{(3,\infty)} \tilde{\cap} C_V = C_{(3,\infty) \cap (\mathbb{R} - \mathbb{Q})} \in (\mathcal{Y}_{\omega_\delta})_V$. Suppose that $C_{(3,\infty) \cap (\mathbb{R} - \mathbb{Q})} \in (\mathcal{Y}_V)_{\omega_\delta}$. Let $a = 1$. Then we find $W \in \lambda$ and $K \in C(V, A)$ such that $a_{\sqrt{11}} \in C_W$ and $C_W - K \tilde{\subseteq} Int_{(\mathcal{Y}_V)_\delta} (C_{(3,\infty) \cap (\mathbb{R} - \mathbb{Q})}) \tilde{\subseteq} Int_{\mathcal{Y}_V} (C_{(3,\infty) \cap (\mathbb{R} - \mathbb{Q})}) = 0_Z$. Thus, $C_W \tilde{\subseteq} K$, and hence $C_W \in C(V, A)$. Therefore, W is a countable set, which is impossible. This shows that $C_{(3,\infty) \cap (\mathbb{R} - \mathbb{Q})} \notin (\mathcal{Y}_V)_{\omega_\delta}$.

Theorem 13. Let (M, \mathcal{Y}, Z) be soft Lindelof. Then for every $W \in \mathcal{Y}_{\omega_\delta} \cap \mathcal{Y}^c$, we have $W - Int_{\mathcal{Y}_\delta}(W) \in C(M, Z)$.

Proof. Let $W \in \mathcal{Y}_{\omega_\delta} \cap \mathcal{Y}^c$. Since $W \in \mathcal{Y}_{\omega_\delta}$, for every $z_m \tilde{\in} W$, we find $T_{z_m} \in \mathcal{Y}$ such that $z_m \tilde{\in} T_{z_m}$ and $T_{z_m} - Int_{\mathcal{Y}_\delta}(W) \in C(M, Z)$. Since $W \in \mathcal{Y}^c$, W is a soft Lindelof subset of (M, \mathcal{Y}, Z) . Set $\Theta = \{T_{z_m} : z_m \tilde{\in} W\}$. Since $W \tilde{\subseteq} \tilde{\cup}_{z_m \tilde{\in} W} T_{z_m}$, then we find a countable subfamily $\Theta_1 \subseteq \Theta$ such that $W \tilde{\subseteq} \tilde{\cup}_{S \in \Theta_1} S$. Since Θ_1 is countable, then $\tilde{\cup}_{S \in \Theta_1} (S - Int_{\mathcal{Y}_\delta}(W)) \in C(M, Z)$. Since $W - Int_{\mathcal{Y}_\delta}(W) \tilde{\subseteq} \tilde{\cup}_{S \in \Theta_1} (S - Int_{\mathcal{Y}_\delta}(W))$, $W - Int_{\mathcal{Y}_\delta}(W) \in C(M, Z)$. \square

Theorem 14. Let (M, \mathcal{Y}, Z) be a soft topological space, and $K \in (\mathcal{Y}_{\omega_\delta})^c$. Then we find $H \in \mathcal{Y}^c$ and $T \in C(M, Z)$ such that $Cl_{\mathcal{Y}_\delta}(K) \tilde{\subseteq} H \tilde{\cup} T$.

Proof. If $K = 1_Z$, then $K \tilde{\subseteq} 1_Z \tilde{\cup} 0_Z$ with $1_Z \in \mathcal{Y}^c$ and $0_Z \in C(M, Z)$. If $K \neq 1_Z$, then we find $z_m \tilde{\in} 1_Z - K \in \mathcal{Y}_{\omega_\delta}$. So, we find $G \in \mathcal{Y}$ and $T \in C(M, Z)$ such that $z_m \tilde{\in} G$ and $G - T \tilde{\subseteq} Int_{\mathcal{Y}_\delta}(1_Z - K) = 1_Z - Cl_{\mathcal{Y}_\delta}(K)$ and thus $Cl_{\mathcal{Y}_\delta}(K) \tilde{\subseteq} 1_Z - (G - T) = (1_Z - G) \tilde{\cup} T$. Let $H = 1_Z - G$. Then $H \in \mathcal{Y}^c$ and $Cl_{\mathcal{Y}_\delta}(K) \tilde{\subseteq} H \tilde{\cup} T$. \square

Theorem 15. A soft topological space (M, \mathcal{Y}, Z) is soft A-L-C if and only if $(M, \mathcal{Y}_{\omega_\delta}, A)$ is soft A-L-C.

Proof. *Necessity.* Let (M, \mathcal{Y}, Z) be soft A-L-C. To show that $(M, \mathcal{Y}_{\omega_\delta}, A)$ is soft A-L-C, on the contrary, we find $K \in (\mathcal{Y}_{\omega_\delta} \cap C(M, Z)) - \{0_Z\}$. Pick $z_m \tilde{\in} K$. Since $K \in \mathcal{Y}_{\omega_\delta}$, then we find $T \in \mathcal{Y}$ and $N \in C(M, Z)$ such that $z_m \tilde{\in} T$ and $T - N \tilde{\subseteq} Int_{\mathcal{Y}_\delta}(K) \tilde{\subseteq} K$. Thus, $T \tilde{\subseteq} K \tilde{\cup} N$, and hence $T \in C(M, Z)$. Since $z_m \tilde{\in} T$, then $T \in \mathcal{Y} - \{0_Z\}$. Since (M, \mathcal{Y}, Z) is soft A-L-C, then $T \notin C(M, Z)$, a contradiction.

Sufficiency. Clear. \square

Theorem 16. Let (M, \mathcal{Y}, Z) be soft A-L-C. Then, for every $K \in \mathcal{Y}_{\omega_\delta}$, $Cl_{\mathcal{Y}}(K) \tilde{\subseteq} Cl_{\mathcal{Y}_{\omega_\delta}}(K)$.

Proof. Let $K \in \mathcal{Y}_{\omega_\delta}$. By Theorem 5, $\mathcal{Y}_{\omega_\delta} \subseteq \mathcal{Y}_{\omega^0}$, and thus $Cl_{\mathcal{Y}_{\omega^0}}(K) \tilde{\subseteq} Cl_{\mathcal{Y}_{\omega_\delta}}(K)$. Since (M, \mathcal{Y}, Z) is soft A-L-C and $H \in \mathcal{Y}_{\omega_\delta} \subseteq \mathcal{Y}_{\omega^0}$, then by Theorem 21 of [44], $Cl_{\mathcal{Y}_{\omega^0}}(K) = Cl_{\mathcal{Y}}(K)$. Hence, $Cl_{\mathcal{Y}}(K) \tilde{\subseteq} Cl_{\mathcal{Y}_{\omega_\delta}}(K)$. \square

Corollary 4. Let (M, \mathcal{Y}, Z) be soft A-L-C. Then for each $K \in (\mathcal{Y}_{\omega_\delta})^c$, then $Int_{\mathcal{Y}_{\omega_\delta}}(K) \tilde{\subseteq} Int_{\mathcal{Y}}(K)$.

Theorem 17. If (M, \mathcal{Y}, Z) is soft Lindelof, then $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof.

Proof. Let $\mathcal{K} \subseteq \mathcal{Y}_{\omega_\delta}$ such that $1_Z = \tilde{\cup}_{K \in \mathcal{K}} K$. For each $z_m \tilde{\in} 1_Z$, choose $K_{z_m} \in \mathcal{K}$ such that $z_m \tilde{\in} K_{z_m}$. For each $z_m \tilde{\in} 1_Z$, choose $H_{z_m} \in \mathcal{Y}$ and $T_{z_m} \in C(M, Z)$ such that $z_m \tilde{\in} H_{z_m}$ and $H_{z_m} - T_{z_m} \tilde{\subseteq} Int_{\mathcal{Y}_{\omega_\delta}}(K_{z_m}) \tilde{\subseteq} K_{z_m}$. Since (M, \mathcal{Y}, Z) is soft Lindelof and $1_Z = \tilde{\cup}_{z_m \tilde{\in} 1_Z} H_{z_m}$, then there exists a countable subset $R \subseteq SP(M, Z)$ such that $1_Z = \tilde{\cup}_{z_m \tilde{\in} R} H_{z_m}$ and so

$$1_Z = \tilde{\cup}_{z_m \tilde{\in} R} H_{z_m} = (\tilde{\cup}_{z_m \tilde{\in} R} (H_{z_m} - T_{z_m})) \tilde{\cup} (\tilde{\cup}_{z_m \tilde{\in} R} T_{z_m}) \tilde{\subseteq} (\tilde{\cup}_{z_m \tilde{\in} R} K_{z_m}) \tilde{\cup} (\tilde{\cup}_{z_m \tilde{\in} R} T_{z_m}).$$

Put $S = \bigcup_{z_m \in R} T_{z_m}$. Then $S \in C(M, Z)$. For each $b_x \in S$, choose $K_{b_x} \in \mathcal{K}$ such that $b_x \in K_{b_x}$. Put $\mathcal{N} = \{K_{z_m} : z_m \in R\} \cup \{K_{b_x} : b_x \in S\}$. Then \mathcal{N} is a countable subcollection of \mathcal{K} such that $1_Z = \bigcup_{N \in \mathcal{N}} N$. Therefore, $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof. \square

But the converse of Theorem 17 is not always true:

Theorem 18. Let $M = \mathbb{R}, Z = \mathbb{N}$, and $\mathcal{Y} = \{0_Z\} \cup \{K \in SS(M, Z) : (-\infty, 1) \subseteq K(z) \text{ for all } z \in Z\}$. Let $\mathcal{H} = \{C_{(-\infty, 1) \cup \{m\}} : m \in [1, \infty)\}$. Then $\mathcal{H} \subseteq \mathcal{Y}, \bigcup_{H \in \mathcal{H}} H = 1_Z$, and for any countable subcollection $\mathcal{H}_1 \subseteq \mathcal{H}, \bigcup_{H \in \mathcal{H}_1} H \neq 1_Z$. Therefore, (M, \mathcal{Y}, Z) is not soft Lindelof. In contrast, since for any $G \in \mathcal{Y} - \{0_Z\}, Cl_{\mathcal{Y}}(G) = 1_Z$, then $\mathcal{Y}_\delta = \{0_Z, 1_Z\}$ and so $\mathcal{Y}_{\omega_\delta} = \{0_Z, 1_Z\}$. Hence, $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof.

Theorem 19. If $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof, then (M, \mathcal{Y}, Z) is soft nearly Lindelof.

Proof. Let $\mathcal{K} \subseteq RO(\mathcal{Y})$ such that $1_Z = \bigcup_{K \in \mathcal{K}} K$. Then $\mathcal{K} \subseteq \mathcal{Y}_\delta$, and by Theorem 5, $\mathcal{K} \subseteq \mathcal{Y}_{\omega_\delta}$. Since $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof, then we find a countable subfamily $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $1_Z = \bigcup_{K \in \mathcal{K}_1} K$. This shows that (M, \mathcal{Y}, Z) is soft nearly Lindelof. \square

In general, Theorem 19 cannot be reversed:

Theorem 20. Let $M = \mathbb{R}, Z = \mathbb{N}$, and

$$\mathcal{Y} = \{0_Z\} \cup \{K \in SS(M, Z) : 1 \in K(z) \text{ for all } z \in Z\}.$$

Since $\mathcal{Y}_\delta = \{0_Z, 1_Z\}$, then (M, \mathcal{Y}, Z) is soft nearly Lindelof. Since for each $z_m \in SP(M, Z), z_m \in \mathcal{Z}_{\{1, m\}} \in \mathcal{Y} \cap C(M, Z)$, then (M, \mathcal{Y}, Z) is soft locally countable. Thus, by Theorem 7, $\mathcal{Y}_{\omega_\delta} = SS(M, Z)$. Since $1_Z = \bigcup_{z_m \in SP(M, Z)} z_m$ and for any countable subfamily $\mathcal{H} \subseteq SP(M, Z), 1_Z \neq \bigcup_{z_m \in \mathcal{H}} z_m$, then $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is not soft Lindelof.

Theorem 21. If $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft compact, then (M, \mathcal{Y}, Z) is soft nearly compact.

Proof. Let $\mathcal{K} \subseteq RO(\mathcal{Y})$ such that $1_Z = \bigcup_{K \in \mathcal{K}} K$. Then $\mathcal{K} \subseteq \mathcal{Y}_\delta$, and by Theorem 5, $\mathcal{K} \subseteq \mathcal{Y}_{\omega_\delta}$. Since $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft compact, then we find a finite subfamily $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $1_Z = \bigcup_{K \in \mathcal{K}_1} K$. This shows that (M, \mathcal{Y}, Z) is soft nearly compact. \square

In general, Theorem 21 cannot be reversed.

Example 5. Let $M = \mathbb{Q}, Z = \{a, b\}$, and $\mathcal{Y} = \{0_Z, 1_Z\}$. Then $\mathcal{Y}_\delta = \{0_Z, 1_Z\}$, and thus (M, \mathcal{Y}, Z) is soft nearly compact. Since (M, \mathcal{Y}, Z) is soft locally countable, then by Theorem 7, $\mathcal{Y}_{\omega_\delta} = SS(M, Z)$. Since $1_Z = \bigcup_{z_m \in SP(M, Z)} z_m$ and for any finite subfamily $\mathcal{H} \subseteq SP(M, Z), 1_Z \neq \bigcup_{z_m \in \mathcal{H}} z_m$, then $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is not soft compact.

Example 5 and the following example show that the soft compactness of a soft topological space (M, \mathcal{Y}, Z) is neither implied nor imply by the soft compactness of $(M, \mathcal{Y}_{\omega_\delta}, Z)$.

Example 6. Let $M = \mathbb{R}, Z = \{a\}$ and

$$\mathcal{Y} = \{0_Z\} \cup \{K \in SS(M, Z) : \mathbb{R} - K(a) \text{ is countable}\}.$$

Since $\mathcal{Y}_\delta = \{0_Z, 1_Z\} = \mathcal{Y}_{\omega_\delta}$, then $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft compact. In contrast, it is clear that (M, \mathcal{Y}, Z) is not soft compact.

3. Decompositions

Definition 9. Let (M, \mathcal{Y}, Z) be a soft topological space and $K \in SS(M, Z)$. Then K is

- (a) Soft ω_δ^δ -open set in (M, \mathcal{Y}, Z) if $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\delta}(K)$.
- (b) Soft ω_δ^0 -open set in (M, \mathcal{Y}, Z) if $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}}(K)$.
- (c) Soft ω_δ^θ -open set in (M, \mathcal{Y}, Z) if $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\theta}(K)$.
- (d) Soft ω_δ^ω -open set in (M, \mathcal{Y}, Z) if $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\omega}(K)$.

In a soft topological space (M, \mathcal{Y}, Z) , the collections of soft ω_δ^δ -open sets, soft ω_δ^0 -open set, soft ω_δ^θ -open sets, and soft ω_δ^ω -open sets will be denoted by $\omega_\delta^\delta(\mathcal{Y})$, $\omega_\delta^0(\mathcal{Y})$, $\omega_\delta^\theta(\mathcal{Y})$, and $\omega_\delta^\omega(\mathcal{Y})$, respectively.

Theorem 22. Let (M, \mathcal{Y}, Z) be a soft topological space. Then

- (a) $\mathcal{Y}_{\omega_\delta} \subseteq \omega_\delta^\omega(\mathcal{Y})$.
- (b) $\mathcal{Y}_\delta \subseteq \omega_\delta^\omega(\mathcal{Y}) \cap \omega_\delta^\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y})$.
- (c) $\mathcal{Y}_\theta \subseteq \omega_\delta^\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\omega(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})$.
- (d) $\omega_\delta^\theta(\mathcal{Y}) \subseteq \omega_\delta^\delta(\mathcal{Y})$.

Proof. (a) Let $K \in \mathcal{Y}_{\omega_\delta}$. Then $Int_{\mathcal{Y}_{\omega_\delta}}(K) = K$. Also, by Theorem 5 and Theorem 5 of [44], $K \in \mathcal{Y}_\omega$, and so $Int_{\mathcal{Y}_\omega}(K) = K$. Therefore, $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\omega}(K)$. Hence, $K \in \omega_\delta^\omega(\mathcal{Y})$.

(b) Since by Theorem 5, $\mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta}$. Then, by (a), $\mathcal{Y}_\delta \subseteq \omega_\delta^\omega(\mathcal{Y})$. Let $K \in \mathcal{Y}_\delta$. Then $Int_{\mathcal{Y}_\delta}(K) = K$. By Theorem 5, $K \in \mathcal{Y}_{\omega_\delta}$, and thus, $Int_{\mathcal{Y}_{\omega_\delta}}(K) = K$. Also, since $\mathcal{Y}_\delta \subseteq \mathcal{Y}$, then $K \in \mathcal{Y}$, and so $Int_{\mathcal{Y}}(K) = K$. Therefore, we have $Int_{\mathcal{Y}_\omega}(K) = Int_{\mathcal{Y}_\delta}(K) = Int_{\mathcal{Y}}(K)$. This shows that $K \in \omega_\delta^\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y})$.

(c) Since $\mathcal{Y}_\theta \subseteq \mathcal{Y}_\delta$, then by (b), $\mathcal{Y}_\theta \subseteq \omega_\delta^\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\omega(\mathcal{Y})$. Let $K \in \mathcal{Y}_\theta$. Then $Int_{\mathcal{Y}_\theta}(K) = K$. Since $\mathcal{Y}_\theta \subseteq \mathcal{Y}_\delta$, then by Theorem 2.3, $K \in \mathcal{Y}_{\omega_\delta}$, and so $Int_{\mathcal{Y}_{\omega_\delta}}(K) = K$. Therefore, $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\theta}(K)$, and hence $K \in \omega_\delta^\theta(\mathcal{Y})$.

(d) Let $K \in \omega_\delta^\theta(\mathcal{Y})$. Then $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\theta}(K)$. In contrast, by Theorem 5, we have $\mathcal{Y}_\theta \subseteq \mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta}$, then $Int_{\mathcal{Y}_\theta}(K) \subseteq Int_{\mathcal{Y}_\delta}(K) \subseteq Int_{\mathcal{Y}_{\omega_\delta}}(K)$. Therefore, we have $Int_{\mathcal{Y}_\delta}(K) = Int_{\mathcal{Y}_{\omega_\delta}}(K)$ and hence $K \in \omega_\delta^\delta(\mathcal{Y})$. \square

As the next two examples show, in general, none of the inclusions in Theorem 22 can be replaced by equality:

Example 7. Let $M = \mathbb{R}$, $A = \{a\}$, and $\mathcal{Y} = \{0_A, 1_A, a_{\mathbb{R}-\mathbb{Q}}\}$. Let $K = a_{\mathbb{N}}$. Suppose that $Int_{\mathcal{Y}_\omega}(K) \neq 0_A$. Then there exists $x \in M$ such that $a_x \in Int_{\mathcal{Y}_\omega}(K) \in \mathcal{Y}_\omega$. So, we find $G \in \mathcal{Y}$ such that $a_x \in G$ and $G - K \in C(M, A)$, which is impossible. Therefore, $Int_{\mathcal{Y}}(K) = Int_{\mathcal{Y}_\omega}(K) = 0_A$. In contrast, since $\mathcal{Y}_\delta = \mathcal{Y}_\theta = \mathcal{Y}_{\omega_\delta} = \{0_A, 1_A\}$, then $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\delta}(K) = Int_{\mathcal{Y}_\theta}(K) = 0_A$ and $K \notin \mathcal{Y}_\theta \cup \mathcal{Y}_\delta \cup \mathcal{Y}_{\omega_\delta}$. This shows that none of the inclusions in Theorem 22 (a), (b), and (c), cannot be replaced by equality in general.

Example 8. Let $M = \mathbb{R}$, $Z = \{a\}$, and

$$\mathcal{Y} = \{K \in SS(M, Z) : K(a) \in \{\emptyset, M, \mathbb{Q} \cap (1, 2), \mathbb{R} - \mathbb{Q}, (\mathbb{Q} \cap (1, 2)) \cup (\mathbb{R} - \mathbb{Q})\}\}.$$

Then $a_{\mathbb{R}-\mathbb{Q}} \in \omega_\delta^\delta(\mathcal{Y}) - \omega_\delta^\theta(\mathcal{Y})$. As a result, equality in general cannot replace the inclusion in Theorem 22 (d).

For a soft topological space (M, \mathcal{Y}, Z) , the first and second components of each of the ordered pairs of classes of soft sets below are not comparable in general, as demonstrated by the following three examples:

1. $(\mathcal{Y}, \omega_\delta^\delta(\mathcal{Y}))$.
2. $(\mathcal{Y}, \omega_\delta^0(\mathcal{Y}))$.
3. $(\mathcal{Y}, \omega_\delta^\theta(\mathcal{Y}))$.
4. $(\omega_\delta^\delta(\mathcal{Y}), \omega_\delta^0(\mathcal{Y}))$.
5. $(\omega_\delta^\delta(\mathcal{Y}), \omega_\delta^\omega(\mathcal{Y}))$.
6. $(\omega_\delta^\delta(\mathcal{Y}), \omega_\delta^\theta(\mathcal{Y}))$.
7. $(\omega_\delta^\delta(\mathcal{Y}), \omega_\delta^0(\mathcal{Y}))$.
8. $(\omega_\delta^\delta(\mathcal{Y}), \omega_\delta^\theta(\mathcal{Y}))$.
9. $(\omega_\delta^0(\mathcal{Y}), \omega_\delta^\omega(\mathcal{Y}))$.
10. $(\mathcal{Y}_\omega, \omega_\delta^\omega(\mathcal{Y}))$.

Example 9. Let $M = \{1, 2\}$, $Z = \{a\}$, and $\mathcal{Y} = \{K \in SS(M, Z) : K(a) \in \{\emptyset, M, \{1\}\}\}$. Then $a_{\{1\}} \in (\mathcal{Y} \cap \omega_\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\omega(\mathcal{Y})) - (\omega_\delta^\theta(\mathcal{Y}) \cup \omega_\delta^\delta(\mathcal{Y}))$ and $a_{\{2\}} \in \omega_\delta(\mathcal{Y}) - (\omega_\delta^0(\mathcal{Y}) \cup \omega_\delta^\delta(\mathcal{Y}))$.

Example 10. Let (M, \mathcal{Y}, Z) be as in Example 6. Then $a_{\mathbb{R}-\mathbb{N}} \in \omega_\delta^\delta(\mathcal{Y}) - (\omega_\delta^0(\mathcal{Y}) \cup \omega_\delta^\omega(\mathcal{Y}))$.

Example 11. Let $M = \{1, 2\}$, $Z = \{a\}$, β be the usual topology on \mathbb{R} , and $\mathcal{Y} = \{K \in SS(M, Z) : K(a) \in \beta\}$. Then $a_{\{1\}} \in (\omega_\delta^\delta(\mathcal{Y}) \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\omega(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})) - \mathcal{Y}_\omega$.

Theorem 23. Let (M, \mathcal{Y}, Z) be a soft topological space. Then

- (a) $\mathcal{Y}_{\omega_\delta} = \mathcal{Y}_\omega \cap \omega_\delta^\omega(\mathcal{Y})$.
- (b) $\mathcal{Y}_\delta = \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\delta(\mathcal{Y})$.
- (c) $\mathcal{Y}_\theta = \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\theta(\mathcal{Y})$.
- (d) $\mathcal{Y}_\theta = \mathcal{Y}_{\omega_\theta} \cap \omega_\delta^\theta(\mathcal{Y})$.
- (e) $\mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y}_{\omega_\delta}$.
- (f) $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y}$.
- (g) $\mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) = \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y})$.

Proof. (a) By Theorem 5 and Theorem 5 of [44], $\mathcal{Y}_{\omega_\delta} \subseteq \tau_\omega$. In contrast, by Theorem 22 (a), $\mathcal{Y}_{\omega_\delta} \subseteq \omega_\delta^\omega(\mathcal{Y})$. Thus, $\mathcal{Y}_{\omega_\delta} \subseteq \mathcal{Y}_\omega \cap \omega_\delta^\omega(\mathcal{Y})$. To see that $\mathcal{Y}_\omega \cap \omega_\delta^\omega(\mathcal{Y}) \subseteq \mathcal{Y}_{\omega_\delta}$, let $K \in \mathcal{Y}_\omega \cap \omega_\delta^\omega(\mathcal{Y})$. Since $K \in \tau_\omega$, then $K = Int_{\mathcal{Y}_\omega}(K)$. Since $K \in \omega_\delta^\omega(\mathcal{Y})$, then $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\omega}(K)$. Thus, $Int_{\mathcal{Y}_{\omega_\delta}}(K) = K$, and hence $K \in \mathcal{Y}_{\omega_\delta}$.

(b) By Theorem 5 and Theorem 22 (b), we have $\mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\delta(\mathcal{Y})$. To see that $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\delta(\mathcal{Y}) \subseteq \mathcal{Y}_\delta$, let $K \in \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\delta(\mathcal{Y})$. Then $K = Int_{\mathcal{Y}_{\omega_\delta}}(K)$ and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\delta}(K)$. Thus, $K = Int_{\mathcal{Y}_\delta}(K)$, and hence $K \in \mathcal{Y}_\delta$.

(c) By Theorem 5, we have $\mathcal{Y}_\theta \subseteq \mathcal{Y}_\delta \subseteq \mathcal{Y}_{\omega_\delta}$. Also, by Theorem 22 (c), $\mathcal{Y}_\theta \subseteq \omega_\delta^\theta(\mathcal{Y})$. Thus, $\mathcal{Y}_\theta \subseteq \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\theta(\mathcal{Y})$. To see that $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\theta(\mathcal{Y}) \subseteq \mathcal{Y}_\theta$, let $K \in \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\theta(\mathcal{Y})$. Then $K = Int_{\mathcal{Y}_{\omega_\delta}}(K)$ and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\theta}(K)$. Thus, $K = Int_{\mathcal{Y}_\theta}(K)$, and hence $K \in \mathcal{Y}_\theta$.

(d) By Theorem 5 of [45], $\mathcal{Y}_\theta \subseteq \mathcal{Y}_{\omega_\theta}$. Also, by (c), $\mathcal{Y}_\theta \subseteq \omega_\delta^\theta(\mathcal{Y})$. Thus, $\mathcal{Y}_\theta \subseteq \mathcal{Y}_{\omega_\theta} \cap \omega_\delta^\theta(\mathcal{Y})$. In contrast, by Theorem 9 and (c), $\mathcal{Y}_{\omega_\theta} \cap \omega_\delta^\theta(\mathcal{Y}) \subseteq \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^\theta(\mathcal{Y}) = \mathcal{Y}_\theta$.

(e) Let $K \in \mathcal{Y} \cap \omega_\delta^0(\mathcal{Y})$. Then $K = Int_{\mathcal{Y}}(K)$ and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}}(K)$. Thus, $K = Int_{\mathcal{Y}_{\omega_\delta}}(K)$, and hence $K \in \mathcal{Y}_{\omega_\delta}$.

(f) Let $K \in \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y})$. Then $K = Int_{\mathcal{Y}_{\omega_\delta}}(K)$ and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}}(K)$. Thus, $K = Int_{\mathcal{Y}}(K)$, and hence $K \in \mathcal{Y}$.

(g) We have $\mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \omega_\delta^0(\mathcal{Y})$. Also, by (e), $\mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y}_{\omega_\delta}$. Hence, $\mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y})$. In contrast, we have $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \omega_\delta^0(\mathcal{Y})$. Also, by (f), $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y}$. Hence, $\mathcal{Y}_{\omega_\delta} \cap \omega_\delta^0(\mathcal{Y}) \subseteq \mathcal{Y} \cap \omega_\delta^0(\mathcal{Y})$. \square

Corollary 5. Let (M, \mathcal{Y}, Z) be a soft topological space and $K \in \omega_\delta^0(\mathcal{Y})$. Then $K \in \mathcal{Y}$ if and only if $K \in \mathcal{Y}_{\omega_\delta}$.

Proof. The proof follows from Theorem 23 (g). \square

Theorem 24. Let (M, \mathcal{Y}, Z) be a soft topological space. Then (M, \mathcal{Y}, Z) is a soft semi-regularization topology if and only if $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\delta(\mathcal{Y})$.

Proof. Necessity. Let (M, \mathcal{Y}, Z) be a soft semi-regularization topology. Then $\mathcal{Y}_\delta = \mathcal{Y}$. Thus, by Theorem 22 (b), $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\delta(\mathcal{Y})$.

Sufficiency. Let $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\delta(\mathcal{Y})$. To see that $\mathcal{Y} \subseteq \mathcal{Y}_\delta$, let $K \in \mathcal{Y}$. Then $K \in \mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\delta(\mathcal{Y})$. So, we have $Int_{\mathcal{Y}}(K) = K$, $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}}(K)$, and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\delta}(K)$. Thus, $Int_{\mathcal{Y}_\delta}(K) = K$. Hence, $K \in \mathcal{Y}_\delta$. \square

Theorem 25. A soft topological space (M, \mathcal{Y}, Z) is soft regular if and only if $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})$.

Proof. *Necessity.* Let (M, \mathcal{Y}, Z) be soft regular. Then $\mathcal{Y}_\theta = \mathcal{Y}$. Thus, by Theorem 22 (c), $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})$.

Sufficiency. Let $\mathcal{Y} \subseteq \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})$. To see that $\mathcal{Y} \subseteq \mathcal{Y}_\theta$, let $K \in \mathcal{Y}$. Then $K \in \mathcal{Y} \cap \omega_\delta^0(\mathcal{Y}) \cap \omega_\delta^\theta(\mathcal{Y})$. So, we have $Int_{\mathcal{Y}}(K) = K, Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}}(K)$, and $Int_{\mathcal{Y}_{\omega_\delta}}(K) = Int_{\mathcal{Y}_\theta}(K)$. Thus, $Int_{\mathcal{Y}_\theta}(K) = K$. Hence, $K \in \mathcal{Y}_\theta$. \square

4. Soft ω_δ -Continuity

Definition 10. A soft function $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is called soft ω_δ -continuous if $f_{qv}^{-1}(K) \in \mathcal{Y}_{\omega_\delta}$ for every $K \in \mathcal{X}$.

Theorem 26. For a soft function $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$, the following are equivalent:

- (1) $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft ω_δ -continuous.
- (2) $f_{qv}^{-1}(T) \in (\mathcal{Y}_{\omega_\delta})^c$ for every $T \in \mathcal{X}^c$.
- (3) $Cl_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(A)) \subseteq f_{qv}^{-1}(Cl_{\mathcal{X}}(A))$ for each $A \in SS(N, W)$.
- (4) $f_{qv}^{-1}(Int_{\mathcal{X}}(A)) \subseteq Int_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(A))$ for each $A \in SS(N, W)$.
- (5) $f_{qv} : (M, \mathcal{Y}_{\omega_\delta}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft continuous.
- (6) For each $z_m \in SP(M, Z)$ and each $G \in \mathcal{X}$ such that $f_{qv}(z_m) \tilde{\in} G$, we find $H \in \mathcal{Y}_{\omega_\delta}$ such that $z_m \tilde{\in} H$ and $f_{qv}(H) \subseteq G$.

Proof. (1) \rightarrow (2): Let $T \in \mathcal{X}^c$. Then $1_W - T \in \mathcal{X}$. So, by (1), $f_{qv}^{-1}(1_W - T) = 1_Z - f_{qv}^{-1}(T) \in \mathcal{Y}_{\omega_\delta}$. Hence, $f_{qv}^{-1}(T) \in (\mathcal{Y}_{\omega_\delta})^c$.

(2) \rightarrow (3): Let $A \in SS(N, W)$. Then $Cl_{\mathcal{X}}(A) \in \mathcal{X}^c$. So, by (2), $f_{qv}^{-1}(Cl_{\mathcal{X}}(A)) \in (\mathcal{Y}_{\omega_\delta})^c$. Since $f_{qv}^{-1}(A) \subseteq f_{qv}^{-1}(Cl_{\mathcal{X}}(A)) \in (\mathcal{Y}_{\omega_\delta})^c$, then $Cl_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(A)) \subseteq f_{qv}^{-1}(Cl_{\mathcal{X}}(A))$.

(3) \rightarrow (4): Let $A \in SS(N, W)$. Then, by (3),

$$\begin{aligned} 1_Z - Int_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(A)) &= Cl_{\mathcal{Y}_{\omega_\delta}}(1_Z - f_{qv}^{-1}(A)) \\ &= Cl_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(1_W - A)) \\ &\subseteq f_{qv}^{-1}(Cl_{\mathcal{X}}(1_W - A)) \\ &= f_{qv}^{-1}(1_W - Int_{\mathcal{X}}(A)) \\ &= 1_Z - f_{qv}^{-1}(Int_{\mathcal{X}}(A)) \end{aligned}$$

and so $f_{qv}^{-1}(Int_{\mathcal{X}}(A)) \subseteq Int_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(A))$.

(4) \rightarrow (5): Let $K \in \mathcal{X}$. Then $Int_{\mathcal{X}}(K) = K$, and by (4), $f_{qv}^{-1}(K) \subseteq Int_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(K))$. Thus, $f_{qv}^{-1}(K) = Int_{\mathcal{Y}_{\omega_\delta}}(f_{qv}^{-1}(K))$. Hence, $f_{qv}^{-1}(K) \in \mathcal{Y}_{\omega_\delta}$. This shows that $f_{qv} : (M, \mathcal{Y}_{\omega_\delta}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft continuous.

(5) \rightarrow (6): Let $z_m \in SP(M, Z)$ and $G \in \mathcal{X}$ such that $f_{qv}(z_m) \tilde{\in} G$. Then, by (5), $f_{qv}^{-1}(G) \in \mathcal{Y}_{\omega_\delta}$. Put $H = f_{qv}^{-1}(G)$. Then $H \in \mathcal{Y}_{\omega_\delta}$ such that $z_m \tilde{\in} H$ and $f_{qv}(H) = f_{qv}(f_{qv}^{-1}(G)) \subseteq G$.

(6) \rightarrow (1): Let $K \in \mathcal{X}$. To show that $f_{qv}^{-1}(K) \in \mathcal{Y}_{\omega_\delta}$, let $z_m \tilde{\in} f_{qv}^{-1}(K)$. Then $f_{qv}(z_m) \tilde{\in} K$, and by (6), we find $H \in \mathcal{Y}_{\omega_\delta}$ such that $z_m \tilde{\in} H$ and $f_{qv}(H) \subseteq K$. Thus, we have $z_m \tilde{\in} H \subseteq f_{qv}^{-1}(f_{qv}(H)) \subseteq f_{qv}^{-1}(K)$. Hence, $f_{qv}^{-1}(K) \in \mathcal{Y}_{\omega_\delta}$. \square

Theorem 27. If $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft ω_δ -continuous, then $q : (M, \mathcal{Y}_a) \rightarrow (N, \mathcal{X}_{v(a)})$ is ω_δ -continuous for every $a \in Z$.

Proof. Suppose that $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft ω_δ -continuous, and let $a \in Z$. By Theorem 4.2 (5), $f_{qv} : (M, \mathcal{Y}_{\omega_\delta}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft continuous. So, by Proposition 3.8 of [38], $q : (M, (\mathcal{Y}_{\omega_\delta})_a) \rightarrow (N, \mathcal{X}_{v(a)})$ is continuous. Since, by Theorem 10, $(\mathcal{Y}_{\omega_\delta})_a \subseteq$

$(\mathcal{Y}_a)_{\omega_\delta}$, then $q : (M, (\mathcal{Y}_a)_{\omega_\delta}) \rightarrow (N, \mathcal{X}_{v(a)})$ is continuous. Hence, by Theorem 4.2 (5) of [41], $q : (M, \mathcal{Y}_a) \rightarrow (N, \mathcal{X}_{v(a)})$ is ω_δ -continuous. \square

Theorem 28. Let $\{(M, \beta_z) : z \in Z\}$ and $\{(N, \alpha_w) : w \in W\}$ be two collections of topological spaces. Let $q : M \rightarrow N$ and $v : Z \rightarrow W$ be functions where v is bijective. Then $f_{qv} : (M, \bigoplus_{z \in Z} \beta_z, Z) \rightarrow (N, \bigoplus_{w \in W} \alpha_w, W)$ is soft ω_δ -continuous if and only if $q : (M, \beta_a) \rightarrow (N, \alpha_{v(a)})$ is ω_δ -continuous for all $a \in Z$.

Proof. Necessity. Let $f_{qv} : (M, \bigoplus_{z \in Z} \beta_z, Z) \rightarrow (N, \bigoplus_{w \in W} \alpha_w, W)$ be soft ω_δ -continuous. Let $a \in Z$. Then, by Theorem 27, $q : (M, (\bigoplus_{z \in Z} \beta_z)_a) \rightarrow (N, (\bigoplus_{w \in W} \alpha_w)_{v(a)})$ is ω_δ -continuous. But by Theorem 3.11 of [38], $(\bigoplus_{z \in Z} \beta_z)_a = \beta_a$ and $(\bigoplus_{w \in W} \alpha_w)_{v(a)} = \alpha_{v(a)}$. Hence, $q : (M, \beta_a) \rightarrow (N, \alpha_{v(a)})$ is ω_δ -continuous.

Sufficiency. Let $q : (M, \beta_a) \rightarrow (N, \alpha_{v(a)})$ be ω_δ -continuous for all $a \in Z$. Let $K \in \bigoplus_{w \in W} \alpha_w$. By Theorem 11, it is sufficient to show that $(f_{qv}^{-1}(K))(a) \in (\beta_a)_{\omega_\delta}$ for all $a \in Z$. Let $a \in Z$. Since $q : (M, \beta_a) \rightarrow (N, \alpha_{v(a)})$ is ω_δ -continuous and $K(v(a)) \in \alpha_{v(a)}$, then $(f_{qv}^{-1}(K))(a) = q^{-1}(K(v(a))) \in (\beta_a)_{\omega_\delta}$. \square

Corollary 6. Let $q : (M, \xi) \rightarrow (N, \phi)$ and $v : Z \rightarrow W$ be two functions where v is a bijection. Then $q : (M, \xi) \rightarrow (N, \phi)$ is ω_δ -continuous if and only if $f_{qv} : (M, \tau(\xi), Z) \rightarrow (N, \tau(\phi), W)$ is soft ω_δ -continuous.

Proof. For each $z \in Z$ and $w \in W$, put $\beta_z = \xi$ and $\alpha_w = \phi$. Then $\tau(\alpha) = \bigoplus_{z \in Z} \beta_z$ and $\tau(\phi) = \bigoplus_{w \in W} \alpha_w$. By using Theorem 28, we get the result. \square

Theorem 29. Let $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ be soft ω_δ -continuous and surjective. If $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof, then (N, \mathcal{X}, W) is soft Lindelof.

Proof. Let $\mathcal{H} \subseteq \mathcal{X}$ such that $\tilde{\bigcup}_{H \in \mathcal{H}} H = 1_W$. Then $f_{qv}^{-1}(\tilde{\bigcup}_{H \in \mathcal{H}} H) = \tilde{\bigcup}_{H \in \mathcal{H}} f_{qv}^{-1}(H) = f_{qv}^{-1}(1_W) = 1_Z$. Since $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ is soft ω_δ -continuous, then $\{f_{qv}^{-1}(H) : H \in \mathcal{H}\} \subseteq \mathcal{Y}_{\omega_\delta}$. Since $(M, \mathcal{Y}_{\omega_\delta}, Z)$ is soft Lindelof, then we find a countable subfamily $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $\tilde{\bigcup}_{H \in \mathcal{H}_1} f_{qv}^{-1}(H) = f_{qv}^{-1}(\tilde{\bigcup}_{H \in \mathcal{H}_1} H) = 1_Z$. So, $f_{qv}(f_{qv}^{-1}(\tilde{\bigcup}_{H \in \mathcal{H}_1} H)) = f_{qv}(1_Z)$. Since f_{qv} is surjective, then $f_{qv}(1_Z) = 1_W$. Thus, $1_W = f_{qv}(f_{qv}^{-1}(\tilde{\bigcup}_{H \in \mathcal{H}_1} H)) \tilde{\subseteq} \tilde{\bigcup}_{H \in \mathcal{H}_1} H$, and hence $1_W = \tilde{\bigcup}_{H \in \mathcal{H}_1} H$. This shows that (N, \mathcal{X}, W) is soft Lindelof. \square

Corollary 7. Let $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (N, \mathcal{X}, W)$ be soft ω_δ -continuous and onto. If (M, \mathcal{Y}, Z) is soft Lindelof, then (N, \mathcal{X}, W) is soft Lindelof.

Proof. The proof follows from Theorems 17 and 29. \square

Theorem 30. Every soft ω_δ -continuous function is soft ω^0 -continuous.

The following illustration shows that Theorem 30's converse need not always hold true:

Example 12. Let (M, \mathcal{Y}, Z) be as in Example 2.14. Let $q : M \rightarrow M$ and $v : Z \rightarrow Z$ be the identity functions. Since $f_{qv}^{-1}(b_{(0, \infty)}) = b_{(0, \infty)} \in \mathcal{Y}_{\omega^0} - \mathcal{Y}_{\omega_\delta}$, then $f_{qv} : (M, \mathcal{Y}, Z) \rightarrow (M, \mathcal{Y}, Z)$ is soft ω^0 -continuous but not soft ω_δ -continuous.

5. Conclusions

We introduced five types of soft sets. Also, we introduced soft ω_δ -continuous functions as a new class of soft functions. We gave several characterizations, relationships, and decomposition theorems. In addition, we investigated the links between our novel soft topological notions and their classical topological analogs.

We intend to do the following in the next work: (1) To define soft separation axioms via our new classes of soft sets; (2) To define new soft classes of functions via our new classes of soft sets.

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