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On Semi-Infinite Optimization Problems with Vanishing Constraints Involving Interval-Valued Functions

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Abstract: In this paper, we examine a semi-infinite interval-valued optimization problem with vanishing constraints (SIVOPVC) that lacks differentiability and involves constraints that tend to vanish. We give definitions of generalized convex functions along with supportive examples. We investigate duality theorems for the SIVOPVC problem. We establish these theorems by creating duality models, which establish a relationship between SIVOPVC and its corresponding dual models, assuming generalized convexity conditions. Some examples are also given to illustrate the results.

Keywords: vanishing constraints; Mond–Weir-type duality; Wolfe-type duality; semi-infinite interval-valued optimization problems

MSC: 90C29; 65G40; 26B25; 90C26

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1. Introduction

Consider the optimization problem

$$\begin{aligned} &\min f(x) \\ \text{s.t. } &g_i(x) \leq 0 \ \ \forall\, i = 1,2,...,m, \\ &h_j(x) = 0 \ \forall\, j = 1,2,...p, \\ &\psi_i(x) \geq 0 \ \ \forall\, i = 1,2,...l, \\ &\phi_i(x)\psi_i(x) \leq 0 \ \ \forall\, i = 1,2,...l, \end{aligned}$$

with continuously differentiable functions $f, g, h_j, \phi_i, \psi_i : \mathbb{R}^n \to \mathbb{R}$. This category of problem is referred to as a mathematical program with vanishing constraints, abbreviated as MPVC. On the one hand, this terminology is due to the fact that the implicit sign constraint $\phi_i(x) \leq 0$ vanishes as soon as $\psi_i(x) = 0$. The field of mathematical programming involving problems with vanishing constraints is a captivating subject, primarily due to its wide-ranging applications in modern research domains such as economic dispatch problems [1], topology design issues [2], optimal control and structural optimization [3], and even problems related to robot motion planning [4]. Achtziger and Kanzow [5] proposed an appropriate model for the Abadie constraint qualification and corresponding optimality conditions, finding that this revised constraint qualification holds under certain assumptions. Later, the work of Achtziger and Kanzow [5] was extended by Kazemi and Kanzi [6] by introducing additional constraint qualifications tailored for systems with vanishing constraints for nonsmooth functions.

The feasible set in a mathematical program with equilibrium constraints (MPVC) [7] problem can exhibit nonconvex characteristics, including the possibility of being disconnected. Additionally, many common constraint qualifications, such as the Mangasarian–Fromovitz and linearly independent constraint qualifications, may not be applicable in such cases. Consequently, the standard Karush–Kuhn–Tucker conditions cannot be relied upon for solving these problems effectively. In [5], various constraint qualifications and necessary optimality conditions were introduced specifically for mathematical programs with vanishing constraints. The work in [8] delves into first-order sufficient optimality conditions as well as second-order necessary and sufficient optimality conditions. In [9], several stationary conditions were derived under less stringent assumptions regarding constraint qualifications. Furthermore, Hoheisel and Kanzow's research in [10] explored necessary and sufficient optimality conditions using Abadie and Guignard-type constraint qualifications for MPVC. For a more comprehensive understanding of MPVC, we refer the reader to [7,11–13], and the associated references therein.

Several extensions and generalizations of convexity have been explored in the literature. One way to generalize the definition of a convex function is to relax the convexity condition. Lin and Fukushima [14] introduced the concept of higher-order strongly convex functions and applied them to the analysis of mathematical programs with equilibrium constraints. Mishra and Sharma [15] derived inequalities akin to Hermite–Hadamard type for higher-order strongly convex functions. It is worth noting that strongly convex functions were initially introduced and investigated by Polyak [16], which have significant implications in optimization theory and its related domains. For instance, Karmardian [17] utilized strongly convex functions to address the unique existence of solutions in nonlinear complementarity problems. These functions also played a crucial role in the convergence analysis of iterative methods for solving variational inequalities and equilibrium problems, as highlighted by Zu and Marcotte [18].

Hanson [19] introduced the notion of invex (invariant convex) functions for differentiable functions, which played a significant role in mathematical programming. Ben-Israel and Mond [20] introduced the concept of invex sets and preinvex functions. It is established that differentiable preinvex functions are, indeed, invex functions. Moreover, the converse holds under certain conditions, as discussed in [21]. Additionally, Noor and Noor [22] examined the properties of strongly preinvex functions and their variations. Noor et al. [23,24] explored the applications of generalized strongly preinvex functions and their various forms. More recently, Joshi [25] investigated mathematical programs with equilibrium constraints, employing assumptions of higher-order invexity within a differentiable framework.

An increasing number of researchers are directing their focus towards problems related to interval-valued optimization [26,27]. In this regard, Wu [28] developed duality theorems applicable to interval-valued optimization problems that involve continuous differentiability. Sun and Wang [29] introduced a concept of optimal solutions for interval-valued programming problems and further derived necessary and sufficient optimality conditions in the style of Kuhn–Tucker and Fritz–John for interval-valued programming problems that lack differentiability. Ahmad et al. [30] delved into interval-valued variational problems, offering sufficient optimality conditions and Mond–Weir-type duality results based on their research. Following this, Kummari and Ahmad [31] examined nonsmooth interval-valued optimization problems, providing both optimality and duality findings. Jayswal et. al. [32,33] explored generalized convexity in the context of nonsmooth interval-valued optimization problems, studying the associated optimality conditions and duality. Later, Ahmad et al. [34] investigated optimality conditions and Mond–Weir-type duality problems specific to differentiable interval-valued optimization problems that include vanishing constraints (IOPVC). For nondifferentiable scenarios, Wang and Wang [35] established results concerning optimality and duality. Later, in [13], authors discussed and focused exclusively on invex functions, in which a single bifunction was involved, and established some results for invex functions. Note that the invex property denotes the property of being invariantly convex.

Inspired by the prior research mentioned, we explore higher-order generalized convex functions, which constitute a much smaller class. Thus, the functions under consideration and their respective supports differ significantly between [13] and this paper. In this article, we give the definitions of strongly higher order invex functions for nonsmooth settings using Clarke subdifferentials. Under certain assumptions, we will also prove duality theorems in the context of semi-infinite interval-valued optimization problems with vanishing constraints.

Here is the article's structure: In Section 2, we provide established definitions and formulas. In Section 3, we delve into the examination of duality theorems linking the IOPVC and the dual model of the Wolfe type. Moving on to Section 4, we investigate duality theorems connecting the IOPVC with the Mond–Weir-type dual model. To illustrate our findings, we include some examples.

2. Definitions and Preliminaries

Let E denote a Euclidean space which is finite-dimensional, and notation $\langle .,. \rangle$ denotes the inner product in E. For a point $\tilde{u} \in E$, we denote the open ball of radius δ around $\tilde{u}$ by $B(\tilde{u}, \delta) := \{u \in E : |u - \tilde{u}| < \delta\}$.

For a set $\Gamma \subset E$, span Γ, cone Γ denote the linear hull and convex cone of Γ, respectively.

Definition 1. *Let* $\Gamma \neq \phi$, *then*

$$T(\Gamma, u) := \{v \in E | \exists t_n \to 0, \exists v_n \to v, \forall n \in N, u + t_n v_n \in \Gamma\}$$

is called the contingent cone of set Γ *at the point* u.

Let C denote the set of closed intervals in $\mathbb{R}$. For any $U = [a_1, a_2] \in C, V = [b_1, b_2] \in C$, one has

$$U + V = [a_1 + b_1, a_2 + b_2], -V = [-b_2, -b_1],$$
$$U - V = [a_1 - b_2, a_2 - b_1], U + k = [a_1 + k, a_2 + k]; k \in \mathbb{R},$$

where $\mathbb{R}$ denotes the set of real numbers. The partial ordering for intervals can be formulated as follows:

$$U \leq_{LU} V \iff a_1 \leq b_1, a_2 \leq b_2,$$
$$U <_{LU} V \iff U \leq_{LU} V, U \neq V,$$
$$U \nless_{LU} V \text{ is the negation of } U <_{LU} V,$$
$$U <^s_{LU} V \iff a_1 < b_1, a_2 < b_2,$$
$$U \nless^s_{LU} V \text{ is the negation of } U <^s_{LU} V.$$

Consider a mapping F from the set E to the set C that is defined as follows:

$$F(u) = [F^L(u), F^U(u)] \; (\forall u \in E)$$

where F^L, F^U are locally Lipschitz functions on E and $F^L(u) \leq F^U(u)$.

We consider a semi-infinite interval-valued optimization problem with vanishing constraints (SIVOPVC) as follows:

$$\begin{aligned} LU - \min\ & F(u) \\ & s.t.\ g_j(u) \leq 0,\ j \in J, \\ & h_k(u) = 0,\ k = 1, 2, .., n, \\ & \psi_e(u) \geq 0,\ e = 1, 2, ..., l, \\ & \varphi_e(u)\psi_e(u) \leq 0,\ e = 1, 2, ..., l, \end{aligned}$$

we consider $g_j : E \to \mathbb{R} \cup \{+\infty\}, h_k : E \to \mathbb{R}, \psi_e : E \to \mathbb{R}, \varphi_e : E \to \mathbb{R}$ as locally Lipschitz functions on E and J is an arbitrary index set. Let $\tau_n := \{1, ..., n\}, \tau_l := \{1, ..., l\}$. The feasible set of problem SIVOPVC is

$$Q := \{u \in E; g_j(u) \leq 0 (j \in J), h_k(u) = 0 (k \in \tau_n), \psi_e(u) \geq 0, \varphi_e(u)\psi_e(u) \leq 0 (e \in \tau_l)\}.$$

Let $\mathbb{R}_+^{|J|}$ denote the collection of all function $\pi : J \to \mathbb{R}$ taking values $\pi_j > 0$ only at finitely many points within the set J, while being zero at all other points.

For any $\tilde{u}$ belonging to the set Q, we define two sets as follows:

(1) The index set of all active constraints at $\tilde{u}$ is represented as $\tau_g(\tilde{u})$, which consists of those indices j in the set J for which $g_j(\tilde{u})$ equals zero.

(2) The set $k(\tilde{u})$ is defined as the collection of non-negative multipliers π_j from $\mathbb{R}_+^{|j|}$ such that $\pi_j g_j(\tilde{u}) = 0$ holds for all j in the set J.

With these definitions in place, we can now proceed to define optimal solutions for the problem SIVOPVC.

Definition 2 ([35])**.** *Let $\tilde{u} \in Q$.*
(i) $\tilde{u}$ is considered a locally LU optimal solution for the problem SIVOPVC if there exists a neighborhood $\mathbb{B}(\tilde{u}; \delta)$ such that no other point u in the intersection of the set Q and the neighborhood $\mathbb{B}(\tilde{u}, \delta)$ satisfies the following condition:

$$F(u) <_{LU} F(\tilde{u}).$$

(ii) $\tilde{u}$ is considered a locally weakly LU optimal solution for the problem SIVOPVC if there exists a neighborhood $\mathbb{B}(\tilde{u}; \delta)$ such that no other point u in the intersection of the set Q and the neighborhood $\mathbb{B}(\tilde{u}, \delta)$ satisfies the following condition:

$$F(u) <_{LU}^{s} F(\tilde{u}).$$

Definition 3 (see Clarke [36])**.** *The Clarke directional derivative of f around $\tilde{u}$ in the direction $v \in E$ is given by*

$$f_c'(\tilde{u}, v) := \lim_{u \to \tilde{u}} \sup_{t \downarrow 0} \frac{f(u + tv) - f(u)}{t}$$

and the Clarke subdifferential of f at $\tilde{u}$ is given by

$$\partial_c f(\tilde{u}) := \{\varsigma \in E : \langle \varsigma, v \rangle \leq f_c'(\tilde{u}; v), \forall v \in E\}.$$

Based on the definition of invex function [37] and generalized invex functions [38], Joshi [7] introduced the definition of higher-order strongly invex function for differentiable framework. We are defining strongly pseudoinvex and strongly quasiinvex functions of order $\alpha > 0$ for a nondifferentiable framework.

Definition 4. *Let $f : E \to \mathbb{R}$. Then*
(i) A function f which is locally Lipschitz around $\tilde{u}$ is said to be strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $\tilde{u}$ with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ if, for each $u \in E$ and any $\varsigma \in \partial_c(\tilde{u})$, there exist $\mathcal{L} > 0$, such that

$$f(u) - f(\tilde{u}) < 0 \Rightarrow \langle \varsigma, \varpi(u, \tilde{u}) \rangle + \mathcal{L}\|u - \tilde{u}\|^{\alpha} < 0.$$

(ii) A function f which is locally Lipschitz around $\tilde{u}$ is said to be strongly ∂_c-quasiinvex of order $\alpha > 0$ at $\tilde{u}$ with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ if, for each $u \in E$ and any $\varsigma \in \partial_c(\tilde{u})$, there exist $\mathcal{L} > 0$, such that

$$f(u) - f(\tilde{u}) \leq 0 \Rightarrow \langle \varsigma, \varpi(u, \tilde{u}) \rangle + \mathcal{L}\|u - \tilde{u}\|^{\alpha} \leq 0.$$

Now, we provide some examples to illustrate the given definitions.

Example 1. *Consider that the function $f : \mathbb{R} \to \mathbb{R}$ is given by $f(u) = -|u|$. We have that the Clarke subdifferential of f at 0 is given by $\partial_c(0) = \varsigma = \{-1, 1\}$. Then, the function is strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $\tilde{u} = 0$ with respect to the kernel function*

$$\varpi(u, \tilde{u}) = \begin{cases} -1 - \frac{u}{4}\,; & u \geq 0, \\ 1 + u^2\,; & u < 0. \end{cases}$$

Example 2. *Consider that the function $f : \mathbb{R} \to \mathbb{R}$ is given by*

$$f(u) = \begin{cases} -1 - u\,; & u \geq 0, \\ \frac{u}{3} - 1\,; & u < 0. \end{cases}$$

We have that the Clarke subdifferential of f at 0 is given by $\partial_c(0) = \varsigma = \{-1, \frac{1}{3}\}$. Then, the function is strongly ∂_c-quasiinvex of order $\alpha > 0$ at $\tilde{u} = 0$ with respect to the kernel function

$$\varpi(u, \tilde{u}) = \begin{cases} \sin|u|\,; & u \geq 0, \\ -u^2\,; & u < 0. \end{cases}$$

Now, for $u \in Q$, we give the following sets of indicators.

$$\begin{aligned}
&\tau_{+}(u) := \{e \in \tau_l | \psi_e(u) > 0\}, \\
&\tau_{0}(u) := \{e \in \tau_l | \psi_e(u) = 0\}, \\
&\tau_{+0}(u) := \{e \in \tau_l | \psi_e(u) > 0, \varphi_e(u) = 0\}, \\
&\tau_{+-}(u) := \{e \in \tau_l | \psi_e(u) > 0, \varphi_e(u) < 0\}, \\
&\tau_{0+}(u) := \{e \in \tau_l | \psi_e(u) = 0, \varphi_e(u) > 0\}, \\
&\tau_{00}(u) := \{e \in \tau_l | \psi_e(u) = 0, \varphi_e(u) = 0\}, \\
&\tau_{0-}(u) := \{e \in \tau_l | \psi_e(u) = 0, \varphi_e(u) < 0\}.
\end{aligned}$$

Definition 5 ([39])**.** *Let $\tilde{u} \in Q$ be a feasible point of SIVOPVC.*
(i) The Abadie constraint qualification (ACQ) is said to hold at $\tilde{u}$ iff $T(Q, \tilde{u}) = L(\tilde{u})$, where $L(\tilde{u})$ is the linearized cone of SIVOPVC at $\tilde{u}$, and

$$\begin{aligned}
L(\tilde{u}) :=& \{v \in E | \langle \varsigma_j^g, v \rangle \leq 0, \forall \varsigma_j^g \in \partial_c g_j(\tilde{u}), j \in \tau_g(\tilde{u}); \langle \varsigma_k^h, v \rangle = 0, \forall \varsigma_k^h \in \partial_c h_k(\tilde{u}), \\
& k \in \tau_n; \langle \varsigma_e^\psi, v \rangle = 0, \forall \varsigma_e^\psi \in \partial_c \psi_e(\tilde{u}), e \in \tau_{0+}; \langle \varsigma_e^\psi, v \rangle \geq 0, \forall \varsigma_e^\psi \in \partial_c \psi_e(\tilde{u}), \\
& e \in \tau_{00} \cup \tau_{0-}; \langle \varsigma_e^\varphi, v \rangle \leq 0, \forall \varsigma_e^\varphi \in \partial_c \varphi_e(\tilde{u}), e \in \tau_{+0}\}.
\end{aligned} \tag{1}$$

(ii) The VC-ACQ is said to hold at $\tilde{u}$ iff $L_{VC}(\tilde{u}) \subseteq T(Q, \tilde{u})$, where $L_{VC}(\tilde{u})$ is the corresponding VC-linearized cone of SIVOPVC at $\tilde{u}$, and

$$\begin{aligned} L_{VC}(\tilde{u}) := \{ v \in E | & \langle \varsigma_j^g, v \rangle \leq 0, \forall \varsigma_j^g \in \partial_c g_j(\tilde{u}), j \in \tau_g(\tilde{u}); \\ & \langle \varsigma_k^h, v \rangle = 0, \forall \varsigma_k^h \in \partial_c h_k(\tilde{u}), k \in \tau_n, \\ & \langle \varsigma_e^\psi, v \rangle = 0, \forall \varsigma_e^\psi \in \partial_c \psi_e(\tilde{u}), e \in \tau_{0+}, \\ & \langle \varsigma_e^\psi, v \rangle \geq 0, \forall \varsigma_e^\psi \in \partial_c \psi_e(\tilde{u}), e \in \tau_{00} \cup \tau_{0-}, \\ & \langle \varsigma_e^\varphi, v \rangle \leq 0, \forall \varsigma_e^\varphi \in \partial_c \varphi_e(\tilde{u}), e \in \tau_{+0} \cup \tau_{00} \}. \end{aligned} \tag{2}$$

Theorem 1 ([35])**.** *Let $\tilde{u} \in Q$ be a locally weakly LU optimal solution of SIVOPVC such that VC-ACQ holds at $\tilde{u}$ and*

$$\Delta := cone\left(\bigcup_{j \in \tau_g(\tilde{u})} \partial_c g_j(\tilde{u}) \cup \bigcup_{e \in \tau_{00} \cup \tau_{0-}} -\partial_c \psi_e(\tilde{u}) \cup \bigcup_{e \in \tau_{+0} \cup \tau_{00}} \partial_c \varphi_e(\tilde{u}) \right) + span\left(\bigcup_{k \in \tau_n} \partial_c h_k(\tilde{u}) \cup \bigcup_{e \in \tau_{0+}} \partial_c \psi_e(\tilde{u}) \right)$$

is closed. Then, there exist Lagrange multipliers $\omega^L, \omega^U \in \mathbb{R}_+, \pi^g \in k(\tilde{u}), \pi^h \in \mathbb{R}^n, \pi^\psi, \pi^\varphi \in \mathbb{R}^l$

$$0 \in \omega^L \partial_c F^L(\tilde{u}) + \omega^U \partial_c F^U(\tilde{u}) + \sum_{j \in J} \pi_j^g \partial_c g_j(\tilde{u}) + \sum_{k=1}^{n} \pi_k^h \partial_c h_k(\tilde{u}) - \sum_{e=1}^{l} \pi_e^\psi \partial_c \psi_e(\tilde{u}) + \sum_{e=1}^{l} \pi_e^\varphi \partial_c \varphi_e(\tilde{u}) \tag{3}$$

and

$$\begin{aligned} & \omega^L + \omega^U = 1, h_k(\tilde{u}) = 0 \ (k \in \tau_n), \\ & \pi_j^g \geq 0, g_j(\tilde{u}) \leq 0, \ \pi_j^g g_j(\tilde{u}) = 0 \ (j \in J), \\ & \pi_e^\psi = 0 \ (e \in \tau_+(\tilde{u})), \pi_e^\psi \geq 0 \ (e \in \tau_{00}(\tilde{u}) \cup \tau_{0-}(\tilde{u})), \\ & \pi_e^\psi \in \mathbb{R} \ (e \in \tau_{0+}(\tilde{u})), \\ & \pi_e^\varphi = 0 \ (e \in \tau_{0+}(\tilde{u}) \cup \tau_{0-}(\tilde{u}) \cup \tau_{+-}(\tilde{u})), \ \pi_e^\varphi \geq 0 \ (e \in \tau_{00}(\tilde{u}) \cup \tau_{+0}(\tilde{u})). \end{aligned} \tag{4}$$

Definition 6 ([39])**.** *A point u is known as a VC-stationary point for the problem SIVOPVC if there exist Lagrange multipliers $\omega^L, \omega^U \in \mathbb{R}_+, \pi^g \in k(u), \pi^h \in \mathbb{R}^n, \pi^\psi, \pi^\varphi \in \mathbb{R}^l$ such that Equations (3) and (4) hold.*

Consider u as a VC-stationary point for the problem SIVOPVC, along with the associated multipliers $\pi^g \in \mathbb{R}_+^{|j|}, \pi^h \in \mathbb{R}^n, \pi^\psi, \pi^\varphi \in \mathbb{R}^l$. Then, we provide the following index sets:

$$\begin{aligned} & \tau_g^+(u) := \{ j \in \tau_g(u) | \pi_j^g > 0 \}, \\ & \tau_h^+(u) := \{ k \in \tau_n(u) | \pi_k^h > 0 \}, \\ & \tau_h^-(u) := \{ k \in \tau_n(u) | \pi_k^h < 0 \}, \\ & \tau_{00}^+(u) := \{ e \in \tau_{00}(u) | \pi_e^\psi > 0 \}, \\ & \tau_{0-}^+(u) := \{ e \in \tau_{0-}(u) | \pi_e^\psi > 0 \}, \\ & \tau_{0+}^+(u) := \{ e \in \tau_{0+}(u) | \pi_e^\psi > 0 \}, \\ & \tau_{0+}^-(u) := \{ e \in \tau_{0+}(u) | \pi_e^\psi < 0 \}, \end{aligned}$$

$$\tau_+^+(u) := \{e \in \tau_+(u) | \pi_e^\psi > 0\},$$
$$\tilde{I}_+^+(u) := \{e \in \tau_+(u) | \pi_e^\varphi > 0\}.$$

In the next section, we will provide duality models.

3. Wolfe-Type Duality

In this section, we give the following Wolfe-type dual model [2]. Here, $\pi^g \in \mathbb{R}_+^{|j|}$, $\pi^h \in \mathbb{R}^n, \pi^\psi, \pi^\varphi \in \mathbb{R}^l$,

$$\rho(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) = F(.) + \sum_{j\in J} \pi_j^g g_j(.) + \sum_{k=1}^{n} \pi_k^h h_k(.) - \sum_{e=1}^{l} \pi_e^\psi \psi_e(.) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(.) \tag{5}$$

is an interval-valued function, and

$$\Delta(.) := \omega^L \partial_c F^L(.) + \omega^U \partial_c F^U(.) + \sum_{j\in J} \pi_j^g \partial_c g_j(.) + \sum_{k=1}^{n} \pi_k^h \partial_c h_k(.) - \sum_{e=1}^{l} \pi_e^\psi \partial_c \psi_e(.) + \sum_{e=1}^{l} \pi_e^\varphi \partial_c \varphi_e(.).$$

Now, we present the dual model in the style of Wolfe type of SIVOPVC, which is taken from [35]. For $u \in Q$, we have

$$\begin{aligned}
(D_W(u)) \;\; LU - \max\, & \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) \\
& s.t.\ 0 \in \Delta(v), \\
& \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1, \\
& \pi_j^g \geq 0, \forall j \in J, \\
& \pi_e^\varphi = \mu_e \psi_e(u),\ \mu_e \geq 0,\ \forall e \in \tau_l, \\
& \pi_e^\psi = \mathcal{B}_e - \mu_e \varphi_e(u), \mathcal{B}_e \geq 0,\ \forall e \in \tau_l.
\end{aligned} \tag{6}$$

$$\begin{aligned}
Q_W(u) :=& (v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) : \\
& 0 \in \Delta(v), v \in E, \\
& \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1, \\
& \pi_j^g \geq 0, \forall j \in J, \\
& \pi_e^\varphi = \mu_e \psi_e(u), \mu_e \geq 0,\ \forall e \in \tau_l, \\
& \pi_e^\psi = \mathcal{B}_e - \mu_e \varphi_e(u), \mathcal{B}_e \geq 0,\ \forall e \in \tau_l.
\end{aligned}$$

denotes the feasible set of $D_W(u)$ and $prQ_w(u) := \{v \in E := (v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_W(u)\}$ represents the projection of the set $Q_W(u)$ on E.

To remove dependence on SIVOPVC, we introduce an alternative dual model in the Wolfe type:

$$\begin{aligned}
(D_w)\ LU \max\, & \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) \\
& s.t.\ (v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_w := \cap_{u\in Q} Q_w(u).
\end{aligned}$$

Here, Q_W represents the collection of all feasible points for the problem DW, and prQ_W signifies the projection of the set Q_W onto the space E.

Definition 7 ([39]). *Let $u \in Q$.*
(i) A point $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_w(u)$ is known as a locally LU optimal solution of $DW(u)$ if there exists $B(\bar{v}; \delta)$ such that there is no $v \in Q_W(u) \cap B(\bar{v}; \delta)$ satisfying

$$\rho(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) <_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

(ii) $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_w(u)$ *is known as a locally weakly LU optimal solution of* $DW(u)$ *if there exists* $B(\bar{v};\delta)$ *such that there is no* $v \in Q_W(u) \cap B(\bar{v};\delta)$ *satisfying*

$$\rho(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) <^s_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

Theorem 2. *(Weak duality) Let* $u \in Q$, $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B})$ *be feasible points for the SIVOPVC and DW. If* $\rho^L(., \omega^L, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \rho^U(., \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ *are strongly* ∂_c*-pseudoinvex of order* $\alpha > 0$ *with respect to the kernel function* $\varpi : E \times E \to \mathbb{R}$ *at* $v \in Q \cup prQ_W$*, then*

$$F(u) \not<^s_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

Proof. Suppose $F(u) <^s_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$, then

$$F(u) <^s_{LU} F(v) + \sum_{j\in J} \pi^g_j g_j(v) + \sum_{k=1}^{n} \pi^h_k h_k(v) - \sum_{e=1}^{l} \pi^\psi_e \psi_e(v) + \sum_{e=1}^{l} \pi^\varphi_e \varphi_e(v). \tag{7}$$

Since $u \in Q$ and $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_w(u)$, we have

$$\begin{aligned}
& g_j(u) < 0,\ \pi^g_j \geq 0, j \notin \tau_g(u), \\
& g_j(u) = 0,\ \pi^g_j \geq 0, j \in \tau_g(u), \\
& h_k(u) = 0,\ \pi^h_k \in \mathbb{R}, k \in \tau_n, \\
& -\psi_e(u) < 0,\ \pi^\psi_e \geq 0, e \in \tau_+(u), \\
& -\psi_e(u) = 0,\ \pi^\psi_e \in \mathbb{R}, e \in \tau_0(u), \\
& \varphi_e(u) > 0,\ \pi^\varphi_e = 0,\ e \in \tau_{0+}(u), \\
& \varphi_e(u) = 0,\ \pi^\varphi_e \geq 0,\ e \in \tau_{00}(u) \cup \tau_{+0}(u), \\
& \varphi_e(u) < 0,\ \pi^\varphi_e \geq 0,\ e \in \tau_{0-}(u) \cup \tau_{+-}(u).
\end{aligned} \tag{8}$$

The above formulas imply that

$$\sum_{j\in J} \pi^g_j g_j(u) + \sum_{k=1}^{n} \pi^h_k h_k(u) - \sum_{e=1}^{l} \pi^\psi_e \psi_e(u) + \sum_{e=1}^{l} \pi^\varphi_e \varphi_e(u) \leq 0. \tag{9}$$

Equation (9) together with (7) shows that

$$\rho(u, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) <^s_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

By the strong ∂_c-pseudoinvexity of $\rho^L(., \omega^L, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$, $\rho^U(., \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ of order $\alpha > 0$ at $v \in Q \cup prQ_W$. For some $\mathcal{L}^L > 0, \mathcal{L}^U > 0, \mathcal{L}^g_j > 0, \mathcal{L}^h_k > 0, \mathcal{L}^\psi_e > 0$, and $\mathcal{L}^\varphi_e > 0$, we have

$$\begin{aligned}
&\Big\langle \varsigma^L + \sum_{j\in J} \pi^g_j \varsigma^g_j + \sum_{k=1}^{n} \pi^h_k \varsigma^h_k - \sum_{e=1}^{l} \pi^\psi_e \varsigma^\psi_e + \sum_{e=1}^{l} \pi^\varphi_e \varsigma^\varphi_e, \varpi(u,v) \Big\rangle + \mathcal{L}^L \|u-v\|^\alpha + \\
&\qquad \mathcal{L}^g_j \|u-v\|^\alpha + \mathcal{L}^h_k \|u-v\|^\alpha + \mathcal{L}^\psi_e \|u-v\|^\alpha + \mathcal{L}^\varphi_e \|u-v\|^\alpha < 0, \qquad (10) \\
&\Big\langle \varsigma^U + \sum_{j\in J} \pi^g_j \varsigma^g_j + \sum_{k=1}^{n} \pi^h_k \varsigma^h_k - \sum_{e=1}^{l} \pi^\psi_e \varsigma^\psi_e + \sum_{e=1}^{l} \pi^\varphi_e \varsigma^\varphi_e, \varpi(u,v) \Big\rangle + \mathcal{L}^U \|u-v\|^\alpha + \\
&\qquad \mathcal{L}^g_j \|u-v\|^\alpha + \mathcal{L}^h_k \|u-v\|^\alpha + \mathcal{L}^\psi_e \|u-v\|^\alpha + \mathcal{L}^\varphi_e \|u-v\|^\alpha < 0,
\end{aligned}$$

where $\varsigma^L \in \partial_c F^L(v), \varsigma^U \in \partial_c F^U(v), \varsigma_j^g \in \partial_c g_j(v), j \in J, \varsigma_k^h \in \partial_c h_k(v), k \in \tau_n, \varsigma_e^\psi \in \partial_c \psi_e(v)$, $e \in \tau_l, \varsigma_e^\varphi \in \partial_c \varphi_e(v), e \in \tau_l$. Adding ω^L and ω^U to both sides of inequalities (10), we obtain

$$\langle \varsigma, \varpi(u, v) \rangle < 0 \; \forall \varsigma \in \Delta(v),$$

contradicting $0 \in \Delta(v)$, hence the result. □

Theorem 3 (Weak duality). *Let $u \in Q$, $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_W$ be feasible points for the SIVOPVC and DW. If $\rho^L(., \omega^L, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \rho^U(., \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $v \in Q \cup prQ_W$, then*

$$F(u) \not<_{LU} \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

Proof. The demonstration of this theorem closely resembles the proof of Theorem 2. □

Theorem 4 (Strong duality). *Suppose we have $\bar{u}$ in set Q, which serves as a locally weakly LU optimal solution for SIVOPVC. In this case, if the VC-ACQ condition is satisfied at $\bar{u}$, and the set Δ is closed, then there are Lagrange multipliers $\bar{\omega}^L, \bar{\omega}^U \in \mathbb{R}_+, \bar{\pi}_g \in \mathbb{R}_+^{|j|}, \bar{\pi}_h \in \mathbb{R}^n, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mathcal{B}}, \bar{\mu} \in \mathbb{R}^l$ such that $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a feasible point of $D_W(\bar{u})$, and*

$$F(\bar{u}) = \rho(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi).$$

Moreover, if $\rho^L(., \omega^L, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \rho^U(., \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $v \in Q \cup prQ_W(\bar{u})$, then $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a locally weakly LU optimal solution of $D_W(\bar{u})$.

Proof. Using, Theorem 1, we have Lagrange multipliers $\bar{\omega}^L, \bar{\omega}^U \in \mathbb{R}_+, \bar{\pi}_g \in k(\bar{u})$, $\bar{\pi}_h \in \mathbb{R}^n, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mathcal{B}}, \bar{\mu} \in \mathbb{R}^l$, such that Euqations (3) and (4) are satisfied. Using the definition of $D_W(\bar{u})$, one has that $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a feasible point of $D_W(\bar{u})$,

$$\sum_{j \in J} \bar{\pi}_j{}^g g_j(\bar{u}) + \sum_{k=1}^{n} \bar{\pi}_k^h h_k(\bar{u}) - \sum_{e=1}^{l} \bar{\pi}_e^\psi \psi_e(\bar{u}) + \sum_{e=1}^{l} \bar{\pi}_e^\varphi \varphi_e(\bar{u}) = 0$$

and

$$F(\bar{u}) = \rho(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi).$$

Then, from Theorem 2 we have, for any $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mu, \mathcal{B}) \in Q_W(\bar{u})$,

$$\rho(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi) = F(\bar{u}) \not<_{LU}^s \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

Therefore, $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a locally weakly LU optimal solution of $D_W(\bar{u})$. □

Theorem 5 (Converse duality). *Consider u as an arbitrary point within the scope of SIVOPVC, and let $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ in Q_W be a feasible point of D_W, such that*

$$\begin{aligned} \pi_j^g g_j(\bar{v}) &\geq 0, \forall j \in J, \\ \pi_k^h h_k(\bar{v}) &= 0, \forall k \in \tau_n, \\ -\pi_e^\psi \psi_e(\bar{v}) &\geq 0, \forall e \in \tau_l, \\ \pi_e^\varphi \varphi_e(\bar{v}) &\geq 0, \forall e \in \tau_m. \end{aligned} \tag{11}$$

If one of the following conditions holds:
(i) $\rho^L(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \rho^U(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $\bar{v} \in Q \cup prQ_W$;

(ii) $F^L(.), F^U(.)$ *are* ∂_c*-pseudoinvex at* $\bar{v} \in Q \cup prQ_W(u), g_j(j \in \tau_g^+(u)), h_k(k \in \tau_h^+(u))$, $-\ h_k(k \in \tau_h^-(u)), -\psi_e(e \in \tau_+^+(u) \cup \tau_{00}^+(u) \cup \tau_{0-}^+(u) \cup \tau_{0+}^+(u))$, $\psi_e(e \in \tau_{0+}^-(u)), \varphi_e(e \in \tilde{I}_+^+(u))$ *are strongly* ∂_c*-quasiinvex of order* $\alpha > 0$ *at* $\bar{v} \in Q \cup prQ_W$*;*
Then $\bar{v}$ *is the locally weakly LU optimal solution of SIVOPVC.*

Proof. Assume, contrary to what was stated, that $\bar{v}$ does not serve as a locally weakly LU optimal solution for SIVOPVC. In that case, there must be a $\bar{u}$ within the intersection of set Q and the ball centered at $\bar{v}$ with radius δ such that

$$F(\bar{u}) <_{LU}^s F(\bar{v}). \tag{12}$$

(i) Since $\bar{u} \in Q$ and $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ are feasible points for the SIVOPVC and the D_W, respectively. Combining (8) and (11), we have

$$\sum_{j\in J}\pi_j^g g_j(\bar{u}) + \sum_{k=1}^{n}\pi_k^h h_k(\bar{u}) - \sum_{e=1}^{l}\pi_e^\psi \psi_e(\bar{u}) + \sum_{e=1}^{l}\pi_e^\varphi \varphi_e(\bar{u}) \le 0 \le$$
$$\sum_{j\in J}\pi_j^g g_j(\bar{v}) + \sum_{k=1}^{n}\pi_k^h h_k(\bar{v}) - \sum_{e=1}^{l}\pi_e^\psi \psi_e(\bar{v}) + \sum_{e=1}^{l}\pi_e^\varphi \varphi_e(\bar{v}). \tag{13}$$

By (12) and (13), one has

$$\rho(\bar{u}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) <_{LU}^s \rho(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi).$$

And by the strong ∂_c-pseudoinvexity of $\rho^L(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \rho^U(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ at $\bar{v} \in Q \cup prQ_W$, of order $\alpha > 0$. For some $\mathcal{L}^L > 0, \mathcal{L}^U > 0, \mathcal{L}_j^g > 0, \mathcal{L}_k^h > 0, \mathcal{L}_e^\psi > 0$, and $\mathcal{L}_e^\varphi > 0$, one has

$$\Big\langle \varsigma^L + \sum_{j\in J}\pi_j^g\varsigma_j^g + \sum_{k=1}^{n}\pi_k^h\varsigma_k^h - \sum_{e=1}^{l}\pi_e^\psi\varsigma_e^\psi + \sum_{e=1}^{l}\pi_e^\varphi\varsigma_e^\varphi, \varpi(\bar{u},\bar{v})\Big\rangle + \mathcal{L}^L\|\bar{u}-\bar{v}\|^\alpha +$$
$$\mathcal{L}_j^g\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_k^h\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\psi\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\varphi\|\bar{u}-\bar{v}\|^\alpha < 0, \tag{14}$$
$$\Big\langle \varsigma^U + \sum_{j\in J}\pi_j^g\varsigma_j^g + \sum_{k=1}^{n}\pi_k^h\varsigma_k^h - \sum_{e=1}^{l}\pi_e^\psi\varsigma_e^\psi + \sum_{e=1}^{l}\pi_e^\varphi\varsigma_e^\varphi, \varpi(\bar{u},\bar{v})\Big\rangle + \mathcal{L}^U\|\bar{u}-\bar{v}\|^\alpha +$$
$$\mathcal{L}_j^g\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_k^h\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\psi\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\varphi\|\bar{u}-\bar{v}\|^\alpha < 0,$$

where $\varsigma^L \in \partial_c F^L(\bar{v}), \varsigma^U \in \partial_c F^L(\bar{v}), \varsigma_j^g \in \partial_c g_j(\bar{v}), j \in J, \varsigma_k^h \in \partial_c h_k(\bar{v}), k \in \tau_n, \varsigma_e^\psi \in \partial_c\psi_e(\bar{v}), e \in \tau_l, \varsigma_e^\varphi \in \partial_c\varphi_e(\bar{v}), e \in \tau_l$. Combining (14) with ω^L and ω^U, we obtain

$$\langle \varsigma, \varpi(\bar{u},\bar{v})\rangle < 0, \forall \varsigma \in \Delta(\bar{v})$$

This contradicts the fact that 0 is an element of $\Delta(\bar{v})$, thereby confirming the validity of the conclusion.
(ii) Since $\bar{u} \in Q$ and $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu) \in Q_W$, by (8) and (11), one has

$$\pi_j^g g_j(\bar{u}) \le \pi_j^g g_j(\bar{v}),\ \forall j \in J,$$
$$\pi_k^h h_k(\bar{u}) = \pi_k^h h_k(\bar{v}),\ \forall k \in \tau_n,$$
$$\pi_e^\psi \psi_e(\bar{u}) \le \pi_e^\psi \psi_e(\bar{v}),\ \forall e \in \tau_l,$$
$$\pi_e^\varphi \varphi_e(\bar{u}) \le \pi_e^\varphi \varphi_e(\bar{v}),\ \forall e \in \tau_l,$$

and using the definition of the index sets, we have

$$\begin{aligned}&g_j(\bar{u}) \le g_j(\bar{v}),\ \forall j \in \tau_g^+(\bar{u}),\\&h_k(\bar{u}) = h_k(\bar{v}),\ \forall k \in \tau_h^+(\bar{u}) \cup \tau_h^-(\bar{u}),\\&-\psi_e(\bar{u}) \le -\psi_e(\bar{v}),\ \forall e \in \tau_+^+(\bar{u}) \cup \tau_{00}^+(\bar{u}) \cup \tau_{0-}^+(\bar{u}) \cup \tau_{0+}^+(\bar{u}),\\&-\psi_e(\bar{u}) \ge -\psi_e(\bar{v}),\ \forall e \in \tau_{0+}^-(\bar{u}),\\&\varphi_e(\bar{u}) \le \varphi_e(\bar{v}),\ \forall e \in \tilde{I}_+^+(\bar{u}),\end{aligned} \tag{15}$$

By the strong ∂_c-quasiinvexity of order $\alpha > 0$ of the functions in assumption (ii) and (15), it shows that

$$\begin{aligned}&\langle \varsigma_j^g, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_j^g\|\bar{u}-\bar{v}\|^\alpha \le 0, \pi_j^g > 0, \exists\, \mathcal{L}_j^g > 0,\ \forall \varsigma_j^g \in \partial_c g_j(\bar{v}),\ j \in \tau_g^+(\bar{u}),\\&\langle \varsigma_k^h, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_k^h\|\bar{u}-\bar{v}\|^\alpha \le 0, \pi_k^h > 0, \exists\, \mathcal{L}_k^h > 0,\ \forall \varsigma_k^h \in \partial_c h_k(\bar{v}),\ k \in \tau_h^+(\bar{u}),\\&\langle \varsigma_k^h, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_k^h\|\bar{u}-\bar{v}\|^\alpha \ge 0, \pi_k^h < 0, \exists\, \mathcal{L}_k^h > 0,\ \forall \varsigma_k^h \in \partial_c h_k(\bar{v}),\ k \in \tau_h^-(\bar{u}),\\&\langle -\varsigma_e^\psi, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_e^\psi\|\bar{u}-\bar{v}\|^\alpha \le 0, \pi_e^\psi \ge 0, \exists\, \mathcal{L}_e^\psi > 0,\ \forall \varsigma_e^\psi \in \partial_c \psi_e(\bar{v}),\ e \in \tau_+^+(\bar{u})\\&\qquad\qquad \cup \tau_{00}^+(\bar{u}) \cup \tau_{0-}^+(\bar{u}) \cup \tau_{0+}^+(\bar{u}),\\&\langle -\varsigma_e^\psi, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_e^\psi\|\bar{u}-\bar{v}\|^\alpha \ge 0, \pi_e^\psi \le 0, \exists\, \mathcal{L}_e^\psi > 0,\ \forall \varsigma_e^\psi \in \partial_c \psi_e(\bar{v}),\ e \in \tau_{0+}^-(\bar{u}),\\&\langle \varsigma_e^\varphi, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_e^\varphi\|\bar{u}-\bar{v}\|^\alpha \le 0, \pi_e^\varphi \ge 0, \exists\, \mathcal{L}_e^\varphi > 0,\ \forall \varsigma_e^\varphi \in \partial_c \varphi_e(\bar{v}),\ e \in \tilde{I}_+^+(\bar{u})\end{aligned}$$

that is,

$$\begin{aligned}&\Big\langle \sum_{j\in J} \pi_j^g \varsigma_j^g + \sum_{k=1}^{n} \pi_k^h \varsigma_k^h - \sum_{e=1}^{l} \pi_e^\psi \varsigma_e^\psi + \sum_{e=1}^{l} \pi_e^\varphi \varsigma_e^\varphi, \varpi(\bar{u},\bar{v})\Big\rangle + \mathcal{L}_j^g\|\bar{u}-\bar{v}\|^\alpha +\\&\qquad \mathcal{L}_k^h\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\psi\|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\varphi\|\bar{u}-\bar{v}\|^\alpha \le 0.\end{aligned}$$

By the above inequality and $0 \in \Delta(\bar{v})$, there exist $\varsigma^L \in \partial_c F^L(\bar{v})$ and $\varsigma^U \in \partial_c F^U(\bar{v})$ such that

$$\langle \omega^L \varsigma^L + \omega^U \varsigma^U, \varpi(\bar{u},\bar{v})\rangle \ge 0. \tag{16}$$

By (12) and the strong ∂_c-pseudoinvexity of order $\alpha > 0$ of $F^L(.)$ and $F^U(.)$, it follows that

$$\begin{aligned}&\langle \varsigma^L, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}^L\|\bar{u}-\bar{v}\|^\alpha < 0,\ \forall \varsigma^L \in \partial_c F^L(\bar{v}),\\&\langle \varsigma^U, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}^U\|\bar{u}-\bar{v}\|^\alpha < 0,\ \forall \varsigma^U \in \partial_c F^U(\bar{v}),\end{aligned}$$

then $\langle \omega^L \varsigma^L + \omega^U \varsigma^U, \varpi(\bar{u},\bar{v})\rangle < 0, \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1$, contradicting (16); hence, the result holds. □

Theorem 6. *(Restricted converse duality). Let $\bar{u} \in Q$ be a feasible point of SIVOPVC, and let $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ be feasible points of D_W such that $F(\bar{u}) = \Phi(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$. If $\Phi^L(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi), \Phi^U(., \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $v \in Q \cup prQ_W$, then $\bar{u}$ is the locally weakly LU optimal solution of SIVOPVC.*

Proof. If $\bar{u}$ does not qualify as a locally weakly LU optimal solution for SIVOPVC, then there exists an element $\tilde{u}$ in the set Q and within a neighborhood $B(\bar{u};\delta)$ such that $F(\tilde{u}) <_{LU}^s F(\bar{u})$. By $F(\bar{u}) = \Phi(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$, we obtain $F(\tilde{u}) <_{LU}^s \Phi(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi)$ contradicting Theorem 2. Thus, $\bar{u}$ is the locally weakly LU optimal solution of SIVOPVC. □

Now we provide an example in order to show the conclusion of Theorem 6.

Example 3. *Let $E = \mathbb{R}^2, n = 0, l = J = 1$, and consider:*

$$\begin{aligned} SIVOPVC1 \min F(u) =& [F^L(u), F^U(u)] = [|u_1| - |u_2|, u_1^2] \\ & s.t.\ g_1(u) = -u_1 \leq 0, \\ & \psi_1(u) = u_1 - u_2 \geq 0, \\ & \varphi_1(u)\psi_1(u) = u_1(u_1 - u_2) \leq 0. \end{aligned}$$

The feasible set of problem SIVOPVC1 is given by

$$Q_1 := \{u \in \mathbb{R} | u_1 > 0, u_1 - u_2 = 0\} \cup \{u \in \mathbb{R} | u_1 = 0, u_2 \leq 0\}.$$

For any $u \in Q_1$, the Wolfe-type dual model to SIVOPVC1 is given by

$$\begin{aligned} D_W(u) \max\ & \rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) \\ & s.t. \omega^L(e_1, e_2) + \omega^U(2v_1, 0) + \pi_1^g(-1, 0) - \pi_1^\psi(1, -1) + \pi_1^\varphi(1, 0) = (0, 0), \\ & \pi_1^\psi \geq 0, \text{if } 1 \in \tau_+(u) \cup \tau_{0-}(u) \cup \tau_{00}(u); \\ & \pi_1^\psi \in \mathbb{R}, \text{if } 1 \in \tau_{0+}(u) \\ & \pi_1^\varphi \in \mathbb{R}, \text{if } 1 \in \tau_+(u);\ \pi_1^\varphi = \mathbb{R}, \text{if } 1 \in \tau_0(u), \\ & e_1 \in [-1, 1],\ e_2 \in \{-1, 1\}, \end{aligned}$$

where $\rho(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) = [|v_1| - |v_2| - \pi_1^g v_1 - \pi_1^\psi(v_1 - v_2) + \pi_1^\varphi v_1, v_1^2 - \pi_1^g v_1 - \pi_1^\psi(v_1 - v_2) + \pi_1^\varphi v_1]$. Hence, we can obtain the feasible set for problem D_W and this set is independent of u.

$$\begin{aligned} Q_2 := \{&(v_1, v_2, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi) : e_1\omega^L - \pi_1^g - \pi_1^\psi + \pi_1^\varphi = 0, e_2\omega^L + \pi_1^\psi = 0, \\ & v_1, v_2 \in E, \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1, \pi_1^g \geq 0, \pi_1^\varphi = 0, \pi_1^\psi \geq 0\}, \end{aligned}$$

Let $\omega^L = 0, \omega^U = 1, \pi_1^g = \pi_1^h = \varrho(\varrho \geq 0)$ one has $v_1 = \varrho, v_2 = \varrho$, and by $F(u) = \rho(v, \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi)$, we obtain

$$\begin{aligned} F^L(u) = \rho^L(v, \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi) = -\varrho_1^2 \leq 0 \implies |u_1| - |u_2| \leq 0, \\ F^U(u) = \rho^U(v, \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi) = 0 \implies u_2^2 = 0. \end{aligned}$$

that is, $u = (0, 0)$. By $\pi_1^g g_1(u) \leq 0, \pi_1^\varphi \varphi_1(u) = 0, -\pi_1^\psi \psi_1(u) \leq 0$ and the strong ∂_c-pseudoinvexity of $\rho^L(., \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi)$ and $\rho^U(., \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi)$ $v \in Q_1 \cup prQ_2$, we obtain $u = (0, 0)$ as the locally weakly LU optimal solution of SIVOPVC1.

In the upcoming section, we present the dual model of the Mond–Weir type for the SIVOPVC problem with reference to [2].

4. Mond–Weir Duality

For $u \in Q$, we have

$$
\begin{aligned}
(D_{MW}(u))\ LU-\max F(v) & \\
& s.t.\ 0\in\Delta(v),\\
& \omega^{L},\omega^{U}\in\mathbb{R}_{+},\omega^{L}+\omega^{U}=1,\\
& \pi_{j}^{g}\geq 0,\pi_{j}^{g}g_{j}(v)\geq 0,\forall j\in J,\\
& \pi_{k}^{h}\in\mathbb{R},\ \pi_{k}^{h}h_{j}(v)=0,\forall k\in\tau_{n},\\
& \pi_{e}^{\varphi}\varphi_{e}(v)\geq 0,\pi_{e}^{\varphi}=\mu_{e}\psi_{e}(u),\mu_{e}\geq 0,\ \forall e\in\tau_{l},\\
& -\pi_{e}^{\psi}\psi_{e}(v)\geq 0,\pi_{e}^{\psi}=\mathcal{B}_{e}-\mu_{e}\varphi_{e}(u),\mathcal{B}_{e}\geq 0,\ \forall e\in\tau_{l}.
\end{aligned}
\tag{17}
$$

Let $Q_{MW}(u)$ denote the feasible set of $D_{MW}(u)$, $prQ_{MW}(u):=\{v\in E:=(v,\omega^{L},\omega^{U},\pi^{g},\pi^{h},\pi^{\psi},\pi^{\varphi},\mu,\mathcal{B})\in Q_{MW}(u)\}$ represents the projection of the set $Q_{MW}(u)$ on E.
To eliminate the dependence on SIVOPVC, we present an alternative Mond–Weir-type dual model, sourced from [35]:

$$
\begin{aligned}
(D_{MW})\ LU-\max F(v) & \\
& s.t.(v,\omega^{L},\omega^{U},\pi^{g},\pi^{h},\pi^{\psi},\pi^{\varphi},\mu,\mathcal{B})\in Q_{MW}:=\cap_{u\in Q}Q_{MW}(u).
\end{aligned}
$$

Here, Q_{MW} represents the collection of all feasible points for D_{MW}, and prQ_{MW} indicates the projection of the set Q_{MW} onto the space E.

Definition 8 ([39])**.** *Let $u\in Q$*
(i) A point $(\bar{v},\omega^{L},\omega^{U},\pi^{g},\pi^{h},\pi^{\psi},\pi^{\varphi},\mu,\mathcal{B})\in Q_{w}(u)$ is known as locally LU optimal solution of $D_{MW}(u)$ if there exists $B(\bar{v};\delta)$ such that there is no $v\in Q_{MW}(u)\cap B(\bar{v};\delta)$ satisfying

$$F(\bar{v})<_{LU}F(v).$$

(ii) $(\bar{v},\omega^{L},\omega^{U},\pi^{g},\pi^{h},\pi^{\psi},\pi^{\varphi},\mu,\mathcal{B})\in Q_{w}(u)$ is known as locally weakly LU optimal solution of $D_{MW}(u)$ if there exists $B(\bar{v};\delta)$ such that there is no $v\in Q_{MW}(u)\cap B(\bar{v};\delta)$ satisfying

$$F(\bar{v})<_{LU}^{s}F(v).$$

Theorem 7 (Weak duality)**.** *Suppose we have an element u in the set Q, and we also have feasible points $(v,\omega^{L},\omega^{U},\pi^{g},\pi^{h},\pi^{\psi},\pi^{\varphi},\mu,\mathcal{B})$ for both the SIVOPVC and D_{MW} problems. If any of the following conditions are met:*
(i) $F^{L}(.),F^{U}(.)$ are strongly ∂_{c}-pseudoinvex of order $\alpha>0$ at $v\in Q\cup prQ_{MW}$, $\sum_{j\in J}\pi_{j}^{g}g_{j}(.)+\sum\limits_{k=1}^{n}\pi_{k}^{h}h_{k}(.)-\sum\limits_{e=1}^{l}\pi_{e}^{\psi}\psi_{e}(.)+\sum\limits_{e=1}^{l}\pi_{e}^{\varphi}\varphi_{e}(.)$ is strongly ∂_{c}-quasiinvex of order $\alpha>0$ at $v\in Q\cup prQ_{MW}$;
(ii) $F^{L}(.),F^{U}(.)$ are strongly ∂_{c}-pseudoinvex of order $\alpha>0$ at $v\in Q\cup prQ_{MW}$, $g_{j}(j\in\tau_{g}^{+}(u))$, $h_{k}(k\in\tau_{h}^{+}(u))$, $-h_{k}(k\in\tau_{h}^{-}(u))$, $-\psi_{e}(e\in\tau_{+}^{+}(u)\cup\tau_{00}^{+}(u)\cup\tau_{0-}^{+}(u)\cup\tau_{0+}^{+}(u))$, $\psi_{e}(e\in\tau_{0+}^{-}(u))$, $\varphi_{e}(e\in\tilde{I}_{+}^{+}(u))$ are strongly ∂_{c}-quasiinvex of order $\alpha>0$ with respect to the kernel function $\varpi:E\times E\rightarrow\mathbb{R}$ at $v\in Q\cup prQ_{MW}$. Then,
$F(u)\not<_{LU}^{s}F(v)$.

Proof. Suppose that $F(u)<_{LU}^{s}F(v)$; there exists

$$[F^{L}(u),F^{U}(u)]<_{LU}^{s}[F^{L}(v),F^{U}(v)]. \tag{18}$$

(i) Since $u \in Q$ and $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ are feasible points for the SIVOPVC and the Q_{MW}, one has (8). Utilizing Equations (17) and (8), we obtain

$$\sum_{j \in J} \pi_j{}^g g_j(u) + \sum_{k=1}^{n} \pi_k^h h_k(u) - \sum_{e=1}^{l} \pi_e^\psi \psi_e(u) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(u) \le \sum_{j \in J} \pi_j{}^g g_j(v) + \sum_{k=1}^{n} \pi_k^h h_k(v) - \sum_{e=1}^{l} \pi_e^\psi \psi_e(v) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(v)$$

and by the strong ∂_c-quasiinvexity of order $\alpha > 0$ of the above functions, we obtain

$$\langle \sum_{j \in J} \pi_j^g \varsigma_j^g + \sum_{k=1}^{n} \pi_k^h \varsigma_k^h - \sum_{e=1}^{l} \pi_e^\psi \varsigma_e^\psi + \sum_{e=1}^{l} \pi_e^\varphi \varsigma_e^\varphi, \varpi(u,v) \rangle + \mathcal{L}_j^g \|u-v\|^\alpha + \mathcal{L}_k^h \|u-v\|^\alpha + \mathcal{L}_e^\psi \|u-v\|^\alpha + \mathcal{L}_e^\varphi \|u-v\|^\alpha \le 0, \tag{19}$$

where $\varsigma_j^g \in \partial_c g_j(v), j \in J, \varsigma_k^h \in \partial_c h_k(v), k \in \tau_n, \varsigma_e^\psi \in \partial_c \psi_e(v), e \in \tau_l, \varsigma_e^\varphi \in \partial_c \varphi_e(v), e \in \tau_l$. Using the above inequality and $0 \in \Delta(v)$, there exist $\varsigma^L \in \partial_c F^L(v)$ and $\varsigma^U \in \partial_c F^U(v)$ such that

$$\langle \omega^L \varsigma^L + \omega^U \varsigma^U, \varpi(u,v) \rangle \ge 0. \tag{20}$$

By (18) and the strong ∂_c-pseudoinvexity of $F^L(.)$ and $F^U(.)$, it follows that

$$\begin{aligned} &\langle \varsigma^L, \varpi(u,v) \rangle + \mathcal{L}^L \|u-v\|^\alpha < 0, \ \forall \varsigma^L \in \partial_c F^L(v), \\ &\langle \varsigma^U, \varpi(u,v) \rangle + \mathcal{L}^U \|u-v\|^\alpha < 0, \ \forall \varsigma^U \in \partial_c F^U(v), \end{aligned}$$

then $\langle \omega^L \varsigma^L + \omega^U \varsigma^U, \varpi(u,v) \rangle < 0, \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1$, contradicting (20), so the result also holds.

(ii) By (17) and (8), one has

$$\begin{aligned} &g_j(u) \le g_j(v), \ \forall j \in \tau_g^+(u), \\ &h_k(u) = h_k(v), \ \forall k \in \tau_h^+(u) \cup \tau_h^-(u), \\ &-\psi_e(u) \le -\psi_e(v), \ \forall e \in \tau_+^+(u) \cup \tau_{00}^+(u) \cup \tau_{0-}^+(u) \cup \tau_{0+}^+(u), \\ &-\psi_e(u) \ge -\psi_e(v), \ \forall e \in \tau_{0+}^-(u), \\ &\varphi_e(u) \le \varphi_e(v), \ \forall e \in \tilde{I}_+^+(u). \end{aligned} \tag{21}$$

Combining (21) with the ∂_c-quasiinvexity of the above functions, it follows that

$$\begin{aligned} &\langle \varsigma_j^g, \varpi(u,v) \rangle + \mathcal{L}_j^g \|u-v\|^\alpha \le 0, \pi_j^g > 0, \exists\, \mathcal{L}_j^g > 0, \ \forall \varsigma_j^g \in \partial_c g_j(v), \ j \in \tau_g^+(u), \\ &\langle \varsigma_k^h, \varpi(u,v) \rangle + \mathcal{L}_k^h \|u-v\|^\alpha \le 0, \pi_k^h > 0, \exists\, \mathcal{L}_k^h > 0, \ \forall \varsigma_k^h \in \partial_c h_k(v), \ k \in \tau_h^+(u), \\ &\langle \varsigma_k^h, \varpi(u,v) \rangle + \mathcal{L}_k^h \|u-v\|^\alpha \ge 0, \pi_k^h < 0, \exists\, \mathcal{L}_k^h > 0, \ \forall \varsigma_k^h \in \partial_c h_k(v), \ k \in \tau_h^-(u), \\ &\langle -\varsigma_e^\psi, \varpi(u,v) \rangle + \mathcal{L}_e^\psi \|u-v\|^\alpha \le 0, \pi_e^\psi \ge 0, \exists\, \mathcal{L}_e^\psi > 0 \ \forall \varsigma_e^\psi \in \partial_c \psi_e(v), \ e \in \tau_+^+(u) \\ &\qquad \cup \tau_{00}^+(u) \cup \tau_{0-}^+(u) \cup \tau_{0+}^+(u), \\ &\langle -\varsigma_e^\psi, \varpi(u,v) \rangle + \mathcal{L}_e^\psi \|u-v\|^\alpha \ge 0, \pi_e^\psi \le 0, \exists\, \mathcal{L}_e^\psi > 0, \ \forall \varsigma_e^\psi \in \partial_c \psi_e(v), \ e \in \tau_{0+}^-(u), \\ &\langle \varsigma_e^\varphi, \varpi(u,v) \rangle + \mathcal{L}_e^\varphi \|u-v\|^\alpha \le 0, \pi_e^\varphi \ge 0, \exists\, \mathcal{L}_e^\varphi > 0, \ \forall \varsigma_e^\varphi \in \partial_c \varphi_e(v), \ e \in \tilde{I}_+^+(u) \end{aligned}$$

that is,

$$\langle \sum_{j\in J} \pi_j^g \varsigma_j^g + \sum_{k=1}^{n} \pi_k^h \varsigma_k^h - \sum_{e=1}^{l} \pi_e^\psi \varsigma_e^\psi + \sum_{e=1}^{l} \pi_e^\varphi \varsigma_e^\varphi, \varpi(u,v)\rangle + \mathcal{L}_j^g \|u-v\|^\alpha + \mathcal{L}_k^h \|u-v\|^\alpha + \mathcal{L}_e^\psi \|u-v\|^\alpha + \mathcal{L}_e^\varphi \|u-v\|^\alpha \leq 0.$$

The remainder of the proof follows the same steps as outlined in part (i). □

Theorem 8 (Strong duality)**.** *Suppose $\bar{u}$ is a locally weakly LU optimal solution within the context of SIVOPVC, with the condition VC-ACQ being satisfied at $\bar{u}$, and assuming that the set Δ is closed. Under these circumstances, there exist Lagrange multipliers $\bar{\omega}^L, \bar{\omega}^U \in \mathbb{R}_+, \bar{\pi}_g \in \mathbb{R}_+^{|J|}, \bar{\pi}_h \in \mathbb{R}^n, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mathcal{B}}, \bar{\mu} \in \mathbb{R}^l$ such that $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a feasible point of $D_{MW}(\bar{u})$, Furthermore, if the conditions stated in Theorem 7 are met, then $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a locally weakly LU optimal solution of $D_{MW}(\bar{u})$.*

Proof. Because $\bar{u}$ is the locally weakly LU optimal solution for SIVOPVC and the VC-ACQ condition is satisfied at $\bar{u}$, based on Theorem 1, we can conclude that there exist Lagrange multipliers $\bar{\omega}^L, \bar{\omega}^L \in \mathbb{R}_+, \bar{\pi}_g \in \mathbb{R}_+^{|J|}, \bar{\pi}_h \in \mathbb{R}^n, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mathcal{B}}, \bar{\mu} \in \mathbb{R}^l$, such that (3) and (4) are satisfied. Combined with the definition of $D_{MW}(\bar{u})$, one has that $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a feasible point of $D_{MW}(\bar{u})$. By Theorem 7, we know

$$F(\bar{u}) \not<_{LU}^s F(v) \;\; \forall (v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu) \in Q_{MW}(\bar{u})$$

so $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is a locally weakly LU optimal solution of $D_{MW}(\bar{u})$. □

Theorem 9 (Converse duality)**.** *Let $u \in Q$ be any feasible point of SIVOPVC and $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu) \in Q_{MW}$ be a feasible point of D_{MW}. If any of the following conditions are met:*
(i) $F^L(.), F^U(.)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ at $\bar{v} \in Q \cup prQ_{MW}$, $\sum_{j\in J} \pi_j^g g_j(.) + \sum_{k=1}^{n} \pi_k^h h_k(.) - \sum_{e=1}^{l} \pi_e^\psi \psi_e(.) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(.)$ is ∂_c-quasiinvex with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ at $\bar{v} \in Q \cup prQ_{MW}$;
(ii) $F^L(.), F^U(.)$ are strongly ∂_c-pseudoinvex of order $\alpha > 0$ with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ at $\bar{v} \in Q \cup prQ_{MW}$, $g_j(j \in \tau_g^+(u)), h_k(k \in \tau_h^+(u)), -h_k(k \in \tau_h^-(u)), -\psi_e(e \in \tau_+^+(u) \cup \tau_{00}^+(u) \cup \tau_{0-}^+(u) \cup \tau_{0+}^+(u)), \psi_e(e \in \tau_{0+}^-(u)), \varphi_e(e \in \tilde{I}_+^+(u))$ are strongly ∂_c-quasiinvex of order $\alpha > 0$ with respect to the kernel function $\varpi : E \times E \to \mathbb{R}$ at $\bar{v} \in Q \cup prQ_{MW}$. Then Then $\bar{v}$ is the locally weakly LU optimal solution of SIVOPVC.

Proof. (i) Assume that $\bar{v}$ does not represent the locally weakly LU optimal solution for SIVOPVC. In that case, we would have Equation (12).

As both $\bar{u}$ belonging to Q and $(\bar{v}, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ are feasible points for SIVOPVC and D_{MW}, utilizing Equations (17) and (8), we can conclude that (13) is satisfied. By the strong ∂_c-quasiinvexity of $\sum_{j\in J} \pi_j{}^g g_j(.) + \sum_{k=1}^{n} \pi_k^h h_k(.) - \sum_{e=1}^{l} \pi_e^\psi \psi_e(.) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(.)$ at $\bar{v} \in Q \cup prQ_{MW}$, we obtain

$$\langle \sum_{j\in J} \pi_j^g \varsigma_j^g + \sum_{k=1}^{n} \pi_k^h \varsigma_k^h - \sum_{e=1}^{l} \pi_e^\psi \varsigma_e^\psi + \sum_{e=1}^{l} \pi_e^\varphi \varsigma_e^\varphi, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}_j^g \|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_k^h \|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\psi \|\bar{u}-\bar{v}\|^\alpha + \mathcal{L}_e^\varphi \|\bar{u}-\bar{v}\|^\alpha \leq 0,$$

where $\varsigma_j^g \in \partial_c g_j(\bar{v}), j \in J, \varsigma_k^h \in \partial_c h_k(\bar{v}), k \in \tau_n, \varsigma_e^\psi \in \partial_c \psi_e(\bar{v}), e \in \tau_l, \varsigma_e^\varphi \in \partial_c \varphi_e(\bar{v}), e \in \tau_l$. Combining this with $0 \in \Delta(\bar{v})$, one has (16). Utilizing (12) and the strong ∂_c-pseudoinvexity of $F^L(.)$ and $F^U(.)$, one has

$$\langle \varsigma^L, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}^L\|\bar{u}-\bar{v}\|^\alpha < 0, \ \forall \varsigma^L \in \partial_c F^L(\bar{v}),$$
$$\langle \varsigma^U, \varpi(\bar{u},\bar{v})\rangle + \mathcal{L}^U\|\bar{u}-\bar{v}\|^\alpha < 0, \ \forall \varsigma^U \in \partial_c F^U(\bar{v}),$$

and $\langle \omega^L\varsigma^L + \omega^U\varsigma^U, \varpi(\bar{u},\bar{v})\rangle < 0, \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1$. This contradicts (16),thus establishing the validity of the outcome.

(ii) The proof is not provided since it closely resembles the proof of Theorem 5(ii). □

Theorem 10 (Restricted converse duality)**.** *Consider $\bar{u}$ as a feasible point within the scope of SIVOPVC, and let $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu)$ in Q_{MW} be valid points within the context of D_{MW}, satisfying the condition $F(\bar{u}) = F(v)$. If the conditions stated in Theorem 7 are met, then $\bar{u}$ qualifies as the locally weakly LU optimal solution for SIVOPVC.*

Proof. Assume that $\bar{u}$ does not represent a locally weakly LU optimal solution for SIVOPVC. Then there exist $\tilde{u} \in Q \cap B(\bar{u}, \delta)$ such that $F(\tilde{u}) <^s_{LU} F(\bar{u})$. By $F(\bar{u}) = F(v)$, one has $F(\tilde{u}) <^s_{LU} F(v)$, contradicting Theorem 7. □

Next, we will examine the following example in order to demonstrate the findings of Theorem 10.

Example 4. *Let $E = \mathbb{R}^2, n = 0, l = J = 1$, consider the SIVOPVC1 problem as follows:*

$$\begin{aligned} SIVOPVC1 \min F(u) =& [F^L(u), F^U(u)] = [|u_1| - |u_2|, u_1^2] \\ & s.t.\ g_1(u) = -u_1 \le 0, \\ & \psi_1(u) = u_1 - u_2 \ge 0, \\ & \varphi_1(u)\psi_1(u) = u_1(u_1 - u_2) \le 0. \end{aligned}$$

One can easily see that

$$Q_3 := \{u \in \mathbb{R} | u_1 > 0, u_1 - u_2 = 0\} \cup \{u \in \mathbb{R} | u_1 = 0, u_2 \le 0\}$$

gives feasible set of SIVOPVC2. The Mond–Weir dual model to SIVOPVC2 for any $x \in Q_1$ is given by

$$\begin{aligned} D_{MW}(u) \max F(v) =& [F^L(v), F^U(v)] = [|v_1| - |v_2|, v_1^2] \\ & s.t. \omega^L(e_1, e_2) + \omega^U(2v_1, 0) + \pi_1^g(-1, 0) - \pi_1^\psi(1, -1) + \pi_1^\varphi(1, 0) = (0, 0), \\ & \pi_1^g \ge 0, \pi_1^g g_1(v) \ge 0, \\ & \pi_1^\psi \ge 0, \text{if } 1 \in \tau_+(u) \cup \tau_{0-}(u) \cup \tau_{00}(u); \\ & \pi_1^\psi \in \mathbb{R}, \text{if } 1 \in \tau_{0+}(u),\ -\pi_1^\psi \psi_1(v) \ge 0, \\ & \pi_1^\varphi \ge 0, \text{if } 1 \in \tau_+(u);\ \pi_1^\varphi = 0, \text{if } 1 \in \tau_0(u), \pi_1^\varphi \varphi_1(v) \ge 0. \\ & e_1 \in [-1, 1],\ \textit{and}\ e_2 \in \{-1, 1\}. \end{aligned}$$

Hence, we possess a feasible set for problem D_{MW}, and this set is independent of u.

$$\begin{aligned} Q_4 := \{(v_1, v_2, \omega^L, \omega^U, \pi_1^g, \pi_1^\psi, \pi_1^\varphi) : e_1\omega^L - \pi_1^g - \pi_1^\psi + \pi_1^\varphi = 0, e_2\omega^L + \pi_1^\psi = 0, \\ v_1, v_2 \in E, \omega^L, \omega^U \in \mathbb{R}_+, \omega^L + \omega^U = 1, -\pi_1^g v_1 \ge 0, \pi_1^\varphi v_1 \ge 0 \\ -\pi_1^\psi(v_1 - v_2) \ge 0, \pi_1^\psi \ge 0, \pi_1^\varphi = 0\}. \end{aligned}$$

Let $\omega^L = 0, \omega^U = 1, \pi_1^g = \pi_1^h = \varrho(\varrho \geq 0)$ one has $v_1 = \varrho, v_2 = \varrho$, and by $F(u) = F(v)$, we obtain

$$F^L(u) = F^L(v) = 0 \implies |u_1| - |u_2| = 0,$$
$$F^U(u) = F^U(v) = \varrho^2 \geq 0 \implies u_1^2 \geq 0.$$

then we obtain $\pi_1^g g_1(u) \leq 0, \pi_1^\varphi \varphi_1(u) = 0, -\pi_1^\psi \psi_1(u) \leq 0$ and by the strong ∂_c-pseudoinvexity of $F^L(.)$ and $F^U(.)$ at $v \in Q_3 \cup prQ_4$ and the strong quasiinvexity of $\pi_1^g g_1(.) - \pi_1^\psi \psi_1(.) + \pi_1^\varphi \varphi_1(.)$. We determine that $u = (0,0)$ represents the locally weakly LU optimal solution for SIVOPVC2.

Theorem 11 (Strict converse duality)**.** *Consider an element $\bar{u}$ belonging to the set Q, which serves as a locally weakly LU optimal solution for the SIVOPVC problem. This solution satisfies the VC-ACQ condition at the point $\bar{u}$, and the set Δ is closed. Assuming that the criteria specified in Theorem 9 are met, and $(v, \omega^L, \omega^U, \pi^g, \pi^h, \pi^\psi, \pi^\varphi, \mathcal{B}, \mu) \in Q_{MW}(\bar{u})$ is the locally weakly LU optimal solution of $D_{MW}(\bar{u})$. If one of the following conditions holds:*
(i) $F^L(.), F^U(.)$ are strictly strongly ∂_c-pseudoinvex of order $\alpha > 0$ at $\bar{v} \in Q \cup prQ_{MW}$, $\sum_{j\in J} \pi_j^g g_j(.) + \sum_{k=1}^{n} \pi_k^h h_k() - \sum_{e=1}^{l} \pi_e^\psi \psi_e(.) + \sum_{e=1}^{l} \pi_e^\varphi \varphi_e(.)$ is ∂_c-quasiinvex at $\bar{v} \in Q \cup prQ_{MW}$;
(ii) $F^L(.), F^U(.)$ are strictly strongly ∂_c-pseudoinvex v at $\bar{v} \in Q \cup prQ_{MW}$, $g_j(j \in \tau_g^+(u))$, $h_k(k \in \tau_h^+(u)), -h_k(k \in \tau_h^-(u)), -\psi_e(e \in \tau_+^+(u) \cup \tau_{00}^+(u) \cup \tau_{0-}^+(u) \cup \tau_{0+}^+(u)), \psi_e(e \in \tau_{0+}^-(u))$, $\varphi_e(e \in \tilde{I}_+^+(u))$ are strongly ∂_c-quasiinvex of order $\alpha > 0$ at $\bar{v} \in Q \cup prQ_W$. Then, $\bar{u} = \bar{v}$.

Proof. Suppose that $\bar{u} \neq \bar{v}$. By Theorem 9, there exist Lagrange multipliers $\bar{\omega}^L, \bar{\omega}^L \in \mathbb{R}_+, \bar{\pi}_g \in \mathbb{R}_+^{|j|}, \bar{\pi}_h \in \mathbb{R}^n, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mathcal{B}}, \bar{\mu} \in \mathbb{R}^l$. such that $(\bar{u}, \bar{\omega}^L, \bar{\omega}^U, \bar{\pi}^g, \bar{\pi}^h, \bar{\pi}^\psi, \bar{\pi}^\varphi, \bar{\mu}, \bar{\mathcal{B}})$ is the locally weakly LU optimal solution of $D_W(\bar{u})$; it shows that

$$F(\bar{u}) = F(\bar{v}).$$

The remaining portions resemble (i) and (ii) from Theorem 10 and are, therefore, not included here. □

5. Conclusions

In this research paper, we gave duality theorems concerning a semi-infinite interval valued optimization problem involving vanishing constraints. We provided a set of duality theorems, including weak, strong, converse, restricted converse, and strict converse duality, establishing relationships between SIVOPVC and its corresponding dual models of Wolfe and Mond–Weir types. These theorems were derived under the conditions of higher-order ∂_c-pseudoinvexity, strict ∂_c-pseudoinvexity, and ∂_c-quasiinvexity. Some examples were also given to illustrate the obtained results. Additionally, alternative dual models such as the mixed-type dual can be explored by using the univexity and generalized univexity assumptions to obtain the duality results. However, some interesting topics for further research remain. It would also be compelling to establish analogous optimality and duality findings for multiobjective optimization problems. We shall investigate these questions in forthcoming papers.

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