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Dual Toeplitz Operators on the Orthogonal Complement of the Generalized Fock Space

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Abstract: We characterize the boundedness and compactness of dual Toeplitz operators on the orthogonal complement of the generalized Fock space. We study the problem when the finite sum of the dual Toeplitz products is compact. Additionally, we also consider when the sum of the dual Toeplitz operators is equal to another dual Toeplitz operator.

Keywords: generalized Fock space; dual Toeplitz operators; compactness

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1. Introduction

For any integer $d > 0$, let $\mathbb{C}^d$ be the complex d-space and let dv be the ordinary volume measure on $\mathbb{C}^d$. For points $z = (z_1, \cdots, z_d)$ and $w = (w_1, \cdots, w_d)$ in $\mathbb{C}^d$, we write $z\overline{w} = \sum_{i=1}^{d} z_i\overline{w_i}$, and $|z| = \sqrt{z\overline{z}}$. Let $H(\mathbb{C}^d)$ be the family of all holomorphic functions on $\mathbb{C}^d$. Given the real numbers $m \geq 1$ and $\alpha > 0$, the Lebesgue measure $d\mu_{m,\alpha}$ is defined by

$$d\mu_{m,\alpha}(z) = c_{m,\alpha} e^{-\alpha|z|^{2m}} dv(z),$$

where $c_{m,\alpha} = \frac{m\alpha^{\frac{d}{m}}}{\pi^d} \frac{\Gamma(d)}{\Gamma\left(\frac{d}{m}\right)}$ is the normalizing constant so that $d\mu_{m,\alpha}$ is a probability measure on $\mathbb{C}^d$. Let $L^2_{m,\alpha}$ be the space of measurable functions $f : \mathbb{C}^d \to \mathbb{C}$, such that

$$\|f\|^2_{L^2_{m,\alpha}} := \int_{\mathbb{C}^d} |f(z)|^2 d\mu_{m,\alpha}(z) < \infty.$$

The generalized Fock space is denoted by $F^2_{m,\alpha} = L^2_{m,\alpha} \cap H(\mathbb{C}^d)$. In particular, $F^2_{1,\alpha}$ is the Fock space when $m = 1$ (see [1]). $F^2_{m,\alpha}$ is a Hilbert space under the inner product

$$\langle f, g\rangle_{m,\alpha} = \int_{\mathbb{C}^d} f(w)\overline{g(w)} d\mu_{m,\alpha}(w).$$

For any $f \in F^2_{m,\alpha}$, there exists a constant C, such that

$$|f(z)| \leq C\|f\|_{F^2_{m,\alpha}} (1 + |z|)^{d(m-1)} e^{\frac{\alpha}{2}|z|^{2m}}, \; z \in \mathbb{C}^d$$

according to [2] (Corollary 2.9), which implies that each point evaluation is bounded on $F^2_{m,\alpha}$. Thus, for each $z \in \mathbb{C}^d$, there exists a unique reproducing kernel function $K_{m,\alpha}(\cdot, z) \in F^2_{m,\alpha}$, such that

$$f(z) = \langle f, K_{m,\alpha}(\cdot, z)\rangle_{m,\alpha}$$

for every $f \in F^2_{m,\alpha}$. The orthogonal projection $P_{m,\alpha} : L^2_{m,\alpha} \to F^2_{m,\alpha}$ is defined by

$$P_{m,\alpha} f(z) = \int_{\mathbb{C}^d} f(w) K_{m,\alpha}(z,w) d\mu_{m,\alpha}(w), \quad f \in L^2_{m,\alpha},\ z \in \mathbb{C}^d, \tag{1}$$

where $K_{m,\alpha}$ is the reproducing kernel in $F^2_{m,\alpha}$.

For multi-index $j = (j_1, j_2, \cdots, j_d) \in \mathbb{N}^d$, we write $|j| = j_1 + j_2 + \cdots + j_d$ and $j! = j_1! j_2! \cdots j_d!$. We also write $z^j = z_1^{j_1} z_2^{j_2} \cdots z_d^{j_d}$ for $z = (z_1, z_2, \cdots, z_d) \in \mathbb{C}^d$. Since the weight $e^{-\alpha|z|^{2m}}$ depends only on $|z|$, the monomials $z^j (j \in \mathbb{N}^d)$ form an orthogonal basis in $F^2_{m,\alpha}$. Its integration into spherical coordinates gives

$$E_{m,\alpha}(j) := \|z^j\|^2_{F^2_{m,\alpha}} = \frac{j!\Gamma(d)\Gamma\left(\frac{d+|j|}{m}\right)}{\Gamma\left(\frac{d}{m}\right)\Gamma(d+|j|)\alpha^{\frac{|j|}{m}}}.$$

Then, the set $\mathfrak{E} = \left\{ e_j(z) = [E_{m,\alpha}(j)]^{-1/2} z^j : j \in \mathbb{N}^d \right\}$ is an orthonormal basis for $F^2_{m,\alpha}$. Using the theory from Aronszajn [3] to compute the reproducing kernel of $F^2_{m,\alpha}$, we obtain

$$K_{m,\alpha}(z,w) = \sum_j \frac{z^j \overline{w}^j}{E_{m,\alpha}(j)}, \quad z, w \in \mathbb{C}^d. \tag{2}$$

Given $1 \le p < \infty$ and $b > 0$, as a consequence of [4] (Corollary 2.11), we obtain the following useful estimate

$$\begin{aligned} C_1(1+|z|)^{2d(m-1)\left(1-\frac{1}{p}\right)} e^{\frac{p\alpha^2}{4b}|z|^{2m}} &\le \int_{\mathbb{C}^d} |K_{m,\alpha}(z,w)|^p e^{-b|z|^{2m}} dv(w) \\ &\le C_2(1+|z|)^{2d(m-1)\left(1-\frac{1}{p}\right)} e^{\frac{p\alpha^2}{4b}|z|^{2m}}, \ z \in \mathbb{C}^d \end{aligned} \tag{3}$$

for the positive constants C_1 and C_2.

Let $\mathcal{M}_{m,\alpha}$ be the set of all bounded functions in $(F^2_{m,\alpha})^\perp$ with a compact support on $\mathbb{C}^d$; it is easy to show that $\mathcal{M}_{m,\alpha}$ is dense in $(F^2_{m,\alpha})^\perp$ by using a similar argument to [5]. Let $\varphi \in L^2_{m,\alpha}$; we define the dual Toeplitz operator S_φ with the symbol φ as follows:

$$S_\varphi f = (I - P_{m,\alpha})(\varphi f), \ f \in (F^2_{m,\alpha})^\perp$$

where I is the identity operator. If φ is bounded, then S_φ is bounded on $(F^2_{m,\alpha})^\perp$. However, S_φ may not be bounded when the symbols are more general or even densely defined on $\mathcal{M}_{m,\alpha}$. To ensure that the product of two dual Toeplitz operators is well defined, for a given $\delta > 0$, let $\mathcal{L}^\infty_{\alpha\delta,m}$ be the space of all Lebesgue measurable functions φ on $\mathbb{C}^d$, such that

$$\operatorname{ess\,sup}\left\{ |\varphi(z)| e^{-\delta\alpha\frac{|z|^{2m}}{2}} : z \in \mathbb{C}^d \right\} < \infty.$$

Let $f \in \mathcal{M}_{m,\alpha}$, and suppose that $\varphi \in \mathcal{L}^\infty_{\alpha\delta_1,m}$ and $\psi \in \mathcal{L}^\infty_{\alpha\delta_2,m}$ for some $0 < \delta_1, \delta_2 < \frac{1}{2}$. By calculating (3), we have

$$|P_{m,\alpha}(\psi f)(z)| \le K e^{\frac{\alpha}{4\left(1-\frac{\delta_2}{2}\right)}|z|^{2m}}, \quad z \in \mathbb{C}^d$$

for some positive constant K.

Furthermore, we obtain $\varphi P_{m,\alpha}(\psi f) \in L^2_{m,\alpha}$; this implies that $\varphi S_\psi f \in L^2_{m,\alpha}$, and then, $S_\varphi S_\psi f$ is well defined as a function in $L^2_{m,\alpha}$. Hence, for φ and ψ, the product $S_\varphi S_\psi$ is densely defined on $(F^2_{m,\alpha})^\perp$ and can be expressed as

$$S_\varphi S_\psi f = \varphi\psi f - \varphi P_{m,\alpha}(\psi f) - P_{m,\alpha}(\varphi\psi f) + P_{m,\alpha}[\varphi P_{m,\alpha}(\psi f)], \quad f \in \mathcal{M}_{m,\alpha}. \tag{4}$$

Let

$$\mathbb{SYM}_m = \bigcup_{0<\delta<\frac{1}{2}} \mathcal{L}^\infty_{\alpha\delta,m}$$

be the the symbol space. Under pointwise multiplication, $\mathbb{SYM}_m$ becomes an algebra.

Dual Toeplitz operators have been widely studied on the orthogonal complement of classical function spaces. For example, in the Bergman space over the unit disk setting, Stroethoff and Zheng [6] first studied the algebraic and spectral properties of the dual Toeplitz operator. Also, they characterized commuting dual Toeplitz operators. Yu and Wu [7] studied commuting dual Toeplitz operators with harmonic symbols on the orthogonal complement of the Dirichlet space. Chen, Yu and Zhao [8] characterized when two dual Toeplitz operators are commuting and semi-commuting on the orthogonal complement of the harmonic Dirichlet space, where the spectral properties of these operators were also studied. Later, their results were extended to a multiple-variable situation. Kong and Lu [9] characterized the algebraic properties of dual Toeplitz operators on Bergman spaces on a unit ball. Furthermore, they studied when the sum of the products of two dual Toeplitz operators is equal to a dual Toeplitz operator, which yielded the results mentioned above concerning the commutativity or product problem. Ding, Wu and Zhao [10] performed complete characterization for the hyponormality of dual Toeplitz operators with bounded harmonic symbols on the orthogonal complement of the Bergman space over an open unit disk. Lee [11] characterized when the finite sum of products of two dual Toeplitz operators is equal to zero on the orthogonal complement of the Dirichlet space. The corresponding problem for dual Toeplitz operators on the Hardy–Sobolev space and Fock space $(m = 1)$ has also been studied (see [12,13]). For more details on the study of dual Toeplitz operators, please refer to [6,14–17].

At the beginning of this century, some scholars began to pay attention to the structure of the generalized Fock space and its operators. Bommier-Hato, Engliš and Youssfi [18] proposed criteria for determining the boundedness of the associated Bergman-type projections on the generalized Fock space over $\mathbb{C}^n$. Schneider [19] studied Hankel operators with anti-holomorphic L^2-symbols on generalized Fock spaces A^2_m in one complex dimension. Bommier-Hato [20] studied the algebraic properties of the Toeplitz operator on the generalized Fock space over $\mathbb{C}^d$. For more details on the generalized Fock space, we refer to [2,21–23].

Motivated by the above results, in this paper, we consider similar problems on the orthogonal complements of the generalized Fock space $F^2_{m,\alpha}$, where m is a positive real number. We generalize the results of [12] to the generalized Fock space $F^2_{m,\alpha}$. That is, we mainly characterize the finite sum of dual Toeplitz products in another dual Toeplitz operator.

Our main results are as follows.

Theorem 1. *Let $\varphi_k, \psi_k \in \mathbb{SYM}_m$ be pluriharmonic for $k = 1, \cdots, N$ and $h \in \mathbb{SYM}_m$. Then, $S_h = \sum_{k=1}^N S_{\varphi_k} S_{\psi_k}$ if and only if $h = \sum_{k=1}^N \varphi_k \psi_k$ if and only if one of the following statement holds:*

(a) $\sum_{k=1}^N \overline{P_{m,\alpha}(\overline{\varphi_k})} P_{m,\alpha}(\psi_k \phi) \in H(\mathbb{C}^d)$ *for all* $\phi \in \mathcal{M}_{m,\alpha}$;

(b) $\sum_{k=1}^N \overline{\mathcal{R} P_{m,\alpha}(\overline{\varphi_k})} P_{m,\alpha}(\psi_k \phi) = 0$ *for all* $\phi \in \mathcal{M}_{m,\alpha}$;

(c) *There exists* $\lambda_k, \nu_k \in \mathbb{C}^N$ *for* $k = 1, \cdots, N$ *with* $\lambda_k \overline{\nu_l} = 0$ *for all* k, l *and*

$$(P_{m,\alpha}\overline{\varphi_1} - \overline{\varphi_1}(0), \cdots, P_{m,\alpha}\overline{\varphi_N} - \overline{\varphi_N}(0)) = \sum_{k=1}^N (\lambda_k P_{m,\alpha}\overline{\varphi_k} - \overline{\varphi_k}(0)),$$

$$(P_{m,\alpha}\psi_1 - \psi_1(0), \cdots, P_{m,\alpha}\psi_N - \psi_N(0)) = \sum_{k=1}^{N}(\nu_k P_{m,\alpha}\psi_k - \psi_k(0)).$$

The organization of this paper is as follows. In Section 2, the boundedness and compactness of the dual Toeplitz operators are characterized, and the necessary condition for the finite sum of the products of two dual Toeplitz operators to be compact is also considered. Section 3 studies the zero sums of the products of two dual Toeplitz operators with pluriharmonic symbols.

2. Boundedness and Compactness

In this section, we characterize the boundedness and compactness of dual Toeplitz operators with symbols in $L^2_{m,\alpha}$.

Let $w \in \mathbb{C}^d$, $0 < s < 1$ define a function on $\mathbb{C}^d$

$$G^m_{w,s}(z) = \overline{(z_1 - w_1)}e^{\alpha|z|^{2m}}\chi_{B(w,s)}(z), \ \ z \in \mathbb{C}^d,$$

where $B(w,s)$ is the Euclidean ball in $\mathbb{C}^d$ centered at $w \in \mathbb{C}^d$ with radius s, and $\chi_{B(w,s)}$ denotes the characteristic function of $B(w,s)$. $G^m_{w,s}$ is usually called test function. Set

$$g^m_{w,s}(z) = \frac{G^m_{w,s}(z)}{\|G^m_{w,s}\|_{L^2_{m,\alpha}}}.$$

For each $f \in (F^2_{m,\alpha})^\perp$ and $w \in \mathbb{C}^d$, by applying the Cauchy–Schwarz inequality, we determine that

$$|\langle f, g^m_{w,s}\rangle_{m,\alpha}|^2 = \left|\int_{B(w,s)} f(z)g^m_{w,s}(z)d\mu_{m,\alpha}(z)\right|^2 \leq \int_{B(w,s)} |f(z)|^2 d\mu_{m,\alpha}(z),$$

and it follows that $g^m_{w,s}$ converges to 0 weakly in $(F^2_{m,\alpha})^\perp$ as $s \to 0$. For $\varphi \in L^2_{m,\alpha}$, the multiplication operator M_φ is defined by $M_\varphi f = \varphi f$ for $f \in L^2_{m,\alpha}$.

Lemma 1. *With the notations above, we have $g^m_{w,s} \in (F^2_{m,\alpha})^\perp$ and*

$$\lim_{s\to 0^+} \|M_\varphi g^m_{w,s}\| = |\varphi(w)|$$

for a.e. $w \in \mathbb{C}^d$ and for each $\varphi \in L^2_{m,\alpha}$.

Proof. For each $f \in F^2_{m,\alpha}$, we have

$$\begin{aligned}\langle f, G^m_{w,s}\rangle &= \int_{\mathbb{C}^d} f(z)\overline{G^m_{w,s}(z)}d\mu_{m,\alpha}(z)\\ &= c_{m,\alpha}\int_{B(w,s)} f(z)(z_1 - w_1)dv(z) = 0,\end{aligned}$$

which implies that $G^m_{w,s} \in (F^2_{m,\alpha})^\perp$, and so, $g^m_{w,s} \in (F^2_{m,\alpha})^\perp$. We next show that the limit

$$\lim_{s\to 0^+} \|M_\varphi g^m_{w,s}\| = |\varphi(w)|$$

holds. Using [24] (Proposition 1.4.9), we determine that

$$\|G_{w,s}\|^2_{L^2_{m,\alpha}} \geq c_{m,\alpha}s^2 v(B(w,s))L$$

for the constant $L > 0$. Hence,

$$
\begin{aligned}
|\|M_\varphi g^m_{w,s}\|^2 - |\varphi(w)|^2| &= \left| \int_{\mathbb{C}^d} |\varphi(z)|^2 |g^m_{w,s}(z)|^2 d\mu_{m,\alpha}(z) - |\varphi(w)|^2 \right| \\
&\leq \int_{\mathbb{C}^d} ||\varphi(z)|^2 - |\varphi(w)|^2| |g^m_{w,s}(z)|^2 d\mu_{m,\alpha}(z) \\
&= \frac{1}{\|G_{w,s}\|^2_{L^2_{m,\alpha}}} \int_{\mathbb{C}^d} ||\varphi(z)|^2 - |\varphi(w)|^2| |G^m_{w,s}(z)|^2 d\mu_{m,\alpha}(z) \\
&= \frac{c_{m,\alpha}}{\|G_{w,s}\|^2_{L^2_{m,\alpha}}} \int_{B(w,s)} ||\varphi(z)|^2 - |\varphi(w)|^2| |z_1 - w_1|^2 dv(z) \\
&\leq \frac{c_{m,\alpha} s^2}{\|G_{w,s}\|^2_{L^2_{m,\alpha}}} \int_{B(w,s)} ||\varphi(z)|^2 - |\varphi(w)|^2| dv(z) \\
&\leq \frac{1}{Lv(B(w,s))} \int_{B(w,s)} ||\varphi(z)|^2 - |\varphi(w)|^2| dv(z).
\end{aligned}
$$

Let

$$
\mathcal{A} = \left\{ w \in \mathbb{C} : \lim_{s \to 0} \frac{\int_{B(w,s)} ||\varphi(z)|^2 - |\varphi(w)|^2| dv(z)}{v(B(w,s))} = 0 \right\}.
$$

We determine that the complement set of $\mathcal{A}$ is a set of measure zero according to Theorem 8.8 of [25].This finishes the proof. □

Given $\varphi \in L^2_{m,\alpha}$, we define the Hankel operators $H_\varphi : F^2_{m,\alpha} \to (F^2_{m,\alpha})^\perp$ and $H^*_{\overline{\varphi}} : (F^2_{m,\alpha})^\perp \to (F^2_{m,\alpha})^\perp$ with $H_\varphi = (I - P_{m,\alpha})M_\varphi$ and $H^*_{\overline{\varphi}} = P_{m,\alpha} M_\varphi$. The following lemma will be useful in our characterization for the boundedness and compactness of the dual Toeplitz operator.

Lemma 2. *For $\varphi \in L^2_{m,\alpha}$, we have*

$$
\lim_{s \to 0} \|S_\varphi g^m_{w,s}\| = |\varphi(w)|
$$

for a.e. $w \in \mathbb{C}^d$.

Proof. Note that

$$
M_\varphi f = P_{m,\alpha}(\varphi f) + (I - P_{m,\alpha})(\varphi f) = H^*_{\overline{\varphi}} f + S_\varphi f
$$

for each $f \in (F^2_{m,\alpha})^\perp$. Thus,

$$
\|M_\varphi g^m_{w,s}\|^2 = \|S_\varphi g^m_{w,s}\|^2 + \|H^*_{\overline{\varphi}} g^m_{w,s}\|^2.
$$

According to (3), we have

$$
\begin{aligned}
\|H^*_{\overline{\varphi}} g^m_{w,s}\|^2 &= \int_{\mathbb{C}^d} |H^*_{\overline{\varphi}} g^m_{w,s}(z)|^2 d\mu_{m,\alpha}(z) \\
&= \int_{\mathbb{C}^d} |P_{m,\alpha}(\varphi g^m_{w,s})(z)|^2 d\mu_{m,\alpha}(z) \\
&= \int_{\mathbb{C}^d} \left| \int_{B(w,s)} \varphi(\zeta) g^m_{w,s}(\zeta) K_{m,\alpha}(z,\zeta) d\mu_{m,\alpha}(\zeta) \right|^2 d\mu_{m,\alpha}(z) \\
&\leq \int_{\mathbb{C}^d} \int_{B(w,s)} |\varphi(\zeta)|^2 |K_{m,\alpha}(z,\zeta)|^2 d\mu_{m,\alpha}(\zeta) d\mu_{m,\alpha}(z) \\
&\leq C \int_{B(w,s)} |\varphi(\zeta)|^2 (1 + |\zeta|)^{2d(m-1)} e^{\alpha|\zeta|^{2m}} d\mu_{m,\alpha}(\zeta) \\
&\leq C(2 + |w|)^{d(m-1)} e^{\alpha(|w|+1)^{2m}} \int_{B(w,s)} |\varphi(\zeta)|^2 d\mu_{m,\alpha}(\zeta).
\end{aligned}
$$

for each $0 < s < 1$. Based on this assumption, we obtain

$$
\lim_{s \to 0} \int_{B(w,s)} |\varphi(\zeta)|^2 d\mu_{m,\alpha}(\zeta) = 0.
$$

Therefore,

$$\lim_{s\to 0} \|H^*_{\overline{\varphi}} g^m_{w,s}\| = 0$$

for each $w \in \mathbb{C}^d$, and this implies that

$$|\varphi(w)|^2 = \lim_{s\to 0} \|M_\varphi g^m_{w,s}\|^2 = \lim_{s\to 0} \|S_\varphi g^m_{w,s}\|^2$$

for *a.e.* $w \in \mathbb{C}^d$ by using Lemma 1. □

Recall that $L^\infty(\mathbb{C}^d)$ is the space of measurable functions f on $\mathbb{C}^d$, such that

$$\|f\|_\infty := \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{C}^d\} < +\infty.$$

Now, we are ready to characterize the boundedness of dual Toeplitz operators on $(F^2_{m,\alpha})^\perp$.

Theorem 2. *If $f \in L^2_{m,\alpha}$, then S_f is bounded on $(F^2_{m,\alpha})^\perp$ if and only if $f \in L^\infty(\mathbb{C}^d)$. In which case, we have $\|S_f\| = \|f\|_\infty$.*

Proof. If $f \in L^\infty(\mathbb{C}^d)$, then $\|S_f\| \le \|f\|_\infty$. Suppose that S_f is bounded on $(F^2_{m,\alpha})^\perp$. Note that

$$\|S_f g^m_{w,s}\| \le \|S_f\|$$

for all $w \in \mathbb{C}^d$ and $0 < s < 1$. Letting $s \to 0$ and using Lemma 2, we have

$$|f(w)| \le \|S_f\|$$

for *a.e.* $w \in \mathbb{C}^d$, so that

$$\|f\|_\infty \le \|S_f\|.$$

This completes the proof. □

Corollary 1. *If $f \in L^\infty(\mathbb{C}^d)$, then S_f is compact on $(F^2_{m,\alpha})^\perp$ if and only if $f(w) = 0$ a.e. $w \in \mathbb{C}^d$.*

Corollary 2. *If $f \in L^\infty(\mathbb{C}^d)$, then $S_f = 0$ on $(F^2_{m,\alpha})^\perp$ if and only if $f(w) = 0$ a.e. $w \in \mathbb{C}^d$.*

We consider the relation between the compactness of the finite sums of finite dual Toeplitz products and their symbols.

Theorem 3. *Let $\varphi_t, \psi_t \in \mathbb{SYM}_m$ for $t = 1, 2, \cdots, N$. If $\sum_{t=1}^N S_{\varphi_t} S_{\psi_t}$ is compact on $(F^2_{m,\alpha})^\perp$, then $\sum_{t=1}^N \varphi_t \psi_t = 0$.*

Proof. If the dual Toeplitz operators are closely related to Hankel operators, we have

$$\sum_{t=1}^N S_{\varphi_t \psi_t} = \sum_{t=1}^N S_{\varphi_t} S_{\psi_t} + \sum_{t=1}^N H_{\varphi_t} H^*_{\overline{\psi_t}}.$$

Lemma 2 and (3) determine that

$$
\begin{aligned}
\|H_{\varphi_t} H^*_{\overline{\psi_t}} g^m_{w,s}\|^2 &= \|(I-P_{m,\alpha})(\varphi_t H^*_{\overline{\psi_t}} g^m_{w,s})\|^2 \\
&\le \int_{\mathbb{C}^d} |\varphi_t(z)|^2 |H^*_{\overline{\varphi}} g^m_{w,s}(z)|^2 d\mu_{m,\alpha}(z) \\
&= \int_{\mathbb{C}^d} |\varphi_t(z)|^2 \int_{B(w,s)} |\psi_t(\zeta)\overline{K_{m,\alpha}(z,\zeta)}|^2 d\mu_{m,\alpha}(\zeta) d_{m,\alpha}\mu(z) \\
&= \int_{B(w,s)} |\psi_t(\zeta)|^2 \int_{\mathbb{C}^d} |\varphi_t(z)|^2 |K_{m,\alpha}(z,\zeta)|^2 d\mu_{m,\alpha}(z) d\mu_{m,\alpha}(\zeta) \\
&\le C \int_{B(w,s)} (1+|\zeta|)^{d(m-1)} e^{\frac{\alpha}{2(1-\frac{\delta}{2})}|\zeta|^{2m}} e^{\frac{\delta\alpha}{2}|\zeta|^{2m}} d\mu_{m,\alpha}(\zeta) \\
&\le C(2+|w|)^{d(m-1)} e^{\frac{\alpha}{2}(\frac{1}{1-\frac{\delta}{2}}+\delta)(|w|+1)^{2m}} \int_{B(w,s)} d\mu_{m,\alpha}(\zeta).
\end{aligned}
$$

for each $0 < s < 1$. Hence,

$$\lim_{s\to 0^+} \|H_{\varphi_j} H^*_{\overline{\psi_t}} g^m_{w,s}\| = 0$$

for $w \in \mathbb{C}^d, t = 1, 2, \cdots, N$. This means that

$$\lim_{s\to 0} \|S_{\sum_{t=1}^N \varphi_t\psi_t} g^m_{w,s}\| = \left|\sum_{t=1}^N \varphi_t\psi_t\right| = 0$$

according to Lemma 1 again. This completes the proof. □

As a simple application of Theorem 3, we determine that a product of several dual Toeplitz operators with harmonic symbols can be compact only in a trivial case. For harmonic functions $\varphi_1, \cdots, \varphi_N \in \mathbb{SYM}_m$ for which $\varphi_1 \cdots \varphi_N = 0$, at least one φ_t must be zero.

Corollary 3. *Let $\varphi_1, \cdots, \varphi_N \in \mathbb{SYM}_m$ be harmonic; then, the following conditions are equivalent:*
(a) $\prod_{t=1}^N S_{\varphi_t}$ *is compact;*
(b) $\prod_{t=1}^N S_{\varphi_t} = 0$;
(c) $\varphi_t = 0$ *for some* t.

Let $\mathcal{B}((F^2_{m,\alpha})^\perp)$ denote the space of all linear bounded operators on $(F^2_{m,\alpha})^\perp$, and $\mathfrak{H}$ be the set of all operators of the form $\sum_{t=1}^M A_t H_\varphi H^*_\psi$, where $M \ge 1$ is an integer, and $A_t \in \mathcal{B}((F^2_{m,\alpha})^\perp)$, $\varphi, \psi \in L^\infty(\mathbb{C}^d)$. It follows from Lemma 2 that

$$\lim_{s\to 0^+} \|A g^m_{w,s}\| = 0 \tag{5}$$

for every $w \in \mathbb{C}^d$ and $A \in \mathfrak{H}$.

Lemma 3. *Let $\varphi_t \in L^\infty(\mathbb{C}^d)$ for $t = 1, 2, \cdots, N$. Then,*

$$S_{\varphi_1} \cdots S_{\varphi_N} = S_{\varphi_1\varphi_2\cdots\varphi_N} + A$$

for some $A \in \mathfrak{H}$.

Proof. As mentioned above, the result is true for $N = 2$. Now, suppose the result holds for $N - 1$; then,

$$
\begin{aligned}
S_{\varphi_1} \cdots S_{\varphi_N} &= S_{\varphi_1}(S_{\varphi_2\cdots\varphi_N} + A) \\
&= S_{\varphi_1} S_{\varphi_2\cdots\varphi_N} + S_{\varphi_1} A \\
&= S_{\varphi_1\varphi_2\cdots\varphi_N} - H_{\varphi_1} H^*_{\overline{\varphi_2\cdots\varphi_N}} + S_{\varphi_1} A.
\end{aligned}
$$

for some $A \in \mathfrak{H}$. Note that $-H_{\varphi_1} H^*_{\overline{\varphi_2\cdots\varphi_N}} + S_{\varphi_1} A \in \mathfrak{H}$. □

Proposition 1. *Let $\varphi_{kt} \in L^\infty(\mathbb{C}^d)$. If $\sum_{k=1}^N \prod_{t=1}^{M_k} S_{\varphi_{kt}}$ is compact, then $\sum_{k=1}^N \prod_{t=1}^{M_k} \varphi_{kt} = 0$.*

Proof. According to Lemma 3, we have

$$\sum_{k=1}^N \prod_{t=1}^{M_k} S_{\varphi_{kt}} = \sum_{k=1}^N \left(S_{\prod_{t=1}^{M_k} \varphi_{kt}} + A_k \right) = S_{\sum_{k=1}^N \prod_{t=1}^{M_k} \varphi_{kt}} + \sum_{k=1}^N A_k$$

for some $A_k \in \mathfrak{H}(k = 1, 2, \cdots, N)$. For $w \in \mathbb{C}^d$, $g_{w,s}^m$ converges to 0 weakly in $(F_{m,\alpha}^2)^\perp$ as $s \to 0$. Combining the assumption, Lemma 2, and equality (5), we obtain

$$\lim_{s \to 0} \| S_{\sum_{k=1}^N \prod_{t=1}^{M_k} \varphi_{kt}}(g_{w,s}^m) \| = 0.$$

□

Remark 1. *The above conclusion holds for bounded symbols. However, we do not know whether it is true for the symbol in $\mathbb{SYM}_m$.*

3. Zero Sum of Products of Dual Toeplitz Operators

In this section, we consider the finite sum of the products of two dual Toeplitz operators on the generalized Fock space, determine when such an operator equals zero, and obtain several applications. Recall that a complex-valued function on $\mathbb{C}^d$ is said to be pluriharmonic when its restriction to an arbitrary complex line is harmonic as a function of one complex variable. It is well known that each pluriharmonic function can be decomposed as $f + \overline{g}$ for some $f, g \in H(\mathbb{C}^d)$. For the two multi-indices $j = (j_1, j_2, \cdots, j_d)$ and $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_d)$, the notation $j \preccurlyeq \gamma$ denotes that $j_k \le \gamma_k$ for all $1 \le k \le d$. For $j \preccurlyeq \gamma$ we assume that $\gamma - j = (\gamma_1 - j_1, \gamma_2 - j_2, \cdots, \gamma_d - j_d)$.

The following proposition will be very useful in our analysis later on.

Proposition 2. *Let $u \in \mathbb{SYM}_m$ be pluriharmonic. Then, the following statements are equivalent:*

(1) $P_{m,\alpha}(u\phi) = 0$ for every $\phi \in \mathcal{M}_{m,\alpha}$;
(2) $\overline{u} \in H(\mathbb{C}^d)$;
(3) $P_{m,\alpha} u$ is constant.

Proof. It is trival that condition (2) $\Leftrightarrow$ (3). We are going to prove that (1) $\Leftrightarrow$ (2). First, assume that (1) holds and write $u = f + \overline{g}$ for some $f, g \in H(\mathbb{C}^d)$. For a multi-index j, we let $\phi_j = \overline{w}^j \chi_B$, where B is the unit ball in $\mathbb{C}^d$. Then, $\phi_j \in \mathcal{M}_{m,\alpha}$ and $P_{m,\alpha}(\overline{g}\phi_j) = 0$ for every j with $|j| \ge 1$. Thus, $P_{m,\alpha}(f\phi_j) = 0$ for every j with $|j| \ge 1$. We let $f(z) = \sum_\gamma a_\gamma z^\gamma$ be its Taylor series. According to (1) and (2), we have

$$\begin{aligned} 0 =& P_{m,\alpha}(f\phi_j)(z) \\ =& \sum_\gamma a_\gamma P_{m,\alpha}(w^\gamma \phi_j)(z) \\ =& \sum_\gamma a_\gamma \sum_\tau \frac{1}{E_{m,\alpha}(\tau)} z^\tau \int_B w^\gamma \overline{w^{\tau+j}} d\mu_{m,\alpha}(w) \\ =& \sum_{j \preccurlyeq \gamma} a_\gamma \frac{1}{E_{m,\alpha}(\gamma - j)} z^{\gamma - j} \int_B |w^\gamma|^2 d\mu_{m,\alpha}(w) \end{aligned}$$

for every j with $|j| \ge 1$ and $z \in \mathbb{C}^d$. Thus, for any given j with $|j| \ge 1$, the above shows that $a_\gamma = 0$ for every γ with $j \preccurlyeq \gamma$, which means that $a_\gamma = 0$ for every $\gamma \ne 0$. So, we determine that (2) holds. Suppose that condition (2) holds. Using (3), $\overline{u}K_{m,\alpha}(\cdot, z) \in F_{m,\alpha}^2$ for all $z \in \mathbb{C}^d$. It follows that

$$P_{m,\alpha}(u\phi)(z) = \langle u\phi, K_{m,\alpha}(\cdot, z) \rangle_{m,\alpha} = \langle \phi, \overline{u}K_{m,\alpha}(\cdot, z) \rangle_{m,\alpha} = 0, \quad z \in \mathbb{C}^d$$

for every $\phi \in \mathcal{M}_{m,\alpha}$. This shows that (1) holds. □

We let

$$\mathcal{R}f(z) = \sum_{i=1}^{d} z_i \frac{\partial f}{\partial z_i}(z), \quad \widetilde{\mathcal{R}}f(z) = \sum_{i=1}^{d} \overline{z_i} \frac{\partial f}{\partial \overline{z_i}}(z)$$

for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$.

We now prove the main result of this section.

Proof of Theorem 1. Write $\varphi_k = u_k + \overline{v_k}$ for some $u_k, v_k \in F^2_{m,\alpha}$. Using (4), we obtain

$$\begin{aligned}\left[S_h - \sum_{k=1}^{N} S_{\varphi_k} S_{\psi_k}\right]\phi =& \phi\left(h - \sum_{k=1}^{N} \varphi_k \psi_k\right) + \sum_{k=1}^{N} \varphi_k P_{m,\alpha}(\psi_k \phi) \\ &- P_{m,\alpha}\left[\phi\left(h - \sum_{k=1}^{N} \varphi_k \psi_k\right)\right] - P_{m,\alpha}\left[\sum_{k=1}^{N} \varphi_k P_{m,\alpha}(\psi_k \phi)\right]\end{aligned} \tag{6}$$

for every $\phi \in \mathcal{M}_{m,\alpha}$. We note that

$$\overline{v_k} = \overline{P_{m,\alpha}\overline{\varphi_k}} - u_k(0)$$

according to (1) and (2) for each k. Then, according to Theorem 3, we see that $S_h = \sum_{k=1}^{N} S_{\varphi_k} S_{\psi_k}$ if and only if $h = \sum_{k=1}^{N} \varphi_k \psi_k$ and (a) holds. Thus, in order to complete the proof, it is sufficient to show that (a), (b) and (c) are all equivalent.

Implication $(a) \Rightarrow (b)$. By taking $\widetilde{R}$ in (a), we determine that $(a) \Rightarrow (b)$ holds.

Equivalence $(b) \Leftrightarrow (c)$. We see from [17] (Theorem 3.2) that (b) holds if and only if there exists $\lambda_k, \nu_k \in \mathbb{C}^N$ for $k = 1, \cdots, N$, such that $\lambda_k \overline{\nu_l} = 0$ for all k, l and

$$\begin{aligned}(\widetilde{\mathcal{R}}P_{m,\alpha}(\overline{\varphi_1}), \cdots, \widetilde{\mathcal{R}}P_{m,\alpha}(\overline{\varphi_N})) &= \sum_{k=1}^{N} \lambda_k \widetilde{\mathcal{R}}P_{m,\alpha}(\overline{\varphi_k}), \\ (P_{m,\alpha}(\psi_1 \phi), \cdots, P_{m,\alpha}(\psi_N \phi)) &= \sum_{k=1}^{N} \nu_k P_{m,\alpha}(\psi_k \phi)\end{aligned} \tag{7}$$

for all $\phi \in \mathcal{M}_{m,\alpha}$. Writing $\lambda_k = (\lambda_k^1, \cdots, \lambda_k^N)$ and $\nu_k = (\nu_k^1, \cdots, \nu_k^N)$ for each k, we note that (7) is equivalent to

$$\begin{aligned}\mathcal{R}\left[P_{m,\alpha}\left(\overline{\varphi_k} - \sum_{k=1}^{N} \lambda_k^l \overline{\varphi_k}\right)\right] &= \mathcal{R}P_{m,\alpha}(\overline{\varphi_k}) - \sum_{k=1}^{N} \lambda_k^l \mathcal{R}P_{m,\alpha}(\overline{\varphi_k}) = 0, \\ P_{m,\alpha}\left[\left(\psi_k - \sum_{k=1}^{N} \nu_k^l \psi_k\right)\phi\right] &= P_{m,\alpha}(\psi_k \phi) - \sum_{k=1}^{N} \nu_k^l P_{m,\alpha}(\psi_k \phi) = 0\end{aligned} \tag{8}$$

for each l and all $\phi \in \mathcal{M}_{m,\alpha}$. According to Proposition 2, we know that (8) is equivalent to

$$\begin{aligned}P_{m,\alpha}\left(\overline{\varphi_k} - \sum_{k=1}^{N} \lambda_k^l \overline{\varphi_k}\right) &= P_{m,\alpha}\left(\overline{\varphi_k} - \sum_{k=1}^{N} \lambda_k^l \overline{\varphi_k}\right)(0), \\ P_{m,\alpha}\left(\psi_k - \sum_{k=1}^{N} \nu_k^l \psi_k\right) &= P_{m,\alpha}\left(\psi_k - \sum_{k=1}^{N} \nu_k^l \psi_k\right)(0)\end{aligned} \tag{9}$$

for each l. Note that $P_{m,\alpha}\overline{\varphi_k}(0) = \overline{\varphi_k}(0)$ and $P_{m,\alpha}\psi_k(0) = \psi_k(0)$ for each k, so (9) is equivalent to (c). Hence we conclude that $(b) \Leftrightarrow (c)$.

Implication $(c) \Rightarrow (a)$. Suppose now that (c) holds. From Proposition 2 and $\lambda_k \cdot \overline{\nu_l} = 0$ for all k, l, we see that

$$\sum_{k=1}^{N} \overline{P_{m,\alpha}(\overline{\varphi_k})} P_{m,\alpha}(\psi_k \phi) = \sum_{k=1}^{N} \varphi_k(0) P_{m,\alpha}(\psi_k \phi) \in H(\mathbb{C}^d)$$

for all $\phi \in \mathcal{M}_{m,\alpha}$, which shows that (a) holds. This completes the proof of the theorem. □

We now have several consequences of Theorem 1. Firstly, in the special case when $N = 2$, we obtain a more concrete solution in the next corollary. In the course of the proof, we use the well-known complexification lemma:

Lemma 4. *Let Ω be a domain in $\mathbb{C}^d$ and assume that Φ is holomorphic on $\Omega \times \Omega^*$, where $\Omega^* = \{\bar{z} : z \in \Omega\}$. If $\Phi(z, \bar{z}) = 0$ for all $z \in \Omega$, then $\Phi = 0$ on $\Omega \times \Omega^*$.*

Corollary 4. *Let $f, g, u, v \in \mathbb{SYM}_m$ be pluriharmonic and $h \in \mathbb{SYM}_m$. Then $S_h = S_f S_g + S_u S_v$ on $(F^2_{m,\alpha})^\perp$ if and only if $h = fg + uv$ and one of the following conditions holds:*

(I) $f, u \in H(\mathbb{C}^d)$;
(II) $\bar{g}, \bar{v} \in H(\mathbb{C}^d)$;
(III) $f, \bar{v} \in H(\mathbb{C}^d)$;
(IV) $\bar{g}, u \in H(\mathbb{C}^d)$;
(V) $f + \lambda u \in H(\mathbb{C}^d)$ *and* $\bar{v} - \overline{\lambda g} \in H(\mathbb{C}^d)$ *for some constant* $\lambda \neq 0$.

Proof. According to Theorem 1, it suffices to prove that (b) of Theorem 1 holds if and only if one of $(I) - (V)$ holds. According to Lemma 4, we determine that (b) of Theorem 1 holds if and only if
$$\overline{\mathcal{R}\bar{u}(z)} P_{m,\alpha}(v\phi)(w) = -\overline{\mathcal{R}\bar{f}(z)} P_{m,\alpha}(g\phi)(w) \tag{10}$$
for all $z, w \in \mathbb{C}^d$ and $\phi \in \mathcal{M}_{m,\alpha}$. First, suppose that (10) holds. Recall that for a pluriharmonic function $\mathscr{P}$, $\mathcal{R}\overline{\mathscr{P}} = 0$ if and only if $\mathscr{P} \in H(\mathbb{C}^d)$. If $u \in H(\mathbb{C}^d)$, then $\mathcal{R}\bar{u} = 0$, and together with (10), we determine that either $\mathcal{R}\bar{u} = \mathcal{R}\bar{f} = 0$ or $\mathcal{R}\bar{u} = P_{m,\alpha}(g\phi) = 0$ for all $\phi \in \mathcal{M}_{m,\alpha}$. Thus, the first case implies that $f, u \in H(\mathbb{C}^d)$, and hence, (I) holds. According to Proposition 2, the second one implies that (IV) holds. Still, if $\bar{g} \in H(\mathbb{C}^d)$, then $P_{m,\alpha}(g\phi) = 0$ for all $\phi \in \mathcal{M}_{m,\alpha}$ according to Proposition 2 again, and (II) or (IV) holds based on a similar argument.

Next, assume that $u, \bar{g}$ are not holomorphic. Then, $\mathcal{R}\bar{u}(z_0) \neq 0$ and $P_{m,\alpha}(g\phi_0)(w_0) \neq 0$ for some $z_0, w_0 \in \mathbb{C}^d$ and $\phi_0 \in \mathcal{M}_{m,\alpha}$. According to (10), we have $P_{m,\alpha}(v\phi)(w) = -\bar{\eta} P_{m,\alpha}(g\phi)(w)$ and $\overline{\mathcal{R}\bar{f}(z)} = -\lambda \overline{\mathcal{R}\bar{u}(z)}$ for all $z, w \in \mathbb{C}^d$ and $\phi \in \mathcal{M}_{m,\alpha}$, where

$$\eta = \frac{\mathcal{R}\bar{f}(z_0)}{\mathcal{R}\bar{u}(z_0)}, \quad \lambda = \frac{P_{m,\alpha}(v\phi_0)(w_0)}{P_{m,\alpha}(g\phi_0)(w_0)}.$$

Therefore, $P_{m,\alpha}(v\phi + \bar{\eta} g\phi) = 0$ for all $\phi \in \mathcal{M}_{m,\alpha}$ and $\mathcal{R}(\overline{f + \lambda u}) = 0$. Notice that $\lambda = -\bar{\eta}$ according to (10). So, $\bar{v} - \overline{\lambda g} \in H(\mathbb{C}^d)$ and $f + \lambda u \in H(\mathbb{C}^d)$ according to Proposition 2. If $\lambda = 0$, then (III) holds. If $\lambda \neq 0$, then (V) holds.

For the converse implication, we assume that $(I) - (V)$ holds. Then, we use Proposition 2 to determine that (10) holds. □

As a simple application of Corollary 4, we characterize the commutativity of dual Toeplitz operators with pluriharmonic symbols.

Corollary 5. *Let $f, g \in \mathbb{SYM}_m$ be pluriharmonic . Then, $S_f S_g = S_g S_f$ on $(F^2_{m,\alpha})^\perp$ if and only if one of the following conditions holds:*

(I) $f, g \in H(\mathbb{C}^d)$;
(II) $\bar{f}, \bar{g} \in H(\mathbb{C}^d)$;
(III) *There exist constants λ, η, not both 0, such that $\lambda f + \eta g$ is constant on $\mathbb{C}^d$.*

Proof. We take $u = g$ and $v = -f$ in Corollary 4, as desired. □

Corollary 6. *Let $f \in \mathbb{SYM}_m$ be pluriharmonic. Then, $S_f S_{\overline{f}} = S_{\overline{f}} S_f$ on $(F^2_{m,\alpha})^\perp$ if and only if f is constant.*

If we take $u = v = 0$ in Corollary 4, we obtain the following corollary.

Corollary 7. *Let $f, g \in \mathbb{SYM}_m$ be pluriharmonic and $h \in \mathbb{SYM}_m$. Then $S_h = S_f S_g$ on $(F^2_{m,\alpha})^\perp$ if and only if $h = fg$, and either $f \in H(\mathbb{C}^d)$ or $\overline{g} \in H(\mathbb{C}^d)$.*

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