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A Quantum Formalism for Abstract Dynamical Systems

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Abstract: This paper presents a quantum formulation for classical abstract dynamical systems (ADS), defined by coupled sets of first-order differential equations. They are referred to as "abstract" because their dynamical variables can be of different interrelated natures, not necessarily corresponding to physics, such as populations, socioeconomic variables, behavioral variables, etc. A classical linear Hamiltonian can be derived for ADS by using Dirac's dynamics for singular Hamiltonian systems. Also, a corresponding first-order Schrödinger equation (which involves the existence of a system Planck constant particular of each system) can be derived from this Hamiltonian. However, Madelung's reinterpretation of quantum mechanics, followed by the Bohm and Hiley work, produces no further information about the mathematical formulation of ADS. However, a classical quadratic Hamiltonian can also be derived for ADS, as well as a corresponding second-order Schrödinger equation. In this case, the Madelung reinterpretation of quantum mechanics provides a quantum Hamiltonian that does provide the quantum formulation for ADS, which provides new quantum variables interrelated dynamically with the classical variables. An application case is presented: the one-dimensional autonomous system given by the logistic dynamics. The differences between the classical and the quantum trajectories are highlighted in the context of this application case.

Keywords: abstract dynamical systems; classical dynamics; Dirac Hamiltonian; Schrödinger equation; Madelung quantum interpretation; Bohm and Hiley interpretation; quantum dynamics; logistic function

MSC: 81Q65; 93A10

1. Introduction

General System Theory was proposed by Bertalanffy [1] as a new complementary approach to science in order to provide it with a mathematical universal language. Since that work, some attempts to state a General System Theory have been proposed. For a summary of these proposals, see [2]. However, the objective of stating that universal language seems too ambitious, and it has been steered to state general systems as theories that unify different fields of science. An example is provided in work [3], which proposes a unified mathematical view of energy in psychology, similar to the energy of physics. The method followed in [3] is the so-called isomorphism of systems proposed by Ferrer [4], i.e., translating a known contrasted theory from one field to another field where similar problems try to be solved.

The present paper's objective is to take the isomorphism of systems proposal as a reference: translating the quantum formalism from dynamics of physical systems to abstract dynamical systems. Note that the dynamical systems in physics are generally defined by Newton's equations, i.e., by coupled sets of second-order differential equations. However, abstract dynamical systems are here referred to as dynamical models given by coupled sets of first-order differential equations, where the unknown temporal variables have different interrelated natures, not necessarily corresponding to physics, such as populations, socioeconomic variables, behavioral variables, etc. Actually, these systems are known in the literature about the subject simply as dynamical systems (see, for example, the work by Anosov and Arnold [5]). However, including the adjective abstract presents

the transdisciplinary aspect of this kind of system. This question must be emphasized because, for instance, a system Planck constant is presented in the formalism, and it is not the physical Planck constant but a singular constant of the particular system under study.

A first attempt to state a (non-relativistic) quantum formalism for abstract dynamical systems was presented by the author of this paper in [6], but he did not reach any significant solution. Therefore, this paper tries to present a significant solution to state a quantum formalism for abstract dynamical systems from a rigorous mathematical approach, although no empirical support has yet been found. Therefore, this paper tries to state a new formalism that opens a new field of research, which lies between General System Theory and mathematical physics, by presenting the theoretical findings obtained.

To reach that formalism, the well-known method to obtain the Schrödinger equation from the Hamiltonian is also followed. A crucial step of formalism is, therefore, to obtain the Hamiltonian for an abstract dynamical system. The method followed is what was discovered by Dirac [7], who provides a method to obtain a Hamiltonian for which the momentum variables either cannot be explicitly isolated from the general velocities (i.e., the time derivatives of the configuration variables) or the momentum variables vanish. This last case is also known as a singular system, and it corresponds to an abstract dynamical system, as presented in this work.

Actually, the real objective of Dirac in the work [7] was to quantize the electromagnetic field due to it being given by first-order partial differential equations. Therefore, it was an approach steered to solve the problem for fields rather than for abstract dynamical systems. However, obtaining the Hamiltonian for abstract dynamical systems can also be solved with the work by Dirac [7]. See the works [8–10] for that solution, although only in [6,9] the subsequent quantization method is presented in two different ways: from a linear Hamiltonian in [6] and from a quadratic Hamiltonian in [9]. The differences between the linear Hamiltonian and the quadratic Hamiltonian are clarified throughout this paper.

In addition to the Dirac Hamiltonian approach [7], there exists a Lagrangian approach, as stated by Havas [11], which obtains results equivalent to the Hamiltonian approach by Dirac [7]. Govaerts highlighted that equivalence for first-order fields in [12], and that equivalence was also highlighted for abstract dynamical systems in [8]. Both problems are known in mathematical physics, respectively, as the inverse Lagrange problem and as the inverse Hamiltonian problem. In other words, the inverse Lagrange problem consists of finding the Lagrangian corresponding to a coupled set of differential equations; therefore, this set must obey the Euler–Lagrange equations. In addition, the inverse Hamiltonian problem consists of finding the Hamiltonian corresponding to a coupled set of differential equations; therefore, this set must obey the Hamilton equations.

Havas solves the inverse Lagrange problem for abstract dynamical systems in Annex B of [11]. He presents this solution from a set of general sufficient equations that solve the inverse Lagrange problem for a more significant number of differential systems. Those sufficient solutions were presented in previous work [13], where a wide range of general inverse Lagrange problems were dealt with and solved. However, Micó in [8] solves the inverse Lagrange problem for abstract dynamical systems by following a simpler way. In [8], Micó also demonstrates the equivalence with the corresponding solution of the inverse Hamiltonian problem provided by the method of Dirac in [7].

In addition, the objective of the paper, enounced as translating the quantum formalism from dynamics of physical systems to abstract dynamical systems, can be brought beyond by the early Madelung reinterpretation of quantum mechanics [14], which was followed subsequently by Bohm and Hiley formalism in the work [15]. This reinterpretation provides a quantum potential correction to the classical equations through the quantum Hamilton–Jacobi. In fact, the quantum Hamilton–Jacobi equation arises from the polar form of the quantum wave function and its split in modulus, which provides the probability conservation, and in phase or action, which provides the quantum Hamilton–Jacobi equation. Bohm and Hiley linked this correction to the existence of a microscopic level underlying the quantum level of description. However, the present paper tries to interpret

a possible quantum correction of abstract dynamical systems as a quantum term that arises from its own system complexity. In addition, a system Planck constant, particular to each system studied and, therefore, with a different value of the physical Planck constant, is presented in the formalism. Consequently, the formalism presented assumes an epistemological correction to the dynamics of an abstract dynamical system, playing a fundamental role in the system Planck constant. Also, that incapability is present in the Bohm and Hiley formalism [15], but as emphasized above, due to the assumption of a microscopic world underlying the quantum level. In this paper, this incapability is only assumed in advance as a hypothetical epistemological principle. In fact, throughout the paper, the usual predictions of abstract dynamical systems are called classical dynamics, while the predictions of abstract dynamical systems with quantum corrections are called quantum dynamics. Also, depending on the context, classical dynamics can be referred to as classical trajectory and quantum dynamics can be referred to as quantum trajectory.

In addition, the Dirac linear Hamiltonian is proved here, which provides, by using the quantization rules, a first-order Schrödinger equation that does not have a Madelung quantum potential correction. Therefore, the solution found is to apply the same method provided by Dirac in [7] to a quadratic Hamiltonian. To perform this, the abstract dynamical system must be previously reformulated as a second-order abstract dynamical system (by taking the time derivative). In addition, some abstract masses are needed to obtain the quadratic Hamiltonian. In fact, the corresponding Hamilton equations are Newtonian-kind equations where the abstract masses play the same role as the masses in the physical systems. In the subsequent step, a canonical transformation is needed to simplify the Hamiltonian. This new Hamiltonian provides, by using the quantization rules again, a second-order Schrödinger equation for which the action does provide the Madelung quantum potential correction. Finally, the first-order abstract dynamical system can be recovered: the quantum potential correction arises under an integral term. This integral term can be represented by a new system of first-order differential equations coupled with the original ones.

In addition, one of the most important theories about abstract dynamical systems is referred to as synergetics, a term coined by Haken [16]. This formalism is probably the closest to what is presented. In fact, it provides a partial differential equation, the Fokker–Planck equation, for a stochastic abstract dynamical equation, given by Ito equations (see also [16]), which plays a similar role to Schrödinger equation in Hamiltonian systems. The Fokker–Planck equation provides the time evolution of the probability density. However, the starting point is stochastic, not deterministic, such as the case presented here.

In fact, the deterministic or stochastic character of the formalism presented must be commented on. In the present work, the approach is deterministic, i.e., the quantum correction of abstract dynamical systems is applied as deterministic dynamics, although the approach could also be stochastic in future works due to the Madelung quantum potential is derived from the previous wave function, which has a stochastic nature. In fact, both differenced cases are considered in the Bohm and Hiley work [15]. This deterministic quantum correction is studied in the application case of Section 5. In this case, the quantum potential presents singularities that highlight the profound differences between the concept of classical trajectory and the quantum trajectory provided here.

In addition, other works must be compared here. The work [17] uses the same denomination, abstract dynamical system, not with the same sense as here provided, but referring to physical systems, which are mathematically described in terms of abstract algebras and their states. In fact, it also deals with classical and quantum dynamics in the context of the Lie Algebras formalism, and this is the reason for the selection of the term abstract. Therefore, no other kinds of dynamical systems that differ from physical systems have been dealt with here. However, the approach of [17] could be an equivalent way to study abstract dynamical systems in future research. In addition, the work [18] deals with the so-called hybrid quantum–classical systems, which are mathematical-physics approaches to systems where quantum degrees of freedom interact with classical ones.

This approach also considers the classical and quantum dynamics relationships in the context of physical systems; therefore, it is closely related to the application case here presented in Section 5. Finally, the work [19] introduces the concept of Schrödingerisation, i.e., how discrete linear dynamical systems can be viewed as discrete-time approximations of dynamical systems that evolve continuously in time. Perhaps it could be a way to overcome the singularities found in the quantum trajectories of the application case of Section 5 in future studies.

Sections 2–4 present the formalism announced: obtaining a quantum formulation for abstract dynamical systems. In particular, Section 2 is devoted to obtaining a Dirac quadratic Hamiltonian for an abstract dynamical system. Section 3 makes use of this Hamiltonian to obtain the corresponding second-order Schrödinger equation, which does present a Madelung quantum potential correction in the Hamiltonian. Section 4 presents the quantum formulation of the abstract dynamical system from the corresponding quantum Hamilton equations. In addition, Section 5 presents an application case: the one-dimensional autonomous system given by the logistic dynamics.

On the other hand, Appendix A provides contents presented previously in the context of different conferences [6,8–10]. In particular, Appendix A is devoted to the Dirac linear Hamiltonian for the abstract dynamical systems, as well as to the corresponding first-order Schrödinger equation derived, showing that no quantum potential correction arises in the Hamiltonian. Note that these contents failed to obtain a quantum formulation for the abstract dynamical systems, but they are necessary for the correct understanding of Sections 2–4. In addition, the inclusion of this appendix contributes to the short length of the paper.

Note also that Sections 2–4 and Appendix A have been developed by following the axiomatic-deductive structure in order to present the contributions in a clearer way. Section 6 is devoted to the paper's discussion and conclusions. In addition, the equations in Appendix A are presented and referred to in the rest of the paper as Equation (A1), Equation (A2), etc. Also, the definitions, theorems, propositions, and corollaries are presented as Definition A1, Definition A2, etc., Theorem A1, Theorem A2, etc.

2. Dirac Quadratic Hamiltonian

Definition 1. *An abstract dynamical system is defined by the following coupled set of differential equations ($1 \leq k \leq n$):*

$$\dot{q}_k(t) = f_k(t, \boldsymbol{q}) \tag{1}$$

In Equation (1) t is the time variable, and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ represent the dynamical variables. Their nature can be arbitrary, not necessarily physical, such as populations, chemical or biochemical species, socioeconomic or behavioral indicators, etc. In addition, the $f_k(t, \boldsymbol{q})$ ($1 \leq k \leq n$) represent their dynamic interactions, considering in advance equipped by all kinds of smoothness properties to develop suitably the formalism. In addition, $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ are called the dynamical velocities.

In the beginning, a quadratic Hamiltonian is derived in this section for the abstract dynamical system of Equation (1). As "quadratic," this section is referred to as quadratic in the momentum variables, unlike Equation (A19) of Appendix A, which is linear in the momentum variables. Then, a corresponding second-order formulation is needed instead of the original Equation (1). This alternative second-order is presented as a Newtonian-kind equation by multiplying Equation (1) by a set of functions ($1 \leq k \leq n$), here called abstract masses $m_k(t, \boldsymbol{q})$ (still unknown), and subsequently taking the time derivative ($1 \leq k \leq n$):

$$\frac{d\left(m_k(t,\boldsymbol{q}) \cdot \dot{q}_k(t)\right)}{dt} = \frac{d\left(m_k(t,\boldsymbol{q}) \cdot f_k(t,\boldsymbol{q})\right)}{dt} = \frac{\partial\left(m_k(t,\boldsymbol{q}) \cdot f_k(t,\boldsymbol{q})\right)}{\partial t} + \sum_{j=1}^{n} \frac{\partial\left(m_k(t,\boldsymbol{q}) \cdot f_k(t,\boldsymbol{q})\right)}{\partial q_j} \dot{q}_j \tag{2}$$

The Dirac method [7] is also applied here to obtain the quadratic Hamiltonian corresponding to Equation (2).

Proposition 1. *The Hamiltonian corresponding to Equation* (2) *is:*

$$H(t, \boldsymbol{q}, \boldsymbol{p}) = h(t, \boldsymbol{q}) + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, \boldsymbol{q})} \cdot \phi_j(t, \boldsymbol{q}, \boldsymbol{p}) =$$
$$= h(t, \boldsymbol{q}) + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, \boldsymbol{q})} \left(p_j - g_j(t, \boldsymbol{q}) \right) \tag{3}$$

Proof. Consider the Lagrangian of Equation (A2) of Appendix A, and also the corresponding Hamiltonian of Equation (A5) with the primary constants $\phi_k(t, \boldsymbol{q}, \boldsymbol{p}) = p_k - g_k(t, \boldsymbol{q})$ $(1 \le k \le n)$. Due to the multipliers $\lambda_k(t, \boldsymbol{q}, \boldsymbol{p})$ $(1 \le k \le n)$ can also depend on the momentum variables (the Dirac method permits it); then, the hypothesis here assumed is that they have the following form $(1 \le k \le n)$:

$$\lambda_k(t, \boldsymbol{q}, \boldsymbol{p}) = \frac{p_k}{2m_k(t, \boldsymbol{q})} \tag{4}$$

Inserting Equation (4) in the Hamiltonian of Equation (A5), the Hamiltonian of Equation (3) holds. □

Note in Equation (3) that also, in addition to the $m_j(t, \boldsymbol{q})$ functions, the $h(t, \boldsymbol{q})$ and $g_j(t, \boldsymbol{q})$ functions are still undetermined, i.e., they are not here obtained by AEquations (A11)–(A13) of Appendix A due to the new hypothesis of Equation (4).

Corollary 1. The corresponding Hamilton equations for Equation (3) are $(1 \le k \le n)$:

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = \frac{p_k}{2m_k(t, \boldsymbol{q})} \tag{5}$$

$$\dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, \boldsymbol{q})} \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} \tag{6}$$

Proof. Taking into account that outside the Hamiltonian expression, the primary constants are $\phi_j(t, \boldsymbol{q}, \boldsymbol{p}) = p_j - g_j(t, \boldsymbol{q}) = 0$ $(1 \le j \le n)$ and considering also Equation (4), the Hamilton equations of Equations (5) and (6) hold. □

Corollary 2. The following equations hold $(1 \le k \le n)$:

$$\frac{d\left(2m_k(t, \boldsymbol{q}) \cdot \dot{q}_k(t)\right)}{dt} = \dot{p}_k(t) = -\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, \boldsymbol{q})} \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} \tag{7}$$

Proof. Equation (34) holds by multiplying Equation (7) by $2m_k(t, \boldsymbol{q})$, taking subsequently the time derivative, and substituting Equation (6). □

Corollary 3. The following equation holds $(1 \le k \le n)$:

$$-\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, \boldsymbol{q})} \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} = \frac{\partial\left(2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q})\right)}{\partial t} + \sum_{j=1}^{n} \frac{\partial\left(2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q})\right)}{\partial q_j} \frac{p_j}{2m_j(t, \boldsymbol{q})} \tag{8}$$

Proof. Equalling the right-hand sides of Equations (2) and (7), multiplying by 2, and substituting $\dot{q}_j(t) = \frac{p_j}{2m_j(t, \boldsymbol{q})}$ from Equations (5) and (8) holds. □

Proposition 2. The $g_k(t, \boldsymbol{q})$ $(1 \leq k \leq n)$ functions hold that:

$$g_k(t, \boldsymbol{q}) = 2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}) \tag{9}$$

Proof. Applying the consistency conditions, $\dot{\phi}_k(t, \boldsymbol{q}, \boldsymbol{p}) = 0$ $(1 \leq k \leq n)$, and taking into account the zero values of the primary constants outside the Hamiltonian expression in the Hamilton equations of Equations (5) and (6) $(1 \leq k \leq n)$:

$$\dot{\phi}_k(t, \boldsymbol{q}, \boldsymbol{p}) = \frac{\partial \phi_k(t, \boldsymbol{q}, \boldsymbol{p})}{\partial t} + \sum_{j=1}^n \frac{\partial \phi_k(t, \boldsymbol{q}, \boldsymbol{p})}{\partial q_j} \dot{q}_j + \sum_{k=1}^n \frac{\partial \phi_k(t, \boldsymbol{q}, \boldsymbol{p})}{\partial p_j} \dot{p}_j = -\frac{\partial g_k(t, \boldsymbol{q})}{\partial t} - $$

$$\sum_{j=1}^n \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} \dot{q}_j + \sum_{j=1}^n \delta_{kj} \dot{p}_j = -\frac{\partial g_k(t, \boldsymbol{q})}{\partial t} - \sum_{j=1}^n \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} \frac{p_j}{2m_j(t, \boldsymbol{q})} - \frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^n \frac{p_j}{2m_j(t, \boldsymbol{q})} \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} = 0 \tag{10}$$

From the last term of Equation (10) equal to zero, the following equation is deduced $(1 \leq k \leq n)$:

$$-\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^n \frac{p_j}{2m_j(t, \boldsymbol{q})} \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} = \frac{\partial g_k(t, \boldsymbol{q})}{\partial t} + \sum_{j=1}^n \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} \frac{p_j}{2m_j(t, \boldsymbol{q})} \tag{11}$$

In the beginning, the left-hand term $(1 \leq k \leq n)$ of Equation (8) is substituted by the right-hand term $(1 \leq k \leq n)$ of Equation (11):

$$\frac{\partial (2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial t} + \sum_{j=1}^n \frac{\partial (2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial q_j} \frac{p_j}{2m_j(t, \boldsymbol{q})} = \frac{\partial g_k(t, \boldsymbol{q})}{\partial t} + \sum_{j=1}^n \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} \frac{p_j}{2m_j(t, \boldsymbol{q})} \tag{12}$$

Note from Equation (12) that Equation (9) holds. $\square$

Corollary 4. The following equation holds:

$$\sum_{j=1}^n \frac{p_j}{2m_j(t, \boldsymbol{q})} \left(\frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} - \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} \right) - \frac{\partial h(t, \boldsymbol{q})}{\partial q_k} - \frac{\partial g_k(t, \boldsymbol{q})}{\partial t} = 0 \tag{13}$$

Proof. Reorganizing the last term of Equation (10) equal to zero, Equation (13) adequately holds. $\square$

Corollary 5. If the following equations

$$\frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} = \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_j} 1 \leq j, k \leq n \tag{14}$$

$$\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \frac{\partial g_k(t, \boldsymbol{q})}{\partial t} = 0 \ 1 \leq k \leq n \tag{15}$$

hold for the $g_j(t, \boldsymbol{q})$ and $h(t, \boldsymbol{q})$ functions, then the consistency equations of Equation (10) hold. In addition, the following equations hold:

$$\frac{\partial (m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q}))}{\partial q_k} = \frac{\partial (m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial q_j} 1 \leq j, k \leq n \tag{16}$$

$$\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} = -\frac{\partial (2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial t} 1 \leq k \leq n \tag{17}$$

Proof. The substitution of Equations (16) and (17) means that Equation (10) holds identically; therefore, the consistency equations of Equation (10) hold. In addition, Equations (43) and (44) are an immediate consequence of Equation (9) substitution in Equations (14) and (15). □

Note in addition that, by Equation (2), these abstract masses can be arbitrary as long as they obey Equations (16) and (17). Moreover, they also could be chosen by a question of practical or theoretical convenience. For instance, if the field $f(t, q)$ derived from a potential, then the abstract masses could be all taken equal to unit or constant values. Also, the $h(t, q)$ can be arbitrary as long as it obeys Equation (17). Also, $h(t, q)$ could be chosen by a question of practical or theoretical convenience. For instance, if the dynamical system was autonomous, i.e., $f(t, q) = f(q)$, then the abstract masses could all be taken as $m_k(q)$ and $h(t, q) = 0$. However, a general solution for the abstract masses can be provided: $m_k(t, q) = \frac{\mu_k}{f_k(t,q)}$, being μ_k constants with the suitable dimensions. Then, Equation (16) holds identically, and the solution $h(t, q) = 0$ can be chosen.

Theorem 1. *The quadratic Hamiltonian of Equation* (3) *becomes:*

$$
\begin{aligned}
H(t, q, p) &= h(t, q) + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, q)} \cdot \phi_j(t, q, p) = \\
&= h(t, q) + \sum_{j=1}^{n} \frac{p_j}{2m_j(t, q)} \left(p_j - 2m_j(t, q) \cdot f_j(t, q) \right) = \\
&= h(t, q) + \sum_{j=1}^{n} \frac{p_j^2}{2m_j(t, q)} - \sum_{j=1}^{n} f_j(t, q) \cdot p_j
\end{aligned}
\tag{18}
$$

Proof. Substitute Equation (9) in Equation (3) to obtain it. □

Proposition 3. The corresponding Hamilton equations to Equation (18) are the following ($1 \leq k \leq n$):

$$
\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = \frac{p_k}{2m_k(t, q)}
\tag{19}
$$

$$
\dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\frac{\partial h(t, q)}{\partial q_k} + \sum_{j=1}^{n} \frac{p_j}{m_j(t, q)} \frac{\partial \left(m_j(t, q) \cdot f_j(t, q) \right)}{\partial q_k}
\tag{20}
$$

Proof. Substitute Equation (9) in the Hamilton equations of Equations (5) and (6). □

Note that Equations (16) and (17) must be obeyed in Equations (18)–(20). However, to obtain the second-order corresponding Schrödinger equation by applying the quantization rules [20] on the Hamiltonian of Equation (18), imaginary terms arise, such as those of the second right hand of this Hamiltonian, due to the terms proportional to p_j ($1 \leq j \leq n$). These imaginary terms make the corresponding time-independent second-order Schrödinger equation non-real. Then, a canonical transformation is needed on Equations (18)–(20) to cancel these terms.

Proposition 4. The transformation ($1 \leq l \leq n$):

$$
\left.
\begin{aligned}
Q_l &= q_l \\
P_l &= p_l - m_l(t, q) \cdot f_l(t, q)
\end{aligned}
\right\}
\tag{21}
$$

is a canonical transformation.

Proof. To prove that Equation (21) is a canonical transformation, the following matrix equation must be obeyed [21]:

$$
M \cdot J \cdot M^T = J
\tag{22}
$$

In Equation (22), M (M^T is the transposed matrix of M), and J are the following $2nx2n$ dimensional matrices:

$$J = \begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix} ; \ M = \begin{bmatrix} \left[\frac{\partial Q}{\partial q}\right]_{nxn} & \left[\frac{\partial P}{\partial q}\right]_{nxn} \\ \left[\frac{\partial Q}{\partial p}\right]_{nxn} & \left[\frac{\partial P}{\partial q}\right]_{nxn} \end{bmatrix} \tag{23}$$

In Equation (23) I_n is the nxn identity matrix, and O_n is the nxn null matrix. Taking into account Equation (21), the M matrix and its transposed matrix M^T become:

$$M = \begin{bmatrix} I_n & -F_n \\ O_n & I_n \end{bmatrix} ; \ M^T = \begin{bmatrix} I_n & O_n \\ -F_n^T & I_n \end{bmatrix} \tag{24}$$

where in Equation (24) $[F_n]_{lr} = \frac{\partial(m_l(t,\boldsymbol{q})\cdot f_l(t,\boldsymbol{q}))}{\partial q_r}$, $1 \leq l, r \leq n$, and $\left[F_n^T\right]_{lr} = \frac{\partial(m_r(t,\boldsymbol{q})\cdot f_r(t,\boldsymbol{q}))}{\partial q_l}$, $1 \leq l, r \leq n$. Then, the left-hand side of Equation (22) becomes:

$$M \cdot J \cdot M^T = \begin{bmatrix} F_n - F_n^T & I_n \\ -I_n & O_n \end{bmatrix} \tag{25}$$

However, in Equation (22), $[F_n]_{lr} - \left[F_n^T\right]_{lr} = \frac{\partial(m_l(t,\boldsymbol{q})\cdot f_l(t,\boldsymbol{q}))}{\partial q_r} - \frac{\partial(m_r(t,\boldsymbol{q})\cdot f_r(t,\boldsymbol{q}))}{\partial q_l}$, $1 \leq l, r \leq n$; but all these terms are zero due to Equation (16). Therefore, Equation (22) holds, and the proposed transformation in Equation (21) is canonical, i.e., it preserves the Hamilton equations. $\square$

Corollary 6. The transformed Hamiltonian $K(t, \boldsymbol{Q}, \boldsymbol{P})$ corresponding to Equation (21) is:

$$
\begin{aligned}
K(t, \boldsymbol{Q}, \boldsymbol{P}) &= \frac{1}{2}h(t, \boldsymbol{Q}) + \sum_{j=1}^{n} \frac{1}{2m_j(t, \boldsymbol{Q})}\left(P_j + m_j(t, \boldsymbol{Q})\cdot f_j(t, \boldsymbol{Q})\right)\varphi_j(t, \boldsymbol{Q}, \boldsymbol{P}) = \\
&= \frac{1}{2}h(t, \boldsymbol{Q}) + \frac{1}{2}\sum_{j=1}^{n} \frac{P_j^2}{m_j(t, \boldsymbol{Q})} - \frac{1}{2}\sum_{j=1}^{n} m_j(t, \boldsymbol{Q})\cdot f_j^2(t, \boldsymbol{Q})
\end{aligned}
\tag{26}
$$

where in Equation (26): $\varphi_j(t, \boldsymbol{Q}, \boldsymbol{P}) = P_j - m_j(t, \boldsymbol{Q})\cdot f_j(t, \boldsymbol{Q})$ $(1 \leq j \leq n)$.

Proof. In order to obtain the transformed Hamiltonian, $K(t, \boldsymbol{Q}, \boldsymbol{P})$, from the previous Hamiltonian, $H(t, \boldsymbol{q}, \boldsymbol{p})$ of Equation (18), the generating function, F, is needed. The three functions are related by the following equations [21]:

$$p_k - \sum_{l=1}^{n} \frac{\partial Q_l}{\partial q_k} P_l = \frac{\partial F}{\partial q_k} 1 \leq k \leq n \tag{27}$$

$$-\sum_{l=1}^{n} \frac{\partial Q_l}{\partial p_k} P_l = \frac{\partial F}{\partial p_k} 1 \leq k \leq n \tag{28}$$

$$-H + K - \sum_{l=1}^{n} \frac{\partial Q_l}{\partial t} P_l = \frac{\partial F}{\partial t} \tag{29}$$

Taking into account Equation (21), $\frac{\partial Q_l}{\partial q_k} = \delta_{lk}$, then, from Equation (27): $p_k - P_k = \frac{\partial F}{\partial q_k}$, i.e., $m_k(t, \boldsymbol{q})\cdot f_k(t, \boldsymbol{q}) = \frac{\partial F}{\partial q_k}$. In addition, from Equation (21) in Equation (28): $\frac{\partial F}{\partial p_k} = 0$, thus, $F(t, \boldsymbol{q}, \boldsymbol{p}) = F_1(t, \boldsymbol{q})$, i.e., $m_k(t, \boldsymbol{q})\cdot f_k(t, \boldsymbol{q}) = \frac{\partial F_1(t,\boldsymbol{q})}{\partial q_k}$, and also, from Equation (21): $\frac{\partial F_1(t,\boldsymbol{Q})}{\partial Q_k} = m_k(t, \boldsymbol{Q})\cdot f_k(t, \boldsymbol{Q})$. In addition, from Equation (21) in Equation (29): $K = H + \frac{\partial F}{\partial t}$; thus, taking $(\boldsymbol{Q}, \boldsymbol{P})$ as the independent variables, the primary constants in Hamiltonian of Equation (18) become, by Equation (21), $\phi_j(t, \boldsymbol{q}, \boldsymbol{p}) = \varphi_j(t, \boldsymbol{Q}, \boldsymbol{P}) = P_j - m_j(t, \boldsymbol{Q})\cdot f_j(t, \boldsymbol{Q})$ $(1 \leq j \leq n)$, and the transformed Hamiltonian, also taking into account Equation (21), becomes:

$$K(t, \mathbf{Q}, \mathbf{P}) = H(t, \mathbf{Q}, \mathbf{P}) + \frac{\partial F_1(t, \mathbf{Q})}{\partial t} =$$

$$= h(t, \mathbf{Q}) + \sum_{j=1}^{n} \frac{1}{2m_j(t, \mathbf{Q})} \left(P_j + m_j(t, \mathbf{Q}) \cdot f_j(t, \mathbf{Q}) \right) \varphi_j(t, \mathbf{Q}, \mathbf{P}) + \frac{\partial F_1(t, \mathbf{Q})}{\partial t} \tag{30}$$

Note in Equation (30) that the term $\frac{\partial F_1(t,\mathbf{Q})}{\partial t}$ must be computed. To perform this, note that from the above result that: $\frac{\partial}{\partial Q_k} \left(\frac{\partial F_1(t,\mathbf{Q})}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial F_1(t,\mathbf{Q})}{\partial Q_k} \right) = \frac{\partial}{\partial t} \left(m_k(t, \mathbf{Q}) \cdot f_k(t, \mathbf{Q}) \right)$. However, from Equations (17) and (21): $\frac{\partial}{\partial t} \left(m_k(t, \mathbf{Q}) \cdot f_k(t, \mathbf{Q}) \right) = \frac{\partial}{\partial Q_k} \left(-\frac{1}{2} h(t, \mathbf{Q}) \right)$; then, comparing both results: $\frac{\partial F_1(t,\mathbf{Q})}{\partial t} = -\frac{1}{2} h(t, \mathbf{Q})$. Therefore, the transformed Hamiltonian of Equation (26) holds. □

Observe that in this transformed Hamiltonian, the $h(t, \mathbf{Q})$ function obeys Equation (17) due to the change in the dynamic variables, which is the identity, i.e., $Q_l = q_l$ ($1 \le l \le n$). For the same reason, the same assertion can be made about the relationships of Equation (16) that the abstract masses must obey.

From now on, for the sake of simplicity, the expressions of the dynamical velocities and of the momentum variables are recovered, i.e., $P_j \to p_j$, $Q_j \to q_j$, and also for the expression of the Hamiltonian, $K(t, \mathbf{Q}, \mathbf{P}) \to H(t, \mathbf{q}, \mathbf{p})$. Therefore, Equation (30) is rewritten as:

$$H(t, \mathbf{q}, \mathbf{p}) = \frac{1}{2} h(t, \mathbf{q}) + \sum_{j=1}^{n} \frac{1}{2m_j(t, \mathbf{q})} \left(p_j + m_j(t, \mathbf{q}) \cdot f_j(t, \mathbf{q}) \right) \varphi_j(t, \mathbf{q}, \mathbf{p}) =$$

$$= \frac{1}{2} h(t, \mathbf{q}) + \frac{1}{2} \sum_{j=1}^{n} \frac{p_j^2}{m_j(t, \mathbf{q})} - \frac{1}{2} \sum_{j=1}^{n} m_j(t, \mathbf{q}) \cdot f_j^2(t, \mathbf{q}) \tag{31}$$

Here, the fact that Equations (16) and (17) must be held in this last Hamiltonian is emphasized. However, if the abstract dynamical system of Equation (1) is autonomous, the abstract masses can be chosen, by Equation (16), as time-independent, and $h(t, \mathbf{q}) = 0$ by Equation (17), as it has been pointed out above. Therefore, the Hamiltonian of Equation (31) becomes a time-conserved magnitude, which, as in the physical context, is called here as energy and represented by E. Its expression form Equation (31) is, therefore:

$$E = \frac{1}{2} \sum_{j=1}^{n} \frac{p_j^2}{m_j(\mathbf{q})} - \frac{1}{2} \sum_{j=1}^{n} m_j(\mathbf{q}) \cdot f_j^2(\mathbf{q}) \tag{32}$$

Also, the Hamilton–Jacobi equation corresponding to the Hamiltonian of Equation (31) is:

$$\frac{\partial S(t, \mathbf{q})}{\partial t} + \frac{1}{2} \sum_{j=1}^{n} \frac{1}{m_j(t, \mathbf{q})} \left(\frac{\partial S(t, \mathbf{q})}{\partial q_j} \right)^2 - \frac{1}{2} \sum_{j=1}^{n} m_j(t, \mathbf{q}) \cdot f_j^2(t, \mathbf{q}) + \frac{1}{2} h(t, \mathbf{q}) = 0 \tag{33}$$

In addition, taking into account that the transformed primary constraints are zero outside the Hamiltonian of Equation (31), i.e., $\varphi_j(t, \mathbf{q}, \mathbf{p}) = p_j - m_j(t, \mathbf{q}) \cdot f_j(t, \mathbf{q}) = 0$ ($1 \le j \le n$), the corresponding Hamilton equations of this Hamiltonian are ($1 \le k \le n$):

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = \frac{p_k + m_k(t, \mathbf{q}) \cdot f_k(t, \mathbf{q})}{2m_k(t, \mathbf{q})} \tag{34}$$

$$\dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\frac{1}{2} \frac{\partial h(t, \mathbf{q})}{\partial q_k} + \sum_{j=1}^{n} \frac{p_j + m_j(t, \mathbf{q}) \cdot f_j(t, \mathbf{q})}{2m_j(t, \mathbf{q})} \frac{\partial \left(m_j(t, \mathbf{q}) \cdot f_j(t, \mathbf{q}) \right)}{\partial q_k} \tag{35}$$

To end this section, the proof that the second-order version (Equation (2)) of the original Equation (1) is recovered from Equations (34) and (35) is provided. Isolating p_k

from Equation (34) and taking the time derivative, it can be equaled to Equation (35) by also using Equation (17), that is ($1 \leq k \leq n$):

$$\frac{d}{dt}\left(2m_k(t,\boldsymbol{q})\cdot\dot{q}_k - m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q})\right) =$$

$$= \frac{1}{2}\frac{\partial(2m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q}))}{\partial t} + \sum_{j=1}^{n}\frac{p_j + m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q})}{2m_j(t,\boldsymbol{q})}\frac{\partial(m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q}))}{\partial q_k} \tag{36}$$

After handling Equation (36), taking into account Equation (34) ($1 \leq k \leq n$):

$$\frac{d}{dt}\left(2m_k(t,\boldsymbol{q})\cdot\dot{q}_k\right) =$$

$$= \frac{d}{dt}\left(m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q})\right) + \frac{\partial(m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q}))}{\partial t} + \sum_{j=1}^{n}\dot{q}_j\frac{\partial(m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q}))}{\partial q_k} \tag{37}$$

Note in Equation (37) that the first and second terms of the right-hand side are the same by expanding the total time derivative of the first term, therefore:

$$\frac{d}{dt}\left(2m_k(t,\boldsymbol{q})\cdot\dot{q}_k\right) = 2\frac{\partial(m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q}))}{\partial t} + 2\sum_{j=1}^{n}\dot{q}_j\frac{\partial(m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q}))}{\partial q_k} \tag{38}$$

Dropping the term 2 in Equation (38), Equation (2) is recovered. It is important to compare this deduction with the deduction made in the following section. There, the quantum potential (obtained from the second-order Schrödinger equation) changes significantly in Equation (2) and also in the original Equation (1).

3. Second-Order Schrödinger Equation

As "second order," this section refers to the second-order partial derivatives that respect the dynamical variables of the Schrödinger equation. They arise by applying the quantization rules [20] to the momentum variables of the Hamiltonian of Equation (31) once the primary constants are substituted (that is, the second equality of Equation (31)), taking into account that in Equation (31) the momentum variables are now quadratic. Therefore, if $\hat{H}(t,\hat{\boldsymbol{q}},\hat{\boldsymbol{p}})$ is the quantum operator corresponding to the Hamiltonian of Equation (31) (its second equality), the Schrödinger equation is written again as:

$$i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial t} = \hat{H}(t,\hat{\boldsymbol{q}},\hat{\boldsymbol{p}})\Psi(t,\boldsymbol{q}) \tag{39}$$

In Equation (39) $\Psi(t,\boldsymbol{q})$ is the wave function, where σ represents again the system Planck constant, with the same sense commented in Appendix A.

Theorem 2. *The Schrödinger equation corresponding to the Hamiltonian of Equation* (31), *here so-called second-order Schrödinger equation, is written as:*

$$i\sigma\frac{\partial\Psi(t,\boldsymbol{q})}{\partial t} =$$

$$-\frac{1}{2}\frac{\sigma^2}{2}\left(\sum_{j=1}^{n}\frac{1}{m_j(t,\boldsymbol{q})}\frac{\partial^2\Psi(t,\boldsymbol{q})}{\partial q_j^2} + \sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2}\left(\frac{1}{m_j(t,\boldsymbol{q})}\Psi(t,\boldsymbol{q})\right)\right) - \tag{40}$$

$$-\frac{1}{2}\sum_{j=1}^{n}m_j(t,\boldsymbol{q})\cdot f_j^2(t,\boldsymbol{q})\cdot\Psi(t,\boldsymbol{q}) + \frac{1}{2}h(t,\boldsymbol{q})\cdot\Psi(t,\boldsymbol{q})$$

Proof. Following the quantization rules provided by the Copenhagen formalism of the quantum theory [20], $\hat{H}(t, \hat{q}, \hat{p})$ is an operator that acts on the wave function as:

$$\hat{H}(t, \hat{q}, \hat{p})\Psi(t, q) = \frac{1}{2}\sum_{j=1}^{n} \frac{1}{2}\left(\frac{1}{m_j(t, \hat{q})}\hat{p}_j^2 + \hat{p}_j^2 \frac{1}{m_j(t, \hat{q})}\right)\Psi(t, q) +$$
$$+ \left(-\frac{1}{2}\sum_{j=1}^{n} m_j(t, \hat{q}) \cdot f_j^2(t, \hat{q}) + \frac{1}{2}h(t, \hat{q})\right)\Psi(t, q) \tag{41}$$

Such that in Equation (41): $\left(-\frac{1}{2}\sum_{j=1}^{n} m_j(t, \hat{q}) \cdot f_j^2(t, \hat{q})\right)\Psi(t, q)$ $= \left(-\frac{1}{2}\sum_{j=1}^{n} m_j(t, q) \cdot f_j^2(t, q)\right)\Psi(t, q)$, $\frac{1}{2}h(t, \hat{q})\Psi(t, q) = \frac{1}{2}h(t, q) \cdot \Psi(t, q)$ and $\hat{p}_j\Psi(t, q) = -i\sigma\frac{\partial \Psi(t, q)}{\partial q_j}$. Note that the term $\frac{1}{2}\sum_{j=1}^{n}\left(\frac{1}{m_j(t, \hat{q})}\hat{p}_j^2 + \hat{p}_j^2 \frac{1}{m_j(t, \hat{q})}\right)\Psi(t, q)$ provides that the Hamiltonian is a self-adjoint operator (see [22] for this case, which is not dealt with in [20]). In fact, by making explicit this operator in Equation (41) and subsequently in Equation (39), the second-order Schrödinger equation arises. □

If the abstract dynamical system of Equation (1) is autonomous, the abstract masses can be chosen, by Equation (16), as time-independent, and $h(t, q) = 0$ by Equation (17), as it has been pointed out in Section 2. Therefore, the time-independent second-order Schrödinger equation corresponding to Equation (40) can be found by the common substitution $\Psi(t, q) = {}^{-i\frac{E}{\sigma}t} \cdot \psi(\mathbf{q})$, where E corresponds to the system energy of Equation (32), that is:

$$E \cdot \psi(\mathbf{q}) =$$
$$-\frac{1}{2}\frac{\sigma^2}{2}\left(\sum_{j=1}^{n} \frac{1}{m_j(q)} \frac{\partial^2 \psi(\mathbf{q})}{\partial q_j^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2}\left(\frac{1}{m_j(q)}\psi(\mathbf{q})\right)\right) - \tag{42}$$
$$-\frac{1}{2}\sum_{j=1}^{n} m_j(q) \cdot f_j^2(t, q) \cdot \psi(\mathbf{q})$$

In order to find out if some quantum potential arises from the second-order Schrödinger equation (Equation (40)), then, as in Equation (A24) of Appendix A, the wave function split in its amplitude $A(t, q)$ and its phase $B(t, q)$ is completed:

$$\Psi(t, q) = A(t, q)^{i\frac{B(t, q)}{\sigma}} \tag{43}$$

Proposition 5. *The* $A(t, q)$ *and* $B(t, q)$ *functions of Equation (43) obey the following equations:*

$$\frac{\partial B(t, q)}{\partial t} + \frac{1}{2}\sum_{j=1}^{n} \frac{1}{m_j(t, q)}\left(\frac{\partial B(t, q)}{\partial q_j}\right)^2 - \frac{1}{2}\sum_{j=1}^{n} m_j(t, q) \cdot f_j^2(t, q) + \frac{1}{2}h(t, q) + V_q(t, q) = 0 \tag{44}$$

$$\frac{\partial A^2(t, q)}{\partial t} + \sum_{j=1}^{n}\left(\frac{1}{m_j(t, q)}\frac{\partial B(t, q)}{\partial q_j}A^2(t, q)\right) = 0 \tag{45}$$

where in Equation (44) $V_q(t, q)$ *is the quantum potential, whose expression is:*

$$V_q(t, q) = -\frac{\sigma^2}{4}\sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2}\left(\frac{1}{m_j(t, q)}\right) - \frac{\sigma^2}{2A(t, q)}\sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(\frac{1}{m_j(t, q)}\frac{\partial A(t, q)}{\partial q_j}\right) \tag{46}$$

Proof. Such as has been performed in Appendix A, the substitution of Equation (43) in Equation (40) provides an equation for the real part and another equation for the imaginary part (after some operations and canceling the term $i^{\frac{B(t,q)}{\sigma}}$):

$$
\begin{aligned}
-\mathrm{A}(t,\boldsymbol{q})\frac{\partial \mathrm{B}(t,\boldsymbol{q})}{\partial t} = {} & -\frac{\sigma^2}{2}\sum_{j=1}^{n}\frac{1}{m_j(\boldsymbol{q})}\frac{\partial^2 \mathrm{A}(t,\boldsymbol{q})}{\partial q_j^2} + \frac{\mathrm{A}(t,\boldsymbol{q})}{2}\sum_{j=1}^{n}\frac{1}{m_j(\boldsymbol{q})}\left(\frac{\partial \mathrm{B}(t,\boldsymbol{q})}{\partial q_j}\right)^2 - \\
& -\frac{\sigma^2}{2}\sum_{j=1}^{n}\frac{\partial}{\partial q_j}\left(\frac{1}{m_j(\boldsymbol{q})}\right)\frac{\partial \mathrm{A}(t,\boldsymbol{q})}{\partial q_j} - \mathrm{A}(t,\boldsymbol{q})\frac{\sigma^2}{4}\sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2}\left(\frac{1}{m_j(\boldsymbol{q})}\right) - \\
& -\frac{1}{2}\sum_{j=1}^{n}m_j(t,\boldsymbol{q})\cdot f_j^2(t,\boldsymbol{q})\cdot\mathrm{A}(t,\boldsymbol{q}) + \frac{1}{2}h(t,\boldsymbol{q})\cdot\mathrm{A}(t,\boldsymbol{q})
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
\sigma\frac{\partial \mathrm{A}(t,\boldsymbol{q})}{\partial t} = {} & -\sigma\sum_{j=1}^{n}\frac{\partial \mathrm{A}(t,\boldsymbol{q})}{\partial q_j}\frac{\partial \mathrm{B}(t,\boldsymbol{q})}{\partial q_j} - \\
& -\frac{\sigma}{2}\mathrm{A}(t,\boldsymbol{q})\sum_{j=1}^{n}\frac{1}{m_j(\boldsymbol{q})}\frac{\partial^2 \mathrm{B}(t,\boldsymbol{q})}{\partial q_j^2} - \frac{\sigma}{2}\mathrm{A}(t,\boldsymbol{q})\sum_{j=1}^{n}\frac{\partial}{\partial q_j}\left(\frac{1}{m_j(\boldsymbol{q})}\right)\frac{\partial \mathrm{B}(t,\boldsymbol{q})}{\partial q_j}
\end{aligned}
\tag{48}
$$

Dividing the first one by $\mathrm{A}(t,\boldsymbol{q})$ and dividing the second one by σ and subsequently multiplying it by $2\mathrm{A}(t,\boldsymbol{q})$, Equations (44)–(46) hold. $\square$

Therefore, Equation (44) represents the quantum Hamilton–Jacobi equation [14,15] due to the difference with the Hamilton–Jacobi equation of Equation (33) is the quantum potential of Equation (44). In addition, Equation (45) represents the conservation of the probability density provided by $\mathrm{A}^2(t,\boldsymbol{q})$, being the vector of coordinates $\frac{1}{m_j(t,\boldsymbol{q})}\frac{\partial \mathrm{B}(t,\boldsymbol{q})}{\partial q_j}$ $(1 \leq j \leq n)$ the current density.

4. The Quantum Hamilton Equations and the Quantum Formulation of the Abstract Dynamical Systems

By comparing the Hamilton–Jacobi equation of Equation (33) and the quantum Hamilton–Jacobi equation of Equation (45), the difference is the quantum potential $V_q(t,\boldsymbol{q})$. Then, following the Madelung interpretation of quantum mechanics [14], assumed also by Bohm and Hiley in [15], a quantum version of the Hamiltonian can be found, $H_q(t,\boldsymbol{q},\boldsymbol{p})$, differenced from the Hamiltonian of Equation (31) by the quantum potential, that is:

$$
\begin{aligned}
H_q(t,\boldsymbol{q},\boldsymbol{p}) = {} & H(t,\boldsymbol{q},\boldsymbol{p}) + V_q(t,\boldsymbol{q}) = \\
= {} & \frac{1}{2}h(t,\boldsymbol{q}) + \sum_{j=1}^{n}\frac{1}{2m_j(t,\boldsymbol{q})}\left(p_j + m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q})\right)\varphi_j(t,\boldsymbol{q},\boldsymbol{p}) + V_q(t,\boldsymbol{q}) \\
= {} & \frac{1}{2}\sum_{j=1}^{n}\frac{p_j^2}{m_j(t,\boldsymbol{q})} - \frac{1}{2}\sum_{j=1}^{n}m_j(t,\boldsymbol{q})\cdot f_j^2(t,\boldsymbol{q}) + \frac{1}{2}h(t,\boldsymbol{q}) + V_q(t,\boldsymbol{q})
\end{aligned}
\tag{49}
$$

In Equation (49) $V_q(t,\boldsymbol{q})$ is the quantum potential of Equation (46).

Proposition 6. The Hamilton equations corresponding to the Hamiltonian of Equation (49), here so-called *quantum Hamilton equations*, are $(1 \leq k \leq n)$:

$$
\dot{q}_k(t) = \frac{\partial H_q}{\partial p_k} = \frac{p_k + m_k(t,\boldsymbol{q})\cdot f_k(t,\boldsymbol{q})}{2m_k(t,\boldsymbol{q})}
\tag{50}
$$

$$
\dot{p}_k(t) = -\frac{\partial H_q}{\partial q_k} = -\frac{1}{2}\frac{\partial h(t,\boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n}\frac{p_j + m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q})}{2m_j(t,\boldsymbol{q})}\frac{\partial\left(m_j(t,\boldsymbol{q})\cdot f_j(t,\boldsymbol{q})\right)}{\partial q_k} - \frac{\partial V_q(t,\boldsymbol{q})}{\partial q_k}
\tag{51}
$$

Proof. Taking into account that the transformed primary constraints are zero outside the Hamiltonian of Equation (49) ($\varphi_j(t, \boldsymbol{q}, \boldsymbol{p}) = p_j - m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q}) = 0, 1 \leq j \leq n$), and considering the Hamilton equations of the classical mechanics, Equations (50) and (51) hold. $\square$

Theorem 3. *The quantum Hamilton equations of Equations (50) and (51) provide the following 2n-dimensional differential system ($1 \leq k \leq n$):*

$$\left.\begin{array}{c} \dot{q}_k(t) = f_k(t, \boldsymbol{q}) - \frac{x_k}{2m_k(t, \boldsymbol{q})} \\ \dot{x}_k(t) = \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k} \end{array}\right\} \tag{52}$$

where $x_k(t)$ ($1 \leq k \leq n$) are additional dynamical variables defined as $x_k(t) = \int_{t_0}^{t} \frac{\partial V_q}{\partial q_k}(s, \boldsymbol{q}(s)) ds$ ($1 \leq k \leq n$).

Proof. First, isolating p_k from Equation (50) and taking the time derivative, it can be equaled to Equation (51) by also using Equations (16) and (17), that is ($1 \leq k \leq n$):

$$\frac{d}{dt}\big(2m_k(t, \boldsymbol{q}) \cdot \dot{q}_k - m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q})\big) =$$
$$= \frac{1}{2} \frac{\partial(2m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial t} + \sum_{j=1}^{n} \frac{p_j + m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q})}{2m_j(t, \boldsymbol{q})} \frac{\partial(m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q}))}{\partial q_k} - \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k} \tag{53}$$

Second, handling Equation (53) and taking into account Equation (50) ($1 \leq k \leq n$):

$$\frac{d}{dt}\big(2m_k(t, \boldsymbol{q}) \cdot \dot{q}_k\big) =$$
$$= \frac{d}{dt}\big(m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q})\big) + \frac{\partial(m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial t} + \sum_{j=1}^{n} \dot{q}_j \frac{\partial(m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q}))}{\partial q_k} - \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k} \tag{54}$$

Third, note in Equation (54) that the first and second terms of the right-hand equation are the same by expanding the total time derivative of the first term, therefore:

$$\frac{d}{dt}\big(2m_k(t, \boldsymbol{q}) \cdot \dot{q}_k\big) = 2\frac{\partial(m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{\partial t} + 2\sum_{j=1}^{n} \dot{q}_j \frac{\partial(m_j(t, \boldsymbol{q}) \cdot f_j(t, \boldsymbol{q}))}{\partial q_k} - \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k} \tag{55}$$

In the beginning, dividing by 2 in Equation (55), Equation (2) is recovered with the additional quantum term $-\frac{1}{2} \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k}$. However, trying to recover the original Equation (1), Equation (55) can be rewritten more simplified as:

$$\frac{d}{dt}\big(m_k(t, \boldsymbol{q}) \cdot \dot{q}_k\big) = \frac{d(m_k(t, \boldsymbol{q}) \cdot f_k(t, \boldsymbol{q}))}{dt} - \frac{1}{2} \frac{\partial V_q(t, \boldsymbol{q})}{\partial q_k} \tag{56}$$

Followed by taking the time integral in Equation (56) and subsequently dividing by $m_k(t, \boldsymbol{q})$ ($1 \leq k \leq n$), the quantum version of Equation (1) is obtained that is ($1 \leq k \leq n$):

$$\dot{q}_k(t) = f_k(t, \boldsymbol{q}) - \frac{1}{2m_k(t, \boldsymbol{q})} \int_{t_0}^{t} \frac{\partial V_q}{\partial q_k}(s, \boldsymbol{q}(s)) ds \tag{57}$$

Finally, considering the additional variables $x_k(t) = \int_{t_0}^{t} \frac{\partial V_q}{\partial q_k}(s, \boldsymbol{q}(s)) ds$ ($1 \leq k \leq n$) in Equation (57), Equation (52) holds. $\square$

Note in Equation (52) that the quantum potential $V_q(t, \boldsymbol{q})$ is provided by Equation (46), taking into account that $A(t, \boldsymbol{q}) = |\Psi(t, \boldsymbol{q})| = \sqrt{\Psi(t, \boldsymbol{q}) \cdot \Psi^*(t, \boldsymbol{q})}$, after being solved the second-order Schrödinger equation of Equation (40) for the wave function $\Psi(t, \boldsymbol{q})$, or its

time-independent version (Equation (42)) for the autonomous case for the wave function $\psi(\mathbf{q})$, such that $\Psi(t, \boldsymbol{q}) = ^{-i\frac{E}{\sigma}t} \cdot \psi(\mathbf{q})$. In this last case, $A(\boldsymbol{q}) = |\Psi(t, \boldsymbol{q})| = |\psi(\mathbf{q})|$.

In addition, if the second-order Schrödinger equation of Equation (40) (or its time-independent version of Equation (42) for the autonomous case) provides quantized wave functions depending on an integer n, $\Psi_n(t, \boldsymbol{q})$, the base of a Hilbert space; this quantization can be translated to the quantum formulation of Equation (52).

Note that, in order to compute the quantum dynamics by Equation (52), the initial values are needed: if $t = t_0 \geq 0$ is the initial time, then $q_k(t_0) = q_k^{(0)}$ ($1 \leq k \leq n$) must be known and $x_k(t_0) = 0$ ($1 \leq k \leq n$), following the above integral definition of $x_k(t)$.

5. The Autonomous One-Dimensional Case: The Logistic Function Dynamics

To better support the formalism provided and its limitations, an application case is presented: the one-dimensional case particularized to the logistic function dynamics. Then, Equation (1) can be written as:

$$\left.\begin{array}{l} \dot{q}(t) = f(q) = a \cdot q - b \cdot q^2 \\ q(t_0) = q_0 \end{array}\right\} \tag{58}$$

Equation (58) can describe population dynamics with parameter $b > 0$, which avoids infinite population growth. Parameter a can be positive or negative and represents the population growth rate (if positive) or population decay rate (if negative). If $a > 0$, the $q = 0$ value is a repulsor from which the dynamics escapes, and the $q = c = a/b$ value is a saturation population or attractor to which the dynamics tends asymptotically (growing toward $q = c$ if the initial value is $q_0 < c$, and decaying toward $q = c$ if the initial value is $q_0 > c$). If $a < 0$, the $q = c$ value is a repulsor from which the dynamics escape, and the value $q = 0$ is an attractor to which the dynamics tend asymptotically (decaying toward $q = 0$ if the initial value is $q_0 < c$, and infinitely growing if the initial value is $q_0 > c$).

For the application case, the following values are taken: $t_0 = 0.0$ $q_0 = 2.0$, $a = 0.01$, and $b = 0.001$, thus $c = \frac{a}{b} = 10.0$. Therefore, the dynamics represent the case of an asymptotic growth toward $q = c = \frac{a}{b} = 10.0$. In addition, to compute the classical logistic dynamics given by Equation (58), the integration of the equation differential provides:

$$q(t) = \frac{c}{1 + \left(\frac{c}{q_0} - 1\right) \cdot \mathrm{e}^{-a(t - t_0)}} \tag{59}$$

It must be highlighted that all figures and computations were obtained with the MATHEMATICA 13.2 software except for the quantized energies, which were obtained by a C++ program.

First of all, Equation (59) provides Figure 1 for the classical logistic dynamics with the above values chosen. In order to provide a realistic application of these dynamics, consider that the population studied is that corresponding to a biological population for an evolution period of $t = 1000.0$ years. In addition, the $q(t)$ variable could be measured by thousands of population in an extreme environment. However, if the application case changed, the units could also change (for instance, in microbiologic populations, the time units could be hours). Note that for a long time term of $t = 1000.0$ years, the dynamics tend to the attractor $c = \frac{a}{b} = 10.0$ population thousands.

Figure 1 is important to be presented because it is compared along all sections with the quantum dynamics provided by the corresponding formulation of Equation (52). Note that the case is one-dimensional; the corresponding abstract mass can be taken equal to the unit ($m(q) = 1$). Then, Equation (52) becomes for this case ($t_0 = 0.0$ $q_0 = 2.0$, $a = 0.01$, and $b = 0.001$):

$$\left.\begin{array}{l} \dot{q}(t) = f(q) - \frac{x}{2} = a \cdot q - b \cdot q^2 - \frac{x}{2} \\ \dot{x}(t) = V_q'(q) \\ q(t_0) = q_0 \\ x(t_0) = 0 \end{array}\right\} \tag{60}$$

Note that the q-derivative of the quantum potential $V_q'(q)$ already contains the abstract Planck system σ (see Equation (40)). Then, its value should be provided here. However, it is provided above for the reasons explained above. In addition, to obtain this quantum potential $V_q(q)$ and its q-derivative $V_q'(q)$, the corresponding time-independent Schrödinger equation of Equation (42) must be solved. After some manipulations, it becomes (note that $m(q) = 1$):

$$\psi''(q) + \left(\frac{f^2(q)}{\sigma^2} + \frac{2E}{\sigma^2} \right) \psi(q) = 0 \tag{61}$$

Note in Equation (61) that $f(q) = a \cdot q - b \cdot q^2$. The way to solve it is through two different steps. In the first step, the approach is analytical, and it is inspired by the method provided in [23] (concretely in that of the IV section called the *transformation-group method*) to reduce a differential equation that models a forced time-dependent oscillator to an autonomous differential equation. The second step is a numerical approach.
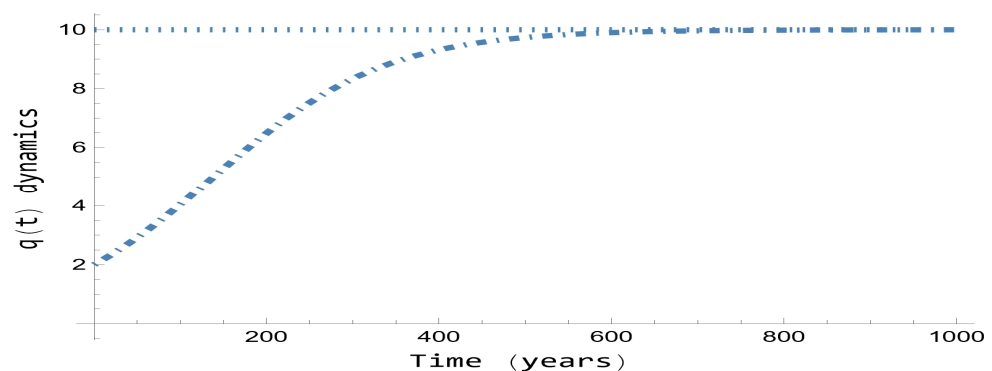

Figure 1. Classical logistic dynamics by Equation (59), solution of Equation (58), with the particular values provided ($t_0 = 0.0$ $q_0 = 2.0$, $a = 0.01$, and $b = 0.001$), with $t = 1000.0$ years. The dotted straight line value represents the saturation value $c = \frac{a}{b} = 10.0$ population thousands and the dot-dashed curve the classical logistic dynamics.

The first step consists of two consecutive changes: first on the dependent variable, $\psi(q) = {}^{\theta(q)}\Omega(q)$, and second on the independent variable $z = z(q)$, then $w(z) = \Omega(q(z))$. After some calculations, including the ${}^{\theta(q)}$ cancellation, both changes provide:

$$(z'(q))^2 w''(z) + z'(q)(z''(q) + 2\theta'(q)) w'(z) + $$
$$+ \left(\theta''(q) + (\theta'(q))^2 + \frac{f^2(q)}{\sigma^2} + \frac{2E}{\sigma^2} \right) w(z) = 0 \tag{62}$$

In Equation (62), the term multiplying $w'(z)$ can vanish if $z''(q) + 2 \cdot \theta'(q) = 0$, i.e., if $z'(q) = -2\theta(q)$ or $z(q) = -2\int_{q_0}^{q} \theta(y)\, dy$, being q_0 an arbitrary value. Thus, Equation (62) becomes:

$$4\theta^2(q) \cdot w''(z) + \left(\theta''(q) + (\theta'(q))^2 + \frac{f^2(q)}{\sigma^2} + \frac{2E}{\sigma^2} \right) w(z) = 0 \tag{63}$$

Now, Equation (63) is forced to hold that:

$$\theta''(q) + (\theta'(q))^2 + \frac{f^2(q)}{\sigma^2} + \frac{2E}{\sigma^2} = k^2 \cdot 4\theta^2(q) \tag{64}$$

In Equation (64), the constant $k^2 \neq 0$. Thus, Equation (63) becomes:

$$w''(z) + k^2 \cdot w(z) = 0 \tag{65}$$

The solution of Equation (65) is then trivial:

$$w(z) = C \cdot sin(k \cdot z + \gamma) \tag{66}$$

In Equation (66), C and γ are two arbitrary constants. Undoing now the proposed changes, the analytical solution of Equation (61) is:

$$\psi(q) = C \cdot e^{\theta(q)} sin\left(-2k \int_{q_0}^{q} \theta(y) \, dy + \gamma\right) \tag{67}$$

Some additional considerations about the solution $\psi(q)$ of Equation (67) must be completed. The first one is the choice of the k value. Note that in the neighboring of the points $q \cong 0$ and $q \cong c = a/b$, $f(q) \cong 0$, therefore, Equation (67) becomes approximately:

$$\theta''(q) + \left(\theta'(q)\right)^2 + \frac{2E}{\sigma^2} \cong 4k^2\theta^2(q) \tag{68}$$

A solution of Equation (68) is an arbitrary constant $\theta(q) = R$, then $\theta'(q) = \theta''(q) = 0$ and $k = \pm\frac{\sqrt{2E}}{2R\sigma}$. Note that the energy must be positive, $E > 0$. In addition, in the neighboring of the critical points of $f(q)$ ($f(q) \cong 0$), Equation (66) solution becomes:

$$\psi(q) \cong C \cdot^R sin\left(\mp\frac{\sqrt{2E}}{R\sigma} \int_{q_0}^{q} R \, dy + \gamma\right) = C \cdot e^R sin\left(\mp\frac{\sqrt{2E}}{\sigma}(q - q_0) + \gamma\right) \tag{69}$$

In Equation (69), the sign of $\mp\frac{\sqrt{2E}}{\sigma}$ can be chosen to take into account that, also in the neighboring of the critical points of $f(q)$ ($f(q) \cong 0$), Equation (61) becomes:

$$\psi''(q) + \frac{2E}{\sigma^2}\psi(q) \cong 0 \tag{70}$$

With the Equation (70) solution (with $E > 0$):

$$\psi(q) \cong D \cdot sin\left(\frac{\sqrt{2E}}{\sigma}q + \delta\right) \tag{71}$$

Comparing Equations (69) and (71), the conclusion is that the sign considered in Equation (69) must be positive, i.e., $+\frac{\sqrt{2E}}{\sigma}$. Moreover, the respective constants are related as $D = C \cdot e^R$ and $-\frac{\sqrt{2E}}{\sigma}q_0 + \gamma = \delta$. Therefore, $k = -\frac{\sqrt{2E}}{2R\sigma}$, and in Equation (67): $-2k = \frac{\sqrt{2E}}{R\sigma}$. The R constant value can be arbitrary, and it has been considered as $R = 1$. Consequently, Equation (67) can be rewritten as:

$$\psi(q) = C \cdot e^{\theta(q)} sin\left(\frac{\sqrt{2E}}{\sigma} \int_{q_0}^{q} \theta(y) \, dy + \gamma\right) \tag{72}$$

In addition, Equation (64) becomes:

$$\theta''(q) + \left(\theta'(q)\right)^2 + \frac{f^2(q)}{\sigma^2} = \frac{2E}{\sigma^2}\left(\theta^2(q) - 1\right) \tag{73}$$

This author's paper has not been able to find the analytical solution of Equation (73). Therefore, at this point, a numerical approach must be taken. This numerical approach needs the system Planck constant value σ and the boundary conditions. In order to better understand the following deductions and computations, Table 1 provides in advance the boundary conditions for Equations (72) and (73) and the initial conditions for Equation (60).

Table 1. Boundary conditions for Equations (72) and (73) and initial conditions for Equation (60).

Function	$\psi(q)$	$\theta(q)$	$\theta'(q)$	$q(t)$	$x(t)$
Boundary $-$ Intitial conditions	$\psi(0) = \psi(c) = 0$	$\theta(0) = -1$	$\theta'(0) = 0$	$q(0) = 2$	$x(0) = 0$

In addition, several numerical essays with different $E > 0$ values provide an adequate $\sigma = 10$ value, although strangely, it can look similar to a big value. In addition, following Table 1, the boundary conditions considered have been: $\theta(0) = -1$, trying to obtain a negative domain for $\theta(q)$, and $\theta'(0) = 0$, trying that $q = 0$ be a maximum and that $\theta(q) \to -\infty$ as $q \to \pm\infty$. Therefore, in Equation (72), $\psi(q) \to 0$, as $q \to \pm\infty$. These assumptions are confirmed below with the numerical solutions of Equations (72) and (73).

In addition, under the hypothesis that the wave function cancels in the two critical points of $f(q)$ (see Table 1), the wave function becomes quantized. On the one hand, if $\psi(0) = 0$:

$$\frac{\sqrt{2E}}{\sigma} \int_{q_0}^{0} \theta(y)\, dy + \gamma = \pi \cdot n_1 \; ; \; n_1 \in \mathbb{Z} \tag{74}$$

On the other hand, if $\psi(c) = 0$:

$$\frac{\sqrt{2E}}{\sigma} \int_{q_0}^{c} \theta(y)\, dy + \gamma = \pi \cdot n_2 \; ; \; n_2 \in \mathbb{Z} \tag{75}$$

Subtracting Equation (75) minus Equation (74), the condition of quantization arises:

$$\frac{\sqrt{2E_n}}{\sigma} \int_{0}^{c} \theta_n(y)\, dy = \pi(n_2 - n_1) = \pi \cdot n \; ; \; n = n_2 - n_1 \in \mathbb{Z}_{<0} \tag{76}$$

Note in Equation (76) that $n = n_2 - n_1 < 0$ due to $\int_{0}^{c} \theta_n(y)\, dy < 0$, under the $\theta_n(q)$ above assumptions and below confirmed. Therefore, in order to compute the E_n energies, with $n < 0$, Equation (76) must be considered jointly with the quantized version of Equation (73) with the boundary conditions of Table 1:

$$\left. \begin{array}{c} \theta_n''(q) + (\theta_n'(q))^2 + \frac{f^2(q)}{\sigma^2} = \frac{2E_n}{\sigma^2}\left(\theta_n^2(q) - 1\right) \\ \theta_n(0) = -1 \\ \theta_n'(0) = 0 \end{array} \right\} \; ; \; n \in \mathbb{Z}_{<0} \tag{77}$$

Once the E_n energies are obtained from Equation (76): $\gamma = \pi \cdot n_1 + \frac{\sqrt{2E}}{\sigma} \int_{0}^{q_0} \theta(y)\, dy = \pi(n_2 - n) + \frac{\sqrt{2E}}{\sigma} \int_{0}^{q_0} \theta(y)\, dy$. However, $\pi \cdot n_2$ can be removed due to the fact that this term's influence does not provide further mathematical information. The substitution in Equation (72) provides the quantized wave functions:

$$\psi_n(q) = C_n \cdot e^{\theta_n(q)} \sin\left(\frac{\sqrt{2E_n}}{\sigma} \int_{0}^{q} \theta_n(y)\, dy - \pi \cdot n\right) \tag{78}$$

Note that the C_n constants must be positive as a consequence that the set $\left\{\Psi_n(t,q) = e^{-i\frac{E_n}{\sigma}t}\psi_n(q)\right\}_{n=-1}^{-\infty}$ be orthonormal, for the scalar product $\int_{-\infty}^{+\infty} \Psi_m^*(t,q) \cdot \Psi_n(t,q)\, dq = \int_{-\infty}^{+\infty} \psi_m(q) \cdot \Psi_n(t,q)\, dq < +\infty$. In other words: $\int_{-\infty}^{+\infty} \Psi_m^*(t,q) \cdot \Psi_n(t,q)\, dq = \int_{-\infty}^{+\infty} \psi_m(q) \cdot \Psi_n(t,q)\, dq = C_n \cdot C_m \cdot \delta_{nm}$. Therefore, $\Psi_n(t,q) \in L^2(\mathbb{R}) \; \forall n \in \mathbb{Z}_{<0}$, such as it is numerically shown below for the first three negative integers.

In order to compute the quantum dynamics by Equation (60), the quantum potential must be computed. First of all, the modulus of the wave function becomes quantized, that is:

$$A_n(q) = |\Psi_n(t,q)| = B_n e^{\theta_n(q)} \left| \sin\left(\frac{\sqrt{2E_n}}{\sigma} \int_{0}^{q} \theta_n(y)\, dy - \pi \cdot n\right) \right| \tag{79}$$

Note in Equation (79) that the absolute value vanishes due to the quantum potential computation is finally divided by $A_n(q)$. Effectively, from Equation (46) for the present application case:

$$V_q^{(n)}(q) = -\frac{\sigma^2}{2A_n(q)}\frac{d^2A_n(q)}{dq^2} =$$

$$= \frac{f^2(q)}{2} + E_n - \frac{\sigma^2}{2}\frac{\sqrt{2E_n}}{\sigma}\theta'_n(q)(2\theta_n(q)+1)cotg\left(\frac{\sqrt{2E_n}}{\sigma}\int_0^q \theta_n(y)\,dy - \pi\cdot n\right) \tag{80}$$

In Equation (80) derivation, the quantized differential equation of Equation (77) has been taken into account to simplify it. Then, the q-derivative of $V_q^{(n)}(q)$ becomes:

$$\frac{dV_q^{(n)}(q)}{dq} = f(q)\cdot f'(q)-$$

$$-\frac{\sigma\sqrt{2E_n}}{2}\left(\theta''_n(q)(2\theta_n(q)+1)+2(\theta'_n(q))^2\right)cotg\left(\frac{\sqrt{2E_n}}{\sigma}\int_0^q \theta_n(y)\,dy - \pi\cdot n\right)+ \tag{81}$$

$$+E_n\frac{\theta_n(q)\cdot\theta'_n(q)(2\theta_n(q)+1)}{sin^2\left(\frac{\sqrt{2E_n}}{\sigma}\int_0^q \theta_n(y)\,dy - \pi\cdot n\right)}$$

Taking into account Equation (81), Equation (60) provides the quantum dynamics depending on any integer $n < 0$, with the initial conditions provided in Table 1:

$$\left.\begin{array}{c}\dot{q}_n(t) = 0.01\cdot q_n - 0.001\cdot q_n^2 - \frac{x_n}{2}\\ \dot{x}_n(t) = \frac{dV_q^{(n)}(q)}{dq}\\ q_n(0) = 2\\ x_n(0) = 0\end{array}\right\} \tag{82}$$

Let us now present the results. First, the quantized energies have been computed by setting up a C++ program for Equation (77) plus the phase $\varphi_n(q) = \frac{\sqrt{2E_n}}{\sigma}\int_0^q \theta_n(y)\,dy$ of Equation (78) due to the energies involved in both equations. This program has used the 4th-order Runge–Kutta method to solve the differential equations, such that $\varphi_n(q)$ is rewritten as a differential equation as $\varphi'_n(q) = \frac{\sqrt{2E_n}}{\sigma}\theta_n(q)$ with $\varphi_n(0) = 0$. The C++ program includes the condition that the energies E_n must hold $\varphi_n(c) = \frac{\sqrt{2E_n}}{\sigma}\int_0^c \theta_n(y)\,dy = \pi\cdot n$, with an error bound of 10^{-3}, for each $n < 0$ considered. The energy outcomes are presented in Table 2 for the 10 first negative integers.

Table 2. Quantized energies.

n	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
E_n	4.838	19.350	43.538	77.401	120.939	174.153	237.041	309.604	391.843	483.757

The numerical results for $n = -1$, $\theta_1(q)$ of Equation (77) and the corresponding $\varphi_1(q) = \frac{\sqrt{2E_1}}{\sigma}\int_0^q \theta_1(y)\,dy$, are presented in Figure 2, for the interval $q \in [-15\cdot c,\ 15\cdot c] = [-150,\ 150]$. Note that $\theta_1(q) < 0$ and that $\theta_1(q) \to -\infty$ as $q \to \pm\infty$, which implies that, by Equation (78), $\Psi_1(t,q) \to 0$ as $q \to \pm\infty$. In addition, as it is shown below numerically, $\int_{-\infty}^{+\infty}\Psi_1(t,q) < \infty$, therefore $\Psi_1(t,q) \in L^2(\mathbb{R})$.
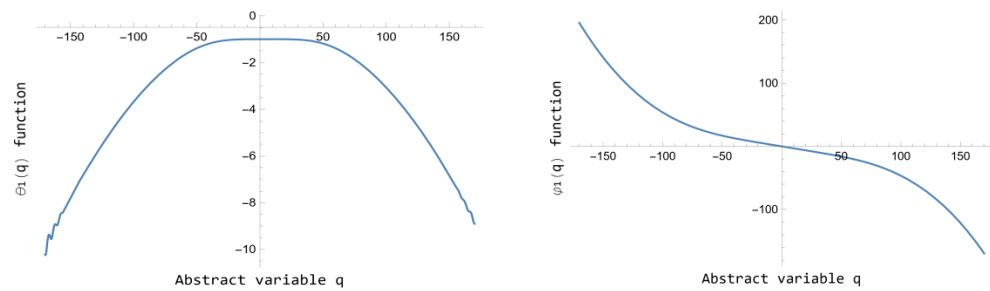
Figure 2. $\theta_1(q)$ **(left)** and the corresponding $\varphi_1(q) = \frac{\sqrt{2E_1}}{\sigma} \int_0^q \theta_1(y)dy$ function **(right)** for the interval $q \in [-15 \cdot c, 15 \cdot c] = [-150, 150]$.

Similar patterns to those of Figure 2 present $\theta_n(q)$ and $\varphi_n(q)$ for the subsequent negative integers. Therefore, from now on, the attention is focused on the wave functions of Equation (78) and the quantum dynamics provided by Equation (82) for the three first negative integers. All the results have been obtained from the previous results of $\theta_n(q)$ and $\varphi_n(q) = \frac{\sqrt{2E_n}}{\sigma} \int_0^q \theta_n(y)\, dy$ in the interval $q \in [-15 \cdot c, \ 15 \cdot c] = [-150, \ 150]$. Note that the q-derivative of the quantum potential of Equation (81) presents singularities when $\frac{\sqrt{2E_n}}{\sigma} \int_0^q \theta_n(y)\, dy - \pi \cdot n = 0$. These singularities represent the fundamental difference between the classical dynamics given by Equation (1) and the quantum dynamics given by Equation (82). The way that these singularities are overcome is explained below. To perform this, both the wave function and the corresponding quantum dynamics are presented in the following three figures in the same interval $q \in [-15 \cdot c, \ 15 \cdot c] = [-150, \ 150]$.

Figure 3 presents the normalized wave function $\psi_1(q)$ of Equation (78) jointly the corresponding quantum dynamics $q_1(t)$ for $t \in [0, 1000]$ of Equation (82), plotted jointly the Figure 1 classical dynamics. On the one hand, note that, as announced above, $\psi_1(q) \to 0$, as $q \to \pm\infty$. The C_1 constant is computed, as usual, as $C_1^2 = \int_{-\infty}^{+\infty} \psi_1^2(q)\, dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_1^2(q)\, dq = 7.348$. On the other hand, no singularity arises in the quantum dynamics $q_1(t)$, at least in the interval $t \in [0, 1000]$. Therefore, the quantum dynamics $q_1(t)$ represents a correction of the classical logistics dynamics of Figure 1 that should be taken into account for empirical studies. However, some singularities arise in the following two cases.
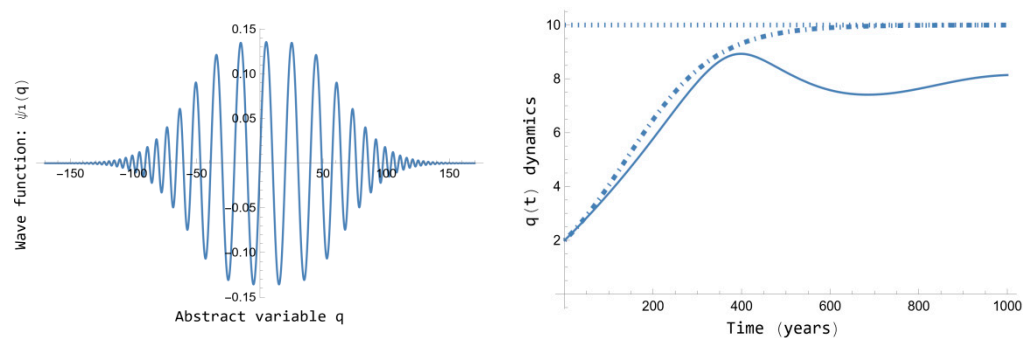

Figure 3. Normalized wave function $\psi_1(q)$ for $q \in [-15 \cdot c, \ 15 \cdot c] = [-150, \ 150]$ **(left)** and the corresponding quantum dynamics $q_1(t)$ for $t \in [0, 1000]$ **(right)**. The quantum dynamics corresponds to the continuous curve, while the dotted straight line value represents the saturation value $c = \frac{a}{b} = 10.0$ population thousands and the dot-dashed curve the classical logistic dynamics.

Figure 4 presents the normalized wave function $\psi_2(q)$ of Equation (78) jointly the corresponding quantum dynamics $q_2(t)$ for $t \in [0, 1000]$ of Equation (82), plotted jointly the Figure 1 classical dynamics. On the one hand, note that, as announced above, $\psi_2(q) \to 0$, as $q \to \pm\infty$. The C_2 constant is computed, as usual, as $C_2^2 = \int_{-\infty}^{+\infty} \psi_2^2(q)\, dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_2^2(q)\, dq = 10.345$. On the other hand, a singularity arises in the quantum dynamics $q_2(t)$ in the time $t \cong 136.53$, corresponding to $q \cong 5.050$. It is represented in Figure 4 with a vertical line. The solution to

represent the quantum dynamics in all the overall interval $t \in [0, 1000]$ has been to consider the *characteristic time interval* $\Delta \tau \cong \frac{\sigma}{|E_1 - E_2|} = 0.689$, provided in [20]. This time represents an approximation for the Energy-Time uncertainty relation, and then it can be interpreted as a time interval for which no information about the quantum dynamics is known. Then, the quantum dynamics is computed first in the interval $t \in \left[0,\, 136.53 - \frac{\Delta \tau}{2} \right]$ with the same initial conditions, and in a second interval $t \in \left[136.53 + \frac{\Delta \tau}{2}, 1000 \right]$ with $q\left(136.53 + \frac{\Delta \tau}{2} \right) = 5.1$ and $x\left(136.53 + \frac{\Delta \tau}{2} \right) = 0.0$. Avoiding this singularity, the quantum dynamics $q_2(t)$ represents again a correction of the classical logistics dynamics of Figure 1 that should be taken into account for empirical studies.
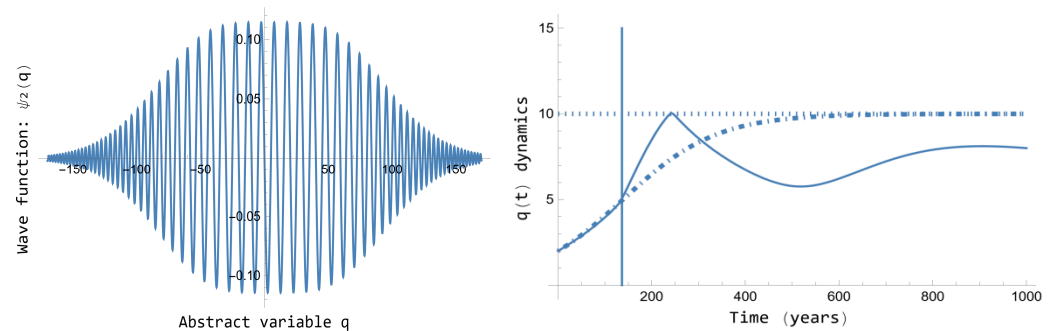

Figure 4. Normalized wave function $\psi_2(q)$ for $q \in [-15 \cdot c,\, 15 \cdot c] = [-150,\, 150]$ (**left**) and the corresponding quantum dynamics $q_2(t)$ for $t \in [0, 1000]$ (**right**). The quantum dynamics corresponds to the continuous curve, where the vertical line in $t \cong 136.53$ represents the singularity, while the dotted straight line value represents the saturation value $c = \frac{a}{b} = 10.0$ population thousands and the dot-dashed curve the classical logistic dynamics.

Figure 5 presents the normalized wave function $\psi_3(q)$ of Equation (78) jointly the corresponding quantum dynamics $q_3(t)$ for $t \in [0, 1000]$ of Equation (82), represented jointly in Figure 1 classical dynamics. Again, as announced above, $\psi_3(q) \to 0$, as $q \to \pm \infty$. The C_3 constant is computed, as usual, as $C_3^2 = \int_{-\infty}^{+\infty} \psi_3^2(q) \, dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_3^2(q) \, dq = 14.492$. On the other hand, also a singularity arises in the quantum dynamics $q_3(t)$ in the time $t \cong 63.81$, corresponding to $q \cong 3.367$. It is represented in Figure 5 with a vertical line. The solution to represent the quantum dynamics in all the overall interval $t \in [0, 1000]$ has been to consider again the *characteristic time interval* $\Delta \tau \cong \frac{\sigma}{|E_1 - E_2|} = 0.689$, provided in [20], with the meaning mentioned above in the context of Figure 4. Then, the quantum dynamics is computed first in the interval $t \in \left[0,\, 63.81 - \frac{\Delta \tau}{2} \right]$ with the same initial conditions, and in a second interval $t \in \left[63.81 + \frac{\Delta \tau}{2}, 1000 \right]$ with $q\left(63.81 + \frac{\Delta \tau}{2} \right) = 3.4$ and $x\left(63.81 + \frac{\Delta \tau}{2} \right) = 0.0$. Avoiding this singularity, The quantum dynamics $q_3(t)$ represents now a very different periodic pattern, not a correction of the classical logistics dynamics of Figure 1, which could also be taken into account for empirical studies.

To complete the results presented, note that $\int_{-\infty}^{+\infty} \psi_1(q) \cdot \psi_2(q) dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_1(q) \cdot \psi_2(q) dq \cong 0.00003$, $\int_{-\infty}^{+\infty} \psi_1(q) \cdot \psi_3(q) dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_1(q) \cdot \psi_3(q) dq \cong -0.00001$ and $\int_{-\infty}^{+\infty} \psi_2(q) \cdot \psi_3(q) dq \cong \int_{-15 \cdot c}^{+15 \cdot c} \psi_2(q) \cdot \psi_3(q) dq \cong 0.003$. These outcomes point out that, numerically, it can be asserted that the set $\left\{ \Psi_n(t, q) = e^{-i \frac{E_n}{\sigma} t} \psi_n(q) \right\}_{n=-1}^{-\infty}$ is orthonormal. However, the approximation to the zero value is lesser in the third integral than for the other two first integrals. Note that this outcome is related to the density of curves in the same interval $q \in [-15 \cdot c, 15 \cdot c] = [-150, 150]$, and consequently, with the number of zeros of the wave function $\psi_3(q)$ in Figure 5, vs. $\psi_2(q)$ in Figure 4 and $\psi_1(q)$ in Figure 3. In fact, subsequent results not here presented support this trend: the density of curves in the same interval $q \in [-15 \cdot c, 15 \cdot c] = [-150, 150]$, and thus, the number of zeros increases as the quantum integer becomes more negative.
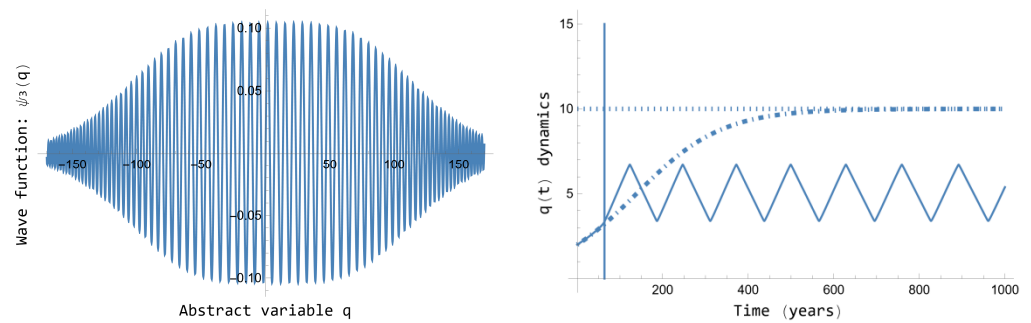
Figure 5. Normalized wave function $\psi_3(q)$ for $q \in [-15 \cdot c, \ 15 \cdot c] = [-150, 150]$ (**left**) and the corresponding quantum dynamics $q_3(t)$ for $t \in [0, 1000]$ (**right**). The quantum dynamics corresponds to the continuous curve, where the vertical line in $t \cong 63.81$ represents the singularity, while the dotted straight line value represents the saturation value $c = \frac{a}{b} = 10.0$ population thousands and the dot-dashed curve the classical logistic dynamics.

That pattern indicates that, as the quantum integer becomes more negative, the number of singularities increases for the quantum dynamics of Equation (82). For instance, for $n = -4$, a singularity arises at $t \cong 24.82$, becoming also periodical the dynamics after the singularity; and for $n = -5$ a first singularity arises at $t \cong 0.53$ and a second one at $t \cong 72.61$. For this last case, the pattern between the first and second singularity is of a growing kind, while the pattern after the second singularity is also periodical, similar to that of Figure 5.

The general conclusions of this section can be summarized as follows:

1. The system energy is positive, and it becomes quantized as a function of the non-zero negative integers over the hypotheses that the quantum wave vanishes in the critical points of the logistic function. Table 2 shows these energy outcomes.
2. The set of quantized wave functions given by Equations (77) and (78), $\left\{ \Psi_n(t,q) = e^{-i\frac{E_n}{\sigma}t} \psi_n(q) \right\}_{n=-1}^{-\infty}$, define an orthonormal set of functions, i.e., $\int_{-\infty}^{+\infty} \Psi_m^*(t,q) \cdot \Psi_n(t,q) \, dq = C_n \cdot C_m \cdot \delta_{nm}$.
3. It is expectable that, as the quantum integer becomes more negative, the density of curves of $\psi_n(q)$ by Equation (77) increases in all the domain $q \in (-\infty, +\infty)$, as well as the number of zeros in the same domain.
4. As a consequence of item 3, it is also expected that, as the quantum integer becomes more negative, the number of singularities will increase in the time interval of prediction for the quantum dynamics given by Equation (78).
5. As a consequence of item 4, the conception of quantum trajectory given by Equation (78) becomes radically different from the classical trajectory given by Equation (58). The growth of singularities as the negative integers become more negative seems to be the key point of this radical difference.
6. In future studies, the approaches presented in the works [18,19] could provide a way to overcome the singularities of the quantum trajectories found in this application case.

In addition, a general conclusion is that a fundamental objective of research must be to study if the singularities can be avoided. Avoiding the singularities will allow a better comparison between classical and quantum dynamics, as Figure 3 provides. However, on the other hand, the singularities could be unavoidable under the quantization hypotheses stated, and other quantization hypotheses should be stated instead. Note that these quantization hypotheses provided in Equations (74) and (75) have been $\psi(0) = 0$ and $\psi(c) = 0$, but other hypotheses could provide quantum dynamics that avoid the singularities. In addition, although the quantization hypotheses $\psi(0) = 0$ and $\psi(c) = 0$ maintain, the boundary conditions $\theta_n(0) = -1$ and $\theta_n'(0) = 0$ in Equation (77) could be different in order to avoid the singularities.

Finally, the assumption of the system Planck constant as $\sigma = 10$ has been chosen by numerical computations' convenience. However, finding a way to relate the system Planck constant with other significant constants of other formalisms could be necessary. For instance, in the work [24], Feigenbaum provides significant constants in the context of the logistic dynamics and similar functions. Consequently, could the Planck constant in the system be related to those constants? In addition, if the singularities discovered were unavoidable, could they also represent a route to chaos as the integers of the quantum approach become more negative?

Therefore, only in the context of this section is the future research area enormous, and it has to be investigated.

6. Discussion and Conclusions

A first discussion and the corresponding conclusions have been presented in the final paragraphs of Section 5. It is unavoidable to present them in the context of Section 5 because the formalism presented has been put into practice for the application case of this section. Therefore, for concrete discussion and conclusions about the quantum dynamics of the logistic function, Section 5 must be addressed.

However, before starting the general discussion, the important results of the paper must be provided:

1. The first-order Schrödinger equation of Appendix A (Equation (A22)), derived from the linear Hamiltonian of Equation (A19) by applying the quantization rules, does not provide any new quantum formulation to abstract dynamical systems defined by Equation (1).

2. The second-order Schrödinger Equation of Equation (40), derived from the quadratic Hamiltonian of Equation (31) by applying the quantization rules, does provide an alternative quantum formulation to abstract dynamical systems defined by Equation (1). This quantum formulation is provided by Equation (52).

3. Equation (52) needs the previous knowledge of the abstract masses, which are provided by Equations (50) and (51).

4. The application case of Section 5 demonstrates that formalism works despite the arising singularities of quantum dynamics for logistic functions.

About the general discussion, note that the application case of Section 5 can also enlighten the general discussion and conclusions paper. The first question that any researcher of General Systems Theory or mathematical physics could ask is if there exists a need for such formalism. The answer is positive in the sense that an understanding of the world's complexity needs new formalisms, such as those presented here. However, this is only an epistemological assumption, and only future empirical support could provide validity for formalism.

In addition, from a mathematical point of view, note that the work by Dirac [7] was what allowed him to obtain his equation for fermionic quantum fields [25]. That equation is also first-order in the partial derivatives, such as the first-order Schrödinger equation of Equation (A22) of Appendix A. However, Equation (A22) is non-relativistic, unlike the Dirac equation. However, a magnitude such as the information speed, similar to the light speed, has never been defined in the context of abstract dynamical systems. If this magnitude could be discovered and empirically supported, may be a kind of Dirac equation could be discovered. In this context, the $F_{lk}(t, \boldsymbol{q})$ ($k, l = 1, 2, \ldots, n$) of Equation (A12) (Appendix A) could play the role of the electromagnetic field and the $g_l(t, \boldsymbol{q})$ ($l = 1, 2, \ldots, n$) and $h(t, \boldsymbol{q})$ functions could play the role of, respectively, its vector potential and its scalar potential. However, that theoretical advance has not been produced, and, in addition, the first-order Schrödinger equation of Equation (A22) presented in Appendix A has no further information to provide to the dynamics of abstract dynamical systems.

The last disappointing conclusion is what motivates the second-order formulation presented in Sections 2–4. It must be highlighted that both Hamiltonians of Section 2, those corresponding to Equations (18) and (31), could provide significant information in addition

to its use to obtain a second-order Schrödinger equation in Section 3. For instance, the energy conservation of autonomous systems for both Hamiltonians could help us better understand the classical dynamics of the abstract dynamical systems. This point should also be investigated. In addition, could there be a universal way to define the abstract masses? The answer has also been provided in Section 2: the abstract masses could be defined as $m_k(t, \boldsymbol{q}) = \frac{\mu_k}{f_k(t,\boldsymbol{q})}$ ($k = 1, 2, \ldots, n$), being μ_k constants with the suitable dimensions. In this case, Equation (43) holds identically, and it can also be considered that $h(t, \boldsymbol{q}) = 0$. Therefore, the application case of Section 5 could have been presented similar to this, i.e., as $m(q) = \frac{\mu}{f(q)}$ instead $m(q) = 1$. Then, the conclusions of the application case could be different. This hypothesis must also be investigated.

However, the most important contributions of this paper are in Section 3, the second-order Schrödinger equation of Equation (40) or this time-independent version of Equation (42) for autonomous systems, and in Section 4, the quantum formulation of the abstract dynamical systems of Equation (52). Note that more application cases must be investigated in addition to that of Section 5.

On the one hand, similar theoretical contributions to that of Section 5 must be investigated, trying to discover if the abstract masses' choice can be universal, such as discussed above, or their choice can be arbitrary (while they obey Equations (16) and (17)). On the other hand, formalism needs a great amount of empirical support. The systems that could support all the formalism presented can be of a different nature. For instance, population dynamics or chemical reaction dynamics can provide a great amount of empirical data to support formalism, basically because both kinds of dynamical systems are modeled by systems of first-order differential equations.

A question that must also be studied in future research is the stochastic formulation of Equation (52). This question was highlighted by Bohm and Hiley in [15]. In fact, in that work, the deterministic and the stochastic approaches are considered. The stochastic approach can provide reliable quantum dynamics, i.e., outcomes with confidence intervals, which are not possible with a deterministic approach. To support this possible approach, note that the wave functions of Equation (78) are probability densities. In addition, the stochasticity consideration could provide quantum dynamics similar to those of physical systems, such as the dynamics of an electron around the proton in a hydrogen atom [20]. However, the stochastic approach must take part in a subsequent phase of research after having found better theoretical and empirical support for the formalism presented.

Finally, the author of this paper is conscious that the inclusion of the Planck constant system in formalism is problematic. In fact, this inclusion makes epistemological and mathematical questions difficult to explain without previous empirical or theoretical evidence. Some ways to support its existence and its value for a particular abstract dynamical system have been discussed in the conclusions of Section 5. However, from a theoretical approach, the system Planck constant arises with the translation of the quantum formalism to the abstract dynamical systems, such as has been shown throughout this paper. However, as it happens in all newly proposed theories, the existence of the system Planck constant, as well as all the formalism background, take part in a hypothetical proposal that must be supported in the future by empirical evidence.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

The definition of an abstract dynamical system is also presented in this appendix, as it is presented in Section 2. The reason is to maintain coherence with the rest of the appendix.

Definition A1. *An abstract dynamical system is defined by the following coupled set of differential equations ($1 \leq k \leq n$):*

$$\dot{q}_k(t) = f_k(t, \boldsymbol{q}) \tag{A1}$$

In Equation (A1) t is the time variable and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ represent the dynamical variables. Their nature can be arbitrary, not necessarily of physical nature, such as populations, chemical or biochemical species, socioeconomic or behavioral indicators, etc. In addition, the $f_k(t, \boldsymbol{q})$ ($1 \leq k \leq n$) represent their dynamic interactions, considering that they are equipped with all kinds of smoothness properties to develop suitably formalism. In addition, $\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$ are called the dynamical velocities.

Following the methodology developed by Dirac [7] for fields, the works [8,10] applied it to obtain a Hamiltonian to abstract dynamical systems. The method presented in those works is also presented here because they are needed in the subsequent sections.

Proposition A1. The Lagrangian for AEquation (A1) is provided by:

$$L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{j=1}^{n} g_j(t, \boldsymbol{q}) \cdot \dot{q}_j - h(t, \boldsymbol{q}) \tag{A2}$$

Proof. It is trivial, taking into account that, as noted by Dirac in [7], a Lagrangian for AEquation (A1) is only possible if it is linear in the dynamical velocities. □

Note in AEquation (A2) that the $g_j(t, \boldsymbol{q})$ and $h(t, \boldsymbol{q})$ functions are still undetermined. The corresponding momentum variables are then defined as ($1 \leq k \leq n$):

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = g_k(t, \boldsymbol{q}) \tag{A3}$$

Note that in AEquation (A3), the first problem is that the momentum variables do not depend on the dynamical velocities; thus, the dynamical velocities cannot be isolated from the momentum variables.

Corollary A1. As a consequence of the definition of Hamiltonian in classical mechanics, the Hamiltonian for AEquation (A1) becomes:

$$H_0(t, \boldsymbol{q}, \boldsymbol{p}) = \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \cdot \dot{q}_j - L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = h(t, \boldsymbol{q}) \tag{A4}$$

Note in AEquation (A4) that the momentum variables are represented as $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$. In addition to the first problem provided by AEquation (A3), AEquation (A4) shows a second problem: the H_0 Hamiltonian does not depend on the momentum variables. Due to these two problems $H_0(t, \boldsymbol{q}, \boldsymbol{p})$ is referred to as *singular*. The Dirac method [7] consists in solving both problems.

Proposition A2. The Hamiltonian, that is, the function of the momentum variables, is provided by:

$$H(t, \boldsymbol{q}, \boldsymbol{p}) = h(t, \boldsymbol{q}) + \sum_{j=1}^{n} \lambda_j(t, \boldsymbol{q}, \boldsymbol{p})(p_j - g_j(t, \boldsymbol{q})) \tag{A5}$$

Proof. Defining the *primary constants* $\phi_k(t, \boldsymbol{q}, \boldsymbol{p})$, such as stated in [7] ($1 \leq k \leq n$):

$$\phi_k(t, \boldsymbol{q}, \boldsymbol{p}) = p_k - g_k(t, \boldsymbol{q}) \tag{A6}$$

and subsequently reinserting the primary constants in the Hamiltonian $H_0(t, \boldsymbol{q}, \boldsymbol{p})$ by the usual way of the use of the (still undetermined) *Lagrange multipliers* $\lambda_k(t, \boldsymbol{q}, \boldsymbol{p})$ ($1 \leq k \leq n$) as $H(t, \boldsymbol{q}, \boldsymbol{p}) = H_0(t, \boldsymbol{q}, \boldsymbol{p}) + \sum_{j=1}^{n} \lambda_j(t, \boldsymbol{q}, \boldsymbol{p}) \phi_k(t, \boldsymbol{q}, \boldsymbol{p})$, the Hamiltonian of AEquation (A5) arises. □

Note that the Dirac method to reinsert the momentum variables implies that, in the context of the Hamiltonian $H(t, \boldsymbol{q}, \boldsymbol{p})$, the primary constants are not identically zero, but $\phi_j(t, \boldsymbol{q}, \boldsymbol{p}) = 0$ $(1 \leq j \leq n)$ when they are applied in the Hamilton equations.

Proposition A3. The corresponding Hamilton equations of the Hamiltonian of AEquation (A5) are $(1 \leq k \leq n)$:

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = \sum_{j=1}^{n} \frac{\partial \lambda_j(t, \boldsymbol{q}, \boldsymbol{p})}{\partial p_k} \cdot \phi_j(t, \boldsymbol{q}, \boldsymbol{p}) + \sum_{j=1}^{n} \lambda_j(t, \boldsymbol{q}, \boldsymbol{p}) \cdot \frac{\partial \phi_j(t, \boldsymbol{q}, \boldsymbol{p})}{\partial p_k} = \lambda_k(t, \boldsymbol{q}, \boldsymbol{p}) \tag{A7}$$

$$\dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\sum_{j=1}^{n} \frac{\partial \lambda_j(t, \boldsymbol{q}, \boldsymbol{p})}{\partial q_k} \cdot \phi_j(t, \boldsymbol{q}, \boldsymbol{p}) - \sum_{j=1}^{n} \lambda_j(t, \boldsymbol{q}, \boldsymbol{p}) \cdot \frac{\partial \phi_j(t, \boldsymbol{q}, \boldsymbol{p})}{\partial q_k} - \frac{\partial h(t, \boldsymbol{q})}{\partial q_k} =$$
$$= -\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n} \lambda_j(t, \boldsymbol{q}, \boldsymbol{p}) \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} \tag{A8}$$

Proof. AEquations (A7) and (A8) are a consequence of the Hamilton equations in classical mechanics and of AEquation (A6) assumption for the primary constants. $\square$

Corollary A2. The Hamilton AEquations (A7) and (A8) become $(1 \leq k \leq n)$:

$$\dot{q}_k(t) = \frac{\partial H}{\partial p_k} = f_k(t, \boldsymbol{q}) \tag{A9}$$

$$\dot{p}_k(t) = -\frac{\partial H}{\partial q_k} = -\frac{\partial h(t, \boldsymbol{q})}{\partial q_k} + \sum_{j=1}^{n} f_j(t, \boldsymbol{q}) \frac{\partial g_j(t, \boldsymbol{q})}{\partial q_k} \tag{A10}$$

Proof. Comparing AEquations (A1) and (A7): $\lambda_k(t, \boldsymbol{q}, \boldsymbol{p}) = f_k(t, \boldsymbol{q})$ $(1 \leq k \leq n)$. $\square$

In order to determine the $g_j(t, \boldsymbol{q})$ and $h(t, \boldsymbol{q})$ functions, the so-called Dirac as *consistency conditions*, $\dot{\phi}_l(t, \boldsymbol{q}, \boldsymbol{p}) = 0$ $(1 \leq l \leq n)$, must be applied below.

Definition A2. For $k, l = 1, 2, \ldots, n$, the $F_{kl}(t, \boldsymbol{q})$ functions are defined as:

$$F_{kl}(t, \boldsymbol{q}) = -F_{lk}(t, \boldsymbol{q}) = -\frac{\partial g_l(t, \boldsymbol{q})}{\partial q_k} + \frac{\partial g_k(t, \boldsymbol{q})}{\partial q_l} \tag{A11}$$

Theorem A1. *The $g_j(t, \boldsymbol{q})$ $(1 \leq j \leq n)$ and $h(t, \boldsymbol{q})$ functions obey the following equations $(1 \leq j, l \leq n)$:*

$$\sum_{k=1}^{n} F_{lk}(t, \boldsymbol{q}) f_k(t, \boldsymbol{q}) = -\frac{\partial g_l(t, \boldsymbol{q})}{\partial t} - \frac{\partial h(t, \boldsymbol{q})}{\partial q_l} \tag{A12}$$

$$\frac{\partial F_{jl}(t, \boldsymbol{q})}{\partial t} = \frac{\partial}{\partial q_j} \left(\sum_{k=1}^{n} F_{lk}(t, \boldsymbol{q}) f_k(t, \boldsymbol{q}) \right) - \frac{\partial}{\partial q_l} \left(F_{jk}(t, \boldsymbol{q}) f_k(t, \boldsymbol{q}) \right) \tag{A13}$$

Proof. From AEquation (A6) $(1 \leq l \leq n)$:

$$\dot{\phi}_l(t, \boldsymbol{q}, \boldsymbol{p}) = \frac{\partial \phi_l(t, \boldsymbol{q}, \boldsymbol{p})}{\partial t} + \sum_{k=1}^{n} \frac{\partial \phi_l(t, \boldsymbol{q}, \boldsymbol{p})}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial \phi_l(t, \boldsymbol{q}, \boldsymbol{p})}{\partial p_k} \dot{p}_k = 0 \tag{A14}$$

The substitution of Aquations (A9) and (A10) in AEquation (A14) provides, taking into account the zero value of the primary constraints, and after some calculations $(1 \leq l \leq n)$:

$$-\frac{\partial g_l(t,\boldsymbol{q})}{\partial t} - \frac{\partial h(t,\boldsymbol{q})}{\partial q_l} + \sum_{k=1}^{n}\left(-\frac{\partial g_l(t,\boldsymbol{q})}{\partial q_k} + \frac{\partial g_k(t,\boldsymbol{q})}{\partial q_l}\right)f_k(t,\boldsymbol{q}) = 0 \qquad \text{(A15)}$$

Then, considering AEquation (A11) in AEquation (A15), AEquation (A12) holds. In addition, AEquation (A13) arises from AEquation (A12) by following the steps: 1. Take the partial derivative respect an arbitrary q_j in AEquation (A12); 2. Rewrite AEquation (A12) by replacing l by j; 3. Take the partial derivative respect q_l in the rewritten equation; 4. Subtract both equations, taking into account the equality of both $h(t,\boldsymbol{q})$ crossed derivatives, AEquation (A13) holds. □

Therefore, the process to obtain the $g_j(t,\boldsymbol{q})$ and $h(t,\boldsymbol{q})$ functions of the Hamiltonian given by AEquation (A5) is: 1. Obtain the $F_{lk}(t,\boldsymbol{q})$ functions by AEquation (A13); 2. Substitute these results in AEquation (A12) to obtain the $g_j(t,\boldsymbol{q})$ and $h(t,\boldsymbol{q})$ functions. Take into account in this process that $F_{ll}(t,\boldsymbol{q}) = 0$ and that $F_{kl}(t,\boldsymbol{q}) = -F_{lk}(t,\boldsymbol{q})$ (i.e., it is an antisymmetric functional matrix).

However, two different classes of solutions must be considered depending on the system dimension n. This is due to the antisymmetric definition of the $F_{lk}(t,\boldsymbol{q})$ functions. On the one hand, if n is even, then, in general $det(F_{lk}(t,\boldsymbol{q})) \neq 0$. In this case both sets of AEquation (A12) or AEquation (A13) are independent. On the other hand, if n is odd, then always $det(F_{lk}(t,\boldsymbol{q})) = 0$, and being $det(F_{lk}(t,\boldsymbol{q})) \neq 0$ for the $n-1$ even dimension, one of AEquation (A12) is dependent on the others, which means some $F_{lk}(t,\boldsymbol{q})$ functions become undetermined parameters from which the rest depend on.

Actually, this last case always happens due to AEquation (A12) is a coupled set of n equations and $n+1$ unknown variables: $g_l(t,\boldsymbol{q})$ ($l = 1, 2, \ldots, n$) and $h(t,\boldsymbol{q})$. For instance, let the special one-dimensional ($n = 1$) odd case be. The consistency conditions that AEquation (A12) provides become:

$$\eta(t,q,p) := \frac{\partial g(t,q)}{\partial t} + \frac{\partial h(t,q)}{\partial q} = 0 \qquad \text{(A16)}$$

Note that AEquation (A16) does not provide the $\lambda(t,q,p) = f(t,q)$ multiplier. When this case happens, the Dirac method [7] prescription is to consider equations such as AEquation (A16) as *secondary constraints*. In order to obtain the multiplier in an equation, the time derivative is taken in AEquation (A16):

$$\dot{\eta}(t,q,p) = \frac{\partial \eta(t,q,p)}{\partial t} + \frac{\partial \eta(t,q,p)}{\partial q}\dot{q} + \frac{\partial \eta(t,q,p)}{\partial p}\dot{p} = 0 \qquad \text{(A17)}$$

Taking into account the Hamilton equations of AEquations (A9) and (A10), such as for the primary constraints, AEquation (A17) becomes, after some calculations:

$$\frac{\partial^2 g(t,q)}{\partial t^2} + \frac{\partial^2 h(t,q)}{\partial t\,\partial q} + f(t,q)\left(\frac{\partial^2 g(t,q)}{\partial q\,\partial t} + \frac{\partial^2 h(t,q)}{\partial q^2}\right) = 0 \qquad \text{(A18)}$$

Note in AEquation (A18) that just an equation is provided for two unknown variables, $g(t,q)$ and $h(t,q)$, then one of the two variables becomes undetermined, and therefore, different solutions can be applied.

Now that, considering all the formalism developed, the $g_j(t,\boldsymbol{q})$ ($j = 1, 2, \ldots, n$) and $h(t,\boldsymbol{q})$ functions can be found, and the Hamiltonian can be written from AEquation (A5) as:

$$\begin{aligned} H(t,\boldsymbol{q},\boldsymbol{p}) &= \sum_{j=1}^{n} f_j(t,\boldsymbol{q})\big(p_j - g_j(t,\boldsymbol{q})\big) + h(t,\boldsymbol{q}) = \\ &= \sum_{j=1}^{n} f_j(t,\boldsymbol{q})\cdot p_j - \sum_{j=1}^{n} f_j(t,\boldsymbol{q})\cdot g_j(t,\boldsymbol{q}) + h(t,\boldsymbol{q}) \end{aligned} \qquad \text{(A19)}$$

Observe in addition that if the action is represented by $S(t, \boldsymbol{q})$, the corresponding Hamilton–Jacobi equation for this Hamiltonian is:

$$\frac{\partial S(t, \boldsymbol{q})}{\partial t} + \sum_{j=1}^{n} f_j(t, \boldsymbol{q}) \frac{\partial S(t, \boldsymbol{q})}{\partial q_j} + W(t, \boldsymbol{q}) = 0 \tag{A20}$$

Note that in AEquation (A20): $W(t, \boldsymbol{q}) = -\sum_{j=1}^{n} f_j(t, \boldsymbol{q}) \cdot g_j(t, \boldsymbol{q}) + h(t, \boldsymbol{q})$.

On the other hand, AEquations (A11)–(A13) are deduced by Havas in Annex 2 of [11] for the Lagrangian of AEquation (A2), which provides that both approaches, Lagrangian and Hamiltonian, are equivalent, such as Govaerts demonstrates in [12] for fields. Also, Micó in [8] deduces both formalisms and demonstrates their equivalence for the abstract dynamical systems here dealt. However, the odd n case is solved by Havas [11] and Micó [8] by adding the differential equation $\dot{q}_{n+1}(t) = 0$. With this new equation, $n + 1$ is even and, in general, $det(F_{lk}(t, \boldsymbol{q})) \neq 0$, becoming independent of both sets of AEquation (A12) or AEquation (A13), such as it happens in the even n case.

A first failed attempt to obtain a quantum approach to the abstract dynamical systems given by AEquation (A1) was provided in [6]. However, this attempt is also here provided to motivate the need for the alternative approach in Sections 2–4.

If $\hat{H}(t, \hat{\boldsymbol{q}}, \hat{\boldsymbol{p}})$ is the quantum operator corresponding to a known Hamiltonian $H(t, \boldsymbol{q}, \boldsymbol{p})$, the Schrödinger equation is written as:

$$i\sigma \frac{\partial \Psi(t, \boldsymbol{q})}{\partial t} = \hat{H}(t, \hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}) \Psi(t, \boldsymbol{q}) \tag{A21}$$

In AEquation (A21) $\Psi(t, \boldsymbol{q})$ is the wave function. Due to the Hamiltonian that corresponds to an abstract dynamical system such as AEquation (A19), σ represents the system Planck constant different from the Planck constant of physics. Therefore, the hypothesis assumed is that each abstract dynamical system has its own system Planck constant, whose value represents a limitation of the mathematical knowledge of the system.

Theorem A2. *The Schrödinger equation corresponding to the Hamiltonian of AEquation* (A19), *here so-called first-order Schrödinger equation, is written as:*

$$i\sigma \frac{\partial \Psi(t, \boldsymbol{q})}{\partial t} = -i\frac{\sigma}{2}\sum_{j=1}^{n} f_j(t, \boldsymbol{q}) \frac{\partial \Psi(t, \boldsymbol{q})}{\partial q_j} - i\frac{\sigma}{2}\sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(f_j(t, \boldsymbol{q}) \cdot \Psi(t, \boldsymbol{q})\right) + W(t, \boldsymbol{q})\,\Psi(t, \boldsymbol{q}) \tag{A22}$$

Proof. Following the quantization rules provided for the Copenhagen formalism of the quantum theory [20], $\hat{H}(t, \hat{\boldsymbol{q}}, \hat{\boldsymbol{p}})$ is an operator that acts on the wave function as:

$$\hat{H}(t, \hat{\boldsymbol{q}}, \hat{\boldsymbol{p}}) \Psi(t, \boldsymbol{q}) = \frac{1}{2}\sum_{j=1}^{n}\left(f_j(t, \hat{\boldsymbol{q}})\hat{p}_j + \hat{p}_j f_i(t, \hat{\boldsymbol{q}})\right)\Psi(t, \boldsymbol{q}) + W(t, \hat{\boldsymbol{q}})\,\Psi(t, \boldsymbol{q}) \tag{A23}$$

Such that in AEquation (A23): $W(t, \hat{\boldsymbol{q}}) = -\sum_{j=1}^{n} f_j(t, \hat{\boldsymbol{q}})g_j(t, \hat{\boldsymbol{q}}) + h(t, \hat{\boldsymbol{q}})$, $W(t, \hat{\boldsymbol{q}})\Psi(t, \boldsymbol{q}) = W(t, \boldsymbol{q})\Psi(t, \boldsymbol{q})$, and $\hat{p}_j\Psi(t, \boldsymbol{q}) = -i\sigma\frac{\partial \Psi(t, \boldsymbol{q})}{\partial q_j}$. Note in AEquation (A23) that the term $\frac{1}{2}\sum_{j=1}^{n}\left(f_j(t, \hat{\boldsymbol{q}})\hat{p}_j + \hat{p}_j f_i(t, \hat{\boldsymbol{q}})\right)$ provides that the Hamiltonian is a self-adjoint operator [20].

Therefore, AEquation (A22) is the first-order Schrödinger equation corresponding to the Hamiltonian of AEquation (A19). $\square$

Note that AEquation (A22) is a first-order partial differential equation, contrary to the physical problems where it is a second-order partial differential equation. However, following the Madelung approach [14], does AEquation (A22) provide new information for AEquation (A1) through a quantum potential arising in the Hamiltonian? To answer this question, splitting the wave function in its amplitude $A(t, \boldsymbol{q})$ and its phase $B(t, \boldsymbol{q})$ is needed:

$$\Psi(t, \boldsymbol{q}) = A(t, \boldsymbol{q})\,^{i\frac{B(t, \boldsymbol{q})}{\sigma}} \tag{A24}$$

Proposition A4. The A$(t, \boldsymbol{q})$ and B$(t, \boldsymbol{q})$ functions of AEquation (A24) obey the following equations:

$$\frac{\partial B(t, \boldsymbol{q})}{\partial t} + \sum_{j=1}^{n} f_j(t, \boldsymbol{q})\frac{\partial B(t, \boldsymbol{q})}{\partial q_j} + W(t, \boldsymbol{q}) = 0 \tag{A25}$$

$$\frac{\partial A^2(t, \boldsymbol{q})}{\partial t} + \sum_{j=1}^{n} \frac{\partial}{\partial q_j}\Big(f_j(t, \boldsymbol{q}) \cdot A^2(t, \boldsymbol{q})\Big) = 0 \tag{A26}$$

Proof. The substitution of AEquation (A24) in AEquation (A22) provides, respectively, for the real and the imaginary parts (after some easy operations and canceling the term $i^{\frac{B(t,q)}{\sigma}}$):

$$-A(t, \boldsymbol{q})\frac{\partial B(t, \boldsymbol{q})}{\partial t} = A(t, \boldsymbol{q})\sum_{j=1}^{n} f_j(t, \boldsymbol{q})\frac{\partial B(t, \boldsymbol{q})}{\partial q_j} + A(t, \boldsymbol{q})W(t, \boldsymbol{q}) \tag{A27}$$

$$\sigma\frac{\partial A(t, \boldsymbol{q})}{\partial t} = -\sigma\sum_{j=1}^{n} f_j(t, \boldsymbol{q})\frac{\partial A(t, \boldsymbol{q})}{\partial q_j} - \frac{\sigma}{2}A(t, \boldsymbol{q})\sum_{j=1}^{n} \frac{\partial f_j(t, \boldsymbol{q})}{\partial q_j} \tag{A28}$$

Dividing AEquation (A27) by A$(t, \boldsymbol{q})$, AEquation (A25) holds, and dividing AEquation (A28) by σ and subsequently multiplying it by 2A$(t, \boldsymbol{q})$, AEquation (A26) holds. □

AEquation (A25) provides the Hamilton–Jacobi equation of AEquation (A20), corresponding to the Hamiltonian of AEquation (A19), for the phase B$(t, \boldsymbol{q})$. AEquation (A26) provides the probability conservation for the square amplitude A$^2(t, \boldsymbol{q})$, being the vector of components $f_j(t, \boldsymbol{q})$ $(1 \le j \le n)$ the probability current density. Note that contrary to the physical systems, despite the fact that the probability conservation holds, an additional quantum potential in the Hamiltonian of AEquation (A19) does not arise in this context. This term would produce a further quantum potential (following the Madelung approach) that could change the original abstract dynamical system of AEquation (A1). Thus, the first-order Schrödinger equation AEquation (A22) fails to provide further mathematical information to AEquation (A1).

Therefore, as an alternative formulation, is there any way to obtain a second-order Schrödinger equation for the abstract dynamical systems of AEquation (A1)? The answer is positive, but previously, a reformulation of AEquation (A1) as a second-order differential equation system is necessary, as well as finding the corresponding Hamiltonian that will be quadratic in the momentum variables (referred to as the quadratic Hamiltonian). This is the objective of Sections 2–4.

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