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Asymptotic for Orthogonal Polynomials with Respect to a Rational Modification of a Measure Supported on the Semi-Axis

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Abstract: Given a sequence of orthogonal polynomials $\{L_n\}_{n=0}^{\infty}$, orthogonal with respect to a positive Borel ν measure supported on $\mathbb{R}_+$, let $\{Q_n\}_{n=0}^{\infty}$ be the the sequence of orthogonal polynomials with respect to the modified measure $r(x)d\nu(x)$, where r is certain rational function. This work is devoted to the proof of the relative asymptotic formula $\frac{Q_n^{(d)}(z)}{L_n^{(d)}(z)} \rightrightarrows_n \prod_{k=1}^{N_1}\left(\frac{\sqrt{a_k}+i}{\sqrt{z}+\sqrt{a_k}}\right)^{A_k} \prod_{j=1}^{N_2}\left(\frac{\sqrt{z}+\sqrt{b_j}}{\sqrt{b_j}+i}\right)^{B_j}$, on compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$, where a_k and b_j are the zeros and poles of r, and the A_k, B_j are their respective multiplicities.

Keywords: orthogonal polynomials; asymptotic behavior; rational modifications

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1. Introduction

Let μ be a positive, finite, Borel measure on $\mathbb{R}_+ = [0, +\infty)$, such that for all $n \in \mathbb{Z}_+$ (the set of all non-negative integers)

$$\eta_n = \int_0^{\infty} x^n \, d\mu(x) < \infty. \tag{1}$$

If there is no other measure μ_0, such that $\eta_n = \int_0^{\infty} x^n \, d\mu_0(x)$ for all $n \in \mathbb{Z}_+$, it is said that the moment problem associated with $\{\eta_n\}_{n\in\mathbb{Z}_+}$ is determined (see ([1] Ch. 4)). By a classical result of T. Carleman (see ([1] Th. 4.3)), a sufficient condition in order to the moment problem associated with the sequence $\{\eta_n\}_{n\in\mathbb{Z}_+}$ in (1) to be determined is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\eta_n}} = +\infty. \tag{2}$$

We say that the measure μ belongs to the class $\mathfrak{M}'[\mathbb{R}_+]$ if $\{\eta_n\}_{n\in\mathbb{Z}_+}$ satisfies (2) and $\mu' > 0$ a.e. on $\mathbb{R}_+$ with respect to Lebesgue measure.

Let $r(z) = \dfrac{\alpha(z)}{\beta(z)}$ be a rational function, where α and β are coprime polynomials with respective degrees A and B. We say that $d\mu_r(x) = r(z)d\mu(z)$ is a rational modification (for brevity, modification) of the measure μ. Write

$$\alpha(z) = \prod_{i=1}^{N_1}(z-a_i)^{A_i}, \qquad \beta(z) = \prod_{j=1}^{N_2}(z-b_j)^{B_j},$$

where $a_i, b_j \in \mathbb{C} \setminus \mathbb{R}_+$, $A_i, B_j \in \mathbb{N}$. $A = A_1 + \cdots + A_{N_1}$ and $B = B_1 + \cdots + B_{N_2}$.

We denote by $\{L_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to $d\mu$. Assume that $\{Q_n\}_{n=0}^{\infty}$ is the sequence of monic polynomials of least degree, not identically equal to zero, such that

$$\int_0^{\infty} x^k\, Q_n(x)\, r(x) d\mu(x) = 0, \quad \text{for all} \quad k = 0, 1, 2, \ldots, n-1. \tag{3}$$

The existence of Q_n is an immediate consequence of (3). Indeed, it is deduced solving an homogeneous linear system with n equations and $n+1$ unknowns. Uniqueness follows from the minimality of the degree of the polynomial. We call Q_n the nth monic modified orthogonal polynomial. In ([2] Th.1), explicit formulas are provided in order to compute Q_n when the poles and zeros of the rational modification have a multiplicity of one.

Suppose that $\{a_i\}_{i=1}^{N_1}, \{b_j\}_{j=1}^{N_2} \subset \mathbb{C} \setminus [-1,1]$. If μ is a positive (finite Borel) measure on $[-1,1]$, such that μ is on the Nevai class $\mathfrak{M}(0,1)$, in ([3] Th. 1) the authors prove the following asymptotic formula

$$\frac{Q_n^{(d)}(z)}{L_n^{(d)}(z)} \underset{n}{\rightrightarrows} \prod_{i=1}^{N_1} \left(\frac{\varphi(z) - \varphi(a_i)}{2(z - a_i)} \right)^{A_i} \prod_{j=1}^{N_2} \left(1 - \frac{1}{\varphi(z)\varphi(b_j)} \right)^{B_j}, \tag{4}$$

on $K \subset \overline{\mathbb{C}} \setminus [-1,1]$. The notation $f_n \rightrightarrows_n f, \; K \subset U$ means that the sequence of functions f_n converges to f uniformly on a compact subset K of the region U, $f^{(d)}$ denotes the dth derivative of f, $d \in \mathbb{Z}_+$ is fixed and

$$\varphi(z) = z + \sqrt{z^2 - 1} \quad \left(\left| z + \sqrt{z^2 - 1} \right| > 1, \quad z \in \mathbb{C} \setminus [-1,1] \right).$$

In [3], the asymptotic formula (4) is pivotal in examining the asymptotic properties of orthogonal polynomials across a broad range of inner products, encompassing Sobolev-type inner products

$$\langle f, g \rangle_S = \int f g \, d\mu + \sum_{j=1}^{m} \sum_{i=0}^{d_j} \lambda_{j,i}\, f^{(i)}(\zeta_j)\, g^{(i)}(\zeta_j),$$

where $\lambda_{j,i} \geq 0$, $m, d_j > 0$, μ is certain kind of complex measure with compact support is defined on the real line, and ζ_j represents complex numbers outside the support of μ. The authors compare the Sobolev-type orthogonal polynomials associated with this measure to the orthogonal polynomials with respect to μ. These asymptotic results are of interest for the electrostatic interpretation of zeros of Jacobi–Sobolev polynomials (cf. [4]).

On the other hand, the use of modified measures provides a stable way of computing the coefficients of the recurrence relation associated to a family of orthogonal polynomials (see ([5] Ch. 2)) and in [6,7] the interest of the modified orthogonal polynomials for the study of the multipoint Padé approximation is shown.

For measures supported on $[0, +\infty)$ (or $(-\infty, +\infty)$) that satisfy the Carleman condition, G. López in ([8] Th. 4) (or ([8] Th. 3) for $(-\infty, +\infty)$) proves a quite general version of the relative asymptotic formula (4). In this case, if the modification function, ρ, is a non-negative function on $[0, +\infty)$ in $\mathrm{L}^1(\mu)$, such that there exists an algebraic polynomial G and $k \in \mathbb{N}$ for which $|G|\rho/(1+x)^k$ and $|G|\rho^{-1}/(1+x)^k$ belong to $\mathrm{L}^{\infty}(\mu)$, then

$$\frac{Q_n(z)}{L_n(z)} \underset{n}{\rightrightarrows} \frac{\mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), z)}{\mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), \infty)}, \quad K \subset \mathbb{C} \setminus [0, +\infty); \tag{5}$$

where $\mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), z)$ is the Szegő's function for ρ with respect to $\mathbb{C} \setminus [0, +\infty)$, i.e.,

$$\mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), z) = e^{s(z)}, \quad s(z) = \frac{1}{2\pi} \int_0^\infty \log \rho(x) \left(\frac{\sqrt{-z}}{z - x} \right) \frac{dx}{\sqrt{x}};$$
$$\mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), \infty) = \lim_{r \to +\infty} \mathrm{S}(\rho, \mathbb{C} \setminus [0, +\infty), -r);$$

where the roots are selected from the condition $\sqrt{1} = 1$. Additionally, it is requested that $f(z) = \rho(-((z+1)/(z-1))^2)$ satisfies the Lipschitz condition in $z = 1$ and $f(1) \neq 0$.

Asymptotic results, analogous to those obtained in [3], are obtained in [9] for the particular case of (5), when $d\mu(x) = x^a e^{-x} dx$ with $a > -1$ (the Laguerre measure).

The aim of this paper is to obtain an analog of (4) for measures supported on $\mathbb{R}_+$. We prove the following theorem.

Theorem 1. *Given a measure $\nu \in \mathfrak{M}'[\mathbb{R}_+]$, it holds in compact subsets of $\mathbb{C} \setminus \mathbb{R}$*

$$\frac{Q_n^{(d)}(z)}{L_n^{(d)}(z)} \underset{n}{\rightrightarrows} \prod_{i=1}^{N_1} \left(\frac{\sqrt{a_i} + i}{\sqrt{z} + \sqrt{a_i}} \right)^{A_i} \prod_{j=1}^{N_2} \left(\frac{\sqrt{z} + \sqrt{b_j}}{\sqrt{b_j} + i} \right)^{B_j}, \tag{6}$$

for $d \in \mathbb{Z}_+$.

This situation is not a particular case of (5), because we consider ρ as a rational function with complex coefficients and no necessarily $\rho(x) \geq 0$ on $\mathbb{R}_+$.

The structure of the paper is as follows: Sections 2 and 3 are devoted to prove some preliminary results on varying measures. On the other hand, in Section 4 we obtain an essential theorem that allows us to finally prove Theorem 1 in Section 5.

2. Varying Measures and Carleman's Condition

In this section, we introduce auxiliary results on varying measures and prove some useful lemmas that allow us to extend results that hold for measures with bounded support to the unbounded case. The following notations will be used throughout the paper:

$$\begin{aligned} \Psi(z) =& \frac{1+z}{1-z} \text{ for } z \in \mathbb{C} \setminus [-1, 1]. \\ \Psi^{-1}(z) =& \frac{z-1}{z+1} \text{ for } z \in \mathbb{C} \setminus \mathbb{R}_+. \\ \Phi(z) =& \frac{\sqrt{z} + i}{\sqrt{z} - i} \text{ where } \Phi(-1) = \infty \text{ and } z \in \mathbb{C} \setminus [|z| \leq 1]. \end{aligned} \tag{7}$$

If σ is a finite positive Borel measure on $[-1, 1]$, we denote

$$d\sigma_n(t) = \frac{d\sigma(t)}{(1-t)^{2n}} \quad \text{and} \quad \varsigma_n = \int_{-1}^{1} \frac{d\sigma(t)}{(1-t)^n}. \tag{8}$$

In this paper, we consider the principal branch of the square root, i.e., $\sqrt{re^{i\theta}} = \sqrt{r} e^{i\frac{\theta}{2}}$, where $r > 0$ and $0 \leq \theta < 2\pi$.

Lemma 1. *Let μ be a positive Borel measure supported on $\mathbb{R}_+$ and suppose that $d\sigma(t) = (1 - t)\, d\mu(\Psi(t))$. Then,*

(a) *$\mu' > 0$ a.e. on $\mathbb{R}_+$ implies that $\sigma' > 0$ a.e. on $[-1, 1]$,*

(b) *if $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\eta_n}} = +\infty$, then $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\varsigma_n}} = +\infty$,*

where, as in (1), η_n denotes the nth moment of the measure $d\mu$.

Proof. To prove the first assertion note that if $d\sigma(t) = \dfrac{2}{1-t}\,\mu'(\Psi(t))dt$, then

$$\frac{d\sigma}{dt} = (1-t)\frac{d\mu(\Psi(t))}{dt} = \frac{2}{1-t}\mu'(\Psi(t)) > 0 \quad \text{a.e. on } [-1,1].$$

The second part is derived using the change of variable $t = \Psi^{-1}(x)$ in the integral

$$\begin{aligned} \varsigma_n &= \int_{-1}^{1} \frac{(1-t)}{(1-t)^n} d\mu(\Psi(t)) = \int_0^{\infty} \left(\frac{x+1}{2}\right)^{n-1} d\mu(x) \\ &= \int_0^{1} \left(\frac{x+1}{2}\right)^{n-1} d\mu(x) + \int_1^{\infty} \left(\frac{x+1}{2}\right)^{n-1} d\mu(x) \\ &\leq \eta_0 + \int_1^{\infty} x^{n-1} d\mu(x) \leq \eta_0 + \eta_n. \end{aligned} \tag{9}$$

As $\sum_{n=0}^{\infty}(\eta_n)^{-1/2n} = +\infty$, from (9) we have $\sum_{n=0}^{\infty}(\eta_0+\eta_n)^{-1/2n} = +\infty$, then $\sum_{n=0}^{\infty}(\varsigma_n)^{-1/2n} = +\infty$. □

Lemma 2. *Assume that* $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$, $r_k(x) = \left(\dfrac{x+1}{2}\right)^k$ *and consider the modification* $d\nu_{r_k}(x) = r_k(x)d\nu(x)$. *Then* $d\nu_{r_k}(x) \in \mathfrak{M}'[\mathbb{R}_+]$ *for all* $k \in \mathbb{Z}$.

Proof. We now proceed by induction. Obviously, the initial case $k = 0$ is given by hypothesis.

- *Case* $k > 0$. Assume that $d\nu_{r_j}(x) \in \mathfrak{M}'[\mathbb{R}_+]$ for all $j \leq k-1$. Since $d\nu_{r_k}(x) = \left(\dfrac{x+1}{2}\right) d\nu_{r_{k-1}}(x)$, it is immediate that $d\nu_{r_k}(x)$ is positive and $\frac{d\nu_{r_k}(x)}{dx} > 0$ a.e. on $\mathbb{R}_+$.

  Let $m_{n,k}$ be the nth moment of the measure $d\nu_{r_k}(x)$, then

$$\begin{aligned} m_{n,k} &= \int_0^{\infty} x^n d\nu_{r_k}(x) = \int_0^1 x^n \left(\frac{x+1}{2}\right) d\nu_{r_{k-1}}(x) + \int_1^{\infty} x^n \left(\frac{x+1}{2}\right) d\nu_{r_{k-1}}(x), \\ &\leq \int_0^1 d\nu_{r_{k-1}}(x) + \int_1^{\infty} x^{n+1} d\nu_{r_{k-1}}(x) \leq m_{0,k-1} + m_{n+1,k-1}, \end{aligned}$$

where we use that $x^n\left(\dfrac{x+1}{2}\right) \leq 1$ for $x \in [0,1]$ and $\left(\dfrac{x+1}{2}\right) \leq x$, for $x \in [1,+\infty)$. Then, using induction hypothesis, we obtain that $m_{n,k} < \infty$ and the sequence of moments for $d\nu_{r_k}(x)$ satisfies Carleman's condition.

- *Case* $k < 0$. Repeating the previous arguments, we obtain that if $d\nu_{r_j}(x) \in \mathfrak{M}'[\mathbb{R}_+]$ for all $0 < j \leq k+1$ then $d\nu_{r_k}(x)$ is positive and $\frac{d\nu_{r_k}(x)}{dx} > 0$ a.e. on $\mathbb{R}_+$.

  For the nth moment of the measure $d\nu_{r_k}(x)$, we have

$$\begin{aligned} m_{n,k} &= \int_0^{\infty} x^n d\nu_{r_k}(x) = \int_0^1 x^n \left(\frac{2}{x+1}\right) d\nu_{r_{k+1}}(x) + \int_1^{\infty} x^n \left(\frac{2}{x+1}\right) d\nu_{r_{k+1}}(x) \\ &\leq 2\, m_{0,k+1} + m_{n,k+1}, \end{aligned}$$

where we use that $x^n\left(\dfrac{2}{x+1}\right) \leq 2$ for $x \in [0,1]$ and $\left(\dfrac{2}{x+1}\right) \leq 1$, for $x \in [1,+\infty)$. Then, using induction hypothesis, we obtain that $m_{n,k} < \infty$ and the sequence of moments for $d\nu_{r_k}(x)$ satisfies Carleman's condition. □

Lemma 3. *[7], Th. 4, Cor. 1. Let $P_{n,k}$ be the kth monic orthogonal polynomial with respect to $d\sigma_n$. If $\sigma' > 0$ a.e. on $[-1,1]$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{\varsigma_n}} = +\infty$, then, for each integer k*

$$\frac{P_{n,n-k+1}}{P_{n,n-k}}(z) \underset{n}{\rightrightarrows} \frac{\varphi(z)}{2}; \quad K \subset \mathbb{C} \setminus [-1,1],$$

where $\varphi(z) = z + \sqrt{z^2-1}$ $\left(\left|z + \sqrt{z^2-1}\right| > 1 \quad z \in \mathbb{C} \setminus [-1,1]\right)$.

Lemma 4. *Assume $\mu \in \mathfrak{M}'[\mathbb{R}_+]$ and $d\mu_m(x) = \left(\frac{2}{x+1}\right)^{2m} d\mu(x)$, with $m \in \mathbb{Z}_+$.*

(a) Let $\ell_{m,n}$ be the nth orthogonal polynomial with respect to μ_m, normalized by the condition $\ell_{m,n}(-1) = (-1)^n$, then for $d \in \mathbb{Z}_+$, on $K \subset \mathbb{C} \setminus \mathbb{R}_+$ it holds

$$\frac{\ell^{(d)}_{m,n+m}(z)}{\ell^{(d)}_{k,n+k}(z)} \underset{n}{\rightrightarrows} \left(\frac{z+1}{4}\right)^{m-k} \Phi^{m-k}(z) = \left(\frac{\sqrt{z}+i}{2}\right)^{2(m-k)}. \tag{10}$$

(b) Let $L_{m,n}$ be the nth monic orthogonal polynomial with respect to μ_m, then on $K \subset \mathbb{C} \setminus \mathbb{R}_+$ it holds

$$\frac{L^{(d)}_{m,n+m}(z)}{L^{(d)}_{k,n+k}(z)} \underset{n}{\rightrightarrows} (z+1)^{m-k}\Phi^{m-k}(z) = \left(\sqrt{z}+i\right)^{2(m-k)}. \tag{11}$$

Proof. *(Proof of a).* Taking $d\sigma_n(t) = (1-t)^{1-2n} d\mu(\Psi(t))$, from the assumptions and Lemma 1, we obtain that $d\sigma_n$ is a finite positive Borel measure on $[-1,1]$, $\sigma'_n > 0$ a.e. on $[-1,1]$ and $\sum_{n=1}^{\infty} \varsigma_n^{-1/(2n)} = +\infty$, where ς_n is as in (8).

Let $P_{n,k}$ be the kth monic orthogonal polynomial with respect to $d\sigma_n$ and denote $\ell^*_{m,n+m}(z) = \left(\frac{z+1}{2}\right)^{n+m} P_{n,n+m}\left(\Psi^{-1}(z)\right)$. After a change of variable $x = \Psi(t)$ in the next integral, we obtain

$$\begin{aligned}
\int_0^{\infty} \left(\frac{x+1}{2}\right)^k \ell^*_{m,n+m}(x) d\mu_m(x) &= \int_{-1}^{1} \frac{1}{(1-t)^{n+m+k}} P_{n,n+m}(t)(1-t)^{2m} d\mu(\Psi(t)) \\
&= \int_{-1}^{1} (1-t)^{n+m-1-k} P_{n,n+m}(t) \frac{d\mu(\Psi(t))}{(1-t)^{2n-1}} \\
&= \int_{-1}^{1} (1-t)^{n+m-1-k} P_{n,n+m}(t)\, d\sigma_n(t) = 0,
\end{aligned} \tag{12}$$

for $k = 0, 1, \cdots, n+m-1$.

$$\ell^*_{m,n+m}(-1) = \lim_{z \to -1} \left(\frac{z+1}{2}\right)^{n+m} P_{n,n+m}\left(\Psi^{-1}(z)\right) = (-1)^{n+m}. \tag{13}$$

From (12) and (13), we have $\ell_{m,n+m} = \ell^*_{m,n+m}$. Therefore,

$$\ell_{m,n+m}(z) = \left(\frac{z+1}{2}\right)^{n+m} P_{n,n+m}\left(\Psi^{-1}(z)\right), \tag{14}$$

$$\begin{aligned}
\frac{\ell_{m,n+m}(z)}{(1+z)^{m-k}\, \ell_{k,n+k}(z)} &= \frac{P_{n,n+m}\left(\Psi^{-1}(z)\right)}{2^{m-k} P_{n,n+k}\left(\Psi^{-1}(z)\right)} \\
&= \frac{1}{2^{m-k}} \prod_{j=k}^{m-1} \frac{P_{n,n+j+1}\left(\Psi^{-1}(z)\right)}{P_{n,n+j}\left(\Psi^{-1}(z)\right)}.
\end{aligned}$$

From Lemma 3, for $j = k, \dots, m-1$;

$$\frac{P_{n,n+j+1}\left(\Psi^{-1}(z)\right)}{P_{n,n+j}\left(\Psi^{-1}(z)\right)} \underset{n}{\rightrightarrows} \frac{\varphi\left(\Psi^{-1}(z)\right)}{2}; \quad K \subset \mathbb{C} \setminus \mathbb{R}_+.$$

Thus,

$$\frac{\ell_{m,n+m}(z)}{\ell_{k,n+k}(z)} \underset{n}{\rightrightarrows} \left(\frac{z+1}{4}\right)^{m-k} \varphi^{m-k}\left(\Psi^{-1}(z)\right); \quad K \subset \mathbb{C} \setminus \mathbb{R}_+,$$

which establishes (10) for $d = 0$. In order to proof (10) for $d > 0$, we proceed by induction on d.

$$\frac{\ell^{(d+1)}_{m,n+m}(z)}{\ell^{(d+1)}_{k,n+k}(z)} = \frac{\ell^{(d)}_{m,n+m}(z)}{\ell^{(d)}_{k,n+k}(z)} + \frac{\ell^{(d)}_{k,n+k}(z)}{\ell^{(d+1)}_{k,n+k}(z)} \cdot \left(\frac{\ell^{(d)}_{m,n+m}(z)}{\ell^{(d)}_{k,n+k}(z)}\right)'.$$

Assume that formula (10) holds for $d \in \mathbb{Z}_+$, then $\left(\ell^{(d)}_{m,n+m}/\ell^{(d)}_{0,n}\right)'$ is uniformly bounded on compact subsets $K \subset \mathbb{C} \setminus \mathbb{R}_+$. Note that $\ell^{(d)}_{0,n}/\ell^{(d+1)}_{0,n} \underset{n}{\rightrightarrows} 0$ on $K \subset \mathbb{C} \setminus \mathbb{R}_+$. This is proved using an analogous of ([3] (2.9)), and the Bell's polynomials version of the Faa Di Bruno formula, see ([10] pp. 218, 219). The assertion (a) is proved.

(Proof of b). Write $f_{d,m,n}(z) = \dfrac{\ell^{(d)}_{m,n+m}(z)}{z^m\ \ell^{(d)}_{0,n}(z)}$ and let $\kappa_{m,n+m}$ be the leading coefficient of $\ell_{m,n+m}$. Hence, for $d > 1$

$$f_{d,m,,k,n}(\infty) = \frac{(n+m)\cdots(n+m-d+1)\kappa_{m,n+m}}{(n+k)\cdots(n+k-d+1)\kappa_{k,n+k}}$$
$$f_{0,m,k,n}(\infty) = \frac{\kappa_{m,n+m}}{\kappa_{k,n+k}}.$$

From (10),

$$f_{d,m,k,n}(z) \underset{n}{\rightrightarrows} \left(\frac{z+1}{4z}\right)^{m-k} \Phi^{m-k}(z); \quad K \subset \overline{\mathbb{C}} \setminus \mathbb{R}_+, \quad l \in \mathbb{Z}_+. \tag{15}$$

$$\lim_{n\to\infty} f_{d,m,k,n}(\infty) = \lim_{n\to\infty} \frac{\kappa_{m,n+m}}{\kappa_{k,n+k}} = \left(\frac{1}{2}\right)^{2(m-k)}. \tag{16}$$

As $L^{(d)}_{m,n+m}(z) = \frac{\ell^{(d)}_{m,n+m}(z)}{\kappa_{m,n+m}}$ for $d \geq 1$, from (15) and (16), we get (11). □

Denote by $\mathfrak{M}[-1,1]$ the class of admissible measures in $[-1,1]$ defined in ([11] Sec. 5). Let σ_n a positive varying Borel measure supported on $[-1,1]$ and

$$p_{n,m}(w) = \tau_{n,m}w^m + \cdots, \quad \tau_{n,m} > 0$$

be the mth orthonormal polynomial with respect to σ_n, then ([11] Th. 7)

$$\lim_{n\to\infty} \frac{\tau_{n,n+k+1}}{\tau_{n,n+k}} = 2, \qquad k \in \mathbb{Z}. \tag{17}$$

Lemma 5. *Let σ_n be an admissible measure, then for all $v \in \mathbb{Z}$,*

$$\int_{-1}^{1} \frac{p_{n,n+v}(t)p_{n,n}(t)}{w-t} d\sigma_n(t) \underset{n}{\rightrightarrows} \frac{1}{\varphi^{|v|}(w)\sqrt{w^2-1}}; \qquad K \subset \mathbb{C} \setminus [-1,1]. \tag{18}$$

Proof. This proof is based on the proof of ([3] Lemma 2). Without loss of generality, let us consider $v \in \mathbb{Z}_+$. Applying the Cauchy–Schwarz inequality we have, for $z \in K \subset \overline{\mathbb{C}} \setminus [-1,1]$

$$\left| \int_{-1}^{1} \frac{p_{n,n+v}(t) p_{n,n}(t)}{w-t} d\sigma_n(t) \right| \leq \frac{1}{d(K,[-1,1])} < \infty,$$

where $d(K,[-1,1])$ denotes the Euclidian distance between the two sets. Thus, for (fixed) values of $v \in \mathbb{Z}_+$, the sequence of functions in the left hand side of (18) is normal. Thus, we deduce uniform convergence from pointwise convergence. The pointwise limit follows from ([11] Th. 9)

$$\lim_{n\to\infty} \int_{-1}^{1} \frac{p_{n,n+v}(t) p_{n,n}(t)}{w-t} d\sigma_n(t) = \frac{1}{\pi} \int_{-1}^{1} \frac{T_v(t)}{w-t} \frac{dt}{\sqrt{1-t^2}},$$

here, T_v is the vth Chebyshev orthonormal polynomial of the first kind. Therefore, (18) holds if we prove that

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_v(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\varphi^v(w)\sqrt{w^2-1}}. \tag{19}$$

Note that $T_0(t) = 1, T_1(t) = x$, and, for $v \leq 1$,

$$2tT_v(t) = T_{v+1}(t) + T_{v-1}(t),$$

or equivalently

$$T_{v+1} = 2tT_v - T_{v-1}. \tag{20}$$

Next, proceed by induction. Start at $v = 0$, expression (18), is obtained from the residue theorem and Cauchy's integral formula. Then, for $v = 1$ we have

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^{1} \frac{T_1(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} &= \frac{w}{\pi} \int_{-1}^{1} \frac{1}{w-t} \frac{dt}{\sqrt{1-t^2}} - \frac{1}{\pi} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{w}{w^2-1} - 1 = \frac{1}{\varphi(w)\sqrt{w^2-1}}. \end{aligned}$$

Now, assume (19) holds for $v = 0, 1, \dots, k$; $k \geq 1$, we will prove that it also holds for $v = k+1$. Combining (20) and the hypothesis of induction, we obtain

$$\begin{aligned} \frac{1}{\pi} \int_{-1}^{1} \frac{T_{k+1}(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} &= \frac{1}{\pi} \int_{-1}^{1} \frac{2tT_k(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} - \frac{1}{\pi} \int_{-1}^{1} \frac{T_{k-1}(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{2z}{\pi} \int_{-1}^{1} \frac{T_k(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} - \frac{1}{\pi} \int_{-1}^{1} \frac{T_{k-1}(t)}{w-t} \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{\varphi^{k-1}(w)\sqrt{w^2-1}} \left(\frac{2w}{\varphi(z)} - 1 \right) \\ &= \frac{1}{\varphi^{k+1}(w)\sqrt{w^2-1}}, \end{aligned}$$

which we wanted to prove. □

Lemma 6. *Let* $d\mu(x) = \left(\frac{x+1}{2}\right)^{A-B} d\nu(x)$, *where* $A, B \in \mathbb{Z}_+$, *and* $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$. *We have on compact subsets of* $\mathbb{C} \setminus \mathbb{R}_+$

$$(v-1)!\tau_{n,n-B}^2 \int_0^\infty \left(\frac{x+1}{2}\right)^k \frac{\ell_{A-k,n+A-k}(x)\ell_{-B,n-B}(x)}{(x-z)^v} d\nu(x)$$
$$\underset{n}{\rightrightarrows} \left(\frac{-1}{(1+z)(2\Phi(z))^{A+B-k}\sqrt{(\Psi^{-1}(z))^2-1}}\right)^{(v-1)}.$$

where $\ell_{n,n+m}$ *is defined as in Lemma 4.*

Proof. First, the sequence $\{\ell_{n,n+m}\}_{n\geq 0}$ is well defined because the measure $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$, implies $d\mu \in \mathfrak{M}'[\mathbb{R}_+]$ (see Lemma 2).

Let us use the connection formula (14) and the change of variable (7) to obtain

$$(v-1)!\tau_{n,n-B}^2 \int_0^\infty \left(\frac{x+1}{2}\right)^k \frac{\ell_{A-k,n+A-k}(x)\ell_{-B,n-B}(x)}{(x-z)^v} d\nu(x),$$
$$= (v-1)!\tau_{n,n-B}^2 \int_{-1}^1 \frac{P_{n,n+A-k}(t)P_{n,n-B}(t)}{(\Psi(t)-z)^v} \frac{d\sigma(t)}{(1-t)^{2n+A-B}},$$
$$f_n^{(v-1)}(z) = \frac{(v-1)!\tau_{n,n-B}}{\tau_{n,n+A-k}} \int_{-1}^1 \frac{1}{1-t} \frac{p_{n,n+A-k}(t)p_{n,n-B}(t)}{(\Psi(t)-z)^v} d\sigma_n(t).$$

where we use

$$d\sigma_n(t) = \frac{d\mu(\Psi(t))}{(1-t)^{2n-1}} = \frac{(1-t)^{B-A}d\nu(\Psi(t))}{(1-t)^{2n-1}}$$

Take the $(v-1)$ primitive with respect to z of the previous expression

$$f_n(z) = \frac{\tau_{n,n-B}}{\tau_{n,n+A-k}} \int_{-1}^1 \frac{1}{1-t} \frac{p_{n,n+A-k}(t)p_{n,n-B}(t)}{\Psi(t)-z} d\sigma_n(t). \tag{21}$$

Since we know that

$$(1-t)(\Psi(t)-z) = (1+z)\left(t-\Psi^{-1}(z)\right),$$

we rewrite (21) as

$$\frac{\tau_{n,n-B}^2}{1+z} \int_{-1}^1 \frac{P_{n,n+A-k}(t)P_{n,n-B}(t)}{t-\Psi^{-1}(z)} d\sigma_n(t),$$
$$= \frac{\tau_{n,n-B}}{(1+z)\tau_{n,n+A-k}} \int_{-1}^1 \frac{p_{n,n+A-k}(t)p_{n,n-B}(t)}{t-\Psi^{-1}(z)} d\sigma_n(t).$$

Then, we use Lemma 5 and (17) to obtain on compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$,

$$\frac{\tau_{n,n-B}}{(1+z)\tau_{n,n+A-k}} \int_{-1}^1 \frac{p_{n,n+A-k}(t)p_{n,n-B}(t)}{t-\Psi^{-1}(z)} d\sigma_n(t)$$
$$\underset{n}{\rightrightarrows} \left(\frac{-1}{(1+z)(2\varphi(\Psi^{-1}(z)))^{A+B-k}\sqrt{(\Psi^{-1}(z))^2-1}}\right) = f(z).$$

Note that by the Cauchy–Schwarz inequality we have for $z \in \mathbb{C} \setminus \mathbb{R}_+$

$$\begin{aligned}\left|f_n^{(v-1)}(z)\right| &= \left|\frac{(v-1)!\tau_{n,n-B}}{\tau_{n,n+A-k}} \int_{-1}^{1} \frac{1}{1-t} \frac{p_{n,n+B-k}(t)p_{n,n-A}(t)}{(\psi(t)-z)^v} d\sigma_n(t)\right| \\ &\leq \frac{B}{d(K,\mathbb{R}_+)}.\end{aligned}$$

Then, for each v, the family $\left\{f_n^{(v-1)}\right\}_n$ is uniformly bounded in each $K \subset \mathbb{C} \setminus \mathbb{R}_+$, which means by Montel's theorem (c.f. [12], §5.4, Th. 15) that $\left\{f_n^{(v-1)}\right\}_{n\geq 0}$ is normal (see ([12] §5.1 Def. 2)), i.e., we have that from each sequence $\mathbf{N} \subset \mathbb{N}$ we can take a subsequence $\mathbf{N}_1 \subset \mathbf{N}$ such that

$$f_n^{(v)} \underset{n}{\rightrightarrows} g_{(v)}; \qquad n \in \mathbf{N}_1, \quad K \subset \mathbb{C} \setminus \mathbb{R}_+.$$

Now, taking the $(v-1)$ derivative and using the uniqueness of the limit we obtain

$$\begin{aligned}&\frac{(v-1)!\tau_{n,n-B}}{\tau_{n,n+A-k}} \int_{-1}^{1} \frac{1}{1-t} \frac{p_{n,n+A-k}(t)p_{n,n-B}(t)}{(\Psi(t)-z)^v} d\sigma_n(t) \\ &\qquad \underset{n}{\rightrightarrows} \left(\frac{-1}{(1+z)(2\Phi(z))^{A+B-k}\sqrt{(\Psi^{-1}(z))^2-1}}\right)^{(v-1)} = f^{(v-1)}(z),\end{aligned}$$

on compact subsets $K \subset \mathbb{C} \setminus \mathbb{R}_+$, which establishes the formula. □

3. Relative Asymptotic within Certain Class of Varying Measures

In this section, we obtain the asymptotic relation between orthogonal polynomials with respect to different measures of the class $\left(\frac{x+1}{2}\right)^m d\mu(x)$, where μ is any measure of $\mathfrak{M}'[\mathbb{R}_+]$ and $m \in \mathbb{Z}$. Note that, because of Lemma 2, the elements of this class belong to $\mathfrak{M}'[\mathbb{R}_+]$.

To maintain a general tone in the expositions in this section we use μ and ν as two measures in $\mathfrak{M}'[\mathbb{R}_+]$ having no relation with the previous use of the notation.

Consider $m \in \mathbb{Z}_+$ and let $h_{m,n}(z)$ be the nth orthogonal polynomial with respect to $\left(\frac{x+1}{2}\right)^m d\nu(x)$, normalized as $h_{m,n}(-1) = (-1)^n$. Consider the following relations

$$\int_0^\infty \left(\frac{x+1}{2}\right)^k h_{0,n}(x)d\nu(x) = 0,$$

for $k = 0, \ldots, n-1$. Apply the change of variable $\Psi(t) = z$ given in (7) to obtain

$$\begin{aligned}0 &= \int_{-1}^{1} \left(\frac{1}{1-t}\right)^k h_{0,n}(\Psi(t))d\nu(\Psi(t)) \\ &= \int_{-1}^{1} (1-t)^{n-k-1}(1-t)^n h_{0,n}(\Psi(t)) \frac{(1-t)d\nu(\Psi(t))}{(1-t)^{2n}}.\end{aligned}$$

Note that the polynomial $H_{n,n}(t) = (1-t)^n h_{0,n}(\Psi(t))$ is the nth monic orthogonal polynomial with respect to the varying measure modified by a polynomial term

$$(1-t)d\sigma_n^*(t) = \frac{(1-t)d\nu(\Psi(t))}{(1-t)^{2n}}.$$

Following the same reasoning, we obtain that

$$H_{n,n}^*(t) = (1-t)^n h_{1,n}(\Psi(t)),$$

is the nth monic orthogonal polynomial with respect to $d\sigma_n^*(t)$. It is not hard to prove that the system $\{\sigma, \{(1-t)^{2n}\}, 0\}$ is an admissible system, see ([11] Def. p 213). Therefore, by ([11] Th. 10), we have

$$\frac{H_{n,n}(t)}{H_{n,n}^*(t)} \underset{n}{\rightrightarrows} \frac{\varphi(t)-\varphi(1)}{t-1}; \qquad K \subset \mathbb{C} \setminus [-1,1]. \tag{22}$$

Theorem 2. *Under the previous hypothesis we have on compact subsets of* $\mathbb{C} \setminus \mathbb{R}_+$

$$\frac{h_{0,n}(z)}{h_{1,n}(z)} \underset{n}{\rightrightarrows} \left(\frac{z+1}{4}\right)(1-\Phi(z)), \tag{23}$$

$$\frac{h_{v,n}(z)}{h_{w,n}(z)} \underset{n}{\rightrightarrows} \left(\frac{z+1}{4}\right)^{w-v} (1-\Phi(z))^{w-v}, \tag{24}$$

where $v, w \in \mathbb{Z}$.

Proof. From (22) and taking the change of variable (7) we have

$$\begin{aligned} \frac{h_{0,n}(\Psi(t))}{h_{1,n}(\Psi(t))} &= \frac{(1-t)^n h_{0,n}(\Psi(t))}{(1-t)^n h_{1,n}(\Psi(t))} \\ &= \frac{H_{n,n}(t)}{H_{n,n}^*(t)} \underset{n}{\rightrightarrows} \frac{\varphi(t)-\varphi(1)}{t-1} = \frac{\Phi^{-1}(z)-1}{\Psi(z)-1}. \end{aligned}$$

To prove (24), note that from Lemma 2.

$$d\mu_k = \left(\frac{x+1}{2}\right)^k d\mu \in \mathfrak{M}'[\mathbb{R}_+] \text{ if } \mu \in \mathfrak{M}'[\mathbb{R}_+].$$

The only hypothesis needed to obtain (23) is $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$. Thus if we let now $d\nu = \left(\frac{x+1}{2}\right)^k d\mu = d\mu_k$, then $\left(\frac{x+1}{2}\right) d\nu = \left(\frac{x+1}{2}\right)^{k+1} d\mu = d\mu_{k+1}$, where $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$.

Therefore, $h_{0,n} = \mathfrak{h}_{k,n}$ and $h_{1,n} = \mathfrak{h}_{k+1,n}$, where $\mathfrak{h}_{k,n}$ and $\mathfrak{h}_{k+1,n}$ are the orthogonal polynomials with respect to the measures $d\mu_k$ and $d\mu_{k+1}$, respectively, normalized by having the value $(-1)^k$ at -1. Therefore, we have

$$\frac{\mathfrak{h}_{k,n}(z)}{\mathfrak{h}_{k+1,n}(z)} \underset{n}{\rightrightarrows} -\left(\frac{z+1}{4}\right)(\Phi(z)-1). \tag{25}$$

Note that, without loss of generality, we can asume $w > v$, otherwise the relation between the measures can be reverted, and they still belong to $\mathfrak{M}'[\mathbb{R}_+]$. Stack formula (25) as

$$\frac{\mathfrak{h}_{v_1,n}(z)}{\mathfrak{h}_{w_1,n}} = \frac{\mathfrak{h}_{v_1,n}(z)}{\mathfrak{h}_{v_1+1,n}} \cdot \frac{\mathfrak{h}_{v_1+1,n}(z)}{\mathfrak{h}_{v_1+2,n}} \cdot \dots \cdot \frac{\mathfrak{h}_{w_1-1,n}(z)}{\mathfrak{h}_{w_1,n}},$$

where $v_1 = v + k$ and $w_1 = w + k$. Since the measure $\mu \in \mathfrak{M}'[\mathbb{R}_+]$, (24) holds. □

4. Asymptotic for Orthogonal Polynomials with Respect to a Measure Modified by a Rational Factor

Let $r = \alpha/\beta$, after canceling out common factors, where

$$\begin{aligned} &\alpha(z) = \prod_{i=1}^{N_1}(z-a_i)^{A_i}, \quad \beta(z) = \prod_{j=1}^{N_2}(z-b_j)^{B_j}, \\ &a_i \in \mathbb{C} \setminus (\mathbb{R}_+ \cup \{-1\}),\ b_j \in \mathbb{C} \setminus \mathbb{R}_+,\ A_i, B_j \in \mathbb{N}, \\ &A = \sum_{i=1}^{N_1} A_i, \qquad B = \sum_{j=1}^{N_2} B_j. \end{aligned} \tag{26}$$

Given a measure $\nu \in \mathfrak{M}'[\mathbb{R}_+]$, denote by $d\mu(x) = \left(\frac{x+1}{2}\right)^{A-B} d\nu(x)$ a modified measure, note that according to Lemma 2 it holds $\nu \in \mathfrak{M}'[\mathbb{R}_+]$.

Assume S_n is the polynomial of least degree not identically equal to zero, such that

$$0 = \int_0^\infty p(x) S_n(x) r(x)\, d\nu(x), \qquad p \in \mathbb{P}_{n-1}, \tag{27}$$

normalized such that $S_n(-1) = (-1)^n$, and L_n is the nth orthogonal polynomial with respect to $d\nu$, normalized such that $L_n(-1) = (-1)^n$. We are interested in the asymptotic behavior of $S_n/L_n, n \in \mathbb{Z}_+$ in compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$.

Theorem 3. *Let $\mu \in \mathfrak{M}'[\mathbb{R}_+]$ and α and β defined as before. Then for all sufficiently large n, for all fixed $d \in \mathbb{Z}_+$, in compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$, it holds*

$$\frac{S_n(z)}{\ell_{0,n}(z)} \underset{n}{\rightrightarrows} \frac{(-1)^A \alpha(-1)}{4^A (z+1)^{-A}} \prod_{i=1}^{N_1} \left(\frac{\Phi(z) - \Phi(a_i)}{z - a_i}\right)^{A_i} \prod_{j=1}^{N_2} \left(1 - \frac{1}{\Phi(z)\Phi(b_j)}\right)^{B_j}. \tag{28}$$

Proof. First we focus on (27) for $\alpha(x) = \left(\dfrac{x+1}{2}\right)^k \beta(x)$ where $k = 0, \ldots, n-B-1$, we have

$$0 = \int_0^\infty \left(\frac{x+1}{2}\right)^k S_n(x) \alpha(x) d\nu(x),$$

now, using the change of variables (7) and considering the expression $d\mu(\Psi(t)) = (1-t)^{B-A} d\nu(\Psi(t))$, the previous integral becomes

$$0 = \int_{-1}^1 (1-t)^{n-B-k-1} (1-t)^{n+A} S_n(\Psi(t))\, \alpha(\Psi(t))\, \frac{d\mu(\Psi(t))}{(1-t)^{2n-1}}. \tag{29}$$

for $k = 0, \ldots, n-B-1$. Define the $(n+A)$-degree polynomial R_{n+A} as

$$R_{n+A}(t) := (1-t)^{n+A} S_n(\Psi(t))\, \alpha(\Psi(t)).$$

Thus, we can consider $d\sigma_n(t) = \frac{d\sigma(t)}{(1-t)^{2n-1}}$ with $d\sigma(t) = d\nu(\Psi(t))$. The measure $d\sigma_n(t)$ defines a varying orthogonal polynomial system, satisfying Lemma 3. We denote by $P_{n,n+A-k}$ the $(n+A-k)$th monic orthogonal polynomial with respect to $d\sigma_n(t)$. According to (29), we have the following quasi-orthogonality of order $n-A$

$$R_{n+A}(t) := (1-t)^{n+A} S_n(\Psi(t))\, \alpha(\Psi(t)) \; = \sum_{k=0}^{A+B} \lambda_{n,k} P_{n,n+A-k}(t). \tag{30}$$

Back to (30), we use the connection formula (14) and the change of variables (7) to obtain

$$\begin{aligned}
\left(\frac{2}{z+1}\right)^{n+A} S_n(z)\alpha(z) &= \sum_{k=0}^{A+B} \lambda_{n,k} P_{n,n+A-k}\left(\Psi^{-1}(z)\right) \\
&= \sum_{k=0}^{A+B} \lambda_{n,k} \left(\frac{2}{z+1}\right)^{n+A-k} \ell_{A-k,n+A-k}(z), \\
S_n(z)\alpha(z) &= \sum_{k=0}^{A+B} \lambda_{n,k} \left(\frac{z+1}{2}\right)^k \ell_{A-k,n+A-k}(z).
\end{aligned} \tag{31}$$

Note that $\lambda_{n,0} = \lambda_0 = (-1)^A \alpha(-1)$ or S_n has deg $S_n < n$. Dividing this relation by $\ell_{-B,n-B}$ we get

$$\frac{S_n(z)\alpha(z)}{\ell_{-B,n-B}(z)} = \sum_{k=0}^{A+B} \lambda_{n,k} \left(\frac{z+1}{2}\right)^k \frac{\ell_{A-k,n+A-k}(z)}{\ell_{-B,n-B}(z)}. \tag{32}$$

Set $\lambda_{n,k}^{**} = \lambda_{n,k}/\lambda_0$, $\lambda_n^* = \left(\sum_{k=0}^{A+B} |\lambda_{n,k}^{**}|\right)^{-1} < \infty$ and introduce the polynomials

$$p_n(z) = \sum_{k=0}^{A+B} \lambda_{n,k}^{**} z^{A+B-k}, \qquad p_n^* = \lambda_n^* p_n(z).$$

We will prove that

$$p_n(z) \underset{n}{\rightrightarrows} \hat{p}(z) = \prod_{i=1}^{N_1}\left(z - \frac{\Phi(a_i)}{2}\right) \prod_{j=1}^{N_2}\left(z - \frac{1}{2\Phi(b_j)}\right); \qquad K \subset \mathbb{C}.$$

To this end, it suffices to show that

$$p_n^*(z) \underset{n}{\rightrightarrows} c\hat{p}(z) = c\left(z^{A+B} + \lambda_1^{**} z^{A+B-1} + \cdots + \lambda_{A+B}^{**}\right), \tag{33}$$

where

$$c = \lim_{n\to\infty} \lambda_n^* = \left(\sum_{k=0}^{A+B} |\lambda_k|\right)^{-1}. \tag{34}$$

Now, note that $\{p_n^*\}$, for $n \in \mathbb{Z}_+$ is contained in $\mathbb{P}_{A+B}$ and the sum of the coefficients of p_n^* for each $n \in \mathbb{Z}_+$, is equal to one. Therefore, this family of polynomials is normal. This means that (33) can be prove if we check that, for all $\Lambda \subset \mathbb{Z}_+$ such that

$$\lim_{\substack{n\to\infty \\ n\in\Lambda}} p_n^*(z) = p_\Lambda, \tag{35}$$

$p_\Lambda(z) = c\hat{p}(z)$, where $\hat{p}(z)$ and c are defined as above. Since $p_\Lambda \in \mathbb{P}_{A+B}$ and $p_\Lambda \not\equiv 0$, we can uniquely determine p_Λ if we find its zeros and leading coefficient. Note that the leading coefficient of p_Λ is positive and the sum of the absolute value of its coefficients is one. Therefore, we conclude that the leading coefficient is uniquely determined by the zeros. This automatically implies that $p_\Lambda(z) = c\hat{p}(z)$ if and only if it is divisible by $\hat{p}(z)$.

Note that the factor β is in (32) and all the zeros of $\ell_{-B,n-B}$ concentrate on $\mathbb{R}_+$. Thus, we immediately obtain the following A equations, for $n \geq n_0$:

$$0 = \sum_{k=0}^{A+B} \lambda_n^* \lambda_{n,k}^{**} \left[\left(\frac{z+1}{2}\right)^k \left(\frac{\ell_{A-k,n+A-k}}{\ell_{-B,n-B}}\right)\right]^{(v)}(a_i),$$

for $i = 1, \ldots, N_1$ and $v = 0, \ldots, A_j - 1$.

From Lemma 4 it follows that, for compact subsets $K \subset \mathbb{C} \setminus \mathbb{R}_+$, it holds

$$\left[\left(\frac{z+1}{2}\right)^k \left(\frac{\ell_{n+A,n+A-k}(z)}{\ell_{-B,n-B}(z)}\right)\right]^{(v)} \underset{n}{\rightrightarrows} \left[\left(\frac{z+1}{2}\right)^{A+B} \left(\frac{\Phi(z)}{2}\right)^{A+B-k}\right]^{(v)}. \tag{36}$$

Relations (35) and (36), together with the fact that Φ is holomorphic with $\Phi' \neq 0$ in $\mathbb{C} \setminus \mathbb{R}_+$, imply, using induction on v, that

$$p_\Lambda^{(v)}\left(\frac{\Phi(a_i)}{2}\right) = 0, \qquad i = 1, \ldots, N_1, \quad v = 0, \ldots, A_i - 1; \tag{37}$$

$$p_\Lambda(z) = c\left(\frac{z+1}{2}\right)^{A+B}\sum_{k=0}^{A+B}\lambda_k^{**}\left(\frac{\Phi(z)}{2}\right)^{A+B-k}.$$

On the other hand, take $p(z) = \beta(z)\ell_{-B,n-B}(z)/(z-b_j)^v$ in (27), $j = 1,\dots,N_2$; $v = 1,\dots,B_j$. Using (31) and multiplying by $(v-1)!\frac{\lambda_n^*}{\lambda_0}\tau^2_{n,n-B}$ we have the additional relations

$$\begin{aligned}
0 =&\frac{\lambda_n^*}{\lambda_0}\tau^2_{n,n-B}\int_0^\infty \frac{(v-1)!}{(x-b_j)^v}\,\ell_{-B,n-B}(x)S_n(x)\alpha(x)d\nu(x),\\
=&\tau^2_{n,n-B}\int_0^\infty \frac{(v-1)!}{(x-b_j)^v}\,\ell_{-B,n-B}(x)\\
&\sum_{k=0}^{A+B}\lambda_n^*\lambda_{n,k}^{**}\left(\frac{x+1}{2}\right)^k\ell_{A-k,n+A-k}(x)d\nu(x),\\
0 =& \sum_{k=0}^{A+B}\lambda_n^*\lambda_{n,k}^{**}(v-1)!\tau^2_{n,n-B}\\
&\int_0^\infty\left(\frac{x+1}{2}\right)^k\frac{\ell_{A-k,n+A-k}(x)\ell_{-B,n-B}(x)}{(x-b_j)^v}d\nu(x),
\end{aligned}\tag{38}$$

for each b_j.

Relations (33), (38) and Lemma 6 together with the fact that $1/\Phi$ is holomorphic with $(1/\Phi)' \neq 0$ and $1/\sqrt{(\psi^{-1}(z))^2-1} \neq 0$ in $\mathbb{C}\setminus\mathbb{R}_+$, give by induction

$$p_\Lambda^{(v)}\left(\frac{1}{2\Phi(b_j)}\right) = 0, \qquad j = 1,\dots,N_2, \qquad v = 0,\dots,B_j-1.$$

From the previous expression and (37) it follows that p_Λ is divisible by $p_0(z)$. Therefore (33) and (34) hold and

$$p_n(z) \underset{n}{\rightrightarrows} p_0(z), \qquad K\subset\mathbb{C}.$$

From the previous expression, the definition of p_n, (32), (36) with $v = 0$, we obtain

$$\frac{S_n(z)\alpha(z)}{\ell_{-B,n-B}(z)} \underset{n}{\rightrightarrows} (-1)^A\alpha(-1)\left(\frac{z+1}{2}\right)^{A+B}\hat{p}\left(\frac{\Phi(z)}{2}\right).$$

Use the asymptotic formula (10) in the previous expression and group conveniently to obtain

$$\begin{aligned}
\frac{S_n(z)}{\ell_{-B,n-B}(z)}\cdot\frac{\ell_{-B,n-B}(z)}{\ell_{0,n}(z)} \underset{n}{\rightrightarrows} &\left(\frac{z+1}{2}\right)^A\frac{(-1)^A\alpha(-1)\Phi(z)^{-B}}{\alpha(z)}\\
&\prod_{i=1}^{N_1}\left(\frac{\Phi(z)-\Phi(a_i)}{2}\right)^{A_i}\prod_{i=1}^{N_2}\left(\frac{\Phi(z)}{2}-\frac{1}{2\Phi(b_j)}\right)^{B_j}
\end{aligned}$$

and (28) follows for $v = 0$. To prove the formula for $d \in \mathbb{Z}_+$, apply the same technique of the proof of Lemma 4. □

Remark 1.

1. *The proof depends on the assumption of $\alpha(-1) \neq 0$, we will remove this restriction in Section 5.*
2. *We suppose that α, β are monic. We can remove that restriction without loss of generality due to the fact that orthogonal polynomial systems are invariant under the constant modification of measures.*

Theorem 3 gives the ratio asymptotic between the orthogonal polynomials with respect to a rational modification of kind $r(x)d\nu(x)$ (a general rational modification with no zeros at -1) denoted as S_n and those orthogonal with respect to a modified measure of type $\left(\frac{x+1}{2}\right)^{A-B}$, denoted as $\ell_{0,n}$.

To obtain the general formula we must find the following limit

$$\lim_{n\to\infty} \frac{\ell_{0,n}(z)}{L_n(z)},$$

on compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$, where $L_n(z)$ is the nth orthogonal polynomial with respect to $d\nu \in \mathfrak{M}'[\mathbb{R}_+]$ normalized such that $L_n(-1) = (-1)^n$.

5. Proof of Theorem 1

Next, we obtain an analogous of (4) for measures with support on $\mathbb{R}_+$. Define $\hat{\alpha}$ as

$$\hat{\alpha}(z) = \left(\frac{z+1}{2}\right)^C \alpha(z)$$

wherein α is defined in (26) and $C \in \mathbb{Z}_+$ is the multiplicity of the zero -1 in $\hat{\alpha}/\beta$. Without loss of generality we can assume that there are more zeros than poles on -1, if not $C = 0$. Also, let L_n be the nth orthogonal polynomial with respect to $d\hat{\nu} \in \mathfrak{M}'[\mathbb{R}_+]$, normalized by the condition $L_n(-1) = (-1)^n$. Denote by Q_n the nth orthogonal polynomial with respect to $\hat{r}d\hat{\nu}$, where $r = \hat{\alpha}/\beta$, normalized as usual, $Q_n(-1) = (-1)^n$.

Note that if $C = 0$, $\hat{r} = r$ and $Q_n = S_n$, as defined in Section 4. Under this notation, (6) is written as

$$\frac{Q_n^{(d)}(z)}{L_n^{(d)}(z)} \underset{n}{\rightrightarrows} \left(\frac{2i}{\sqrt{z}+i}\right)^C \prod_{i=1}^{N_1} \left(\frac{\sqrt{a_i}+i}{\sqrt{z}+\sqrt{a_i}}\right)^{A_i} \prod_{j=1}^{N_2} \left(\frac{\sqrt{z}+\sqrt{b_j}}{\sqrt{b_j}+i}\right)^{B_j},$$

in compact subsets of $\mathbb{C} \setminus \mathbb{R}$, for $d \in \mathbb{Z}_+$.

Proof of Theorem 1. Let us first observe that Q_n is orthogonal with respect to $\left(\frac{x+1}{2}\right)^C \frac{\alpha}{\beta} d\hat{\nu}$. Then if we set

$$d\hat{\nu} = \left(\frac{x+1}{2}\right)^{-C} d\nu, \tag{39}$$

we obtain that Q_n is orthogonal with respect to $\frac{\alpha}{\beta} d\nu$, and satisfies the hypotheses of Theorem 3, thus we have on compact subsets of $\mathbb{C} \setminus \mathbb{R}_+$

$$\frac{Q_n(z)}{\ell_{0,n}(z)} \underset{n}{\rightrightarrows} \mathfrak{F}(z),$$

where $\mathfrak{F}(z)$ is given in (28).

On the other hand, $\ell_{0,n}$ is orthogonal with respect to $\left(\frac{x+1}{2}\right)^{A-B} d\nu$. This means by (39) that $\ell_{0,n}$ is orthogonal with respect to $\left(\frac{x+1}{2}\right)^{A-B+C} d\hat{\nu}$. Thus, taking into account Theorem 2, we have

$$\frac{\ell_{0,n}(z)}{L_n(z)} \underset{n}{\rightrightarrows} \left(\frac{z+1}{4}\right)^{B-A-C} (1-\Phi(z))^{B-A-C}.$$

Multiply the expressions corresponding to

$$\left(\frac{z+1}{4}\right)^{B-A-C} (1-\Phi(z))^{B-A-C} \cdot \mathfrak{F}(z), \tag{40}$$

Let us break down this expression into the following terms

$$\mathfrak{F}(z) = \frac{(-1)^A \alpha(-1)}{4^A (z+1)^{-A}} \prod_{i=1}^{N_1} \left(\frac{\Phi(z) - \Phi(a_i)}{z - a_i} \right)^{A_i} \prod_{i=1}^{N_2} \left(1 - \frac{1}{\Phi(z)\Phi(b_j)} \right)^{B_j}.$$

$$\begin{aligned}
(-1)^A \alpha(-1) &= \prod_{i=1}^{N_1} (1 + a_i)^{A_i} \\
(1 - \phi(z)) &= -\frac{2i}{\sqrt{z} - i} \\
\frac{\Phi(z) - \Phi(a_i)}{z - a_i} &= \frac{-2i}{(\sqrt{z} - i)(\sqrt{a_i} - i)(\sqrt{a_i} + \sqrt{z})} \\
1 - \frac{1}{\Phi(z)\Phi(b_j)} &= \frac{2i\left(\sqrt{b_j} + \sqrt{z}\right)}{(\sqrt{z} + i)\left(\sqrt{b_j} + i\right)}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\prod_{i=1}^{N_1} \left(\frac{\Phi(z) - \Phi(a_i)}{z - a_i} \right)^{A_i} &= \left(\frac{-2i}{\sqrt{z} - i} \right)^A \prod_{i=1}^{N_1} \left(\frac{1}{(\sqrt{a_i} - i)(\sqrt{a_i} + \sqrt{z})} \right)^{A_i} \\
\prod_{j=1}^{N_2} \left(1 - \frac{1}{\Phi(z)\Phi(b_j)} \right)^{B_j} &= \left(\frac{2i}{\sqrt{z} + i} \right)^B \prod_{j=1}^{N_2} \left(\frac{\sqrt{b_j} + \sqrt{z}}{\sqrt{b_j} + i} \right)^{B_j}
\end{aligned}$$

Combining these terms in (40) we obtain

$$\begin{aligned}
&\left(\frac{z+1}{4} \right)^{B-A-C} (1 - \Phi(z))^{B-A-C} \cdot \mathfrak{F}(z) \\
=& \frac{1}{4^A} \prod_{i=1}^{N_1} (1 + a_i)^{A_i} \left(\frac{-2i}{\sqrt{z} - i} \right)^{B-C-A} \left(\frac{z+1}{4} \right)^{B-C} \left(\frac{-2i}{\sqrt{z} - i} \right)^A \left(\frac{2i}{\sqrt{z} + i} \right)^B \\
&\cdot \prod_{i=1}^{N_1} \left(\frac{1}{(\sqrt{a_i} - i)(\sqrt{a_i} + \sqrt{z})} \right)^{A_i} \prod_{j=1}^{N_2} \left(\frac{\sqrt{b_j} + \sqrt{z}}{\sqrt{b_j} + i} \right)^{B_j}.
\end{aligned}$$

Finally, taking into account

$$\begin{aligned}
\prod_{i=1}^{N_1} \left(\frac{\sqrt{a_i} + i}{\sqrt{a_i} + \sqrt{z}} \right)^{A_i} &= \prod_{i=1}^{N_1} (1 + a_i)^{A_i} \cdot \prod_{i=1}^{N_1} \left(\frac{1}{(\sqrt{a_i} - i)(\sqrt{a_i} + \sqrt{z})} \right)^{A_i} \\
\left(\frac{2i}{\sqrt{z} + i} \right)^C &= \frac{1}{4^A} \left(\frac{-2i}{\sqrt{z} - i} \right)^{B-C-A} \left(\frac{z+1}{4} \right)^{B-C} \left(\frac{-2i}{\sqrt{z} - i} \right)^A \left(\frac{2i}{\sqrt{z} + i} \right)^B
\end{aligned}$$

we obtain (6) for $d = 0$. To prove (6) for $d \geq 1$, use induction in d and the method from the proof of Lemma 4. The proof is complete. □

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