



Article

Abnormality and Strict-Sense Minimizers That Are Not Extended Minimizers

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Abstract: We consider a constrained optimal control problem and an extension of it, in which the set of strict-sense trajectories is enlarged. Extension is a common procedure in optimal control used to derive necessary and sufficient optimality conditions for the original problem from the extended one, which usually admits a minimizer and has a more regular structure. However, this procedure fails if the two problems have different infima. Therefore, it is relevant to identify such situations. Following on from earlier work by Warga but adopting perturbation techniques developed in nonsmooth analysis, we investigate the relation between the occurrence of an infimum gap and the abnormality of necessary conditions. For the notion of a local minimizer based on control distance and an extension, including the impulsive one, we prove that (i) a local extended minimizer that is not a local minimizer of the original problem, and (ii) a local strict-sense minimizer that is not a local minimizer of the extended problem both satisfy the extended maximum principle in abnormal form. The main novelty is result (ii), as until now, it has only been shown that a strict-sense minimizer that is not an extended minimizer is abnormal for an ‘averaged version’ of the maximum principle.

Keywords: optimal control problems; maximum principle; state constraints; gap phenomena; impulsive optimal control

MSC: 49K15; 34K45; 49N25



Citation: Fusco, G.; Motta, M.

Abnormality and Strict-Sense

Minimizers That Are Not Extended

Minimizers. *Mathematics* **2024**, *12*, 943.

<https://doi.org/10.3390/math12070943>

Academic Editor: Savin Treanta

Received: 21 February 2024

Revised: 19 March 2024

Accepted: 20 March 2024

Published: 22 March 2024



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1. Introduction

It is common practice in the fields of the calculus of variations and optimal control to extend the space of solutions for problems that cannot be solved in, say, an ordinary space, or if the solution is difficult to find, even with numerical approximation. This process, known as *extension*, involves compactifying and regularizing the problem, resulting in a more manageable structure and the possibility of obtaining necessary and sufficient conditions for optimality. However, in order for an extension to be well posed, it is fundamental that the infimum value achievable in the original problem coincides with that of the extended problem. Otherwise, the extended problem will not provide any useful information about the original problem, which is the only one whose strategies we actually want or can implement. So, for instance, determining an extended minimizer and the solution to the Hamilton–Jacobi equation associated with the extended problem (analytically or using numerical methods) is useful only if from them, we can derive a quasi-optimal control and the value function for the original problem, respectively. Clearly, this is only possible if there is no gap between infima. However, in the presence of endpoint and state constraints, a gap often occurs, even in situations where the set of strict-sense original solutions is L^∞ -dense in the set of extended paths. In particular, this problem arises when all strict-sense solutions close to an extended trajectory that satisfies the constraints, for instance, a local minimizer, fail to meet them in turn. Criteria for avoiding an infimum gap have, therefore, been extensively investigated in the literature. In the calculus of variations,

for example, it is well known that, in the absence of suitable coercivity assumptions, the minimum of an integral cost over Lipschitz-continuous functions with assigned initial and final points may not exist or may be greater than the minimum assumed in the largest set of absolutely continuous functions. In this context, the gap issue is called the Lavrentiev phenomenon, and it is still widely studied (see, e.g., [1] and the comprehensive bibliography therein). As far as optimal control is concerned, a classical extension involves relaxation, obtained by either convexifying the set of admissible velocities or introducing relaxed controls that take values in a set of probability measures. Another extension is the impulsive one, in which a non-coercive problem with unbounded controls, i.e., where minimizing sequences of solutions may have increasing velocities and tend in the limit to discontinuous paths, is extended by admitting functions of bounded variation as solutions. A detailed description of these well-known extensions and a wide bibliography can be found, e.g., in [2,3]. The gap phenomenon has also been studied extensively in optimal control, often in correspondence with necessary optimality conditions, known in the literature as the Pontryagin Maximum Principle (see [4,5] for its original formulation and applications). In particular, starting from the seminal work by Warga [6] in the early 1970s, criteria for excluding an infimum gap for different problems and extensions have been expressed in terms of normality conditions for some versions of the Pontryagin Maximum Principle, where normality means that all sets of multipliers have the cost multiplier different from zero (see, e.g., [7–13]).

This paper focuses on the connection between the presence of an infimum gap, *at least in a local sense*, and a nonsmooth version of the maximum principle satisfied in abnormal form, i.e., not normal, for the following problem (P) and its extension (P_e).

Given $T > 0$ and $\check{x}_0 \in \mathbb{R}^n$, we introduce the constrained control system

$$\dot{y}(t) = \mathcal{F}(t, y(t), \omega(t), \alpha(t)) \quad \text{for a.e. } t \in [0, T], \quad y(0) = \check{x}_0, \tag{1}$$

$$h(t, y(t)) \leq 0 \quad \forall t \in [0, T], \quad y(T) \in \mathcal{T}. \tag{2}$$

Here, \mathcal{T} is a closed subset of \mathbb{R}^n , which we call the target; $h : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the state constraint function; and $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \times \bar{V} \times A \rightarrow \mathbb{R}^n$ is the dynamics function, where the compact subset $A \subset \mathbb{R}^q$ and the bounded subset $V \subset \mathbb{R}^m$ are the sets of control values. Indeed, with \bar{V} denoting the closure of V , let us define the sets \mathcal{A} , \mathcal{V} , and \mathcal{W} of admissible control functions as follows

$$\mathcal{A} := L^1([0, T], A) \quad \mathcal{V} := L^1([0, T], V) \quad \mathcal{W} := L^1([0, T], \bar{V}).$$

We call an *extended process* any triple $(\omega, \alpha, y) \in \mathcal{A} \times \mathcal{W} \times W^{1,1}([0, T], \mathbb{R}^n)$ that satisfies the dynamic constraint (1). If $\omega \in \mathcal{V}$ in particular, then we refer to (ω, α, y) as a *strict-sense process*. Any process (either extended or strict-sense) that additionally fulfills the endpoint and the state constraint in (2) is said to be *feasible*. The sets of feasible strict-sense and feasible extended processes are denoted by Γ_s and Γ_e , respectively.

Given a cost function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$, we introduce the *strict-sense optimal control problem*

$$\text{minimize } \Psi(y(T)) \quad \text{over } (\omega, \alpha, y) \in \Gamma_s \tag{P_s}$$

and the *extended optimal control problem*

$$\text{minimize } \Psi(y(T)) \quad \text{over } (\omega, \alpha, y) \in \Gamma_e. \tag{P_e}$$

Note how the controls α and ω play different roles, given that only the control set \mathcal{V} , to which ω belongs, is extended. The opportunity to consider both arises from applications. For example, in impulsive problems, it is common that only certain control components can take values in an unbounded set. In this case, which we clarify in Section 4, ω represents these components while α represents the remaining ones (see, e.g., the model example in [14]). Incidentally, this distinction is reflected in the hypotheses on the dynamics \mathcal{F} ,

which require continuity in α and, instead, a form of uniform continuity for both \mathcal{F} and its Clarke-generalized Jacobian $D_x \mathcal{F}$ in the variable ω , as specified in Section 2.

Since $\Gamma_s \subseteq \Gamma_e$, it immediately follows that $\inf_{\Gamma_s} \Psi(y(T)) \leq \inf_{\Gamma_e} \Psi(y(T))$. In fact, this inequality might be strict, in which case we say that *there is an infimum gap*. In order to introduce the notion of a *local infimum gap*, for any pair of extended processes (ω, α, y) , (ω', α', y') , we define the following control distance:

$$\mathbf{d}((\omega, \alpha, y), (\omega', \alpha', y')) := \|\omega - \omega'\|_{L^1(0,T)} + \ell(\{t \in [0, T] : \alpha(t) \neq \alpha'(t)\}),$$

where ℓ is the Lebesgue measure. Hence, a feasible strict-sense [resp., extended] process $(\bar{\omega}, \bar{\alpha}, \bar{y})$ is a *local minimizer* for (P_s) [resp., (P_e)] if there exists some $\delta > 0$ such that $\Psi(\bar{y}(T)) \leq \Psi(y(T))$ for any (ω, α, y) in Γ_s [resp., Γ_e], satisfying $\mathbf{d}((\bar{\omega}, \bar{\alpha}, \bar{y}), (\omega, \alpha, y)) \leq \delta$.

We distinguish the following two types of local infimum gaps according to whether we focus on the strict-sense problem or the extended problem:

- *Type-E local infimum gap*, when the cost of a local minimizer $(\bar{\omega}, \bar{\alpha}, \bar{y})$ of (P_e) is strictly smaller than the infimum of (P_s) in a \mathbf{d} -neighborhood of $(\bar{\omega}, \bar{\alpha}, \bar{y})$,
- *Type-S local infimum gap*, if a local minimizer of (P_s) is not a local minimizer of (P_e) .

Assuming the hypotheses provided in Section 2, and with reference to the maximum principle of Definition 3 below, our main results are the following:

- (i) If at $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$, there is a type-E local infimum gap, then $(\bar{\omega}, \bar{\alpha}, \bar{y})$ satisfies the maximum principle in abnormal form, i.e., for a set of multipliers with cost multiplier equal to zero;
- (ii) If $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ is a local minimizer of (P) , then it satisfies the same maximum principle as the extended problem. If, in addition, at $(\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$, there is a type-S local infimum gap, then it is an abnormal extremal.

We emphasize that the choice of the distance \mathbf{d} above, which plays a fundamental role in the proof of these results, represents a novelty compared to the works we quoted above. In fact, in these papers, they always consider L^∞ local minimizers, where the L^∞ distance between the trajectories is used instead of \mathbf{d} .

Furthermore, in Section 4, we illustrate a relevant application of the above results to impulsive optimal control. There are significant examples in aerodynamics [14,15], mechanics [16,17], and biology [18,19] where the evolution of the involved variables can be modeled as a control system, in which controls can reach very high intensity in a very short time interval, resulting in an abrupt change in the state of the system. The impulsive extension is, therefore, a limit problem, in which the previous controls and trajectories are replaced with their (suitably defined) limits. It is worth emphasizing that in the above-mentioned applications to real-world problems, impulsive controls are only idealizations of the original controls so results in relation to the impulsive problem are of interest only if they provide information on the original problem, namely only if no gap of any type occurs.

As already mentioned, Warga was the first to study the correlation between the presence of an infimum gap and the validity of the maximum principle in abnormal form for a classical extension through relaxation in the measure of the controls. Specifically, he announced the result for a type-S L^∞ -local infimum gap in his early paper [6], which focused on state constraint-free optimal control problems with smooth data. Then, in his monograph [13], Warga proved the relationship between the gap and abnormality for a type-E L^∞ -local infimum gap in optimal control problems with state constraints (see also [11]). His subsequent work [12] extended this result to include nonsmooth data, utilizing the results in [20]. Vinter and Palladino [10] proved the above-mentioned correlation in the case of both type-E and type-S L^∞ -local infimum gaps for the classical extension through convex relaxation of a class of nonsmooth state-constrained optimal control problems, which subsumed those considered by Warga and under less restrictive hypotheses on data. Their techniques differed significantly from those of Warga, reflecting different approaches to the maximum principle. In more detail, the method adopted in [11,12,20] involved constructing approximating cones to reachable sets and using set separation arguments, whereas the

technique adopted in [10] utilized perturbation and penalization procedures as well as Ekeland's variational principle. When applied to nonsmooth optimal control problems, it is difficult to compare these methods as they require different assumptions on the dynamics and target. More importantly, they give rise to distinct abnormality conditions. Indeed, following Warga's method, these conditions involve the use of 'derivative containers' as generalized gradients from [12], whereas the second method relies on Clarke's version of the maximum principle, in which subdifferentials are considered (see [21,22]). More recently, following the latter approach, results similar to those in [10] were established in [7] for the impulsive extension of optimal control problems with unbounded dynamics and state constraints (see also the references therein). Additionally, in [8,9], an abstract extension including both relaxation and impulsive extension as special cases was addressed. In particular, in [7,8], for the first time, we also provided sufficient conditions for the nondegeneracy of the abnormality condition related to a type-E L^∞ -local infimum gap.

However, besides considering L^∞ -local minimizers, all these works focused primarily on the type-E local infimum gap. Specifically, apart from Warga's initial work, the type-S local infimum gap was only studied in [10] for the extension through convexification of the dynamics and in [9] for a more general extension. In both papers, the results were not entirely satisfactory; however, it was shown that a strict-sense L^∞ -local minimizer that is not also an extended local minimizer satisfies, in abnormal form, an 'averaged version' of the maximum principle, which is much less informative than the actual maximum principle.

In this paper, for the extension under consideration, on the one hand, we fill the gap in the previous literature regarding the results obtained for the type-E and type-S local infimum gaps by showing that in both cases, the local minimizer is abnormal for the maximum principle associated with the extended problem. On the other hand, we extend the previous results for the type-E L^∞ -local infimum gap to the case of the local minimizer based on the distance \mathbf{d} described above. Note that from the continuity property of the input-output map associated with the control system, it follows that the present results imply the previous ones. With regard to the techniques used, we are inspired by the approach proposed in [10], as generalized to the case of an abstract extension in [9]. In particular, this allows us to consider rather weak assumptions, including nonsmooth dynamics and state constraint functions, and a target that is simply a closed set (see Section 2).

This paper is organized as follows. In Section 2, we present the notations used, some useful definitions, and precise assumptions. In Section 3, we rigorously introduce the concepts of type-E and type-S local infimum gaps and state our main results, which are proved in Section 5. Section 4 is devoted to applying these results to the impulsive extension of a control-affine system with unbounded controls. We also give an example. Section 6 contains some concluding remarks.

2. Notations and Basic Assumptions

2.1. Notations and Preliminaries

Given $T > 0$ and $X \subseteq \mathbb{R}^k$, we denote by $W^{1,1}([0, T], X)$, $L^1([0, T], X)$, and $L^\infty([0, T], X)$ the sets of absolutely continuous functions, Lebesgue integrable functions, and essentially bounded functions defined on $[0, T]$ and taking values in X , respectively. We do not write domains and codomains when the meaning is clear, and we adopt $\|\cdot\|_{L^1(0,T)}$, $\|\cdot\|_{L^\infty(0,T)}$, or $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^\infty}$ to denote the L^1 and the ess-sup norm, respectively. Moreover, $\ell(X)$, $\text{co}(X)$, \overline{X} , and ∂X denote the Lebesgue measure, the convex hull, the closure, and the boundary of X , respectively. Given a closed set $\mathcal{C} \subseteq \mathbb{R}^k$ and a point $z \in \mathbb{R}^k$, we define the distance of z from \mathcal{C} as $d_{\mathcal{C}}(z) := \min_{y \in \mathcal{C}} |z - y|$. For any $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$. We employ $NBV^+([0, T], \mathbb{R})$ to denote the set of monotone non-decreasing, real-valued functions μ on $[0, T]$ of bounded variation, vanishing at the point 0 and right continuous on $]0, T[$. Each $\mu \in NBV^+([0, T], \mathbb{R})$ defines a Borel measure on $[0, T]$, denoted by μ ; its total variation is indicated by $\|\mu\|_{TV}$ or $\mu([0, T])$; and its support is denoted by $\text{spt}(\mu)$. If $(\mu_i) \subset$

$NBV^+([0, T], \mathbb{R})$, we say that $\mu_i \rightharpoonup^* \mu \in NBV^+([0, T], \mathbb{R})$ if $\int_{[0, T]} \psi \mu_i(dt) \rightarrow \int_{[0, T]} \psi \mu(dt)$ for any continuous map $\psi : [0, T] \rightarrow \mathbb{R}$.

Let us present some notions from nonsmooth analysis (see [21,22] for more details). A set $K \subseteq \mathbb{R}^k$ is a cone if, given $k \in K$ and $a > 0$, then $ak \in K$. Let C be a closed subset of \mathbb{R}^k , and let $\bar{x} \in C$. Then, the limiting normal cone $N_C(\bar{x})$ of C at \bar{x} is given by

$$N_C(\bar{x}) := \left\{ \eta \in \mathbb{R}^k : \exists (x_i, \eta_i) \in C \times \mathbb{R}^k \text{ s.t. } (x_i, \eta_i) \rightarrow (\bar{x}, \eta), \limsup_{x \rightarrow x_i} \frac{\eta_i \cdot (x - x_i)}{|x - x_i|} \leq 0 \ \forall i \right\}.$$

Let $H : \mathbb{R}^k \rightarrow \mathbb{R}$ be a lower semicontinuous map, and let $\bar{z} \in \mathbb{R}^k$. Then, the limiting subdifferential of H at \bar{z} is

$$\partial H(\bar{z}) := \left\{ \xi : \exists \xi_i \rightarrow \xi, z_i \rightarrow \bar{z} \text{ s.t. } \limsup_{z \rightarrow z_i} \frac{\xi_i \cdot (z - z_i) - H(z) + H(z_i)}{|z - z_i|} \leq 0 \ \forall i \right\}.$$

If $k = h + l$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^h \times \mathbb{R}^l$, $\partial_x H(\bar{x}, \bar{y})$ and $\partial_y H(\bar{x}, \bar{y})$ denote the partial limiting subdifferential of H at (\bar{x}, \bar{y}) with respect to x, y , respectively. When H is differentiable, ∇H is the usual gradient operator, and $\nabla_x H, \nabla_y H$ denote the partial derivatives of H . If H is also locally Lipschitz continuous, the hybrid subdifferential of H at $\bar{z} \in \mathbb{R}^k$ is

$$\partial^> H(\bar{z}) := \text{co} \{ \xi : \exists (z_i)_i \subset \text{diff}(H) \setminus \{ \bar{z} \} \text{ s.t. } z_i \rightarrow \bar{z}, H(z_i) > 0 \ \forall i, \nabla H(z_i) \rightarrow \xi \},$$

where $\text{diff}(H)$ is the set of differentiability points of H . Finally, if $U : \mathbb{R}^k \rightarrow \mathbb{R}^l$ is a locally Lipschitz-continuous map and $\bar{z} \in \mathbb{R}^k$, then $DU(\bar{z})$ stands for the Clarke-generalized Jacobian, given by

$$DU(\bar{a}) := \text{co} \{ \xi : \exists (z_i)_i \subset \text{diff}(U) \setminus \{ \bar{z} \} \text{ s.t. } z_i \rightarrow \bar{z} \text{ and } \nabla U(z_i) \rightarrow \xi \},$$

where ∇U refers to the Jacobian matrix of U . If $k = h + l$ and $\bar{z} = (\bar{x}, \bar{y}) \in \mathbb{R}^h \times \mathbb{R}^l$, $D_x U(\bar{x}, \bar{y}), D_y U(\bar{x}, \bar{y})$ denote the Clarke-generalized Jacobian of U at (\bar{x}, \bar{y}) with respect to x, y , respectively. We recall that the following relation holds:

$$q \cdot DU(z) = \text{co} \partial(q \cdot U)(z) \quad \forall (z, q) \in \mathbb{R}^{k+k}. \tag{3}$$

2.2. Basic Assumptions

Now, we present the hypotheses we assume throughout this paper. In the following, $(\bar{\omega}, \bar{a}, \bar{y})$ is a feasible process, which we refer to as the reference process. Moreover, for a given $\theta > 0$, the set $\Sigma_\theta \subset \mathbb{R}^{1+n}$ is defined as

$$\Sigma_\theta := \{ (t, x) \in \mathbb{R} \times \mathbb{R}^n : t \in [0, T], x \in \bar{y}(t) + \theta \mathbb{B} \}.$$

H1. The Borel set $A \subset \mathbb{R}^q$ is compact, and the Borel set $V \subset \mathbb{R}^m$ is bounded. Moreover, there exists a sequence $(V_i)_i$ of closed subsets of V satisfying

$$V_i \subseteq V_{i+1} \quad \forall i, \quad \bigcup_{i=1}^{+\infty} V_i = V.$$

H2. The cost function Ψ is Lipschitz continuous on a neighborhood of $\bar{y}(T)$. The target $\mathcal{T} \subseteq \mathbb{R}^n$ is closed. The state constraint function h is upper semicontinuous, and for some $K_h > 0$, it satisfies

$$|h(t, x) - h(t, x')| \leq K_h |x - x'| \quad \text{for any } (t, x), (t, x') \in \Sigma_\theta.$$

H3. For all $(x, w, a) \in \{x \in \mathbb{R}^n : (t, x) \in \Sigma_\theta \text{ for some } t \in [0, T]\} \times \bar{V} \times A$, the map $\mathcal{F}(\cdot, x, w, a)$ is Lebesgue measurable on $[0, T]$. Moreover, for some $k \in L^1([0, T], [0, +\infty[)$, one has

$$|\mathcal{F}(t, x, w, a)| \leq k(t), \quad |\mathcal{F}(t, x', w, a) - \mathcal{F}(t, x, w, a)| \leq k(t)|x' - x|, \tag{4}$$

for all $(t, x, w, a), (t, x', w, a) \in \Sigma_\theta \times \bar{V} \times A$. Furthermore, there exists a continuous increasing function $\varphi : [0, +\infty[\rightarrow [0, +\infty[$ vanishing at 0 and satisfying, for all $(t, x, a) \in \Sigma_\theta \times A$, the following relations

$$\begin{aligned} |\mathcal{F}(t, x, w', a) - \mathcal{F}(t, x, w, a)| &\leq k(t)\varphi(|w' - w|) \quad \forall w', w \in \bar{V}, \\ D_x \mathcal{F}(t, x, w', a) &\subseteq D_x \mathcal{F}(t, x, w, a) + k(t)\varphi(|w' - w|)\mathbb{B} \quad \forall w', w \in \bar{V}. \end{aligned}$$

Remark 1. Hypothesis (H1) holds whenever V is a relatively open set. Moreover, we observe that if (H1) is satisfied, then \mathcal{V} is a dense subset of \mathcal{W} in the L^1 -norm. In particular, for any $\bar{\omega} \in \mathcal{W}$ and any $\varepsilon > 0$, there exists an integer i_ε for which $d_H(V_i, \bar{V}) < \varepsilon/T$ for every $i \geq i_\varepsilon$, where $d_H(V_i, \bar{V})$ stands for the Hausdorff distance between V_i and \bar{V} . Therefore, as a consequence of the selection theorem [23] (Theorem 2, p. 91), it is possible to find a measurable function $\omega_\varepsilon(t) \in \text{proj}_{V_{i_\varepsilon}}(\omega(t))$ a.e., satisfying

$$\|\omega_\varepsilon - \bar{\omega}\|_{L^1} \leq T\|\omega_i - \omega\|_{L^\infty} \leq Td_H(V_i, \bar{V}) \leq \varepsilon.$$

Remark 2. A sufficient condition for (H3) to be satisfied is that

$$\mathcal{F}(t, x, w, a) = \mathcal{F}_1(t, x, a) + \mathcal{F}_2(t, x, w, a),$$

provided \mathcal{F}_1 and \mathcal{F}_2 meet relation (4), and \mathcal{F}_2 is continuous on the compact domain $\Sigma_\theta \times \bar{V} \times A$ and continuously differentiable with respect to the state variable. Hypothesis (H3) still holds if, for some integer $d \geq 1$, the dynamics function has the following control-polynomial structure

$$\mathcal{F}(t, x, w, a) := f(t, x, a)(w_1)^d + \sum_{k=1}^d \left(\sum_{2 \leq j_1 \leq \dots \leq j_k \leq m} g_{j_1, \dots, j_k}^k(t, x) w_{j_1} \dots w_{j_k} (w_1)^{d-k} \right),$$

provided f is continuous and locally Lipschitz continuous in (t, x) uniformly with respect to a and all the maps g_{j_1, \dots, j_k}^k are locally Lipschitz continuous.

3. Type-E or Type-S Local Infimum Gap and Abnormality

In this section, we first introduce the precise definitions of the two types of local infimum gaps we may encounter, depending on whether the process we consider is a local minimum of the extended or the strict-sense problem. Then, in Theorem 1 we establish our main result, namely that the presence of any kind of local infimum gap implies the abnormal extremal condition described in the second part of the section.

3.1. Type-E and Type-S Local Infimum Gaps

As already mentioned in Section 1, for any pair of extended processes $z = (\omega, \alpha, y)$, $\hat{z} = (\hat{\omega}, \hat{\alpha}, \hat{y})$, we consider the distance

$$\mathbf{d}(z, \hat{z}) := \|\omega - \hat{\omega}\|_{L^1} + \ell\{t \in [0, T] : \alpha(t) \neq \hat{\alpha}(t)\}. \tag{5}$$

Moreover, Γ_s and Γ_e are the sets of feasible strict-sense and feasible extended processes, respectively.

Definition 1 (Local minimizer). Let $\tilde{\Gamma}$ and (\tilde{P}) denote Γ_e and (P_e) or Γ_s and (P_s) , respectively. A process $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \tilde{\Gamma}$ is a local Ψ -minimizer for problem (\tilde{P}) if, for some $\delta > 0$, one has

$$\Psi(\bar{y}(T)) = \inf \left\{ \Psi(y(T)) : z = (\omega, \alpha, y) \in \tilde{\Gamma}, \mathbf{d}(z, \bar{z}) < \delta \right\}.$$

The process \bar{z} is a Ψ -minimizer for problem (\tilde{P}) if $\Psi(\bar{y}(T)) = \inf_{\tilde{\Gamma}} \Psi(y(T))$.

Remark 3. Under hypothesis (H3), for each extended control $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$ in a suitable \mathbf{d} -neighborhood of the reference control $(\bar{\omega}, \bar{\alpha})$, there is one and only one solution $y := y[\omega, \alpha]$ of (1). Furthermore, the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$ from $\mathcal{W} \times \mathcal{A}$ to C^0 is continuous in this neighborhood, provided $\mathcal{W} \times \mathcal{A}$ is endowed with the distance \mathbf{d} and C^0 is endowed with the distance induced by the sup-norm. Consequently, if the process \bar{z} is an L^∞ -local minimizer, meaning that \bar{z} reaches the minimum over processes $z = (\omega, \alpha, y)$ with $\|y - \bar{y}\|_{L^\infty} < \delta$ for some $\delta > 0$, then it is also a local minimizer according to Definition 1. In general, the contrary is not true. This makes the results in [8,9] concerning L^∞ -local minimizers not directly applicable to the present case.

It is now natural to provide the definitions of the local infimum gaps, depending on whether the reference process is extended or strict-sense.

Definition 2 (Infimum gaps). Let $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function.

(i) If $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$ and for some $\delta > 0$, it holds that

$$\Psi(\bar{y}(T)) < \inf \{ \Psi(y(T)) : z = (\omega, \alpha, y) \in \Gamma_s, \mathbf{d}(z, \bar{z}) < \delta \},$$

we say that at \bar{z} , there is a type-E local Ψ -infimum gap. If $\{z = (\omega, \alpha, y) \in \Gamma_s, \mathbf{d}(z, \bar{z}) < \delta\} = \emptyset$, we set $\inf \{ \Psi(y(T)) : z = (\omega, \alpha, y) \in \Gamma_s, \mathbf{d}(z, \bar{z}) < \delta \} = +\infty$.

(ii) If $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ is a local Ψ -minimizer for problem (P_s) , which is not a local Ψ -minimizer for problem (P_e) , i.e., $\forall \varepsilon > 0 \exists (\omega, \alpha, y) \in \Gamma_e$ satisfying

$$\Psi(y(T)) < \Psi(\bar{y}(T)) \quad \text{and} \quad \mathbf{d}(z, \bar{z}) < \varepsilon,$$

we say that at \bar{z} , there is a type-S local Ψ -infimum gap.

(iii) We say that there is a Ψ -infimum gap if $\inf_{\Gamma_e} \Psi(y(T)) < \inf_{\Gamma_s} \Psi(y(T))$.

In cases where Ψ can easily be inferred from the context, we write *infimum gap* in place of Ψ -infimum gap.

Remark 4. Given the continuity of the input-output map associated with control system (1), it is easy to see that the notion of the type-E local Ψ -infimum gap at \bar{z} does not depend on the cost function Ψ , as it is equivalent to the fact that

$$\{z = (\omega, \alpha, y) \in \Gamma_s : \mathbf{d}(z, \bar{z}) < \delta\} = \emptyset \quad \text{for some } \delta > 0 \tag{6}$$

(see [8], Proposition 2.1). If \bar{z} satisfies (6), we say that it is an isolated process.

3.2. Main Results

We introduce a nonsmooth version of the Pontryagin maximum principle for (P_e) , and we provide the notions of *normal* and *abnormal extremals*. Then, we establish a link between the abnormality and occurrence of a gap phenomenon.

Definition 3 (Pontryagin maximum principle). Let $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$, and let hypotheses (H1)–(H3) be satisfied. We say that \bar{z} is a Ψ -extremal or satisfies the Pontryagin maximum principle

if there exists a path $p \in W^{1,1}([0, T], \mathbb{R}^n)$, $\gamma \geq 0$, $\mu \in NBV^+([0, T], \mathbb{R})$, and a Borel-measurable and μ -integrable function $m : [0, T] \rightarrow \mathbb{R}^n$ satisfying the following conditions:

$$\|p\|_{L^\infty} + \|\mu\|_{TV} + \gamma \neq 0, \tag{7}$$

$$-\dot{p}(t) \in \text{co } \partial_x \{q(t) \cdot \mathcal{F}(t, \bar{y}(t), \bar{\omega}(t), \bar{\alpha}(t))\} \text{ a.e. } t \in [0, T]; \tag{8}$$

$$-q(T) \in \gamma \partial \Psi(\bar{y}(T)) + N_{\mathcal{T}}(\bar{y}(T)); \tag{9}$$

for a.e. $t \in [0, T]$, one has

$$q(t) \cdot \mathcal{F}(t, \bar{y}(t), \bar{\omega}(t), \bar{\alpha}(t)) = \max_{(w,a) \in \bar{V} \times A} q(t) \cdot \mathcal{F}(t, \bar{y}(t), w, a); \tag{10}$$

$$m(t) \in \partial_x^> h(t, \bar{y}(t)) \quad \mu\text{-a.e. } t \in [0, T]; \tag{11}$$

$$\text{spt}(\mu) \subseteq \{t \in [0, T] : h(t, \bar{y}(t)) = 0\}, \tag{12}$$

where

$$q(t) := \begin{cases} p(t) + \int_{[0,t[} m(t')\mu(dt') & t \in [0, T[, \\ p(T) + \int_{[0,T]} m(t')\mu(dt') & t = T. \end{cases}$$

We say that a Ψ -extremal \bar{z} is normal if all sets of multipliers (p, γ, μ, m) , as described above, have $\gamma > 0$. Conversely, we say that \bar{z} is abnormal when it is not normal. Clearly, abnormal Ψ -extremals do not depend on Ψ so we refer to them simply as abnormal extremals.

Remark 5. By the start of the 1970s, it was commonly acknowledged that efforts to expand the usefulness of existing necessary conditions were being hindered by a common problem: a dearth of methods for examining the characteristics of nonsmooth functions and sets with nonsmooth boundaries. One approach to extending the celebrated Pontryagin Maximum Principle [5] in this direction is to use nonsmooth analysis, a branch of analysis that investigates precisely how to locally approximate functions that are non-differentiable and sets with a non-differentiable boundary. The maximum principle above is based on this approach, developed by Clarke and collaborators, for which we refer to the books [21,22].

Theorem 1. Let $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$ and assume that hypotheses (H1)–(H3) hold. Then, consider the following statements:

- (i) If \bar{z} is a local Ψ -minimizer for (P_e) , then \bar{z} is a Ψ -extremal. If at \bar{z} , there is a type-E local Ψ -infimum gap, then \bar{z} is an abnormal extremal;
- (ii) If $\bar{z} \in \Gamma_s$ is a local Ψ -minimizer for (P_s) , then \bar{z} is a Ψ -extremal. If at \bar{z} , there is a type-S local Ψ -infimum gap, then \bar{z} is an abnormal extremal.

The proof of Theorem 1 is given in Section 5.

The main novelty of Theorem 1 is statement (ii), concerning the case where \bar{z} is a local minimizer of the original problem but not of the extended one. Indeed, in the previous literature (see [9,10]), it was proven that in such cases, an L^∞ -local minimizer \bar{z} is an abnormal extremal only for an ‘averaged version’ of the maximum principle, meaning that the adjoint Equation (8) was replaced with the following weaker differential inclusion

$$-\dot{p}(t) \in \text{co} \left\{ \bigcup_{(w,a) \in \bar{V} \times A} \partial_x \left(q(t) \cdot \mathcal{F}(t, \bar{y}(t), w, a) \right) \right\} \quad \text{a.e. } t \in [0, T],$$

in which all information on optimal control is lost. Incidentally, note that the difference between the two adjoint equations still holds even if \mathcal{F} is C^1 in the state variable.

Remark 6. It is worth mentioning that, despite hypothesis (H1) implying that \mathcal{V} is a dense subset of \mathcal{W} in the L^1 -norm, it has been well known since the earliest work by Warga [12] and Kaskovz [11]

that, in general, if only this latter condition is satisfied, the link between the gap and abnormality established in Theorem 1 may fail (see, e.g., the example in [24], Section 9).

A straightforward corollary of Theorem 1 is that the normality of an extremal turns out to be sufficient for any type of local infimum gap not to occur.

Theorem 2. Let $\bar{z} := (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_e$, and assume that hypotheses (H1)–(H3) hold. Then, consider the following statements:

- (i) If \bar{z} is a local Ψ -minimizer for (P_e) , which is a normal Ψ -extremal, at \bar{z} , there is no type-E local Ψ -infimum gap. If, in addition, \bar{z} is a Ψ -minimizer for (P_e) , then there is no Ψ -infimum gap;
- (ii) If $\bar{z} \in \Gamma_s$ is a local Ψ -minimizer for (P_s) , which is a normal Ψ -extremal, at \bar{z} , there is no type-S local Ψ -infimum gap, namely \bar{z} is a local Ψ -minimizer for (P_e) as well.

4. An Application: The Impulsive Extension

In this section, we describe how the previous results can be used to investigate the gap phenomenon in a case relevant to applications: the impulsive extension of an optimal control problem with endpoint and state constraints. We also provide an example of an impulsive problem in which both a type-E and a type-S local infimum gap occur, and we explicitly show the abnormality condition in this case.

4.1. An Impulsive Optimization Problem

Let us consider the following free end-time optimization problem with *unbounded, control-affine* dynamics:

$$(\mathcal{P}) \begin{cases} \text{minimize } \Psi(S, x(S), v(S)) \\ \text{over } S > 0, u \in L^1([0, S], U), (x, v) \in W^{1,1}([0, S], \mathbb{R}^{n+1}), \text{ s.t.} \\ (\dot{x}(s), \dot{v}(s)) = (f(s, x(s)) + \sum_{j=1}^m g_j(s, x(s))u^j(s), |u(s)|) \quad \text{a.e. } s \in [0, S], \\ (x(0), v(0)) = (\check{x}_0, 0), \\ h(s, x(s)) \leq 0 \quad \text{for all } s \in [0, S], \quad (S, x(S)) \in \mathcal{T}^*, \quad v(S) \leq K, \end{cases}$$

in which $U \subseteq \mathbb{R}^m$, $\mathcal{T}^* \subset \mathbb{R}^{1+n}$, $f : \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$, $g_j : \mathbb{R}^{1+n} \rightarrow \mathbb{R}^n$ for any $j = 1, \dots, m$, $\Psi : \mathbb{R}^{1+n+1} \rightarrow \mathbb{R}$, and $h : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$.

We make the following assumptions on the data:

H4. $K \in]0, +\infty[$ (i.e., K might be $+\infty$); the (unbounded) set of control values U is a closed cone; the target \mathcal{T}^* is a closed set; and the dynamics functions f, g_j , the constraint function h , and the cost function Ψ are locally Lipschitz continuous.

Note that $v(s)$, sometimes called *fuel* or *energy*, is simply the L^1 -norm of the control function u on $[0, s]$. Assuming, as usual, that the function $v \mapsto \Psi(s, x, v)$ is merely monotone nondecreasing, this problem is non-coercive, i.e., there are no conditions that prevent a minimizing sequence of trajectories from having increasing velocities and converging to a discontinuous path. It is well known that it is possible to embed the original problem (\mathcal{P}) into the *space-time* or *extended* problem (P_e) below, where the time becomes a new state variable and the trajectories are reparameterizations of the limits of the graphs of the trajectories of (\mathcal{P}) in the L^∞ -norm [25–28] (we recall that (\mathcal{P}) can be analyzed using a distributional approach, meaning that u is substituted by a Radon measure, only if the

coefficients g_i are autonomous and commute, i.e., the Lie brackets $[g_i, g_j]$ are equal to 0 for any $i, j = 1, \dots, m$ (see, e.g., [25,29]):

$$(P_e) \left\{ \begin{array}{l} \text{minimize } \Psi(y^0(T), y(T), v(T)) \\ \text{over } T > 0, (\omega^0, \omega) \in \mathcal{W}(T), (y^0, y, v) \in W^{1,1}([0, T], \mathbb{R}^{1+n+1}), \text{ s.t.} \\ \dot{y}^0(t) = \omega^0(t) \quad \text{a.e. } t \in [0, T], \\ \dot{y}(t) = f(y^0(t), y(t))\omega^0(t) + \sum_{j=1}^m g_j(y^0(t), y(t))\omega^j(t) \quad \text{a.e. } t \in [0, T], \\ \dot{v}(t) = |\omega(t)| \quad \text{a.e. } t \in [0, T], \\ (y^0, y, v)(0) = (0, \check{x}_0, 0), \quad (y^0(T), y(T), v(T)) \in \mathcal{T}^* \times]-\infty, K], \\ h(y^0(t), y(t)) \leq 0 \text{ for all } t \in [0, T], \end{array} \right.$$

where $\mathcal{W}(T) := L^1([0, T], W)$, with W the set of control values given by

$$W := \left\{ (\omega^0, \omega) \in [0, +\infty[\times U : \omega^0 + |\omega| = 1 \right\}.$$

Let (S, u, x, v) be an *original process*, i.e., it satisfies the dynamics constraint together with the initial condition of problem (\mathcal{P}) , and let $\sigma : [0, S] \rightarrow [0, +\infty[$ be defined as follows

$$\sigma(s) := s + v(s) \quad \text{for any } s \in [0, S].$$

We observe that $(T, \omega^0, \omega, y^0, y, v) := (\sigma(S), \dot{y}^0, (u \circ y^0)\dot{y}^0, \sigma^{-1}, x \circ y^0, v \circ y^0)$ results in an *extended process*, i.e., it satisfies the dynamics constraint together with the initial condition of problem (P_e) , and $\omega^0 = \dot{y}^0 > 0$ a.e. Actually, the map that associates with each original process an extended process with $\omega^0 > 0$ a.e. turns out to be a bijection, so that problem (\mathcal{P}) is in correspondence with the *strict-sense problem* (P_s) , namely the optimal control problem that arises when in (P_e) , we limit ourselves to consider *strict-sense processes* only, i.e., extended processes with $\omega^0 > 0$ a.e. Therefore, the extension involves allowing the control variable ω^0 to vanish on some non-trivial intervals contained in $[0, T]$. There, y^0 remains constant, whereas y evolves instantaneously according to $\dot{y} = \sum_{j=1}^m g_j(y^0, y)\omega^j(t)$. This is the reason why (P_e) , despite being an ordinary optimal control problem with controls taking values in compact sets, is usually labeled as the *impulsive extension* of (\mathcal{P}) . Indeed, problem (P_e) is also equivalent to another generalization of (\mathcal{P}) where the controls are vector-valued measures and the trajectories are bounded variation paths [14,30–34].

Adopting the terminology of the present paper, we say that an extended or strict-sense process $(T, \omega^0, \omega, y^0, y, v)$ is *feasible* [resp. an original process (S, u, x, v) is *feasible*] if it additionally fulfills all the endpoint and the state constraint of (P_e) [resp. (\mathcal{P})]. The sets of feasible original, feasible extended, and feasible strict-sense processes are denoted by Γ^* , Γ_e , and Γ_s , respectively. Given $z = (T, \omega^0, \omega, y^0, y, v)$ and $\hat{z} = (\hat{T}, \hat{\omega}^0, \hat{\omega}, \hat{y}^0, \hat{y}, \hat{v}) \in \Gamma_e$, we define the distance:

$$\mathbf{d}_{\text{imp}}(z, \hat{z}) := |T - \hat{T}| + \|(\omega^0, \omega) - (\hat{\omega}^0, \hat{\omega})\|_{L^1(0, T \wedge \hat{T})}. \tag{13}$$

Note that \mathbf{d}_{imp} is equivalent to the distance obtained by replacing $T \wedge \hat{T}$ with $T \vee \hat{T}$ in the L^1 -norm (possibly extending the controls to \mathbb{R} constantly equal to 0), as $\|(\omega^0, \omega) - (\hat{\omega}^0, \hat{\omega})\|_{L^1(0, T \vee \hat{T})} - \|(\omega^0, \omega) - (\hat{\omega}^0, \hat{\omega})\|_{L^1(0, T \wedge \hat{T})} \leq M|T - \hat{T}|$ for some constant $M > 0$. At this point, the definitions of the local minimizer and type-E and type-S local Ψ -infimum gaps (see Definitions 1 and 2) can be easily adapted to the impulsive extension by replacing

the distance \mathbf{d} defined in (5) with the distance \mathbf{d}_{imp} given in (13). The unmaximized Hamiltonian associated with problem (P_e) above is given by

$$H(s, x, p_0, p, \pi, w^0, w) := p_0 w^0 + p \cdot (f(s, x)w^0 + \sum_{j=1}^m g_j(s, x)w^j) + \pi|\omega|$$

for all $(s, x, p_0, p, \pi, w^0, w) \in \mathbb{R}^{1+n+1+n+1} \times W$.

Definition 4. We say that $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v}) \in \Gamma_e$ is a Ψ -extremal if there exists a path $(p_0, p) \in W^{1,1}([0, \bar{T}], \mathbb{R}^{1+n})$, $\gamma \geq 0$, $\pi \leq 0$, $\mu \in NBV^+([0, \bar{T}], \mathbb{R})$, and Borel-measurable and μ -integrable functions $(m_0, m) : [0, \bar{T}] \rightarrow \mathbb{R}^{1+n}$ satisfying the following conditions:

$$\|p_0\|_{L^\infty} + \|p\|_{L^\infty} + \mu([0, \bar{T}]) + \gamma \neq 0 \tag{14}$$

$$-(\dot{p}_0, \dot{p})(t) \in \text{co } \partial_{s,x} H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, \bar{\omega}^0(t), \bar{\omega}(t)) \quad \text{a.e. } t \tag{15}$$

$$-(q_0(\bar{T}), q(\bar{T}), \pi) \in \gamma \partial \Psi(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) + N_{\mathcal{T}^* \times]-\infty, K]}(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) \tag{16}$$

$$\begin{aligned} & H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, \bar{\omega}^0(t), \bar{\omega}(t)) \\ &= \max_{(w^0, w) \in W} H(\bar{y}^0(t), \bar{y}(t), q_0(t), q(t), \pi, w^0, w) = 0 \quad \text{a.e. } t \end{aligned} \tag{17}$$

$$(m_0, m)(t) \in \partial_{s,x}^> h(\bar{y}^0(t), \bar{y}(t)) \quad \mu\text{-a.e. } t \tag{18}$$

$$\text{spt}(\mu) \subseteq \{t \in [0, \bar{T}] : h(\bar{y}^0(t), \bar{y}(t)) = 0\}, \tag{19}$$

where $(q_0, q) : [0, \bar{T}] \rightarrow \mathbb{R}^{1+n}$ is given by

$$(q_0, q)(t) := \begin{cases} (p_0, p)(t) + \int_{[0,t[} (m_0, m)(t') \mu(dt') & t \in [0, \bar{T}[\\ (p_0, p)(\bar{T}) + \int_{[0,\bar{T}]} (m_0, m)(t') \mu(dt') & t = \bar{T}. \end{cases}$$

Moreover, if $\gamma \partial \Psi(\bar{y}^0(\bar{T}), \bar{y}(\bar{T}), \bar{v}(\bar{T})) = 0$ and $\bar{v}(\bar{T}) < K$, then $\pi = 0$. Furthermore, if $\bar{y}^0(0) < \bar{y}^0(\bar{T})$, then (14) can be strengthened with

$$\|p\|_{L^\infty} + \mu([0, \bar{T}]) + \gamma \neq 0. \tag{20}$$

We say that $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v})$ is normal if all sets of multipliers $(p_0, p, \gamma, \pi, \mu, m_0, m)$, as described above, have $\gamma > 0$. Conversely, we say that $(\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v})$ is abnormal when it is not normal.

From Theorem 1, we deduce the following result.

Theorem 3. Let $\bar{z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v}) \in \Gamma_e$, and assume that hypothesis (H4) holds. Then, consider the following statements:

- (i) If \bar{z} is a local Ψ -minimizer for (P_e) , then \bar{z} is a Ψ -extremal. If at \bar{z} , there is a type-E local Ψ -infimum gap, then \bar{z} is an abnormal extremal;
- (ii) If $\bar{z} \in \Gamma_s$ is a local Ψ -minimizer for (P_s) , then \bar{z} is a Ψ -extremal. If at \bar{z} , there is a type-S local Ψ -infimum gap, then \bar{z} is an abnormal extremal.

Proof. The impulsive extended problem (P_e) has a free end time, so the results of the previous sections concerning fixed end-time problems do not apply straightforwardly. However, through a standard time-rescaling procedure that applies to free end-time problems with Lipschitz-continuous time dependence, we can embed problem (P_e) into a fixed end-time optimization problem, satisfying all the assumptions of Theorem 1 and for which,

for example, \bar{z} is still a local minimizer if it was so for (P_e) . Precisely, let $\mathcal{W} := \mathcal{W}(\bar{T})$, $\mathcal{D} := L^1([0, \bar{T}], [-1/2, 1/2])$, and consider the rescaled problem :

$$(P_e^r) \begin{cases} \text{minimize } \Psi(y^0(\bar{T}), y(\bar{T}), v(\bar{T})) \\ \text{over } (\omega^0, \omega) \in \mathcal{W}, d \in \mathcal{D}, (y^0, y, v) \in W^{1,1}([0, \bar{T}], \mathbb{R}^{1+n+1}), \text{ s.t.} \\ \dot{y}^0(t) = (1 + d(t)) \omega^0(t) \quad \text{a.e. } t \in [0, \bar{T}], \\ \dot{y}(t) = (1 + d(t)) \mathcal{F}(y^0(t), y(t), \omega^0(t), \omega(t)) \quad \text{a.e. } t \in [0, \bar{T}], \\ \dot{v}(t) = (1 + d(t)) |\omega(t)| \quad \text{a.e. } t \in [0, \bar{T}], \\ (y^0, y, v)(0) = (t_1, \check{x}_0, 0), \\ h(y^0(t), y(t)) \leq 0 \text{ for all } t \in [0, \bar{T}], \quad (y^0(\bar{T}), y(\bar{T}), v(\bar{T})) \in \mathcal{T}^* \times]-\infty, K], \end{cases}$$

where, for any $(t, x, w^0, w) \in \mathbb{R}^{1+n} \times W$, we have the set

$$\mathcal{F}(t, x, w^0, w) := f(t, x) w^0 + \sum_{j=1}^m g_j(t, x) w^j.$$

Any element $(\omega^0, \omega, d, y^0, y, v)$ satisfying all constraints in (P_e^r) is referred to as a feasible rescaled extended process. If $\omega^0 > 0$ a.e., then $(\omega^0, \omega, d, y^0, y, v)$ is called a feasible rescaled strict-sense process. For any pair of feasible rescaled extended processes $\zeta := (\omega^0, \omega, d, y^0, y, v)$, $\hat{\zeta} := (\hat{\omega}^0, \hat{\omega}, \hat{d}, \hat{y}^0, \hat{y}, \hat{v})$, we define the distance as

$$\mathbf{d}^r(\zeta, \hat{\zeta}) := \|(\omega^0, \omega, d) - (\hat{\omega}^0, \hat{\omega}, \hat{d})\|_{L^1(0, \bar{T})}.$$

Let us associate the (feasible) rescaled process $\bar{\zeta} := (\bar{\omega}^0, \bar{\omega}, \bar{d} = 0, \bar{y}^0, \bar{y}, \bar{v})$ with the given reference process $\bar{z} = (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v})$. From a straightforward application of the chain rule and standard calculations, we deduce that for any $\delta > 0$, there exists some $\varepsilon \in]0, \delta[$ such that with each feasible rescaled extended process $\zeta := (\omega^0, \omega, d, y^0, y, v)$ satisfying $\mathbf{d}^r(\zeta, \bar{\zeta}) < \varepsilon$, using the time change

$$\tau(s) = \int_0^s \frac{ds'}{1 + \bar{d}(s')}, \quad s \in [0, \bar{T}],$$

we can associate the following feasible extended process

$$z = (T, \omega^0, \omega, y^0, y, v) := (\tau(\bar{T}), (\bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}, \bar{v}) \circ \tau).$$

satisfying $\mathbf{d}_{\text{imp}}(z, \bar{z}) < \delta$. Moreover, $\Psi((\bar{y}^0, \bar{y}, \bar{v})(\bar{T})) = \Psi((y^0, y, v)(T))$.

As a consequence, if \bar{z} is a local Ψ -minimizer for (P_e) , then $\bar{\zeta}$ is a local Ψ -minimizer for (P_e^r) , at which there is a type-E local infimum gap as soon as at \bar{z} , there is a type-E local infimum gap. At this point, the proof of Theorem 3 can be derived by applying Theorem 1 to the rescaled problem. We omit the details, which follow the same line as the proofs in [22] (Theorem 8.7.1), and [8] (Theorem 4.1). □

Remark 7. Using similar arguments to those in [8], what we have done in this section can be easily generalized to control-polynomial impulsive problems, by which we mean that the dynamics of the original problem (\mathcal{P}) can be replaced with

$$(\dot{x}, \dot{v})(t) = \left(f(t, x) + \sum_{k=1}^d \left(\sum_{1 \leq j_1 \leq \dots \leq j_k \leq m} g_{j_1, \dots, j_k}^k(t, x) u^{j_1} \dots u^{j_k} \right), |u|^d \right) \text{ a.e. } t,$$

where d is an integer ≥ 1 . This generalization may be relevant for some applications to Lagrangian mechanics, where dynamics are usually control-polynomial with a degree of $d = 2$ (see [17]).

4.2. An Example

The following example tells us that both a type-S local infimum gap and a type-E local infimum gap may occur. Moreover, we exhibit sets of abnormal multipliers, which exist in accordance with Theorem 3.

Consider the optimization problem with scalar, unbounded controls:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{minimize } |x^1(1) - 1| \\ \text{over } u \in L^1([0, 1], [0, +\infty[), (x^1, x^2) \in W^{1,1}([0, 1], \mathbb{R}^2) \text{ s.t.} \\ \dot{x}^1(s), \dot{x}^2(s) = (u(s), 2) \quad \text{a.e. } s \in [0, 1], \\ (x^1, x^2)(0) = (-1, -1), \quad x^2(1) = 1, \quad \int_0^1 u(s) ds \leq 3, \\ h(x^1(s), x^2(s)) := 1 - |x^1(s)| \vee |x^2(s)| \leq 0 \text{ for all } s \in [0, 1]. \end{array} \right.$$

Let $W := \{(w^0, w) \in [0, +\infty[\times [0, +\infty[: w^0 + w = 1\}$. Then, the space-time extension of the above problem is given by

$$(P_\varepsilon) \left\{ \begin{array}{l} \text{minimize } |y^1(T) - 1| \\ \text{over } T > 0, (\omega^0, \omega) \in L^1([0, T], W), (y^0, y^1, y^2, \nu) \in W^{1,1}([0, T], \mathbb{R}^4) \text{ s.t.} \\ \dot{y}^0, \dot{y}^1, \dot{y}^2, \dot{\nu}(t) = (\omega^0, \omega, 2\omega^0, \omega)(t) \quad \text{a.e. } t \in [0, T], \\ (y^0, y^1, y^2, \nu)(0) = (0, -1, -1, 0), \quad y^0(T) = 1, \quad y^2(T) = 1, \quad \nu(T) \leq 3, \\ h(y^1(t), y^2(t)) = 1 - |y^1(t)| \vee |y^2(t)| \leq 0 \text{ for all } t \in [0, T]. \end{array} \right.$$

Type-S local infimum gap. Let $\bar{z} := (\bar{T}, \bar{\omega}^0, \bar{\omega}, \bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{\nu})$ be the following strict-sense process, where $\bar{T} = 1$, the control $(\bar{\omega}^0, \bar{\omega})$ is given by the constant pair

$$(\bar{\omega}^0, \bar{\omega})(t) = (1, 0) \quad \forall t \in [0, 1],$$

and

$$(\bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{\nu})(t) = (t, -1, -1 + 2t, 0) \quad \forall t \in [0, 1].$$

It is easy to see that \bar{z} , which corresponds to the process of (\mathcal{P}) associated with the control $\bar{u} \equiv 0$, is trivially a strict-sense minimizer, as $(\bar{y}^0, \bar{y}^1, \bar{y}^2, \bar{\nu})$ is the unique feasible strict-sense trajectory. However, \bar{z} is not a local minimizer for the extended problem (P_ε) . Indeed, let us fix $\varepsilon > 0$ sufficiently small, and let us consider the extended process $z_\varepsilon = (T_\varepsilon, \omega_\varepsilon^0, \omega_\varepsilon, y_\varepsilon^0, y_\varepsilon^1, y_\varepsilon^2, \nu_\varepsilon)$, where $T_\varepsilon = 1 + \varepsilon$ and $(\omega_\varepsilon^0, \omega_\varepsilon)$ is given by

$$(\omega_\varepsilon^0, \omega_\varepsilon)(t) := \begin{cases} (1, 0) & \text{if } t \in [0, 1] \\ (0, 1) & \text{if } t \in]1, 1 + \varepsilon], \end{cases}$$

so that one has

$$(y_\varepsilon^0, y_\varepsilon^1, y_\varepsilon^2, \nu_\varepsilon)(t) = \begin{cases} (t, -1, -1 + 2t, 0) & \text{if } t \in [0, 1] \\ (1, -2 + t, 1, t - 1) & \text{if } t \in]1, 1 + \varepsilon]. \end{cases}$$

For any $\varepsilon > 0$, this is the description in the state space of a discontinuous state trajectory $(x_\varepsilon^1, x_\varepsilon^2)$ for problem (\mathcal{P}) , which first reaches the point $(-1, 1)$ using the control $u = 0$ and then jumps to the position $(-1 + \varepsilon, 1)$ with an impulse. Note that z_ε is a feasible extended process that satisfies

$$d_{\text{imp}}(z_\varepsilon, \bar{z}) = |T_\varepsilon - \bar{T}| + \|(\omega_\varepsilon^0, \omega_\varepsilon) - (\bar{\omega}^0, \bar{\omega})\|_{L^1(0, 1 \wedge (1 + \varepsilon))} = \varepsilon$$

whose cost is strictly less than the cost corresponding to \bar{z} because it holds that

$$|y_\varepsilon^1(1 + \varepsilon) - 1| = 2 - \varepsilon < 2 = |\bar{y}^1(1) - 1|.$$

Thus, by the arbitrariness of $\varepsilon > 0$, at \bar{z} , there is a type-S local infimum gap. Indeed, a set of abnormal multipliers corresponding to \bar{z} is given by $(p_0, p, \gamma, \pi, \mu, m_0, m)$, where $\gamma = \pi = 0, p_0 \equiv 0, \mu \equiv 0, p = (p_1, p_2) \equiv (0, 1), m_0 \equiv 0$, and $m(t) = (m_1, m_2)(t) \in \partial^>h(\bar{y}^1(t), \bar{y}^2(t))$ for any $t \in [0, 1]$.

Type-E local infimum gap. Now consider the following extended process $\hat{z} := (\hat{\omega}^0, \hat{\omega}, \hat{y}^0, \hat{y}_1, \hat{y}_2, \hat{v})$, where $\hat{T} = 3$ and $(\hat{\omega}^0, \hat{\omega})$ is given by

$$(\hat{\omega}^0, \hat{\omega})(t) := \begin{cases} (1, 0) & t \in [0, 1] \\ (0, 1) & t \in]1, 3], \end{cases}$$

so that one has

$$(\hat{y}^0, \hat{y}^1, \hat{y}^2, \hat{v})(t) = \begin{cases} (t, -1, -1 + 2t, 0) & t \in [0, 1] \\ (1, -2 + t, 1, t - 1) & t \in]1, 3]. \end{cases}$$

It is easy to see that \hat{z} is a minimizer for (P_e) as it is feasible, and its corresponding cost is equal to zero. Moreover, at \hat{z} , there is a type-E local infimum gap since \bar{z} defined in the previous step is the unique feasible strict-sense process. Indeed, a set of abnormal multipliers corresponding to \hat{z} is given by $(p_0, p, \gamma, \pi, \mu, m_0, m)$, where $\gamma = \pi = 0, p_0 \equiv 0, \mu(\{0\}) = 2, \mu(]0, 1]) = 0, p = (p_1, p_2) \equiv (-2, 0), m_0 \equiv 0, m(0) = (m_1, m_2)(0) = (1, 0)$, and $m(t) = (m_1, m_2)(t) \in \partial^>h(\hat{y}^1(t), \hat{y}^2(t))$ for any $t \in]0, 1[$.

5. Proof of Theorem 1

First, we point out that by utilizing standard cutoff procedures, we may assume, without loss of generality, that hypotheses (H2) and (H3) hold, replacing Σ_θ with \mathbb{R}^{1+n} . In the proofs, we utilize extended trajectories lying in an L^∞ -tube around the reference trajectory \bar{y} , and the control functions take values in compact sets. Therefore, the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$ associated with (1) is well defined and continuous (actually, uniformly continuous).

5.1. Proof of Statement (i)

If \bar{z} is a local Ψ -minimizer for (P_e) , the fact that it is an extremal can be easily derived from [22] (Theorem 9.3.1). Proving that whenever there is a type-E local infimum gap at \bar{z} , it is an abnormal extremal, instead requires a careful adaptation of the reasoning employed in the proof in [8] (Theorem 2.1), where the same result was obtained for the notion of a type-E local infimum gap, in which the distance \mathbf{d} between the controls was replaced with the L^∞ -distance of the trajectories. Specifically, the proof is structured as follows. In the first step, we construct a sequence of optimization problems (\hat{P}_i) over strict-sense processes with the controls taking values in $V_i \times A$, where V_i is as in (H1) and the cost function penalizes processes that violate the endpoint and the state constraint. Hence, we build another sequence of optimal control problems, say (P_i) , by suitably perturbing (\hat{P}_i) . Finally, by applying the Ekeland principle, we find a sequence (z_i) of minimizers for (P_i) that converges to the reference process $\bar{z} = (\bar{\omega}, \bar{\alpha}, \bar{y})$. In the second step of the proof, we write the necessary conditions satisfied by each z_i , whereas in the third step, we pass to the limit in these conditions, obtaining a set of abnormal multipliers for \bar{z} .

Step 1. Define the function $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, given by

$$\Phi(x, c) := d_{\mathcal{T}}(x) \vee c$$

and for any $y \in W^{1,1}([0, T], \mathbb{R}^n)$ we set

$$\mathcal{J}(y) := \Phi\left(y(T), \max_{t \in [0, T]} h(t, y(t))\right).$$

Let $(\varepsilon_i)_i$ be a sequence converging to 0, and let $(\rho_i)_i$ be such that

$$\rho_i^2 = \sup\{\mathcal{J}(y) : z = (\omega, \alpha, y) \in \Gamma_s, \mathbf{d}(z, \bar{z}) \leq \varepsilon_i\}.$$

By the uniform continuity of the input-output map and the Lipschitz continuity of Φ , it follows that $\lim_{i \rightarrow +\infty} \rho_i^2 = 0$. Moreover, $\rho_i > 0$ as soon as i is sufficiently large, as \bar{z} is an isolated process by Remark 4.

By (H1) and Remark 1, for any i , there exists a closed subset $V_{\varepsilon_i} \subset V$ and a control $\hat{\omega}_i \in \mathcal{V}_{\varepsilon_i} := L^1([0, T], V_{\varepsilon_i})$ such that $\|\hat{\omega}_i - \bar{\omega}\|_{L^1} \leq \varepsilon_i$. Hence, let $\hat{z}_i = (\hat{\omega}_i, \hat{\alpha}_i, \hat{y}_i)$ be such that $\hat{\alpha}_i \equiv \bar{\alpha}$ and $\hat{y}_i = y[\hat{\omega}_i, \hat{\alpha}_i]$. As a consequence, \hat{z}_i is a ρ_i^2 -minimizer for the optimization problem (\hat{P}_i) , given by

$$(\hat{P}_i) \left\{ \begin{array}{l} \text{Minimize } \mathcal{J}(y) \\ \text{over } z = (\omega, \alpha, y) \in \Gamma^i \end{array} \right.$$

where

$$\Gamma^i := \{ (\omega, \alpha, y) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A} \times W^{1,1}([0, T], \mathbb{R}^n) \text{ satisfying (1)} \}.$$

It is easy to show that if we equip Γ^i with the distance \mathbf{d} , it turns out to be a complete metric space. Accordingly, by applying Ekeland’s variational principle, we deduce that there exists $z_i = (\omega_i, \alpha_i, y_i) \in \Gamma^i$, which is a minimizer for the optimal control problem (P_i) , given by

$$(P_i) \left\{ \begin{array}{l} \text{Minimize } \mathcal{J}(y) + \rho_i \int_0^T [|\omega(t) - \omega_i(t)| + \vartheta_i(t, \alpha(t))] dt \\ \text{over } z = (\omega, \alpha, y) \in \Gamma^i, \end{array} \right.$$

where $\vartheta_i : [0, T] \times A$ is defined as

$$\vartheta_i(t, a) := \begin{cases} 0 & \text{if } a = \alpha_i(t) \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, one has $\mathbf{d}(z_i, \hat{z}_i) \leq \rho_i$ so $\mathbf{d}(z_i, \bar{z}) \leq \rho_i + \varepsilon_i \rightarrow 0$. In particular, it holds that

$$\omega_i \rightarrow \bar{\omega} \text{ in } L^1, \quad \ell(\{t \in [0, T] : \alpha_i(t) \neq \bar{\alpha}(t)\}) \rightarrow 0. \tag{21}$$

Furthermore, since the input-output map $(\omega, \alpha) \mapsto y[\omega, \alpha]$ is continuous, one has

$$y_i \rightarrow \bar{y} \text{ in } L^\infty, \quad \dot{y}_i \rightharpoonup \dot{\bar{y}} \text{ weakly in } L^1. \tag{22}$$

By the previous convergence analysis and, since \bar{z} is isolated, one has $\mathcal{J}(y_i) > 0$ for any i . Therefore, possibly passing to a subsequence, for any i , we have

$$\text{either } d_{\mathcal{T}}(y_i(T)) > 0 \text{ or } c_i := \max_{t \in [0, T]} h(t, y_i(t)) > 0. \tag{23}$$

Step 2. From the above reasoning, it follows that

$$(z_i, c_i) = (\omega_i, \alpha_i, y_i, \max_{t \in [0, T]} h(t, y_i(t)))$$

is a minimizer for the optimal control problem (Q_i) , given by

$$(Q_i) \begin{cases} \text{Minimize } \left(d_{\mathcal{T}}(y(T)) \vee c(T) \right) + \rho_i \int_0^T [|\omega(t) - \omega_i(t)| + \vartheta_i(t, \alpha(t))] dt \\ \text{over } (\omega, \alpha, y, c) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A} \times W^{1,1}([0, T], \mathbb{R}^{n+1}) \text{ satisfying} \\ (\dot{y}(t), \dot{c}(t)) = (\mathcal{F}(t, y(t), \omega(t), \alpha(t)), 0) \quad \text{a.e. } t \in [0, T], \\ y(0) = \check{x}_0, \\ \tilde{h}(t, y(t), c(t)) := h(t, y(t)) - c(t) \leq 0 \quad \forall t \in [0, T]. \end{cases}$$

Possibly passing to a subsequence, only one of the following two cases occurs:

Case (a) : $c_i > 0$ for any i .

Case (b) : $c_i \leq 0$ for any i .

Let us first analyze **Case (a)**. Since from $h(t, y_i(t)) - c_i > 0$ it follows that $h(t, y_i(t)) > 0$, one has $\partial_{x,c}^> \tilde{h}(t, x, c) = \partial_x^> h(t, x) \times \{-1\}$. Moreover, by the max rule for subdifferentials (see, e.g., [22] (Section 5)), if $(\beta_i^1, \beta_i^2) \in \partial\Phi(y_i(T), c_i)$, there exists $\sigma_i^1, \sigma_i^2 \geq 0$ such that $\sigma_i^1 + \sigma_i^2 = 1$, $\beta_i^1 \in \sigma_i^1(\partial d_{\mathcal{T}}(y_i(T)) \cap \partial\mathbb{B})$ and $\beta_i^2 = \sigma_i^2$. Furthermore, $\sigma_i^1 = 0$ [resp. $\sigma_i^2 = 0$] whenever $d_{\mathcal{T}}(y_i(T)) < d_{\mathcal{T}}(y_i(T)) \vee c_i$ [resp. $c_i < d_{\mathcal{T}}(y_i(T)) \vee c_i$]. Thanks to the above reasoning, if we write the necessary conditions of the maximum principle satisfied by the minimizer (z_i, c_i) , we deduce that there exists $(p_i, \pi_i) \in W^{1,1}([0, T], \mathbb{R}^{n+1})$, $\lambda_i \geq 0$, $\mu_i \in NBV^+([0, T], \mathbb{R})$, $\sigma_i^1, \sigma_i^2 \geq 0$ such that $\sigma_i^1 + \sigma_i^2 = 1$ and a Borel-measurable and μ_i -integrable map $m_i : [0, T] \rightarrow \mathbb{R}^n$ satisfying conditions (i)'–(vi)' below:

- (i)' $\|p_i\|_{L^\infty} + \lambda_i + \mu_i([0, T]) + \|\pi_i\|_{L^\infty} = 1$;
- (ii)' $-\dot{p}_i(t) \in \text{co } \partial_x \{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))\}$ and $\dot{\pi}_i(t) = 0$ for a.e. $t \in [0, T]$;
- (iii)' $-q_i(T) \in \lambda_i \sigma_i^1 (\partial\Phi(y_i(T)) \cap \partial\mathbb{B})$, $\pi(0) = 0$, $-\pi(T) + \mu_i([0, T]) = \lambda_i \sigma_i^2$;
- (iv)' $m_i(t) \in \partial_x^> h(t, y_i(t))$ μ_i -a.e. $t \in [0, T]$;
- (v)' $\text{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;
- (vi)' $\int_0^T q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t)) dt$
 $\geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega(t), \alpha(t)) - \rho_i \lambda_i (|\omega_i(t) - \omega(t)| + \vartheta_i(t, \alpha(t)))] dt$
 $\geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega(t), \alpha(t)) - \rho_i \lambda_i (1 + \text{diam}(\bar{V}))] dt$
 for any $(\omega, \alpha) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A}$,

where $\text{diam}(\bar{V})$ is the diameter of the compact set \bar{V} and $q_i : [0, T] \rightarrow \mathbb{R}^n$ is defined as

$$q_i(t) := \begin{cases} p_i(t) + \int_{[0,t]} m_i(t') \mu_i(dt') & \text{if } t \in [0, T[, \\ p_i(T) + \int_{[0,T]} m_i(t') \mu_i(dt') & \text{if } t = T. \end{cases} \tag{24}$$

From (ii)' and (iii)', we deduce that $\pi_i \equiv 0$ and $\mu_i([0, T]) = \lambda_i \sigma_i^2$. Since $\|m_i\|_{L^\infty} \leq K_h$, from (iii)', we also have $\lambda_i \sigma_i^1 = |q_i(T)| \leq \|p_i\|_{L^\infty} + K_h \mu_i([0, T])$. By summing up these relations and (i)', we obtain

$$2\|p_i\|_{L^\infty} + (2 + K_h) \mu_i([0, T]) + \lambda_i \geq 1 + \lambda_i \sigma_i^1 + \lambda_i \sigma_i^2,$$

which implies $\|p_i\|_{L^\infty} + \mu_i([0, T]) \geq \frac{1}{2+K_h}$. By rescaling the multipliers, one obtains $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$ and $\lambda_i \geq 2 + K_h$.

If instead, **Case (b)** occurs, then $d_{\mathcal{T}}(y_i(T)) > 0$ for any i by (23). Hence, for $\delta > 0$ small, the process $(z_i, c_i + \delta)$ is still a minimizer for (Q_i) , and $h(t, y_i(t)) - (c_i + \delta) < 0$ for all $t \in [0, T]$. If we also write in this case the necessary conditions of optimality satisfied by the minimizer $(z_i, c_i + \delta)$, we deduce the existence of $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$ and $\lambda_i > 0$, fulfilling relations (i)'–(vi)' above for $\mu_i \equiv 0$, $\sigma_i^2 = 0$ (hence, $\sigma_i^1 = 1$). Indeed, if it were $\lambda_i = 0$, then $q_i(T) = p_i(T) = 0$, so the linearity of the adjoint equation (ii)' implies $p_i \equiv 0$, contradicting (i)'. In this case, from (iii)', we deduce $0 < \lambda_i = |q_i(T)| \leq \|p_i\|_{L^\infty}$. By summing up this

relation with (i)', we obtain $2\|p_i\|_{L^\infty} + \lambda_i > 1 + \lambda_i$, which implies $\|p_i\|_{L^\infty} > \frac{1}{2}$. By rescaling the multipliers, we have $\|p_i\|_{L^\infty} = 1$ and $\lambda_i \leq 2 \leq 2 + K_h$.

Step 3. For both **Case (a)** and **Case (b)**, we have proved that for any i , there exists $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$, $\mu_i \in NBV^+([0, T], \mathbb{R})$, and a Borel-measurable and μ_i -integrable map $m_i : [0, T] \rightarrow \mathbb{R}^n$ satisfying relations (i)–(vi) below:

- (i) $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$;
- (ii) $-\dot{p}_i(t) \in \text{co } \partial_x \{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))\}$ a.e. $t \in [0, T]$;
- (iii) $-q_i(T) \in [0, 2 + K_h] \left(\partial\Phi(y_i(T)) \cap \partial\mathbb{B} \right)$;
- (iv) $m_i(t) \in \partial_x^> h(t, y_i(t))$ μ_i -a.e. $t \in [0, T]$;
- (v) $\text{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;
- (vi) $\int_0^T q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t)) dt$
 $\geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega(t), \alpha(t)) - \rho_i(2 + K_h)(1 + \text{diam}(\bar{V}))] dt$
 for any $(\omega, \alpha) \in \mathcal{V}_{\varepsilon_i} \times \mathcal{A}$,

where $q_i : [0, T] \rightarrow \mathbb{R}^n$ is given by (24). Employing a standard convergence analysis (see [7] for more details), we deduce the existence of $(p, \mu) \in W^{1,1}([0, T], \mathbb{R}^n) \times NBV^+([0, T], \mathbb{R})$ and a Borel-measurable and μ -integrable map $m : [0, T] \rightarrow \mathbb{R}^n$ satisfying, up to a subsequence, the following conditions:

$$\begin{aligned} \mu_i \rightharpoonup^* \mu, \quad m_i(t)\mu_i(dt) \rightharpoonup^* m(t)\mu(dt), \\ p_i \rightarrow p \text{ in } L^\infty, \quad q_i \rightarrow q \text{ in } L^1, \quad \dot{p}_i \rightharpoonup \dot{p} \text{ weakly in } L^1. \end{aligned} \tag{25}$$

Therefore, using (22) and passing to the limit in conditions (i), (iv), and (v), we obtain

$$\begin{aligned} \|p\|_{L^\infty} + \mu([0, T]) = 1, \quad m(t) \in \partial_x^> h(t, \bar{y}(t)) \mu\text{-a.e. } t \in [0, T], \\ \text{spt}(\mu) \subset \{t \in [0, T] : h(t, \bar{y}(t)) = 0\}. \end{aligned}$$

Moreover, using the basic properties of subdifferentials and the fact that $\partial d_{\mathcal{T}}(x) = N_{\mathcal{T}}(x) \cap \mathbb{B}$ for any $x \in \mathcal{T}$ (see [22]), by (iii), we deduce that

$$-q(T) \in N_{\mathcal{T}}(\bar{y}(T)),$$

where $q : [0, T] \rightarrow \mathbb{R}^n$ is given by

$$q(t) := \begin{cases} p(t) + \int_{[0,t]} m(t')\mu(dt') & \text{if } t \in [0, T[\\ p(T) + \int_{[0,T]} m(t')\mu(dt') & \text{if } t = T. \end{cases}$$

Let us now derive the adjoint Equation (8). Let $\Omega_i := \{t \in [0, T] : \alpha_i(t) = \bar{\alpha}(t)\}$, so that $\ell(\Omega_i) \rightarrow 0$ by (21). Using (3) and hypothesis (H3), for a.e. $t \in \Omega_i$, we obtain

$$\begin{aligned} (-\dot{p}_i(t), \dot{y}_i(t)) &\in \left(\text{co } \partial_x \{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \bar{\alpha}(t))\}, \mathcal{F}(t, y_i(t), \omega_i(t), \bar{\alpha}(t)) \right) \\ &\subseteq \left(q_i(t) \cdot D_x \mathcal{F}(t, y_i(t), \bar{\omega}(t), \bar{\alpha}(t)) + |q_i(t)|k(t)\varphi(|\omega_i(t) - \bar{\omega}(t)|)\mathbb{B}, \right. \\ &\quad \left. \mathcal{F}(t, y_i(t), \bar{\omega}(t), \bar{\alpha}(t)) + k(t)\varphi(|\omega_i(t) - \bar{\omega}(t)|)\mathbb{B} \right) \\ &\subseteq \left(\text{co } \partial_x \{q(t) \cdot \mathcal{F}(t, y_i(t), \bar{\omega}(t), \bar{\alpha}(t)), \mathcal{F}(t, y_i(t), \bar{\omega}(t), \bar{\alpha}(t))\} \right) + r_i(t)\mathbb{B} \end{aligned}$$

where, since $\|q_i\|_{L^\infty} \leq \|p_i\|_{L^\infty} + K_h \mu_i([0, T]) \leq 1 + K_h$, the map $r_i : [0, T] \rightarrow \mathbb{R}$ is given by

$$r_i(t) = |q_i(t) - q(t)|k(t) + 2(1 + K_h)k(t)\varphi(|\omega_i(t) - \bar{\omega}(t)|).$$

By the continuity of φ , (21), and (25), we deduce that, up to a subsequence, $r_i(t) \rightarrow 0$ for a.e. $t \in [0, T]$. Moreover, it holds that

$$|r_i(t)| \leq 2(1 + K_h)(1 + \varphi(\text{diam}(\bar{V})))k(t) \in L^1.$$

Hence, by the dominated convergence theorem, $r_i \rightarrow 0$ in L^1 (in particular, $\varphi(|\omega_i - \bar{\omega}|) \rightarrow 0$ in L^1). From the compactness of trajectories theorem (see [22], Theorem 2.5.3), it follows that for a.e. $t \in [0, T]$, it holds that

$$(-\dot{p}(t), \dot{y}(t)) \in \left(\text{co } \partial_x \{q(t) \cdot \mathcal{F}(t, \bar{y}(t), \bar{\omega}(t), \bar{\alpha}(t))\}, \mathcal{F}(t, \bar{y}(t), \bar{\omega}(t), \bar{\alpha}(t)) \right)$$

Now, we conclude the proof by demonstrating (10). Let $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$ and, as a consequence of hypothesis (H1), let $(v_i)_i \subset \mathcal{V}$ satisfy $v_i \in \mathcal{V}_{\varepsilon_i}$ for each i , and $\|\omega - v_i\|_{L^1} \leq \varepsilon_i \downarrow 0$. Condition (vi) implies that

$$\int_0^T q_i(t) \cdot \dot{y}_i(t) dt \geq \int_0^T [q_i(t) \cdot \mathcal{F}(t, y_i(t), v_i(t), \alpha(t)) - \rho_i(1 + \text{diam}(\bar{V}))(2 + K_h)] dt$$

Up to a subsequence, the term on the right in the above relation converges to $\int_0^T [q(t) \cdot \mathcal{F}(t, \bar{y}(t), \omega(t), \alpha(t))] dt$ by the dominated convergence theorem. At the same time, it holds that

$$\int_0^T q_i(t) \cdot \dot{y}_i(t) dt = \int_0^T q(t) \cdot \dot{y}(t) dt + \int_0^T (q_i(t) - q(t)) \cdot \dot{y}_i(t) dt + \int_0^T q(t) \cdot (\dot{y}_i(t) - \dot{y}(t)) dt.$$

But now the second term on the right tends to zero by the dominated convergence theorem, whereas the third one converges to zero because of (22) and since q is bounded. Therefore, we have proved that for any $(\omega, \alpha) \in \mathcal{W} \times \mathcal{A}$, one has

$$\int_0^T q(t) \cdot \dot{y}(t) dt \geq \int_0^T q(t) \mathcal{F}(t, \bar{y}(t), \omega(t), \alpha(t)) dt.$$

From a measurable selection theorem, (10) immediately follows.

5.2. Proof of Statement (ii)

Let $\bar{z} = (\bar{\omega}, \bar{\alpha}, \bar{y}) \in \Gamma_s$ be a local Ψ -minimizer for (P_s) . We can derive that it is an extremal of the Pontryagin maximum principle from [22] (Theorem 9.3.1). In particular, the maximality condition (10) still holds with the maximum taken over $\bar{V} \times A$ since we assume that the dynamics function is continuous with respect to the w -variable.

If \bar{z} is a local Ψ -minimizer for (P_s) , which is not a local Ψ -minimizer for (P_e) , then, on the one hand, there exists $\delta > 0$ such that $\Psi(\bar{y}(T)) \leq \Psi(y(T))$ for any $z = (\omega, \alpha, y) \in \Gamma_s$ such that $\mathbf{d}(z, \bar{z}) \leq 2\delta$. On the other hand, taken $(\varepsilon_i)_i \subset]0, \delta[$ with $\varepsilon_i \downarrow 0$, for each i , there exists some $z_i = (\omega_i, \alpha_i, y_i) \in \Gamma_e$ such that $\mathbf{d}(z_i, \bar{z}) \leq \varepsilon_i < \delta$ and $\Psi(y_i(T)) < \Psi(\bar{y}(T))$. Hence, for any $z = (\omega, \alpha, y) \in \Gamma_s$ such that $\mathbf{d}(z_i, z) \leq \delta$, one has $\mathbf{d}(z, \bar{z}) \leq 2\delta$, so by construction, we have

$$\Psi(y_i(T)) < \Psi(\bar{y}(T)) \leq \Psi(y(T)).$$

Since the strict-sense process z is arbitrary, this proves that at z_i , there is a type-E local infimum gap for any i . Hence, by Theorem 1.(i), for any i , there exists $p_i \in W^{1,1}([0, T], \mathbb{R}^n)$, $\mu_i \in NBV^+([0, T], \mathbb{R})$, and a Borel-measurable and μ_i -integrable map $m_i : [0, T] \rightarrow \mathbb{R}^n$ satisfying conditions (i)–(vi) below:

- (i) $\|p_i\|_{L^\infty} + \mu_i([0, T]) = 1$;
- (ii) $-\dot{p}_i(t) \in \text{co } \partial_x \{q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t))\}$ a.e. $t \in [0, T]$;
- (iii) $-q_i(T) \in N_{\mathcal{T}}(y_i(T))$;
- (iv) $m_i(t) \in \partial_x^> h(t, y_i(t))$ μ_i -a.e. $t \in [0, T]$;
- (v) $\text{spt}(\mu_i) \subset \{t \in [0, T] : h(t, y_i(t)) - c_i = 0\}$;

$$(vi) \quad q_i(t) \cdot \mathcal{F}(t, y_i(t), \omega_i(t), \alpha_i(t)) = \max_{(w,a) \in \bar{V} \times A} q_i(t) \cdot \mathcal{F}(t, y_i(t), w, a) \text{ a.e. } t,$$

where $q_i : [0, T] \rightarrow \mathbb{R}^n$ is as in (24). We observe that our construction implies $\mathbf{d}(z_i, \bar{z}) \rightarrow 0$, so (21) and (22) hold true. We can thus conclude the proof employing a standard convergence analysis similar to that in Step 3 of the proof of Theorem 1.(i).

6. Concluding Remarks

In this paper, we investigate infimum gap phenomena that may occur when we pass from an optimal control problem with nonsmooth data, endpoint, and state constraints to an extended version of it in a framework that includes the impulsive extension of a class of non-coercive problems with unbounded dynamics. In particular, we consider type-E and type-S local infimum gaps. In the former, an extended minimizer has a cost that is strictly smaller than the infimum cost over close feasible strict-sense processes. In the latter, a local strict-sense minimizer does not locally minimize the extended problem. Following on from Warga's previous research but utilizing more recent perturbation techniques from nonsmooth analysis, which allow us to obtain results for non-differentiable data and an arbitrary closed set as the target, we prove that whenever there is either a type-E or a type-S local infimum gap at a process for a notion of local minimizer based on the control distance \mathbf{d} defined in (5), it satisfies a nonsmooth constrained version of the Pontryagin maximum principle in abnormal form. In contrast to previous results, where there was an 'asymmetry' between the necessary abnormality conditions derived for type-E and type-S local infimum gaps, for the extension under consideration, we obtain the same condition for both.

As a corollary, we provide sufficient conditions in the form of a normality test for the absence of local infimum gap phenomena. Although a normality test for gap avoidance might seem completely theoretical and hardly verifiable, it can actually be very useful because in certain situations, normality follows from easily verifiable criteria. These criteria take the form of constraint and endpoint qualification conditions for normality and have been extensively explored in the literature (see, e.g., [35–38] and the references therein). As shown in [7] (see also the references therein), where several explicit conditions for normality in control-affine impulsive extensions were presented, these criteria are generally weaker than those previously established for directly determining the absence of a gap.

The framework introduced in this paper may have implications for future infimum gap research in several directions. On the one hand, it may be the starting point for some generalizations, including the following: (i) Determining a higher-order maximum principle for local minimizers of the strict-sense problem and proving that in the case of a type-S local infimum gap, abnormality of the higher-order conditions also occurs. So far, results of this kind are only known for extended minimizers and type-E infimum gaps, limited to the impulsive extension case (see [39]). (ii) Exploring infimum gap phenomena for the impulsive extension of optimal control problems involving control-affine systems with time delays. Necessary optimality conditions for such systems were recently established in [40]. We point out that this line of research, conducted in collaboration with R.Vinter, could have important implications for many applications modeled as a sort of impulsive problem with delays, where impulses may occur only at some prescribed instants. For instance, applications in fed-batch fermentation [41,42] and in the impulsive control of delayed neural networks [43].

Another interesting problem might be to consider different extension procedures for classes of control systems not considered in this paper (such as distributed parameters systems or multistage problems).

Author Contributions: Conceptualization, G.F. and M.M.; Writing—original draft, G.F. and M.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the INdAM-GNAMPA Project 2023, CUP E53C22001930001, and by PRIN 2022, Prot. 2022238YY5, CUP C53D23002370006.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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