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3-Complex Symmetric and Complex Normal Weighted Composition Operators on the Weighted Bergman Spaces of the Half-Plane

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Abstract: One of the aims of this paper is to characterize 3-complex symmetric weighted composition operators induced by three types of symbols on the weighted Bergman space of the right half-plane with the conjugation $\mathcal{J}f(z) = \overline{f(\bar{z})}$. It is well known that the complex symmetry is equivalent to 2-complex symmetry for the weighted composition operators studied in the paper. However, the interesting fact that 3-complex symmetry is not equivalent to 2-complex symmetry for such operators is found in the paper. Finally, the complex normal of such operators on the weighted Bergman space of the right half-plane with the conjugation \mathcal{J} is characterized.

Keywords: weighted Bergman space; weighted composition operator; 3-complex symmetric operator; complex normal operator; right half-plane

MSC: 7B38; 47B33; 47B37; 30H05



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1. Introduction

The study of complex symmetric operators was initiated by Garcia and Putinar in [1]. The class of complex symmetric operators included all normal operators, binormal operators, Hankel operators, compressed Toeplitz operators and Volterra integration operators. Interestingly, complex symmetric operators have become particularly important in both theoretic and application aspects (see [2]). At the beginning, research was mainly focused on the study of complex symmetric operators on abstract Hilbert spaces (see [3–5]). As research continued, experts and scholars began to consider some special complex symmetric operators (such as composition operators and weighted composition operators) on analytic function spaces (see [6–15]).

It is worth noting that Noor et al. in [16] studied complex symmetric composition operators on Hardy spaces of the right half-plane. This inspired this work to consider complex symmetric operators on function spaces over unbounded regions. The research of this paper is influenced by the work of [16], in a sense.

We first need to present the definition of conjugation in order to introduce complex symmetric operators. To this end, let H be a separable complex Hilbert space and $\mathcal{B}(H)$ the set of all bounded linear operators on H . Let \mathbb{C} be the complex plane and \bar{z} the usual conjugation of a complex number z . For an operator $T \in \mathcal{B}(H)$, let T^* denote the adjoint operator of T .

Definition 1. An operator $C : H \rightarrow H$ is said to be a conjugation if it satisfies the following conditions:

- (i) *Anti-linear:* $C(\alpha x + \beta y) = \bar{\alpha}C(x) + \bar{\beta}C(y)$, for $\alpha, \beta \in \mathbb{C}$ and $x, y \in H$;
- (ii) *Isometric:* $\|C(x)\| = \|x\|$, for all $x \in H$;
- (iii) *Involutive:* $C^2 = I_d$, where I_d is an identity operator.

Actually, there exist many conjugations on function spaces. For example, $\mathcal{J}f(z) = \overline{f(\bar{z})}$ is a conjugation on weighted Bergman space of the right half-plane. One can see [1] for more information about conjugations.

How are complex symmetric operators defined? See the definition below.

Definition 2. Let C be a conjugation on H . An operator $T \in \mathcal{B}(H)$ is said to be complex symmetric with C if $CTC = T^*$.

Helton in [17] generalized the definition of complex symmetric operators and initiated the study of operators $T \in \mathcal{B}(H)$, which satisfy an identity of the form

$$\sum_{j=0}^m (-1)^{m-j} C_m^j T^{*j} T^{m-j} = 0, \tag{1}$$

where $C_m^j = \frac{m(m-1)\cdots(m-j+1)}{j!}$. In light of complex symmetric operators, Chō et al. in [18] gave the following definition by using the identity (1).

Definition 3. Let m be a positive integer and C a conjugation on H . An operator $T \in \mathcal{B}(H)$ is said to be m -complex symmetric with C if

$$\sum_{j=0}^m (-1)^{m-j} C_m^j T^{*j} C T^{m-j} = 0.$$

Ptak et al. in [19] introduced the complex normal operators and proved that the class of the complex normal operators properly contains complex symmetric operators. See [20,21] for the studies of such operators.

Definition 4. Let C be a conjugation on H . An operator $T \in \mathcal{B}(H)$ is said to be complex normal with C if $CT^*T = TT^*C$.

From the definition, it follows that an operator $T \in \mathcal{B}(H)$ is 3-complex symmetric with the conjugation C if and only if

$$CT^3 - 3T^*CT^2 + 3T^{*2}CT - T^{*3}C = 0.$$

It is clear that a 1-complex symmetric operator is just the complex symmetric operator. Recently, Hu et al. in [22] characterized 2-complex symmetric composition operators on Hardy space on the unit disk. From [18], or direct proof, we see that complex symmetric operators are also 2-complex symmetric operators. Hence, the set of all 2-complex symmetric operators may be larger than the set of all complex symmetric operators. However, for some special weighted composition operators, the author in [23] proved that they are indeed equivalent on weighted Bergman space of the right half-plane with the conjugation \mathcal{J} . This reminds us to question whether 2-complex symmetry is equivalent to 3-complex symmetry for such special weighted composition operators on this space with the conjugation \mathcal{J} ? In this paper, we will see that 2-complex symmetry must be 3-complex symmetry for such special weighted composition operators on this space with this conjugation, but that the converse does not necessarily hold. Furthermore, one can of course continuously consider the analogous problem for $(n - 1)$ -complex symmetric and n -complex symmetric cases.

Let $\Re z$ denote the real part of the complex number z , $\Im z$ the imaginary part of the complex number z , $\Pi = \{z \in \mathbb{C} : \Re z > 0\}$ the right half-plane and $A_\alpha^2(\Pi)$ the weighted Bergman space on Π . In preparation of this paper, we found that Hai et al. in [24] characterized the complex symmetric weighted composition operators induced by the symbols in (I), (II) and (III) with respect to conjugation \mathcal{J} . These symbols, which induce the bounded weighted composition operators on $A_\alpha^2(\Pi)$, are defined as follows:

(I)

$$u(z) = \frac{1}{(z - c)^{\alpha+2}}, \quad \varphi(z) = -a - \frac{b}{z - c},$$

where coefficients satisfy

$$\begin{cases} \text{either } \Re a = \Im b = 0, \Re b < 0, \Re c \leq 0, \\ \text{or } \Re a < 0 \leq -\Re c + \frac{\Re b + |b|}{2\Re a}. \end{cases} \quad (2)$$

(II)

$$u(z) = \frac{\delta}{(z + \mu + i\eta)^{\alpha+2}}, \quad \varphi(z) = \mu,$$

where coefficients satisfy $\delta \in \mathbb{C}$, $\mu \in \Pi$ and $\eta \in \mathbb{R}$.

(III)

$$u(z) = \lambda, \quad \varphi(z) = z + \gamma,$$

where coefficients satisfy $\lambda \in \mathbb{C}$ and $\gamma \in \Pi$.

As stated earlier, one of the main goals of the paper is to characterize 3-complex symmetric weighted composition operators induced by the symbols in (I), (II) and (III) on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . In addition, we will reveal that 2-complex symmetry is not equivalent to 3-complex symmetry for the weighted composition operators induced by the symbols in (I) on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

2. Preliminaries

Here, \mathbb{N} denotes the set of all nonzero integers and $H(\Pi)$ denotes the set of all analytic functions on Π . Throughout the paper, we always assume that $\alpha \in \mathbb{N}$. The assumption is essential, as in general, for any $w, z \in \mathbb{C}$ and $\alpha > 0$, $(wz)^\alpha \neq w^\alpha z^\alpha$ while the equality holds when $\alpha \in \mathbb{N}$.

For $\alpha \in \mathbb{N}$, let dA be the area measure on Π and $dA_\alpha(z) = \frac{2^\alpha(\alpha+1)}{\pi} (\Re z)^\alpha dA(z)$. The weighted Bergman space $A_\alpha^2(\Pi)$ consists of all $f \in H(\Pi)$, such that

$$\|f\|_{A_\alpha^2(\Pi)}^2 = \int_\Pi |f(z)|^2 dA_\alpha(z) < \infty.$$

This norm is induced by the inner product

$$\langle f, g \rangle_{A_\alpha^2(\Pi)} = \int_\Pi f(z) \overline{g(z)} dA_\alpha(z).$$

The space $A_\alpha^2(\Pi)$ is a reproducing kernel Hilbert space with the reproducing kernel

$$K_w^\alpha(z) = \frac{2^\alpha(\alpha+1)}{(z + \overline{w})^{\alpha+2}}, \quad z \in \Pi.$$

That is,

$$f(z) = \langle f, K_z^\alpha \rangle_{A_\alpha^2(\Pi)} = \int_\Pi f(w) \overline{K_z^\alpha(w)} dA_\alpha(w)$$

for any $f \in A_\alpha^2(\Pi)$ and $z \in \Pi$. One can see [25] for more information about this space.

Let $\varphi : \Pi \rightarrow \Pi$ be analytic mapping and $u \in H(\Pi)$. The weighted composition operator induced by the symbols u and φ on (or between) some subspaces of $H(\Pi)$ is defined by

$$W_{u,\varphi}f(z) = u(z)f(\varphi(z)).$$

From the definition, it follows that if $u \equiv 1$, then $W_{u,\varphi}$ is reduced to the composition operator C_φ ; if $\varphi(z) = z$, then $W_{u,\varphi}$ is reduced to the multiplication operator M_u . It can be regarded as the product of M_u and C_φ , since $W_{u,\varphi} = M_u \cdot C_\varphi$. An important thing is to provide a function-theoretic characterization that the symbols u and φ induce bounded or compact weighted composition operators. However, there is no compact composition operators on $A_\alpha^2(\Pi)$ (see [25]). Due to this possible reason, there is relatively little research about composition operators and weighted composition operators on such space. Thus, the paper can be viewed as a supplement of the studies on $A_\alpha^2(\Pi)$.

3. 3-Complex Symmetric Weighted Composition Operators

Since the linear span of the reproducing kernel functions $\{K_w^\alpha : w \in \Pi\}$ is dense in $A_\alpha^2(\Pi)$, the following result holds.

Lemma 1. *Let $T \in \mathcal{B}(A_\alpha^2(\Pi))$ and C be a conjugation on $A_\alpha^2(\Pi)$. Then, the operator T is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation C if and only if*

$$(CT^3 - 3T^*CT^2 + 3T^{*2}CT - T^{*3}C)K_w^\alpha(z) = 0$$

for all $w, z \in \Pi$.

To study 3-complex symmetric operator $W_{u,\varphi}$ induced by the symbols in (I) on $A_\alpha^2(\Pi)$, we need the formula of the adjoint of $W_{u,\varphi}$ on $A_\alpha^2(\Pi)$ (see [26]).

Lemma 2. *Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols defined in (I). Then, on $A_\alpha^2(\Pi)$, it follows that*

$$W_{u,\varphi}^* = W_{\frac{1}{(z-\bar{a})^{\alpha+2}}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}}. \tag{3}$$

Similar to Lemma 2, it is not difficult to obtain the following result (also see [23]).

Lemma 3. (a) *Let $u(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\varphi(z) = \mu$ be the symbols in (II). Then, on $A_\alpha^2(\Pi)$, it follows that*

$$W_{u,\varphi}^* = \bar{\delta}W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu}-i\eta}.$$

(b) *Let $u(z) = \lambda$ and $\varphi(z) = z + \gamma$ be the symbols in (III). Then, on $A_\alpha^2(\Pi)$, it follows that*

$$W_{u,\varphi}^* = \bar{\lambda}C_{z+\bar{\gamma}}.$$

Remark 1. *If we write $a = -\mu, b = 0$ and $c = -\mu - i\eta$, then Lemma 3 (a) can be expressed in the form of (3):*

$$W_{u,\varphi}^* = W_{\frac{\delta}{(z-\bar{a})^{\alpha+2}}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}}.$$

As was said in the Introduction, we know that if $T \in \mathcal{B}(H)$ is complex symmetric with the conjugation C , then it must be 2-complex symmetric. Actually, we have the following more general observation.

Lemma 4. Let $T \in \mathcal{B}(H)$ be complex symmetric with the conjugation C . Then, it is m -complex symmetric with the conjugation C .

Proof. From the definition, it follows that $CTC = T^*$, and then $T^{*j}CT^{m-j}C = CT^mC$. Since, for each $m \in \mathbb{N}$,

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} C_m^{2k} = \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} C_m^{2k+1},$$

we have

$$\sum_{j=0}^m (-1)^{m-j} C_m^j = 0.$$

From this, we obtain

$$\sum_{j=0}^m (-1)^{m-j} C_m^j T^{*j}CT^{m-j}C = \sum_{j=0}^m (-1)^{m-j} C_m^j CT^mC = 0.$$

From Definition 3, it follows that T is m -complex symmetric with the conjugation C . \square

By Lemma 4, 2-complex symmetric weighted composition operators on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} must be 3-complex symmetric. To further discuss their relationships, we need to characterize 3-complex symmetric weighted composition operators. To this end, we have

$$A_1 = a^3c - a^2b + b^2 + ac^3 - bc^2 + 2a^2c^2 - 3abc$$

and

$$A_2 = 2a^3c + 2a^2c^2 - 2a^2b - 3abc + b^2.$$

Now, we give and prove one of the main results in this paper.

Theorem 1. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols in (I). Then, the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $a = c$ or $a = -c$ and $b = 0$.

Proof. For all $w, z \in \Pi$, it follows from Lemma 2 that

$$\begin{aligned} \mathcal{J}W_{u,\varphi}^3 K_w^\alpha(z) &= \mathcal{J}W_{u,\varphi}^3 \left(\frac{2^\alpha(\alpha+1)}{(z+\bar{w})^{\alpha+2}} \right) = \mathcal{J}W_{u,\varphi}^2 \left(\frac{1}{(z-c)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{\left(-a - \frac{b}{z-c} + \bar{w}\right)^{\alpha+2}} \right) \\ &= \mathcal{J}W_{u,\varphi}^2 \left(\frac{2^\alpha(\alpha+1)}{\left[\left(\bar{w}-a\right)z - c\bar{w} + ac - b\right]^{\alpha+2}} \right) \\ &= \mathcal{J}W_{u,\varphi} \left(\frac{1}{(z-c)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{\left[\left(\bar{w}-a\right)\left(-a - \frac{b}{z-c}\right) - c\bar{w} + ac - b\right]^{\alpha+2}} \right) \\ &= \mathcal{J}W_{u,\varphi} \left(\frac{2^\alpha(\alpha+1)}{\left[\left(a^2 - a\bar{w} + ac - c\bar{w} - b\right)z + \left(ac - b + c^2\right)\bar{w} - ac^2 + bc + ab - a^2c\right]^{\alpha+2}} \right) \\ &= \mathcal{J} \left(\frac{2^\alpha(\alpha+1)}{\left[\left(2a\bar{b} - 2a^2c - ac^2 + bc - a^3\right)z + \left(2ac - b + c^2 + a^2\right)\bar{w}z + \left(ab + 2bc - 2ac^2 - c^3 - a^2c\right)\bar{w} + A_1\right]^{\alpha+2}} \right) \\ &= \frac{2^\alpha(\alpha+1)}{\left[\left(2a\bar{b} - 2a^2c - a\bar{c}^2 + b\bar{c} - a^3\right)z + \left(2a\bar{c} - \bar{b} + c^2 + a^2\right)wz + \left(a\bar{b} + 2b\bar{c} - 2a\bar{c}^2 - c^3 - a^2c\right)w + \bar{A}_1\right]^{\alpha+2}}, \end{aligned} \tag{4}$$

$$\begin{aligned}
 W_{u,\varphi}^* \mathcal{J} W_{u,\varphi}^2 K_w^\alpha(z) &= W_{u,\varphi}^* \mathcal{J} W_{u,\varphi} \left(\frac{2^\alpha(\alpha+1)}{[(\bar{w}-a)z - c\bar{w} + ac - b]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \mathcal{J} \left(\frac{1}{(z-c)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{[(\bar{w}-a)(-a - \frac{b}{z-c}) - c\bar{w} + ac - b]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \mathcal{J} \left(\frac{2^\alpha(\alpha+1)}{[(a^2 + ac - b)z - (a+c)\bar{w}z + (ac - b + c^2)\bar{w} + ab - a^2c - ac^2 + bc]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \left(\frac{2^\alpha(\alpha+1)}{[(\bar{a}^2 + \bar{a}\bar{c} - \bar{b})z - (\bar{a} + \bar{c})wz + (\bar{a}\bar{c} - \bar{b} + \bar{c}^2)w + \bar{a}\bar{b} - \bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c}]^{\alpha+2}} \right) \\
 &= W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{[(\bar{a}^2 + \bar{a}\bar{c} - \bar{b})z - (\bar{a} + \bar{c})wz + (\bar{a}\bar{c} - \bar{b} + \bar{c}^2)w + \bar{a}\bar{b} - \bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c}]^{\alpha+2}} \right) \\
 &= \frac{1}{(z-\bar{a})^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{[(\bar{a}^2 + \bar{a}\bar{c} - \bar{b})\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} - w(\bar{a} + \bar{c})\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} + (\bar{a}\bar{c} - \bar{b} + \bar{c}^2)w + \bar{a}\bar{b} - \bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c}]^{\alpha+2}} \\
 &= \frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + (2\bar{a}\bar{c} + 2\bar{c}^2 -)wz + (\bar{b}\bar{c} - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b} - 2\bar{a}\bar{c}^2)w + \bar{A}_2]^{\alpha+2}}, \tag{5}
 \end{aligned}$$

$$\begin{aligned}
 W_{u,\varphi}^{*2} \mathcal{J} W_{u,\varphi} K_w^\alpha(z) &= W_{u,\varphi}^{*2} \mathcal{J} \left(\frac{1}{(z-c)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{(-a - \frac{b}{z-c} + \bar{w})^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^{*2} \mathcal{J} \left(\frac{2^\alpha(\alpha+1)}{[(\bar{w}-a)z - c\bar{w} + ac - b]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{[(w-\bar{a})z - \bar{c}w + \bar{a}\bar{c} - \bar{b}]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \left(\frac{2^\alpha(\alpha+1)}{[(2\bar{a}\bar{c} -)z - 2\bar{c}wz + (2\bar{a}\bar{c} - \bar{b})w - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b}]^{\alpha+2}} \right) \\
 &= W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{[(2\bar{a}\bar{c} -)z - 2\bar{c}wz + (2\bar{a}\bar{c} - \bar{b})w - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b}]^{\alpha+2}} \right) \\
 &= \frac{1}{(z-\bar{a})^{\alpha+2}} \left(\frac{2^\alpha(\alpha+1)}{[(2\bar{a}\bar{c} -)\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} - 2\bar{c}w\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} + (2\bar{a}\bar{c} - \bar{b})w - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b}]^{\alpha+2}} \right) \\
 &= \frac{2^\alpha(\alpha+1)}{[(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + (2\bar{c}^2 + 2\bar{a}\bar{c} -)wz + (2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + \bar{a}\bar{b})w + \bar{A}_2]^{\alpha+2}}, \tag{6}
 \end{aligned}$$

and

$$\begin{aligned}
 W_{u,\varphi}^{*3} \mathcal{J} K_w^\alpha(z) &= W_{u,\varphi}^{*2} W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^{*2} \left(\frac{1}{(z-\bar{a})^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{(\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} + w)^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^{*2} \left(\frac{2^\alpha(\alpha+1)}{(wz - \bar{c}z - \bar{a}w + \bar{a}\bar{c} -)^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{(wz - \bar{c}z - \bar{a}w + \bar{a}\bar{c} -)^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \left(\frac{1}{(z-\bar{a})^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{[(w-\bar{c})\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}} - \bar{a}w + \bar{a}\bar{c} - \bar{b}]^{\alpha+2}} \right) \\
 &= W_{u,\varphi}^* \left(\frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{c} - \bar{b} + \bar{c}^2)z - (\bar{a} + \bar{c})wz + (\bar{a}\bar{c} - \bar{b} + \bar{a}^2)w - \bar{a}\bar{c}^2 - \bar{a}^2\bar{c} + \bar{a}\bar{b} + \bar{b}\bar{c}]^{\alpha+2}} \right) \\
 &= W_{\frac{1}{(z-\bar{a})^{\alpha+2}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} \left(\frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{c} - \bar{b} + \bar{c}^2)z - (\bar{a} + \bar{c})wz + (\bar{a}\bar{c} - \bar{b} + \bar{a}^2)w - \bar{a}\bar{c}^2 - \bar{a}^2\bar{c} + \bar{a}\bar{b} + \bar{b}\bar{c}]^{\alpha+2}} \right) \\
 &= \frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + (2\bar{a}\bar{c} + \bar{c}^2 - \bar{a}^2)wz + (2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)w + \bar{A}_1]^{\alpha+2}}. \tag{7}
 \end{aligned}$$

From (4)–(7) and Lemma 1, it follows that the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if

$$\begin{aligned} & \frac{1}{[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + (2\bar{a}\bar{c} - \bar{b} + \bar{c}^2 + \bar{a}^2)wz + (\bar{a} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{c}^3 - \bar{a}^2\bar{c})w + \bar{A}_1]^{\alpha+2}} \\ & + \frac{3}{[(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + (2\bar{c}^2 + 2\bar{a}\bar{c} - \bar{a}^2)wz + (2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + \bar{a}\bar{b})w + \bar{A}_2]^{\alpha+2}} \\ & = \frac{3}{[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + (2\bar{a}\bar{c} + 2\bar{c}^2 - \bar{a}^2)wz + (\bar{b}\bar{c} - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b} - 2\bar{a}\bar{c}^2)w + \bar{A}_2]^{\alpha+2}} \\ & + \frac{1}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + (2\bar{a}\bar{c} + \bar{c}^2 - \bar{a}^2)wz + (2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)w + \bar{A}_1]^{\alpha+2}} \end{aligned} \tag{8}$$

for all $w, z \in \Pi$.

Assume that the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . Letting w converge to zero in (8), we obtain

$$\begin{aligned} & \frac{1}{[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+2}} + \frac{3}{[(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+2}} \\ & = \frac{3}{[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+2}} + \frac{1}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+2}} \end{aligned}$$

for all $z \in \Pi$. That is,

$$\begin{aligned} & \frac{3}{[(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+2}} - \frac{3}{[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+2}} \\ & = \frac{1}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+2}} - \frac{1}{[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+2}} \end{aligned} \tag{9}$$

for all $z \in \Pi$. Then, by using the elementary formula

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

in (9), we obtain

$$\begin{aligned} & \frac{3\bar{b}(\bar{c} - \bar{a})F(z)}{[(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+2}[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+2}} \\ & = \frac{(\bar{c} - \bar{a})[(\bar{a} + \bar{c})^2 - \bar{b}]G(z)}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+2}[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+2}} \end{aligned} \tag{10}$$

for all $z \in \Pi$, where

$$\begin{aligned} F(z) &= [(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+1} \\ &+ [(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^\alpha [(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2] \\ &+ \dots + [(\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+1} \end{aligned}$$

and

$$\begin{aligned} G(z) &= [(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+1} \\ &+ [(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^\alpha [(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1] \\ &+ \dots + [(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+1}. \end{aligned}$$

Thus, if $a \neq c$, then (10) becomes

$$\frac{3\bar{b}F(z)}{[(\bar{b}c - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+2}[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+2}} = \frac{[(\bar{a} + \bar{c})^2 - \bar{b}]G(z)}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+2}[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+2}} \quad (11)$$

for all $z \in \Pi$. Clearly, if $b \neq (a + c)^2$, then from (11), we have

$$\frac{3\bar{b}}{(\bar{a} + \bar{c})^2 - \bar{b}} \equiv \frac{[(\bar{b}c - 2\bar{a}\bar{c}^2 - 2\bar{a}^2\bar{c} + 2\bar{a}\bar{b})z + \bar{A}_2]^{\alpha+2}[(\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - 2\bar{a}\bar{c}^2 + 2\bar{b}\bar{c})z + \bar{A}_2]^{\alpha+2}G(z)}{[(\bar{a}\bar{b} + 2\bar{b}\bar{c} - 2\bar{a}\bar{c}^2 - \bar{a}^2\bar{c} - \bar{c}^3)z + \bar{A}_1]^{\alpha+2}[(2\bar{a}\bar{b} - 2\bar{a}^2\bar{c} - \bar{a}\bar{c}^2 + \bar{b}\bar{c} - \bar{a}^3)z + \bar{A}_1]^{\alpha+2}F(z)} \quad (12)$$

for all $z \in \Pi$. From the arbitrariness of $z \in \Pi$, it follows that the function on the right-hand of (12) cannot be a constant. So, we obtain $b = (a + c)^2$. From this and (11), we have $b = 0$. From these two equations, we further obtain $a = -c$. Consequently, we prove that $a = -c$ and $b = 0$. Clearly, if $a = c$, then (10) also holds.

Conversely, if $a = c$, or $a = -c$ and $b = 0$, then from a direct calculation, it follows that (8) holds. This shows that the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . \square

In [26], the authors proved the following result.

Lemma 5. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols in (I). Then, the operator $W_{u,\varphi}$ is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $a = c$.

In order to discuss the complex symmetric difference in the weighted composition operators on $A_\alpha^2(\Pi)$, the author in [23] obtained the following result.

Lemma 6. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols in (I). Then the operator $W_{u,\varphi}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $a = c$.

These two results show that the operator $W_{u,\varphi}$ induced by the symbols in (I) is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if it is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

Corollary 1. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols in (I). If the operator $W_{u,\varphi}$ is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} , then it is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

From Theorem 1 and Lemma 5, we can give the following example.

Example 1. (a) Let $u(z) = \frac{1}{z^{\alpha+2}}$ and $\varphi(z) = \frac{b}{z}$, where $b > 0$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric and 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

(b) Let $u(z) = \frac{1}{(z+1+i)^{\alpha+2}}$ and $\varphi(z) = 1 + i - \frac{2i}{z+1+i}$. Then, the operator $W_{u,\varphi}$ is 2-complex symmetric and 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

(c) Let $u(z) = \frac{1}{(z+1+i)^{\alpha+2}}$ and $\varphi(z) = -1 - i$. Then, the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} , but it is not 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

Proof. (a) It is clear that $a = c = 0$. From condition (2), it follows that φ is an analytic self-mapping of Π . From Lemma 5, we obtain that the operator $W_{u,\varphi}$ is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . From Theorem 3.1, it follows that this operator is also 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

(b) Obviously, we have $a = c = -1 - i$. Since $\Re(-1 - i) = -1 < 0$ and $0 = 1 - 1 = -\Re(-1 - i) + \frac{\Re(2i) + |2i|}{2\Re(-1 - i)}$, u and φ satisfy condition (2), which shows that φ is an analytic self-mapping of Π . It follows from Lemma 5 and Theorem 1 that the operator $W_{u,\varphi}$ is 2-complex symmetric and 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation J .

(c) It is clear that $a = 1 + i$ and $c = -1 - i = -a$. Then, from Theorem 1, it follows that the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} , but it is not 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . \square

Corollary 2. Let $u(z) = \frac{\delta}{(z + \mu + i\eta)^{\alpha+2}}$ and $\varphi(z) = \mu$ be the symbols in (II). Then, the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\eta = 0$.

Proof. From Remark 1, we have that $a = -\mu, b = 0$ and $c = -\mu - i\eta$. Then, from Theorem 1, it follows that the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\mu = \mu + i\eta$ or $-\mu = \mu + i\eta$. Thus, we find that $\eta = 0$ or $\Re\mu = 0$. Since $\mu \in \Pi$, it is impossible that $\Re\mu = 0$. So, we find that the operator is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\eta = 0$. \square

The author in [23] also proved that the operator $W_{u,\varphi}$ induced by the symbols in (II) is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\eta = 0$. So, we have

Corollary 3. Let $u(z) = \frac{\delta}{(z + \mu + i\eta)^{\alpha+2}}$ and $\varphi(z) = \mu$ be the symbols in (II). Then, the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if it is 2-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

Theorem 2. Let $u(z) = \lambda$ and $\varphi(z) = \gamma + z$ be the symbols in (III). Then, the operator $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

Proof. For each $w, z \in \Pi$, it follows from Lemma 3 that

$$\begin{aligned} \mathcal{J}W_{u,\varphi}^3K_w^\alpha(z) &= W_{u,\varphi}^* \mathcal{J}W_{u,\varphi}^2K_w^\alpha(z) = W_{u,\varphi}^{*2} \mathcal{J}W_{u,\varphi}K_w^\alpha(z) \\ &= W_{u,\varphi}^{*3} \mathcal{J}K_w^\alpha(z) = \bar{\lambda}^3 \frac{2^\alpha(\alpha + 1)}{(3\bar{\gamma} + z + w)^{\alpha+2}}. \end{aligned}$$

This shows that $W_{u,\varphi}$ is 3-complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . \square

Remark 2. It is not difficult to see that the operator $W_{u,\varphi}$ induced by the symbols in (III) is also m -complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

4. Complex Normal Weighted Composition Operators

First, from the calculations, we have the following result.

Lemma 7. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols defined in (I). Then, on $A_\alpha^2(\Pi)$ the following statements hold.

(a)

$$\mathcal{J}W_{u,\varphi}^* W_{u,\varphi} K_w^\alpha(z) = \frac{2^\alpha(\alpha + 1)}{[(\bar{a}c + \bar{a}\bar{c} -)z - (c + \bar{c})wz + (ac - b + a\bar{c})w + \bar{a}b - |a|^2c + a_-|a|^2\bar{c}]^{\alpha+2}}.$$

(b)

$$W_{u,\varphi} W_{u,\varphi}^* \mathcal{J}K_w^\alpha(z) = \frac{2^\alpha(\alpha + 1)}{[(a\bar{c} + \bar{a}\bar{c} -)z - (a + \bar{a})wz + (\bar{a}c - b + ac)w + b\bar{c} - |c|^2a - |c|^2\bar{a} + \bar{c}]^{\alpha+2}}.$$

From the definition, we have

Lemma 8. Let $T \in \mathcal{B}(A_\alpha^2(\Pi))$ and C be a conjugation on $A_\alpha^2(\Pi)$. Then, the operator T is complex normal on $A_\alpha^2(\Pi)$ with the conjugation C if and only if

$$(CT^*T - TT^*C)K_w^\alpha(z) = 0$$

for all $w, z \in \Pi$.

Theorem 3. Let $u(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\varphi(z) = -a - \frac{b}{z-c}$ be the symbols defined in (1). Then, the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\Re a = \Re c = 0$, or $a = c$.

Proof. From Lemmas 7 and 8, it follows that the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if

$$\frac{1}{[(\bar{a}c + \bar{a}\bar{c} - \underline{z} - (c + \bar{c})wz + (ac - b + a\bar{c})w + \bar{a}b - |a|^2c + a_-|a|^2\bar{c})]^{\alpha+2}} = \frac{1}{[(a\bar{c} + \bar{a}\bar{c} - \underline{z} - (a + \bar{a})wz + (\bar{a}c - b + ac)w + b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c})]^{\alpha+2}} \quad (13)$$

for all $w, z \in \Pi$.

Assume that the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . Letting w and z converge to zero in (13), we obtain

$$[\bar{a}b - |a|^2c + a_-|a|^2\bar{c}]^{\alpha+2} = [b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c}]^{\alpha+2}. \quad (14)$$

Since $\bar{a}b - |a|^2c + a_-|a|^2\bar{c}$ and $b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c}$ are real numbers, it follows from (14) that

$$\bar{a}b - |a|^2c + a_-|a|^2\bar{c} = b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c}, \quad (15)$$

that is,

$$(\bar{a} - \bar{c})b + (a - c) = 2|a|^2\Re c - 2|c|^2\Re a. \quad (16)$$

Let $w = z$ in (13). Then, since the coefficients of z^2 are real numbers, we obtain $a + \bar{a} = c + \bar{c}$, that is, $\Re a = \Re c$. Therefore, (16) becomes

$$(a - c)(b - \underline{c}) = 2(|a|^2 - |c|^2)\Re a.$$

Letting z converge to zero in (13), we see that

$$[(ac - b + a\bar{c})w + \bar{a}b - |a|^2c + a_-|a|^2\bar{c}]^{\alpha+2} = [(\bar{a}c - b + ac)w + b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c}]^{\alpha+2} \quad (17)$$

for all $w \in \Pi$. From (17), it follows that

$$(ac - b + a\bar{c})w + \bar{a}b - |a|^2c + a_-|a|^2\bar{c} = [(\bar{a}c - b + ac)w + b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c}]\zeta(w), \quad (18)$$

where $\zeta(w)$ satisfies $[\zeta(w)]^{\alpha+2} = 1$. From (18), we obtain

$$\bar{a}b - |a|^2c + a_-|a|^2\bar{c} = (b\bar{c} - |c|^2a - |c|^2\bar{a} + \underline{c})\zeta(w).$$

Then, it follows from (15) that $\zeta(w) = 1$. Therefore, from (18) we obtain

$$ac - b + a\bar{c} = \bar{a}c - b + ac,$$

which shows that $\bar{a}c$ is a real number. Let $a = \Re a + i\Im a$ and $c = \Re c + i\Im c$. Since $\bar{a}c$ is a real number, we have

$$\Re a(\Im a - \Im c) = 0.$$

From this and $\Re a = \Re c$, we obtain $\Re a = \Re c = 0$ or $a = c$.

Conversely, if $\Re a = \Re c = 0$ or $a = c$, then it is not difficult to see that (13) holds. This shows that the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . \square

From Lemma 5 and Theorem 3, we can give the following example.

Example 2. Let $u(z) = \frac{1}{(z-i\sigma)^{\alpha+2}}$ and $\varphi(z) = -ix - \frac{y}{z-i\sigma}$, where $x, y, \sigma \in \mathbb{R}$ and $x \neq \sigma$. Then, the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} , but it is not complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

Next, we have the following two results. The first can be directly obtained by Remark 1 and Theorem 3.

Corollary 4. Let $u(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\varphi(z) = \mu$ be the symbols defined in (II). Then, the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} if and only if $\eta = 0$.

It is easy to obtain the following result. Here, the proof is omitted.

Theorem 4. Let $u(z) = \lambda$ and $\varphi(z) = z + \gamma$ be the symbols defined in (III). Then, the operator $W_{u,\varphi}$ is complex normal on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} .

5. Conclusions

The reason why we just consider these special weighted composition operators is that the proper description of the adjoint $W_{u,\varphi}^*$ of the operator $W_{u,\varphi}$ with the general symbols on $A_\alpha^2(\Pi)$ is difficult. For these special operators, we characterized 2-complex symmetry on $A_\alpha^2(\Pi)$ with the conjugation \mathcal{J} . We considered how to characterize m -complex symmetry of such operators on $A_\alpha^2(\Pi)$ with respect to \mathcal{J} or other conjugations. But, it is difficult and cumbersome. Therefore, we chose to characterize 3-complex symmetry in the paper. In addition, we also characterized the complex normality.

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