mathematics

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Edge Odd Graceful Labeling in Some Wheel-Related Graphs

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Abstract: A graph's edge labeling involves the allocation of symbols (colors or numbers) to the edges of a graph governed by specific criteria. Such labeling of a graph G with order n and size m is named edge odd graceful if there is a bijective map φ from the set of edges $E(G) = \{e_1, \ldots, e_m\}$ to the set $\{1, 3, \ldots, 2m-1\}$ in a way that the derived transformation φ^* from the vertex-set $V(G) = \{v_1, \ldots, v_n\}$ to the set $\{0, 1, 2, \ldots, 2m-1\}$ given by $\varphi^*(u) = \sum_{uv \in E(G)} \varphi(uv) \mod (2m)$ is injective. Any graph is named edge odd graceful if it permits an edge odd graceful allocation (Solairaju and Chithra). The primary aim of this study is to define and explore the edge odd graceful labeling of five new families of wheel-related graphs. Consequently, necessary and sufficient conditions for these families to be edge odd graceful are provided.

Keywords: graceful labeling; edge graceful labeling; edge odd graceful labeling; wheel graph

MSC: 05C78

1. Introduction

Let $G = (V(G), E(G))$ be a simple, connected, finite, and undirected graph, where n is the number of its vertices and m is the number of its edges. An edge labeling of G is an allocation of symbols (integers) to the edges of G, governed by specific criteria. Graph labeling, whether it is vertex or edge labeling, plays a significant role in understanding and resolving issues associated with graphs and networks across a wide range of fields. It facilitates effective representation, identification of patterns, optimization, and communication in diverse applications. As a fundamental tool in graph theory, graph labeling holds important practical implications (see [1–3]). Although graph labeling is crucial for applications, it remains an active area of research within graph theory. Two fundamental questions for edge labeling are: What is the family of graphs that admit an edge labeling? What are the necessary and sufficient conditions for these graphs to have such labeling?

The paper is organized into two main sections and multiple subsections to ensure a clear structure of the content. The introductory part offers contextual information concerning graph labeling, elucidating the importance of the present study and introducing the research problem, which focuses on investigating the edge odd graceful labeling in diverse graph families. The results section is further divided into subsections numbered Sections 2.1–2.5. Within each of these subsections, we define specific categories of graphs, namely, closed flower graphs, cog wheel graphs, triangulated wheel graphs, double crown-wheel graphs, and crown-triangulated wheel graphs, respectively. For every graph family, we provide comprehensive conditions that are both necessary and sufficient to establish their edge odd graceful nature. It is worth noting that all the graphs examined in this paper have no loops, no multi-edges, and no weighted edges.

Most of the references in the literature attribute the beginning of graph labeling to the work of Rosa [4] in 1967. In Rosa's paper, a labeling of G called β-valuation was introduced

Citation: Aljohani, M.; Daoud, S.N. Edge Odd Graceful Labeling in Some Wheel-Related Graphs. *Mathematics* **2024**, *12*, 1203. https://doi.org/10.3390/math12081203

Academic Editor: Rasul Kochkarov

Received: 10 March 2024
Revised: 11 April 2024
Accepted: 12 April 2024
Published: 17 April 2024

as an injection φ from the set of vertices $V(G)$ to the set $\{0, 1, 2, \ldots, m\}$ such that when each edge $e = uv$ is designated as $|\varphi(u) - \varphi(v)|$, the derived edges are assigned distinct symbols. The same labeling is named "graceful labeling" by Solomon W. Golomb [5].

Another kind of labeling was defined in 1991 by Gnanajothi [6]. It is called an odd graceful labeling, which is an injection φ from the set of vertices $V(G)$ to the set $\{0, 1, 2, \ldots, 2m - 1\}$ such that when each edge $e = uv$ is designated as $|\varphi(u) - \varphi(v)|$, the derived edges are assigned $\{1, 3, \ldots, 2m - 1\}$.

In 1985, Lo [7] considered a modified version of the graceful labeling of a graph G and called it edge graceful labeling. It is defined as a bijection φ from the set of edges $E(G)$ to the set $\{1, 2, \ldots, m\}$ in a way that the derived transformation φ^* from the set of vertices $V(G)$ to $\{0, 1, 2, \ldots, n - 1\}$ given by $\varphi^*(u) = \sum_{uv \in E(G)} \varphi(uv) \bmod m$ is a bijection.

In 2009, Solairaju and Chithra [8] defined a new labeling of G by combining the ideas of Gnanajothi and Lo and they call it edge odd graceful labeling. This labeling is a bijection φ from the set of edges $E(G) = \{e_1, \ldots, e_m\}$ to the set $\{1, 3, \ldots, 2m - 1\}$ in a way that the derived transformation φ^* from the vertex-set $V(G) = \{v_1, \ldots, v_n\}$ to the set $\{0, 1, 2, \ldots, 2m - 1\}$ given by $\varphi^*(u) = \sum_{uv \in E(G)} \varphi(uv) \mod (2m)$ is injective. A graph is named an edge odd graceful if it admits an edge odd graceful labeling. A graph is called graceful (resp. odd graceful, edge graceful) if it admits a labeling that is graceful (resp. odd graceful, edge graceful). For more results on edge odd graceful graphs, see [9–13].

It is not difficult to see that not all graphs are edge odd graceful. For instance, not all stars are edge odd graceful, as we can observe in the following:

Observation 1. *For $n \geq 3$, the star graph S_n is edge odd graceful if and only if n is an even integer.*

Proof. Let S_n be a star graph with the vertex set $\{u_0, u_1, \ldots, u_n\}$, where $n \geq 2$. Then $|E(S_n)| = |E(K_{1,n})| = n$. Each edge in S_n is incident with u_0 and exactly one leaf vertex u_i for $1 \leq i \leq n$. All the ways to label the edges of S_n are equivalent.

Now, each leaf vertex has the same labeling as the edge incident with it. Therefore, the outer vertices $\{u_1, \ldots, u_n\}$ receive distinct labels from $\{1, 3, \ldots, 2n - 1\}$, while the central vertex u_0 is labeled as $(1 + 3 + \cdots + 2n - 1) \bmod 2n = n^2 \bmod 2n$. Therefore, u_0 is labeled as 0 if n is even, which means that S_n is edge odd graceful in this case. However, if n is odd, $(1 + 3 + \cdots + 2n - 1) \mod 2n = k$, where k is an odd positive integer less than $2m$, this means that S_n is not edge odd graceful in this case. □

Note that identifying the outer vertices $\{u_1, u_2, \ldots, u_n\}$ of a star S_n with the vertices of an n-cycle, C_n, with vertex set $\{v_1, v_2, \ldots, v_n\}$, results in the wheel W_n. For the graph W_n, the vertices v_i and u_i are combined into a single vertex w_i. It was shown in [14] that the wheel graph W_n is edge odd graceful. This manuscript investigates some graphs that are defined in a similar way.

2. Results

2.1. Closed Flower Graphs

The first graph is the ***closed flower graph***, $\mathrm{FC_n}$. It is obtained by connecting a cycle C_n with vertex set $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and a star graph S_n with vertex set $V(S_n) = \{u_0, u_1, u_2, u_3, \ldots, u_n\}$, such that each vertex u_i is connected to vertices v_i and v_{i+1} for $1 \leq i \leq n - 1$. In addition the vertex u_n is adjacent to v_1 and v_n.

Theorem 1. *The closed flower graph, FC_n, is edge odd graceful for n greater than or equal to 3.*

Proof. Let CF_n be a closed flower graph, where n is an integer greater than or equal to 3. The size of CF_n is $|E(FC_n)| = 4n = m$. Assume that the vertex set of CF_n is $\{u_0, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$. We show that this graph is edge odd graceful by considering three different cases:

Case (1): Let FC_n be as in Figure 1 and $n \equiv k(\bmod 8), k \neq 5$.

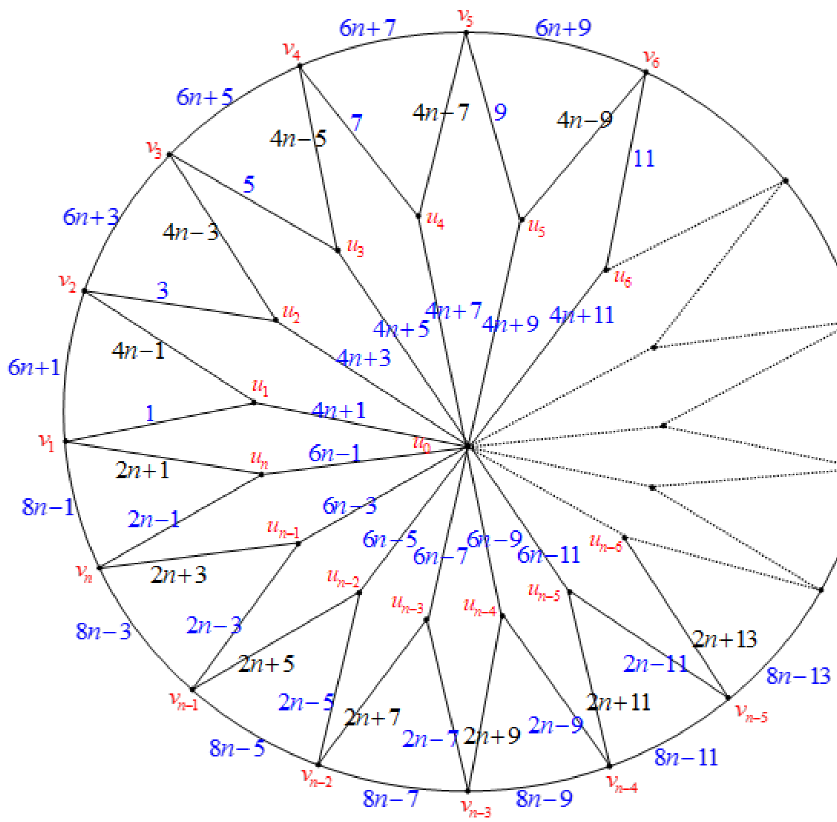

Figure 1. $FC_n, n \equiv k(\bmod 8), k \neq 5$.

Define the map

$$\varphi : E(FC_n) \to \{1, 3, \ldots, 8n-1\} \text{ by}$$

$\varphi(u_i v_i) = 2i - 1, 1 \leq i \leq n;$ $\quad\varphi(u_n v_1) = 2n + 1;$
$\varphi(u_{n-i} v_{n-i+1}) = 2n + 2i + 1, 1 \leq i \leq n - 1;$ $\quad\varphi(u_0 u_i) = 4n + 2i - 1, 1 \leq i \leq n;$
$\varphi(v_i v_{i+1}) = 6n + 2i - 1, 1 \leq i \leq n - 1;$ and $\varphi(v_n v_1) = 8n - 1.$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(u_i) = (2i - 1) \bmod (8n), 1 \leq i \leq n;$$
$$\varphi^*(v_i) = (4i - 2) \bmod (8n), 1 \leq i \leq n; \text{ and}$$
$$\varphi^*(u_0) \equiv [8n + 1 + 4n + 3 + \ldots + 6n - 1] \equiv 5n^2 \bmod (8n)$$

Furthermore,

$n \equiv 0 \bmod 8,\quad \varphi^*(u_0) = 0;$ $n \equiv 1 \bmod 8,\quad \varphi^*(u_0) = 5n;$ $n \equiv 2 \bmod 8,\quad \varphi^*(u_0) = 2n;$
$n \equiv 3 \bmod 8,\quad \varphi^*(u_0) = 7n;$ $n \equiv 4 \bmod 8,\quad \varphi^*(u_0) = 4n;$ $n \equiv 6 \bmod 8,\quad \varphi^*(u_0) = 6\,n$
$n \equiv 7 \bmod 8,\quad \varphi^*(u_0) = 3\,n.$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_n\}$ to $V_{11} = \{2, 6, 10, \ldots, 4n-6, 4n-2\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_1, u_2, \ldots, u_n\}$ to $U_{11} = \{1, 3, 5, \ldots, 2n-3, 2n-1\}$, respectively. It is evident that U_{11} and V_{11} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{01} = \{0, 2n, 3n, 4n, 5n, 6n, 7n\}$, and U_{01} has no elements in common with U_{11} or V_{11}. Consequently, φ^* is a bijective function, and the closed flower graph is edge odd graceful when $n \equiv k(\bmod 8), k \neq 5$.

Case (2): Let FC_n be as in Figure 2 and $n \equiv 5(\bmod 8)$.

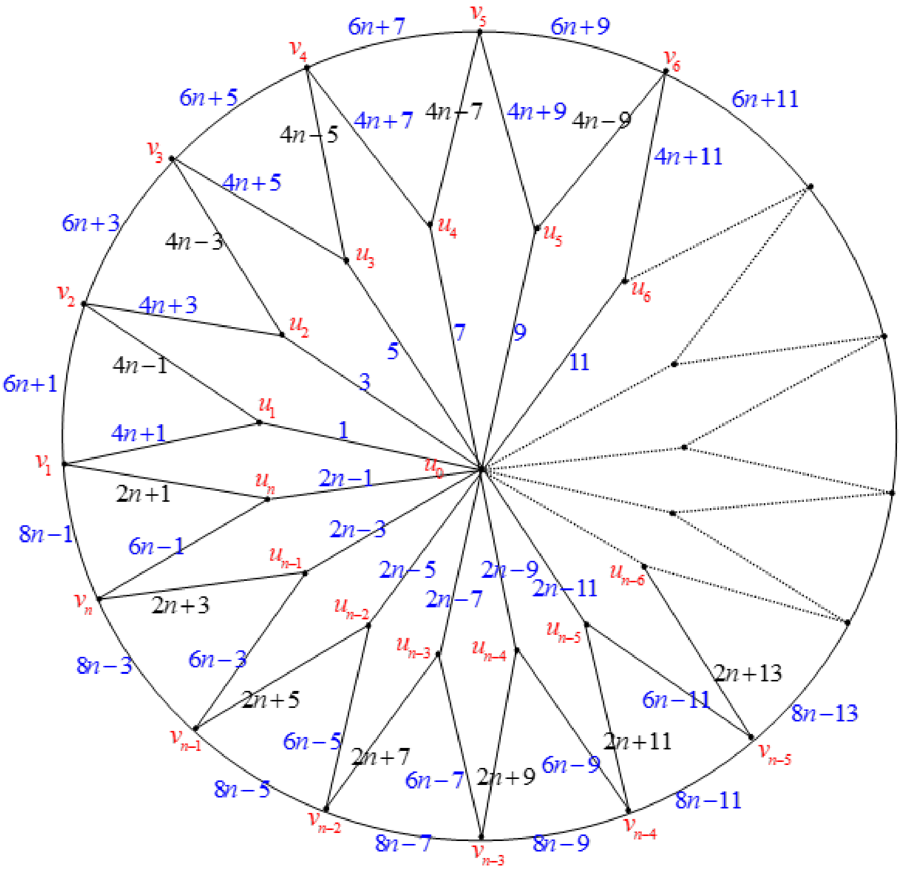

Figure 2. FC_n, $n \equiv 5(\bmod 8)$.

Define the map

$$\varphi : E(FC_n) \to \{1, 3, \ldots, 8n-1\} \text{ by}$$

$$\begin{array}{ll} \varphi(u_0u_i) = 2i-1, 1 \le i \le n; & \varphi(u_nv_1) = 2n+1; \\ \varphi(u_{n-i}v_{n-i+1}) = 2n+2i+1, 1 \le i \le n-1; & \varphi(u_iv_i) = 4n+2i-1, 1 \le i \le n; \\ \varphi(v_iv_{i+1}) = 6n+2i-1, 1 \le i \le n-1; \text{ and} & \varphi(v_nv_1) = 8n-1. \end{array}$$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\begin{aligned} &\varphi^*(u_i) = (2i-1) \bmod (8n), 1 \le i \le n; \\ &\varphi^*(v_i) = (4n+4i-2) \bmod (8n), 1 \le i \le n; \quad \text{and} \\ &\phi^*(u_0) = [1+3+\cdots 2n-1] \equiv n^2 \bmod 8n = 5n. \end{aligned}$$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_n\}$ to $V_{12} = \{4n+2, 4n+6, 4n+10, \ldots, 8n-6, 8n-2\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_0, u_1, u_2, \ldots, \}$ $\{u_n\}$ to $U_{12} = \{5n, 1, 3, 5, \ldots, 2n-3, 2n-1\}$, respectively. It is evident that U_{12} and V_{12} do not share any common elements. Consequently, φ^* is a bijective function, and the closed flower graph exhibits an edge odd graceful property when $n \equiv 5(\bmod 8)$. After considering case (1) and case (2), it can be concluded that every closed flower graph possesses the property of being edge odd graceful for values of n greater than or equal to 3. □

Examples: The closed flower graphs $CF_{14}, CF_{15}, \ldots, CF_{21}$ and their explicit labeling are depicted in Figures 3, 4, 5–10, respectively.

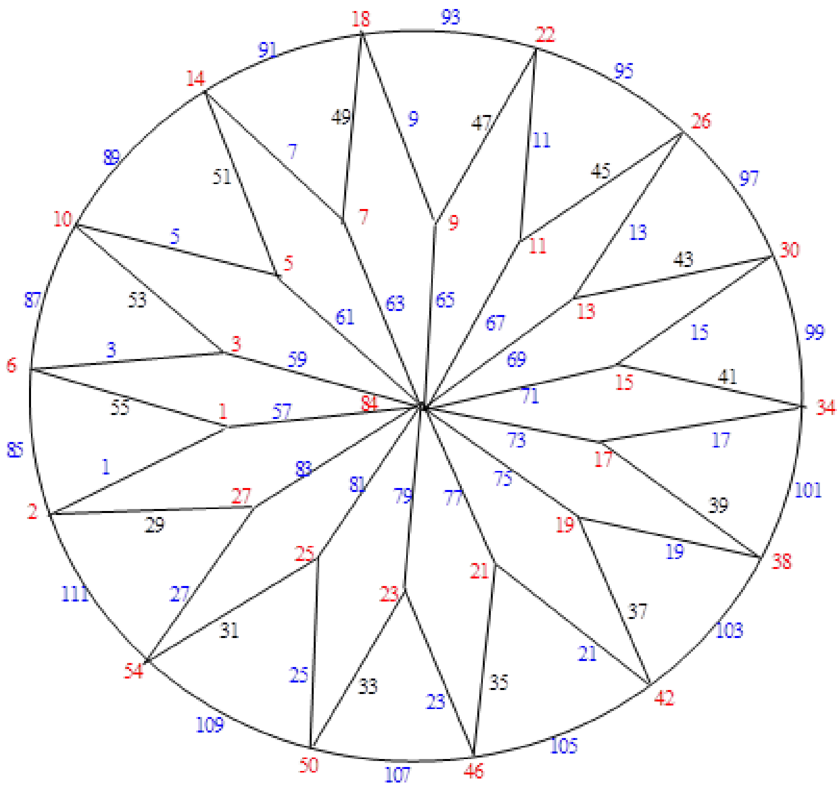

Figure 3. CF_{14}.

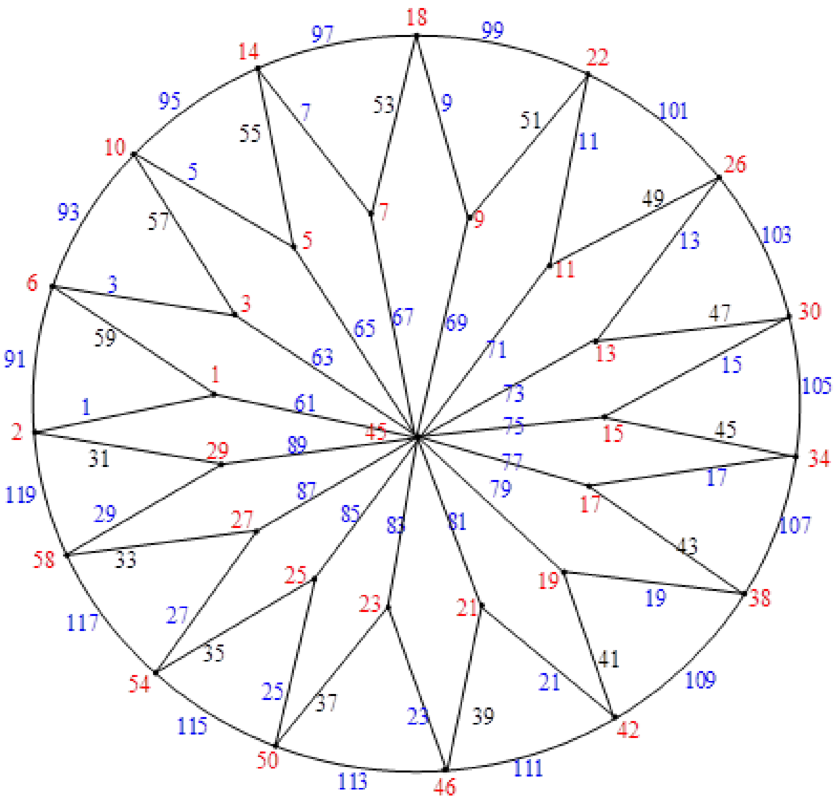

Figure 4. CF_{15}.

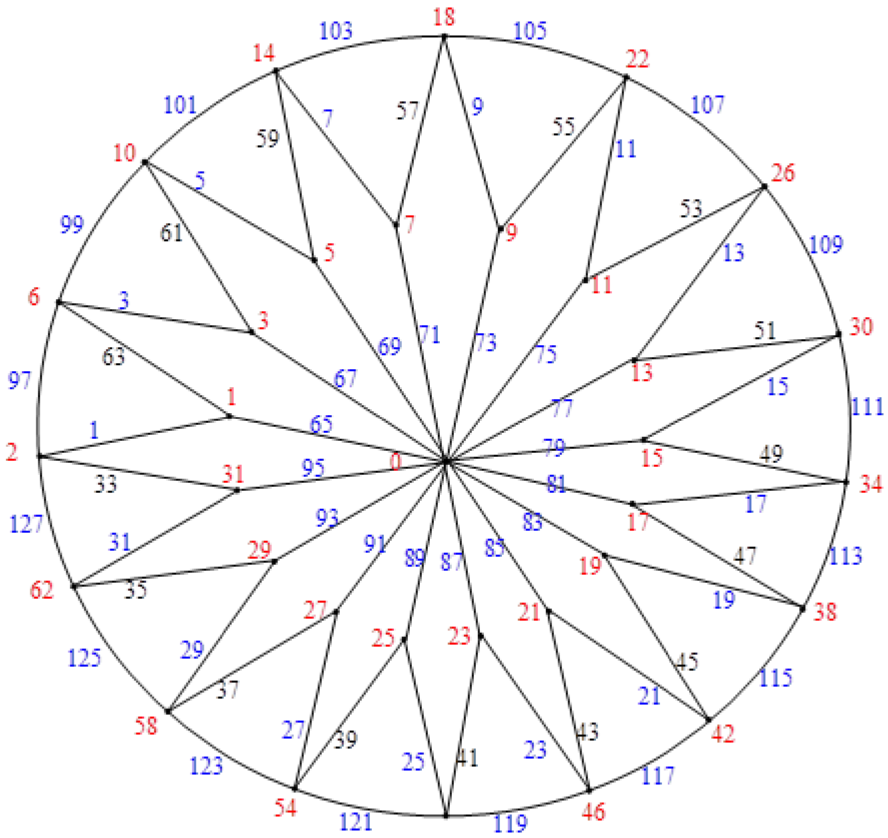

Figure 5. CF_{16}.

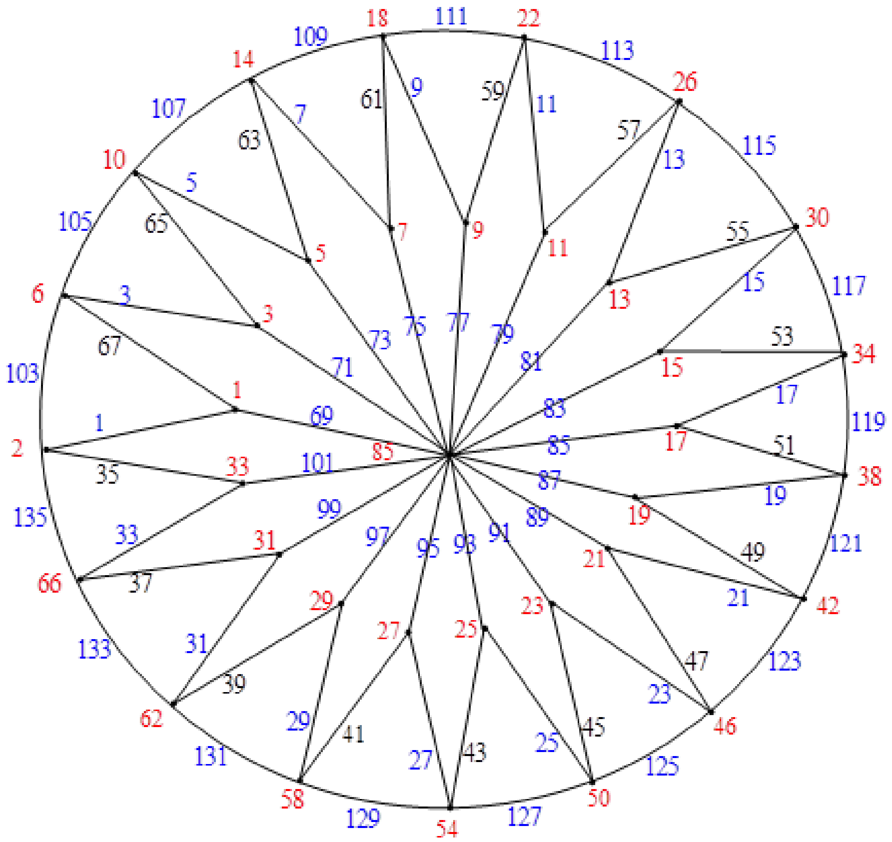

Figure 6. CF_{17}.

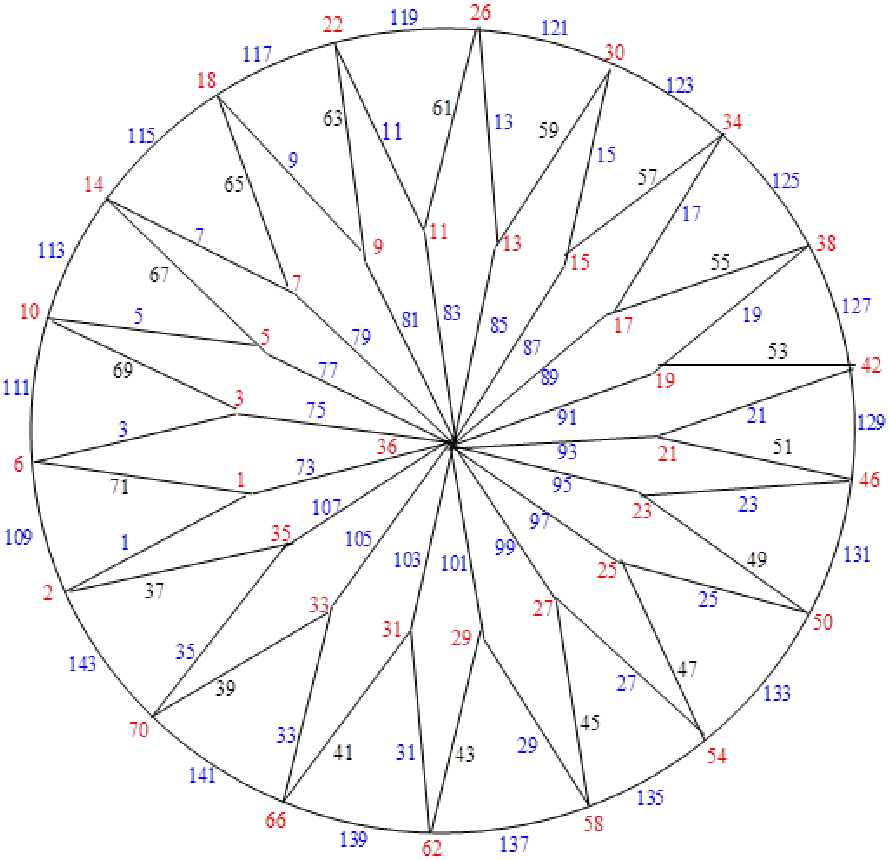

Figure 7. CF_{18}.

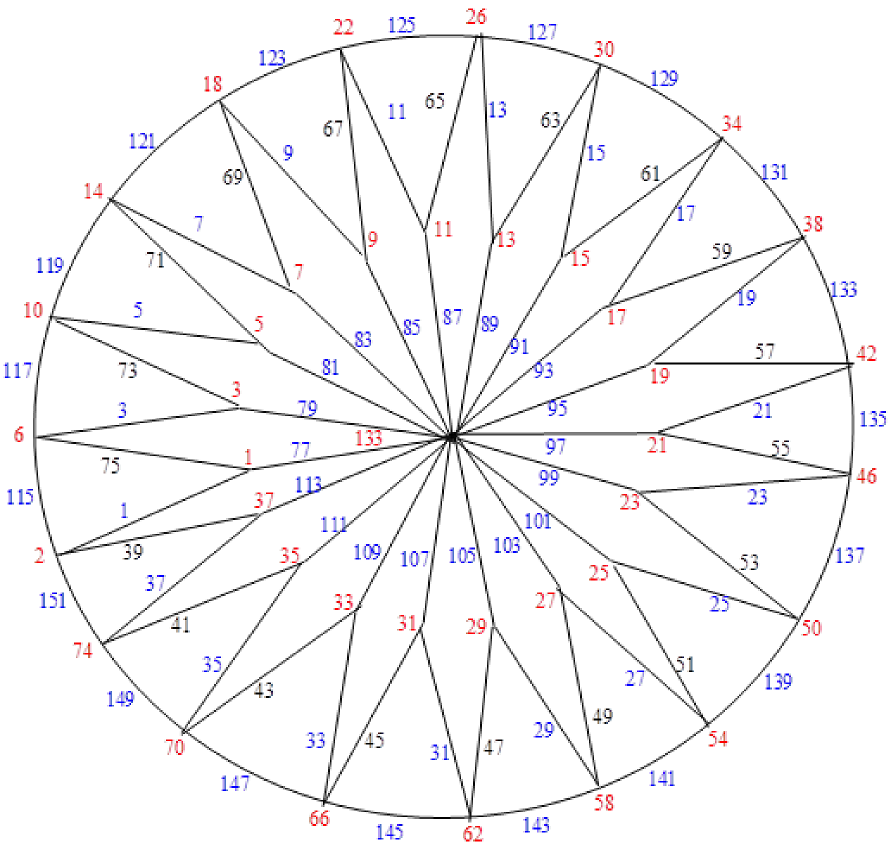

Figure 8. CF_{19}.

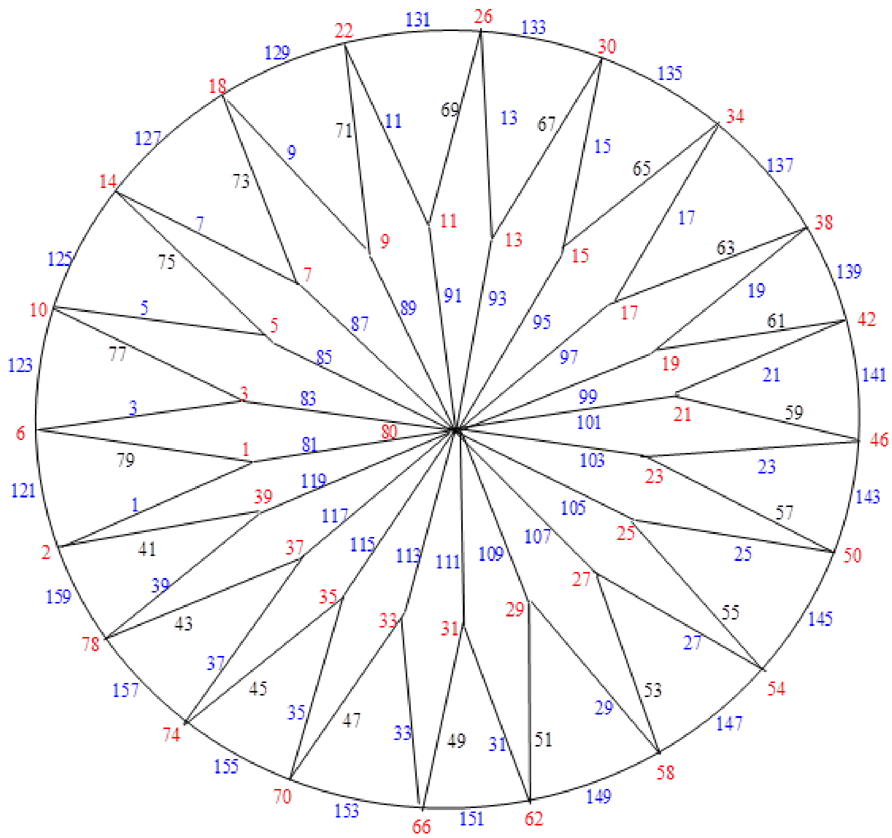

Figure 9. CF_{20}.

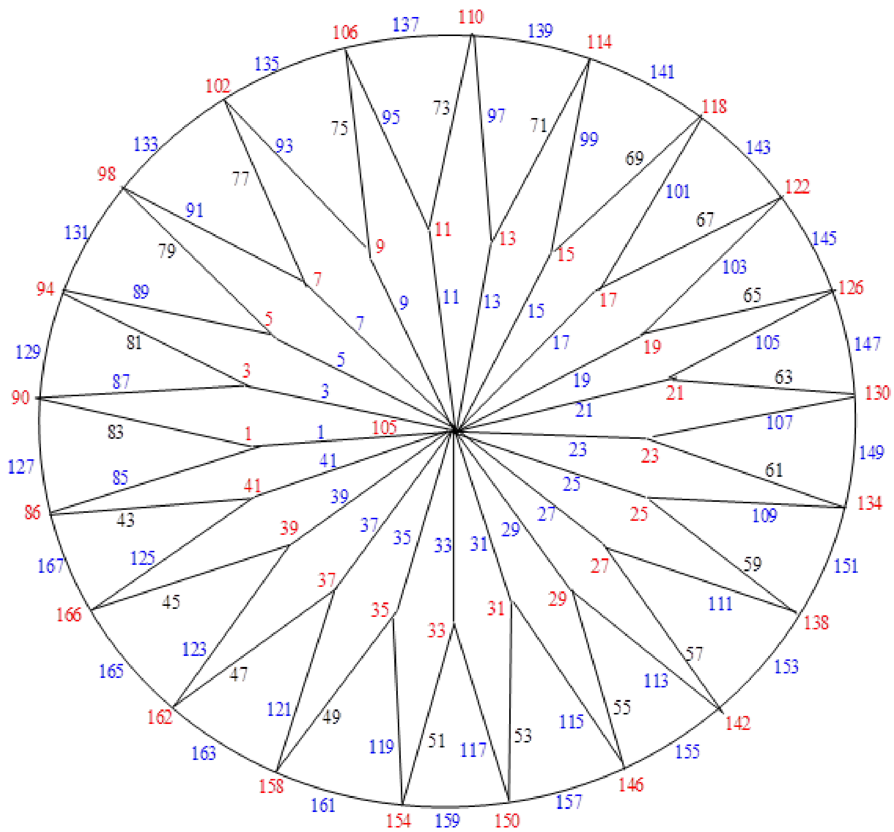

Figure 10. CF_{21}.

2.2. Cog Wheel Graphs

The second graph is the ***cog wheel graph,*** CW_n. It is obtained by combining a wheel graph $W_n = \{u_0, v_1, v_2, v_3, \ldots, v_n\}$ with a set of vertices $\{u_1, u_2, u_3, \ldots, u_n\}$ such that a vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$. Furthermore, the vertex u_n is adjacent to vertices v_1 and v_n.

Theorem 2. *The Cog wheel graph, CW_n, is edge odd graceful for n greater than or equal to* 3.

Proof. Let CW_n be a cog wheel graph where n is an integer greater than or equal to 3. It has a size of $|E(CW_n)| = 4n = m$. Assuming that the vertex set of CW_n is $\{u_0, u_1, u_2, u_3, \ldots, u_n,\}$ $\{v_1, v_2, v_3, \ldots, v_n\}$, we divide the proof into two cases and provide explicit edge labeling in each case.

Case (1): Let CW_n be as in Figure 11 and $n \equiv k(\text{mod}8), k \neq 5$.

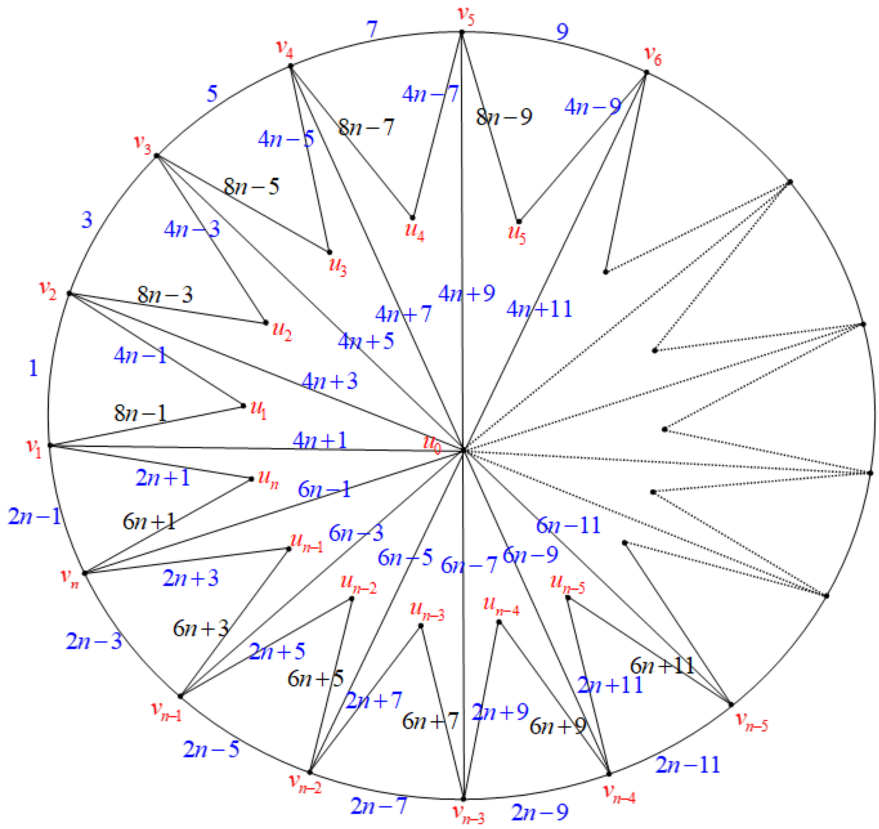

Figure 11. $CW_n, n \equiv k(\text{mod}8), k \neq 5$.

Define the map

$$\varphi : E(CW_n) \to \{1, 3, \ldots, 8n-1\} \text{ by}$$

$\varphi(v_i v_{i+1}) = 2i - 1, 1 \leq i \leq n-1;$ $\varphi(v_n v_1) = 2n - 1;$

$\varphi(u_n v_1) = 2n + 1;$ $\varphi(u_{n-i} v_{n-i+1}) = 2n + 2i + 1, 1 \leq i \leq n-1;$

$\varphi(u_0 v_i) = 4n + 2i - 1, 1 \leq i \leq n;$ and

$\varphi(u_i v_i) = 8n - 2i + 1, 1 \leq i \leq n.$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(u_i) = (4n - 4i + 2) \text{ mod } (8n), 1 \leq i \leq n;$$
$$\varphi^*(v_i) = (2i - 1) \text{ mod } (8n), 1 \leq i \leq n; \text{ and}$$
$$\varphi^*(u_0) \equiv [4n + 1 + 4n + 3 + \ldots + 6n - 1] \equiv 5n^2 \text{ mod } (8n).$$

Furthermore,

$$
\begin{aligned}
&n \equiv 0 \text{ mod } 8, \varphi^*(u_0) = 0,\\
&n \equiv 1 \text{ mod } 8, \varphi^*(u_0) = 5n,\\
&n \equiv 2 \text{ mod } 8, \varphi^*(u_0) = 2n,\\
&n \equiv 3 \text{ mod } 8, \varphi^*(u_0) = 7n,\\
&n \equiv 4 \text{ mod } 8, \varphi^*(u_0) = 4n,\\
&n \equiv 6 \text{ mod } 8, \varphi^*(u_0) = 6n,\\
&n \equiv 7 \text{ mod } 8, \varphi^*(u_0) = 3\,n.
\end{aligned}
$$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_n\}$ to $V_{21} = \{1, 3, 5, \ldots, 2n-1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_1, u_2, \ldots, u_n\}$ to $U_{21} = \{4n-2, 4n-6, \ldots, 6, 2\}$, respectively. It is evident that U_{21} and V_{21} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{02} = \{0, 2n, 3n, 4n, 5n, 6n, 7n\}$, and U_{01} has no elements in common with U_{21} or V_{21}. Consequently, φ^* is a bijective function, and the cog wheel graph is edge odd graceful when $n \equiv k(\text{mod}8), k \neq 5$.

Case (2): Let CW_n be as in Figure 12 and $n \equiv 5(\text{mod}8)$.

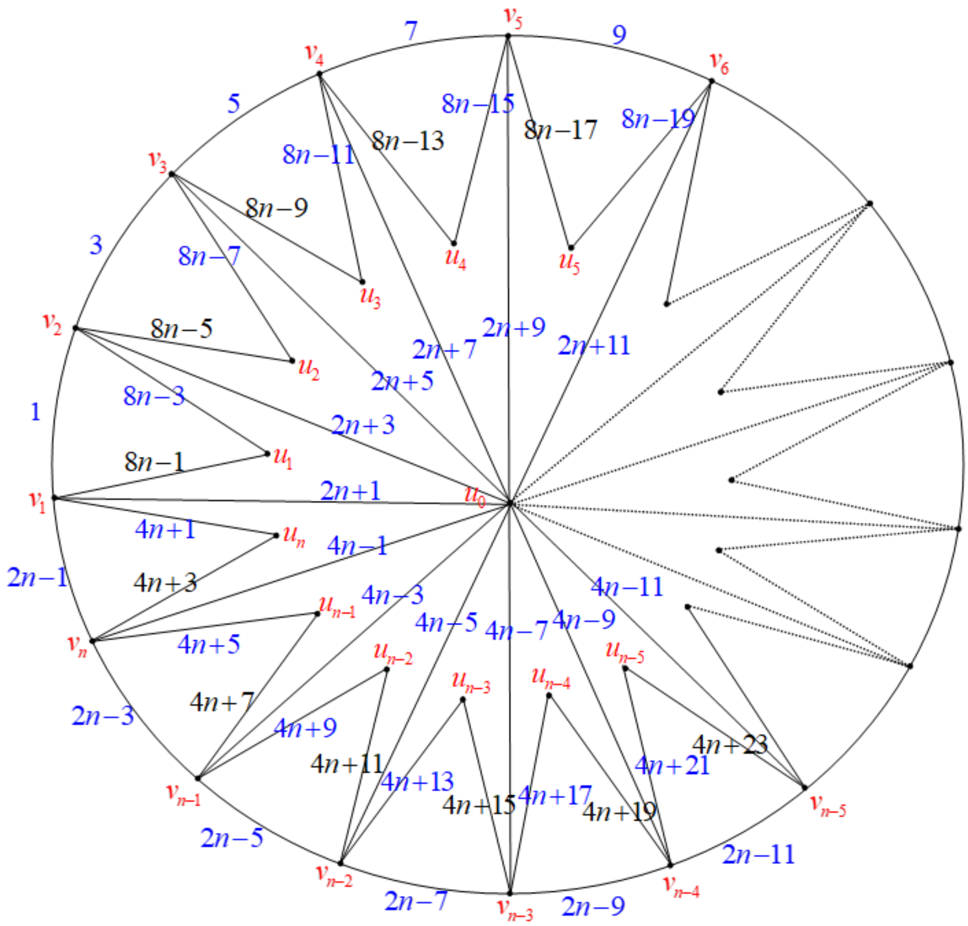

Figure 12. $CW_n, n \equiv 5(\text{mod}8)$.

Define the map

$$\varphi : E(CW_n) \to \{1, 3, \ldots, 8n-1\} \text{ by}$$

$$\varphi(v_i v_{i+1}) = 2i - 1, 1 \leq i \leq n - 1;$$
$$\varphi(v_n v_1) = 2n - 1;$$
$$\varphi(u_0 v_i) = 2n + 2i - 1, 1 \leq i \leq n;$$
$$\varphi(u_n v_1) = 4n + 1$$
$$\varphi(u_{n-i} v_{n-i+1}) = 4n + 4i + 1, 1 \leq i \leq n - 1; \text{and}$$
$$\varphi(u_{n-i} v_{n-i}) = 4n + 4i + 3, 0 \leq i \leq n - 1.$$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(u_i) = (8n - 8i + 4) \bmod (8n), 1 \leq i \leq n;$$
$$\varphi^*(v_1) = 1; \varphi^*(v_i) = ((2n - 2i + 3) \bmod (8n), 2 \leq i \leq n; \text{ and}$$
$$\varphi^*(u_0) \equiv [2n + 1 + 2n + 3 + \ldots + 4n - 1] = 3n^2 \equiv 7n \bmod (8n).$$

In this case, the map φ^* from $\{v_1, v_n, \ldots, v_2\}$ to $V_{22} = \{1, 3, \ldots, 2n - 1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_0, u_1, u_2, \ldots, u_{n-1}, u_n\}$ to $U_{22} = \{7n, 8n - 4, 8n - 12, \ldots, 12, 4\}$, respectively. It is evident that U_{22} and V_{22} do not share any common elements. Consequently, φ^* is a bijective function, and the cog wheel graph exhibits an edge odd graceful property when $n \equiv 5(\bmod 8)$. After considering case (1) and case (2), it can be concluded that every cog wheel graph possesses the property of being edge odd graceful for values of n greater than or equal to 3. □

Examples: The cog wheel graphs $CW_{16}, \cdots, CW_{23}$ and their explicit labeling are depicted in Figures 9, 13–20, respectively.

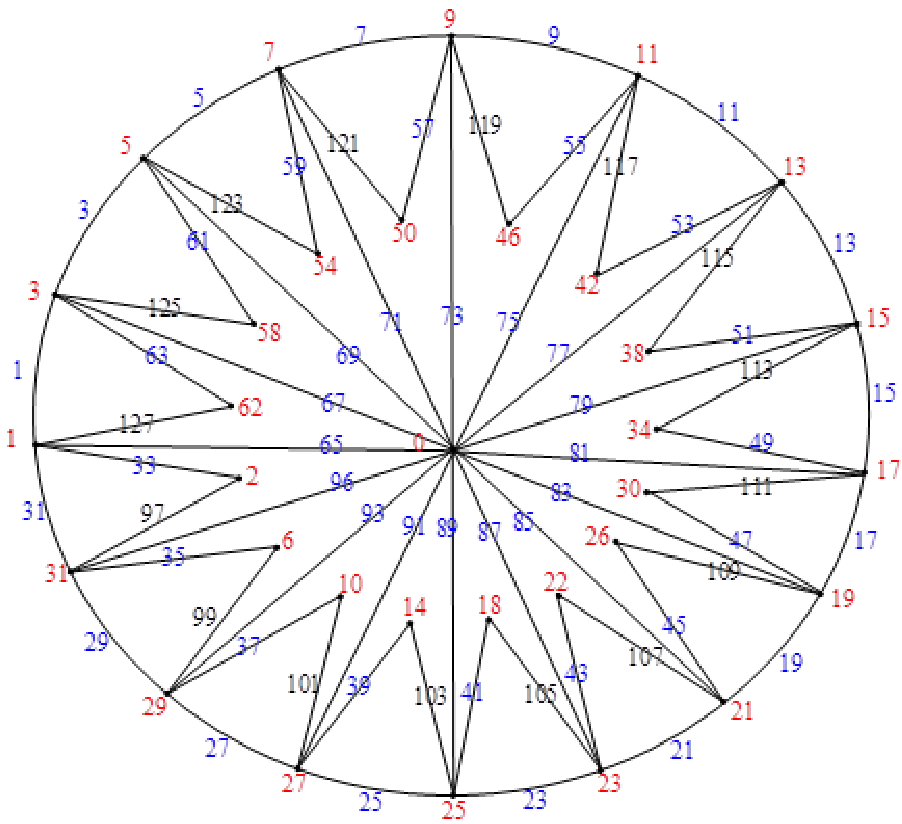

Figure 13. CW_{16}.

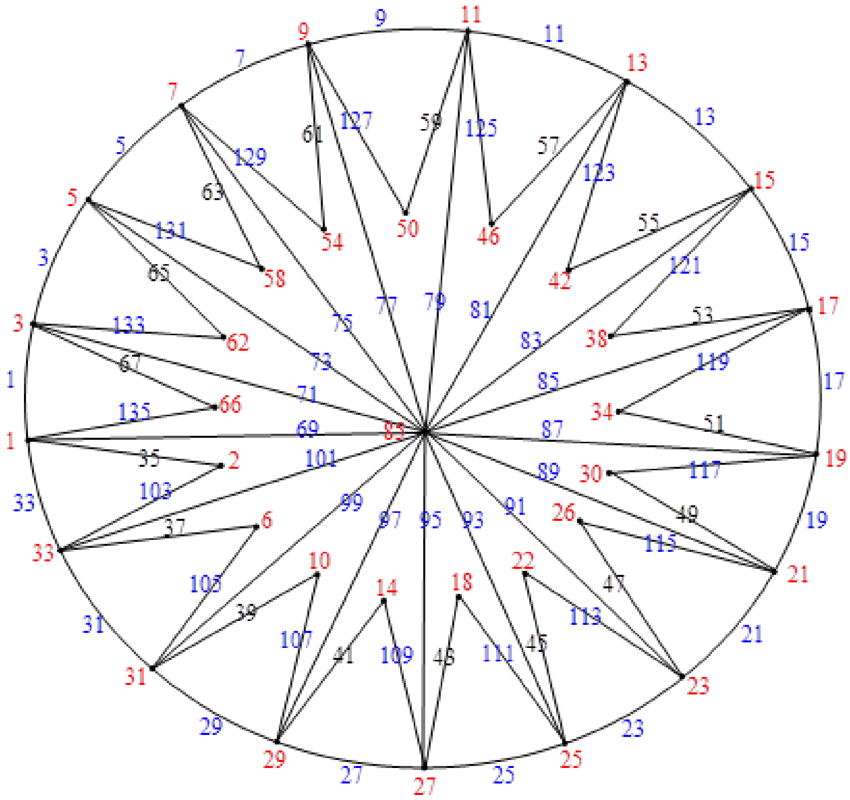

Figure 14. CW_{17}.

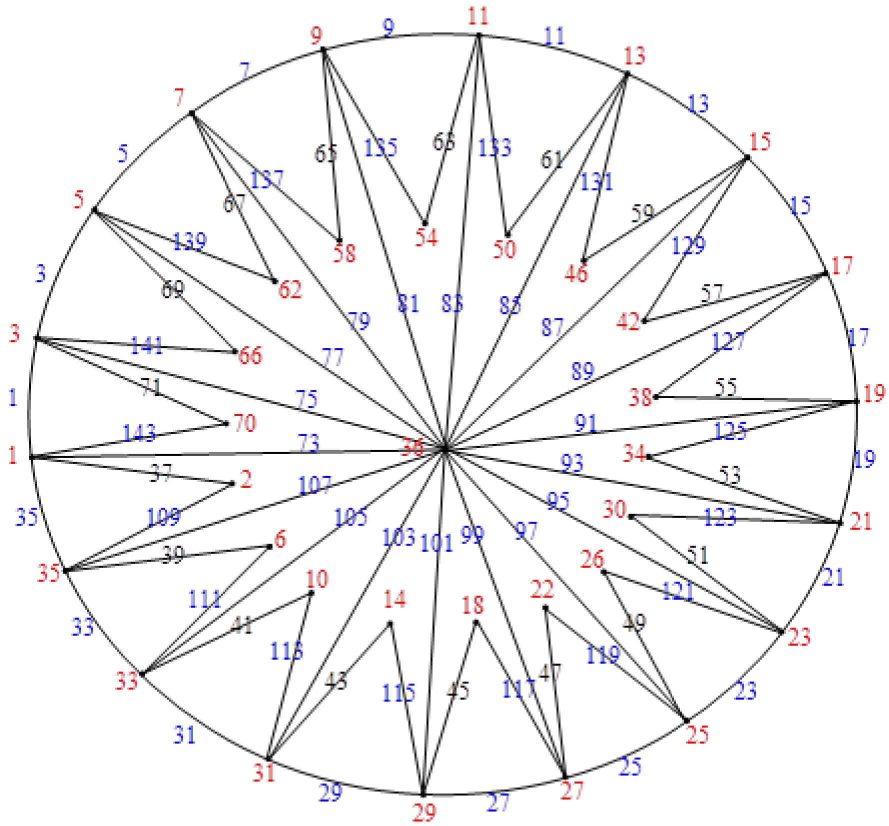

Figure 15. CW_{18}.

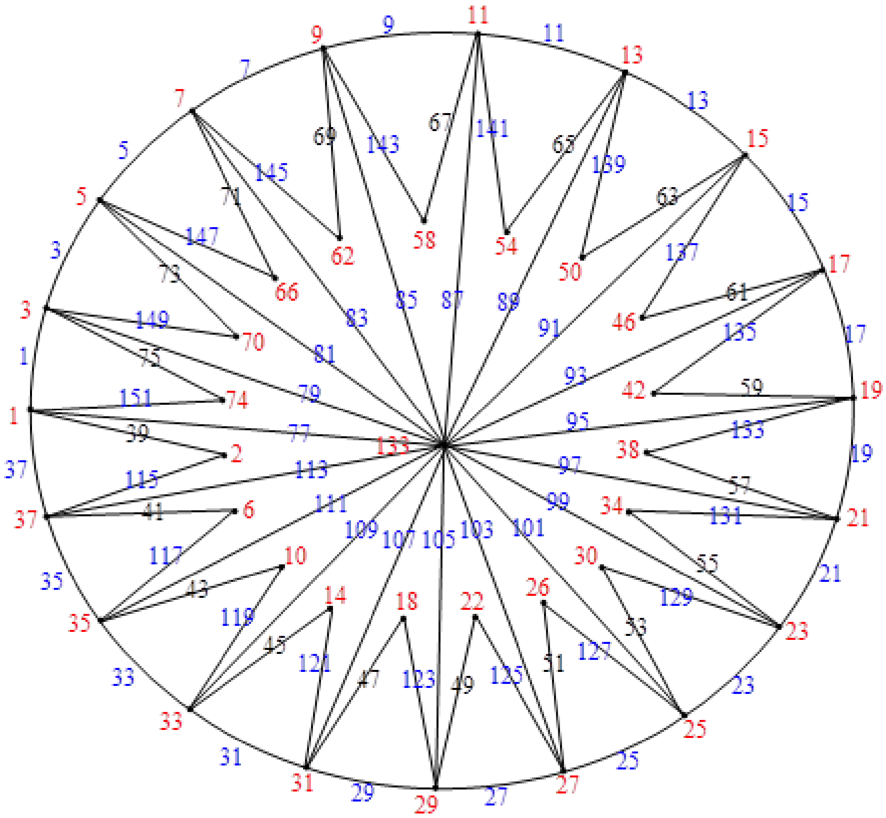

Figure 16. CW_{19}.

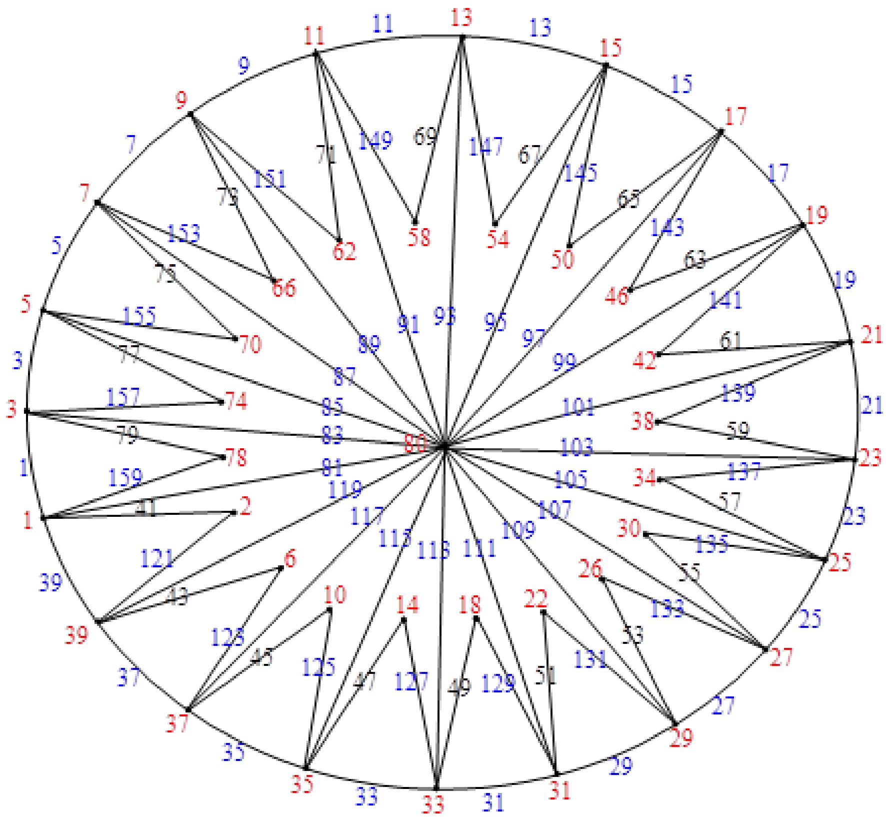

Figure 17. CW_{20}.

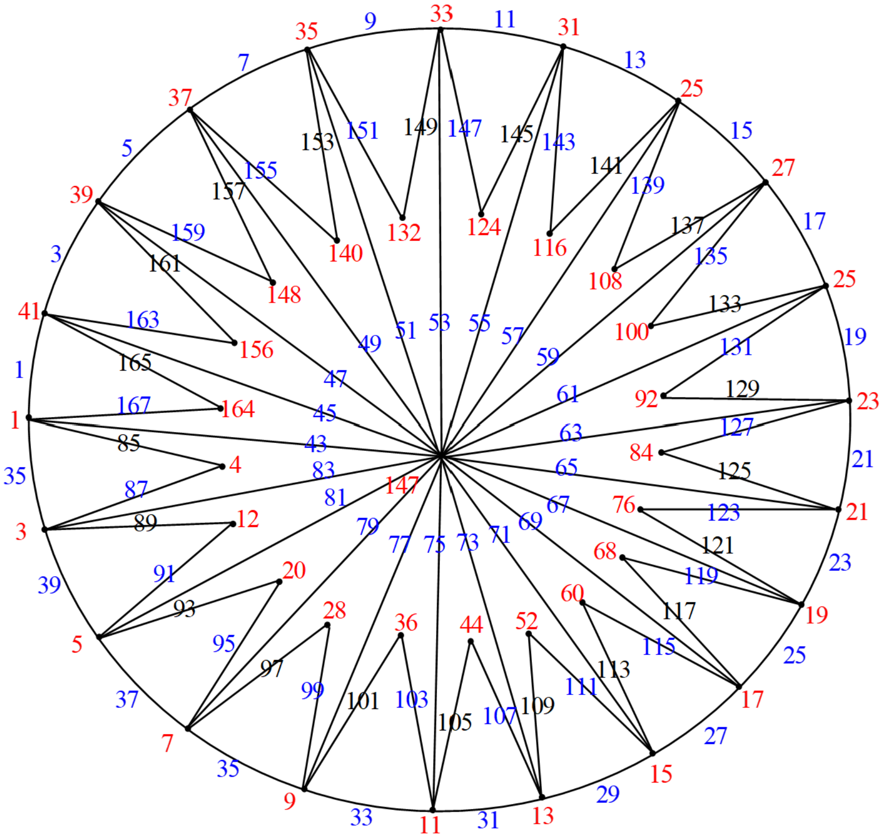

Figure 18. CW_{21}.

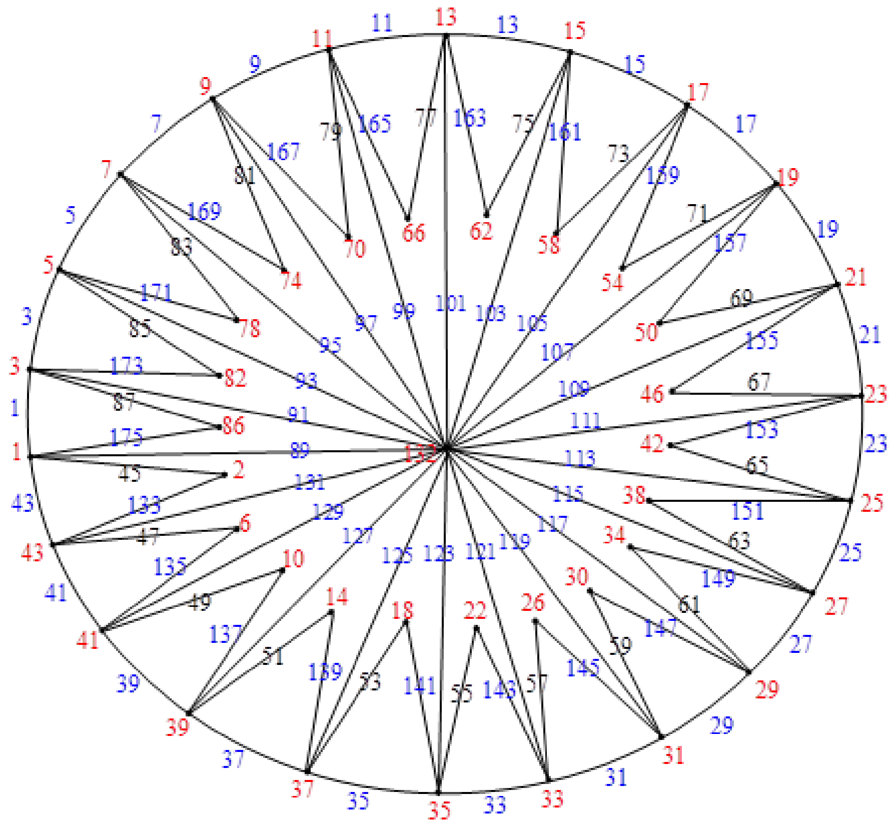

Figure 19. CW_{22}.

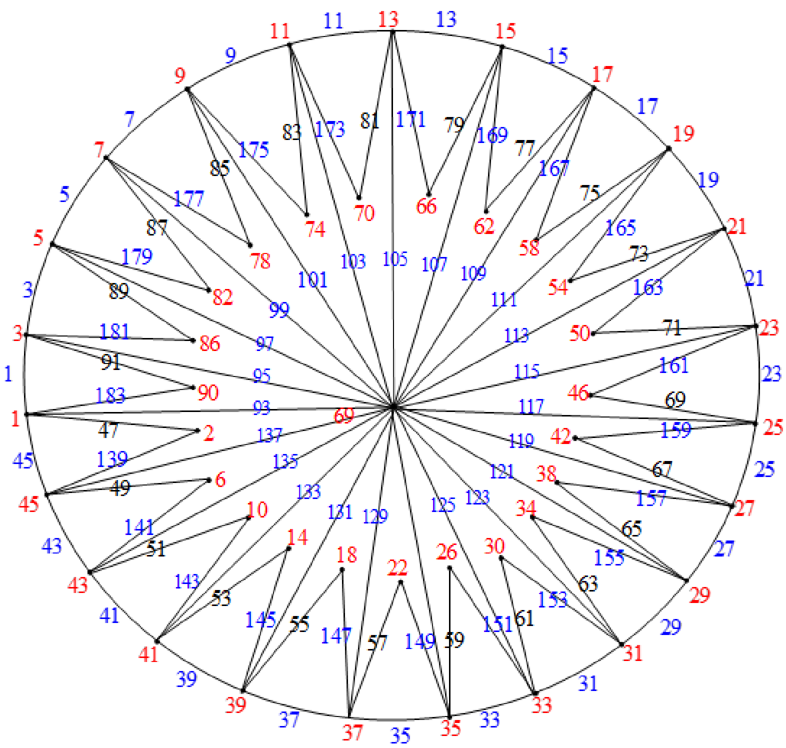

Figure 20. CW_{23}.

2.3. Triangulated Wheel Graphs

The third graph is the ***triangulated wheel graph,*** TW_n. It is obtained by combining the wheel graph $W_n = \{u_0, v_1, v_2, v_3, \ldots, v_n\}$ with a set of vertices $\{u_1, u_2, u_3, \ldots, u_n\}$ such that vertex u_i is adjacent to vertices v_i and v_{i+1}, and vertex u_n is adjacent to vertices v_1 and v_n. Additionally, each vertex u_i is adjacent to the vertex u_0.

Theorem 3. *The triangulated wheel graph, TW_n, is edge odd graceful for n greater than or equal to* 3.

Proof. Let TW_n be a triangulated wheel graph where n is an integer greater than or equal to 3. It has a size of $|E(TW_n)| = 5n = m$. Assuming that the vertex set of TW_n is $\{u_0, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$, we provide an explicit edge odd labeling in three different cases based on the number of vertices.

Case (1): Let TW_n be as in Figure 21 and $n \equiv 0 \bmod 3$.

Define the map

$$\varphi : E(TW_n) \to \{1, 3, \ldots, 10n - 1\} \text{ by}$$

$$\varphi(u_0u_i) = 2i - 1, 1 \le i \le n;$$
$$\varphi(u_0v_1) = 4n + 3;$$
$$\varphi(u_0v_2) = 4n + 1;$$
$$\varphi(u_0v_{n-i}) = 4n + 2i + 5, 0 \le i \le n - 3;$$
$$\varphi(u_{n-i}v_{n-i}) = 2n + 2i + 3, 0 \le i \le n - 2;$$
$$\varphi(u_iv_{i+1}) = 6n + 2i - 3, 2 \le i \le n - 1;$$
$$\varphi(u_nv_1) = 8n - 3 \text{ and}$$
$$\varphi(u_1v_1) = 2n + 1;$$
$$\varphi(u_1v_2) = 8n - 1;$$
$$\varphi(v_iv_{i+1}) = 8n + 2i - 1, 1 \le i \le n - 1;$$
$$\varphi(v_nv_1) = 10n - 1.$$

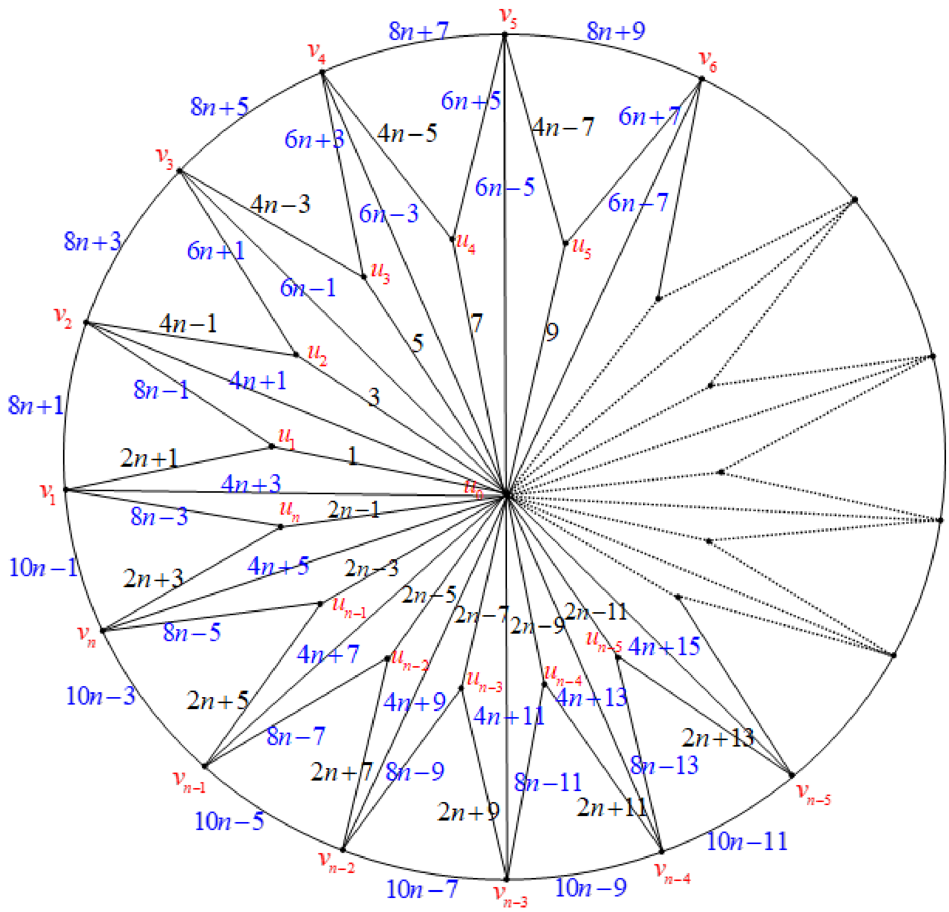

Figure 21. $TW_n, n \equiv 0(\bmod 3)$.

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$\varphi^*(u_i) = (2i-1) \bmod (10n), 1 \leq i \leq n$;
$\varphi^*(v_i) = (2n+2i-1) \bmod (10n), 1 \leq i \leq n$; and
$\varphi^*(u_0) \equiv [1+3+\ldots+2n-1] + [4n+1+4n+3+\cdots+6n-1] \equiv 6n^2 \bmod (10n)$.

Furthermore

$$\begin{aligned}
n &\equiv 0 \bmod 15, \varphi^*(u_0) = 0, \\
n &\equiv 3 \bmod 15, \varphi^*(u_0) = 8n, \\
n &\equiv 6 \bmod 15, \varphi^*(u_0) = 6n, \\
n &\equiv 9 \bmod 15, \varphi^*(u_0) = 4n, \\
n &\equiv 12 \bmod 15, \varphi^*(u_0) = 2n.
\end{aligned}$$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ to $V_{31} = \{2n+1, 2n+3, \ldots, 4n-3, 4n-1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_1, u_2, \ldots, u_n\}$ to $U_{31} = \{1, 3, \ldots, 2n-1\}$, respectively. It is evident that U_{31} and V_{31} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{031} = \{0, 2n, 4n, 6n, 8n\}$, and U_{031} has no elements in common with U_{31} or V_{31}. Consequently, φ^* is a bijective function, and the triangulated wheel graph is edge odd graceful when $n \equiv 0 \bmod 3$.

Case (2): Let TW_n be as in Figure 22 and $n \equiv 1(\mathrm{mod}3)$.

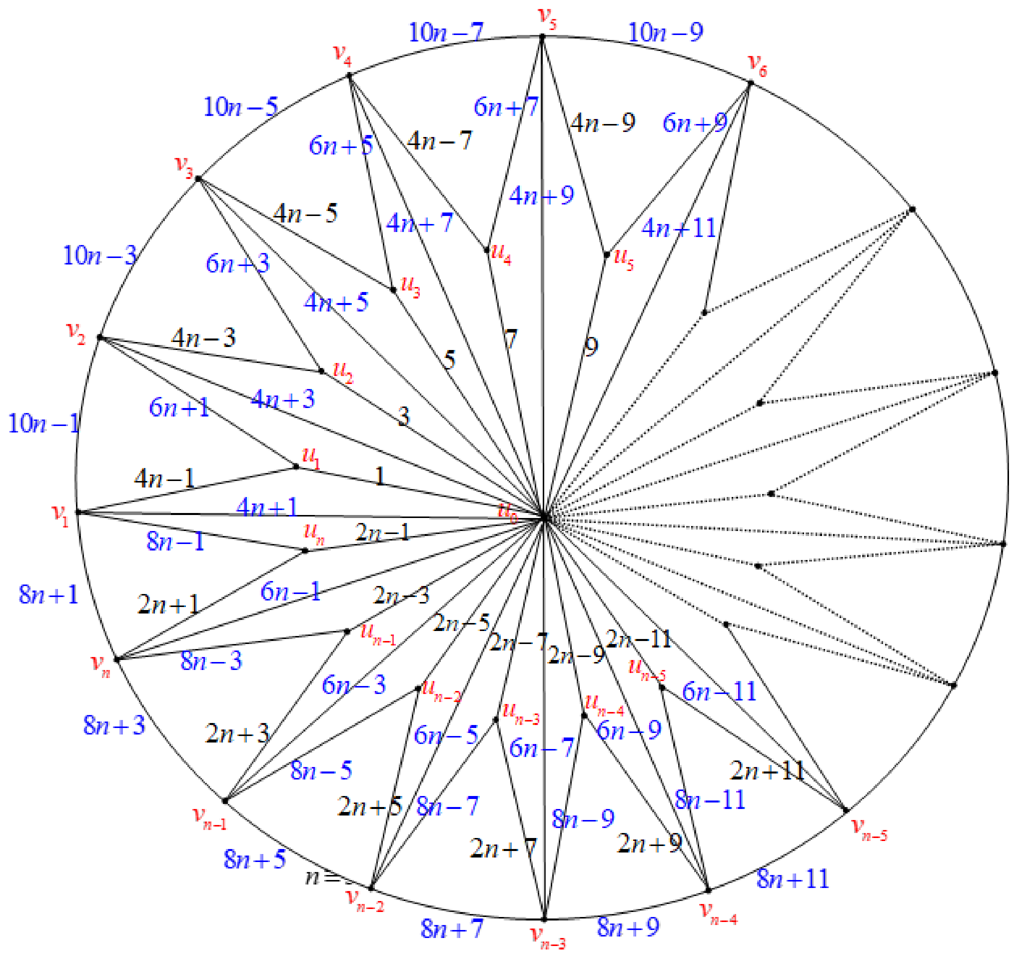

Figure 22. TW_n, $n \equiv 1(\mathrm{mod}3)$.

Define the map

$$\varphi : E(TW_n) \to \{1, 3, \ldots, 10n-1\} \text{ by}$$

$\varphi(u_0u_i) = 2i - 1, 1 \le i \le n$; $\varphi(u_{n-i}v_{n-i}) = 2n + 2i + 1, 0 \le i \le n-1$;
$\varphi(u_0v_i) = 4n + 2i - 1, 1 \le i \le n$; $\varphi(u_iv_{i+1}) = 6n + 2i - 1, 1 \le i \le n-1$;
$\varphi(u_nv_1) = 8n - 1$ $\varphi(v_1v_n) = 8n + 1$; and
$\varphi(v_{n-i}v_{n-i-1}) = 8n + 2i + 3, 0 \le i \le n-2$.

Therefore, the derived transformation ϕ^* on the set of vertices is given in the following manner:

$\varphi^*(u_i) = (2i - 1) \bmod (10n), 1 \le i \le n$;
$\varphi^*(v_{n-i}) = (2n + 2i + 1) \bmod (10n), 0 \le i \le n-1$; and
$\varphi^*(u_0) \equiv [1 + 3 + \ldots + 2n - 1] + [4n + 1 + 4n + 3 + \cdots + 6n - 1] \equiv 6n^2 \bmod (10n)$.

Furthermore,

$$n \equiv 1 \bmod 15, \varphi^*(u_0) = 6n,$$
$$n \equiv 4 \bmod 15, \varphi^*(u_0) = 4n,$$
$$n \equiv 7 \bmod 15, \varphi^*(u_0) = 2n,$$
$$n \equiv 10 \bmod 15, \varphi^*(u_0) = 0,$$
$$n \equiv 13 \bmod 15, \varphi^*(u_0) = 8n.$$

In this case, the map φ^* from $\{v_n, v_{n-1}, \ldots, v_2, v_1\}$ to $V_{32} = \{2n+1, 2n+3, \ldots, 4n-3, 4n-1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_1, u_2, \ldots, u_n\}$ to $U_{32} = \{1, 3, \ldots, 2n-1\}$, respectively. It is evident that U_{32} and V_{32} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{032} = \{0, 2n, 4n, 6n, 8n\}$, and U_{032} has no elements in common with U_{32} or V_{32}. Consequently, φ^* is a bijective function, and the triangulated wheel graph is edge odd graceful when $n \equiv 1 \bmod 3$.

Case (3): Let TW_n be as in Figure 23 and $n \equiv 2(\bmod 3)$.

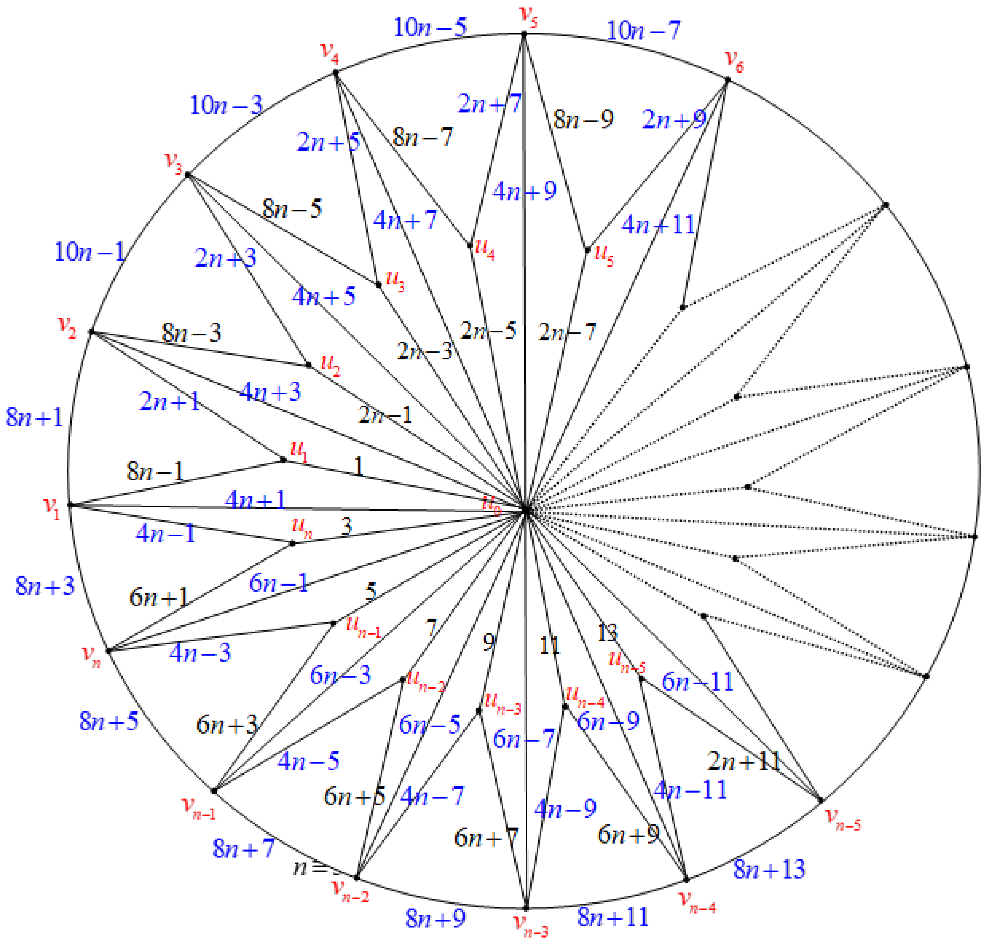

Figure 23. $TW_n, n \equiv 2(\bmod 3)$.

Define the map

$$\varphi : E(TW_n) \to \{1, 3, \ldots, 10n-1\} \text{ by}$$

$\varphi(u_0u_1) = 1$; $\varphi(u_0u_{n-i}) = 2i+3, 0 \le i \le n-2$;
$\varphi(u_0v_i) = 4n+2i-1, 1 \le i \le n$; $\varphi(u_iv_{i+1}) = 2n+2i-1, 1 \le i \le n-1$;
$\varphi(u_nv_1) = 4n-1$; $\varphi(u_{n-i}v_{n-i}) = 6n+2i+1, 0 \le i \le n-1$;
$\varphi(v_2v_1) = 8n+1$; $\varphi(v_1v_n) = 8n+3$; and
$\varphi(v_{n-i}v_{n-i-1}) = 8n+2i+5, 0 \le i \le n-3$.

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$\varphi^*(u_1) = 1$; $\varphi^*(u_i) = (2n-2i+3) \bmod (10n), 2 \le i \le n$;
$\varphi^*(v_1) = 2n+3, \varphi^*(v_2) = 2n+1$;
$\varphi^*(v_i) = (4n-2i+5) \bmod (10n), 3 \le i \le n$; and
$\varphi^*(u_0) \equiv [1+3+\ldots+2n-1] + [4n+1+4n+3+\cdots+6n-1] \equiv 6n^2 \bmod (10n)$.

Furthermore,

$$\begin{aligned}
&n \equiv 2 \bmod 15, \varphi^*(u_0) = 2n,\\
&n \equiv 5 \bmod 15, \varphi^*(u_0) = 0,\\
&n \equiv 8 \bmod 15, \varphi^*(u_0) = 8n,\\
&n \equiv 11 \bmod 15, \varphi^*(u_0) = 6n,\\
&n \equiv 14 \bmod 15, \varphi^*(u_0) = 4n.
\end{aligned}$$

In this case, the map φ^* from $\{v_3, v_4 \ldots, v_{n-1}, v_n, v_1, v_2\}$ to $V_{33} = \{4n-1, 4n-3, \ldots, 2n+7, 2n+5, 2n+3, 2n+1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{u_2, u_3, \ldots, u_n, u_1\}$ to $U_{33} = \{2n-1, 2n-3, \ldots, 3, 1\}$, respectively. It is evident that U_{33} and V_{33} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{033} = \{0, 2n, 4n, 6n, 8n\}$, and U_{033} has no elements in common with U_{33} or V_{33}. Consequently, φ^* is a bijective function, and the triangulated wheel graph is edge odd graceful when $n \equiv 2 \bmod 3$. After considering case (1), case (2), and case (3), it can be concluded that every triangulated wheel graph possesses the property of being edge odd graceful for values of n greater than or equal to 3. $\square$

Examples: The triangulated wheel graphs TW_{15}, TW_{16} and TW_{17}, and their explicit labeling are depicted in Figures 24–26, respectively.

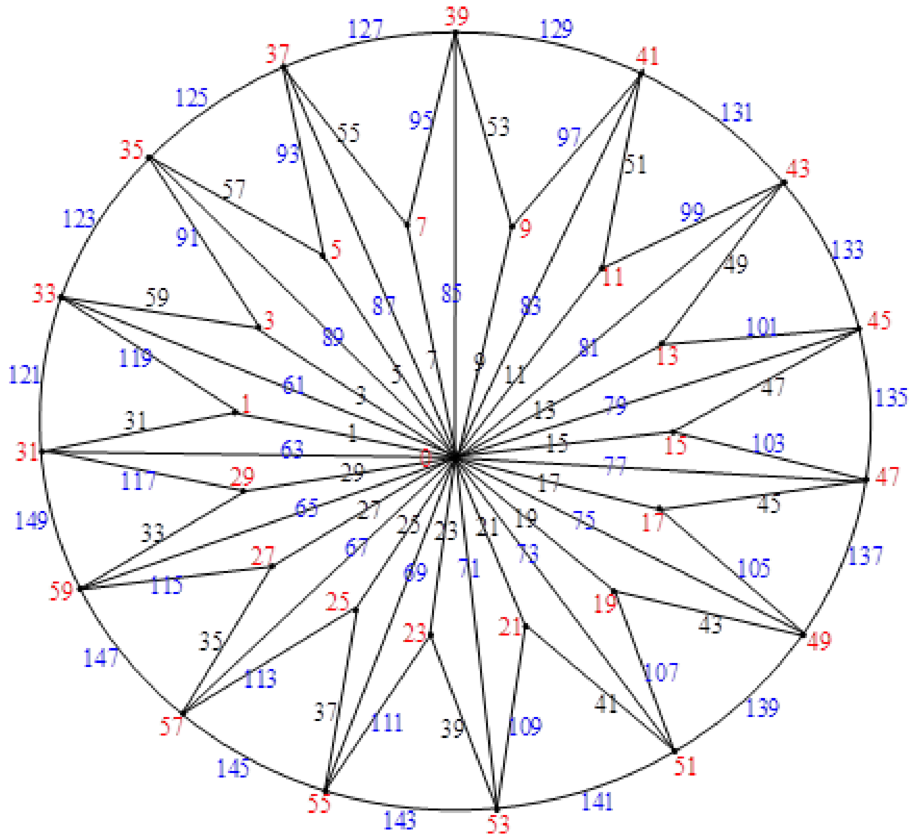

Figure 24. TW_{15}.

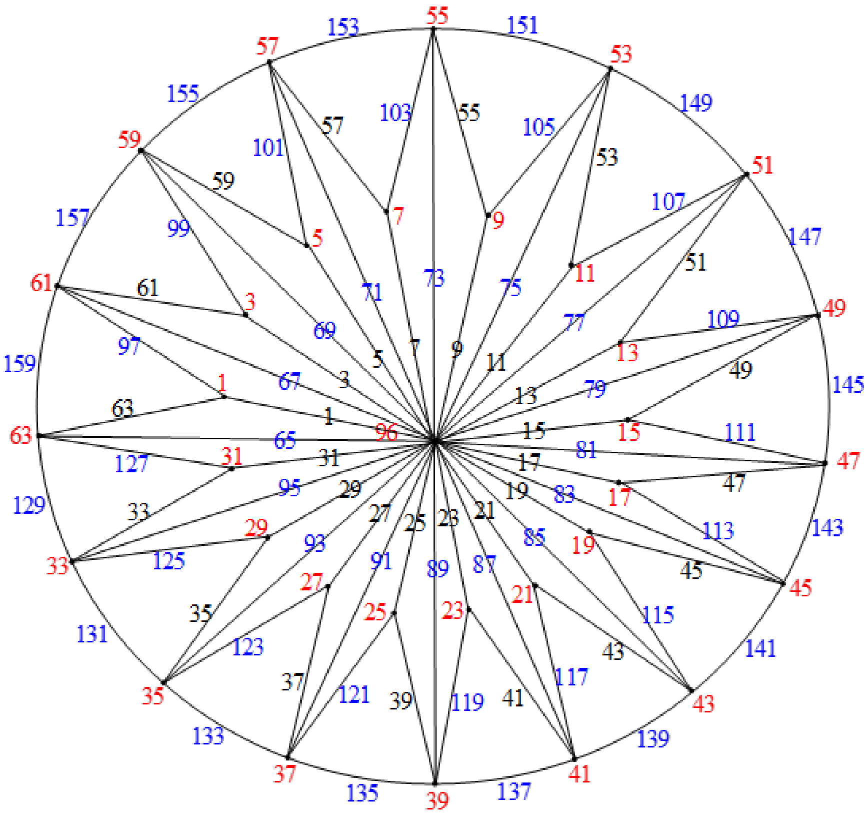

Figure 25. TW_{16}.

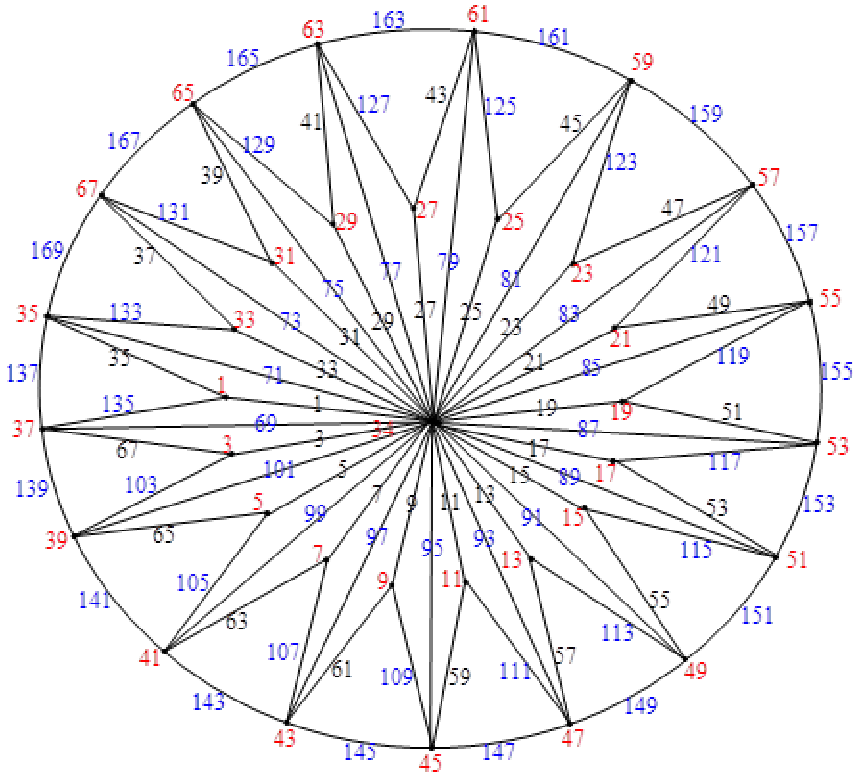

Figure 26. TW_{17}.

2.4. Double Crown-Wheel Graphs

The fourth graph is the ***double crown-wheel graph***, DCW_n. It is obtained by combining a wheel graph $W_n = \{u_0, v_1, v_2, v_3, \ldots, v_n\}$ with two sets of vertices. The first set is $\{w_1, w_2, w_3, \ldots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n. The second set is $\{u_1, u_2, u_3, \ldots, u_n\}$, where each vertex u_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex u_n is adjacent to vertices v_1 and v_n.

Theorem 4. *The double crown-wheel graph, DCW_n is edge odd graceful for n greater than or equal to* 3.

Proof. Clearly, $|E(DCW_n)| = 6n = m$; assuming that the set of vertices of DCW_n is $\{u_0, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_n\}$, we provide an explicit edge odd labeling in two different cases based on the number of vertices.

Case (1): Let DCW_n be as in Figure 27 and $n \equiv 11(\mathrm{mod}\,12)$.

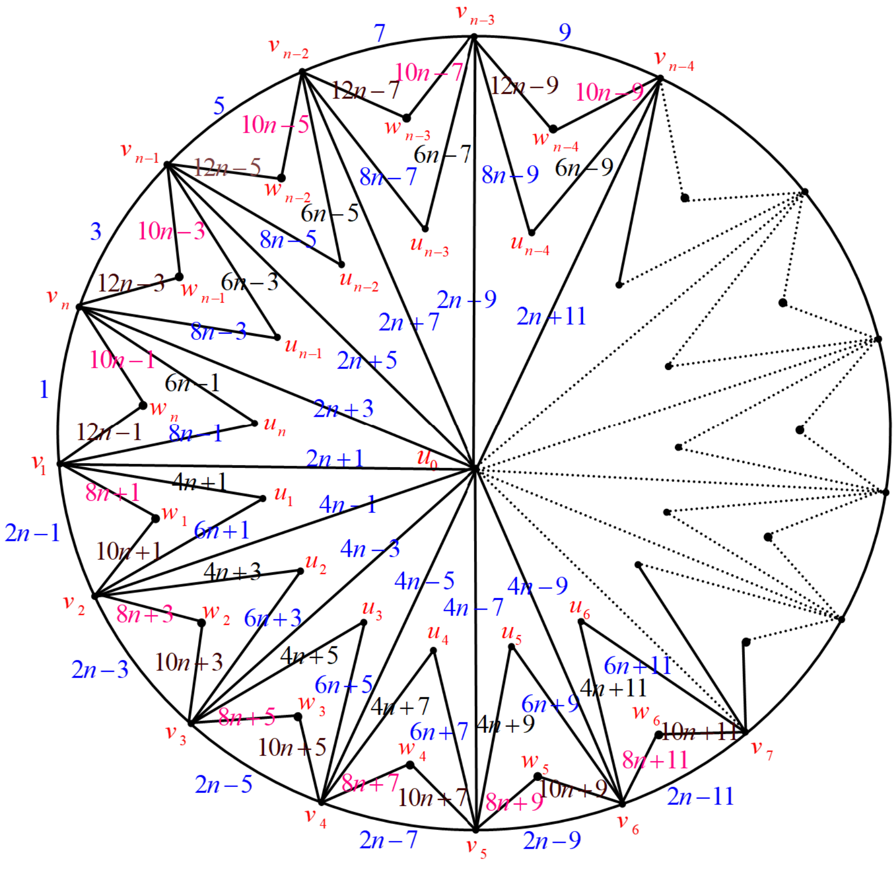

Figure 27. DCW_n, $n \equiv 11(\mathrm{mod}\,12)$.

Define the map

$$\varphi : E(DCW_n) \to \{1, 3, \ldots, 12n-1\} \text{ by}$$

$\varphi(v_1 v_n) = 1$;

$\varphi\left(v_{n-i} v_{n-(i+1)}\right) = 2i + 3, 0 \leq i \leq n-2$;

$\varphi(u_0 v_1) = 2n + 1$;

$\varphi(u_0 v_{n-i}) = 2n + 2i + 3, 0 \leq i \leq n-2$;

$\varphi(u_i v_i) = 4n + 2i - 1, 1 \leq i \leq n$;

$\varphi(u_i v_{i+1}) = 6n + 2i - 1, 1 \leq i \leq n-1$;

$\varphi(u_n v_1) = 8n - 1$;

$\varphi(w_i v_i) = 8n + 2i - 1, 1 \leq i \leq n$;

$\varphi(w_i v_{i+1}) = 10n + 2i - 1, 1 \leq i \leq n-1$; and

$\varphi(w_n v_1) = 12n - 1$.

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$\varphi^*(v_i) = (2i-1) \bmod 12n;$ $\quad\quad\varphi^*(u_i) = (10n+4i-2) \bmod 12\,n$
$\varphi^*(w_i) = (6n+4i-2) \bmod 12n, 1 \le i \le n;$ and
$\varphi^*(u_0) \equiv \dfrac{n}{2}(2n+1+4n-1) \equiv 3n^2 \bmod 12n \equiv 9n.$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_{n-1}, v_n, u_0, u_1, \ldots, u_{n-2}, u_{n-1}, u_n\}$ to $VU_{41} = \{1, 3, \ldots, 2n-3, 2n-1, 9n, (10n+2) \bmod 12n, (10n+6) \bmod 12n, \ldots, (14n-6) \bmod 12n, (14n-2) \bmod 12n\}$, respectively is one to one. Additionally, it is one to one map from $\{w_1, w_2, \ldots, w_{n-1}, w_n\}$ to $W_{41} = \{6n+2, 6n+6, \ldots, 10n-6, 10n-2\}$, respectively. It is evident that VU_{41} and W_{41} do not share any common elements. Consequently, φ^* is a bijective function, and the double crown-wheel graph is edge odd graceful when $n \equiv 11 (\bmod 12)$.

Case (2): Let DCW_n be as in Figure 28 and $n \equiv k(\bmod 12), k \ne 11$.

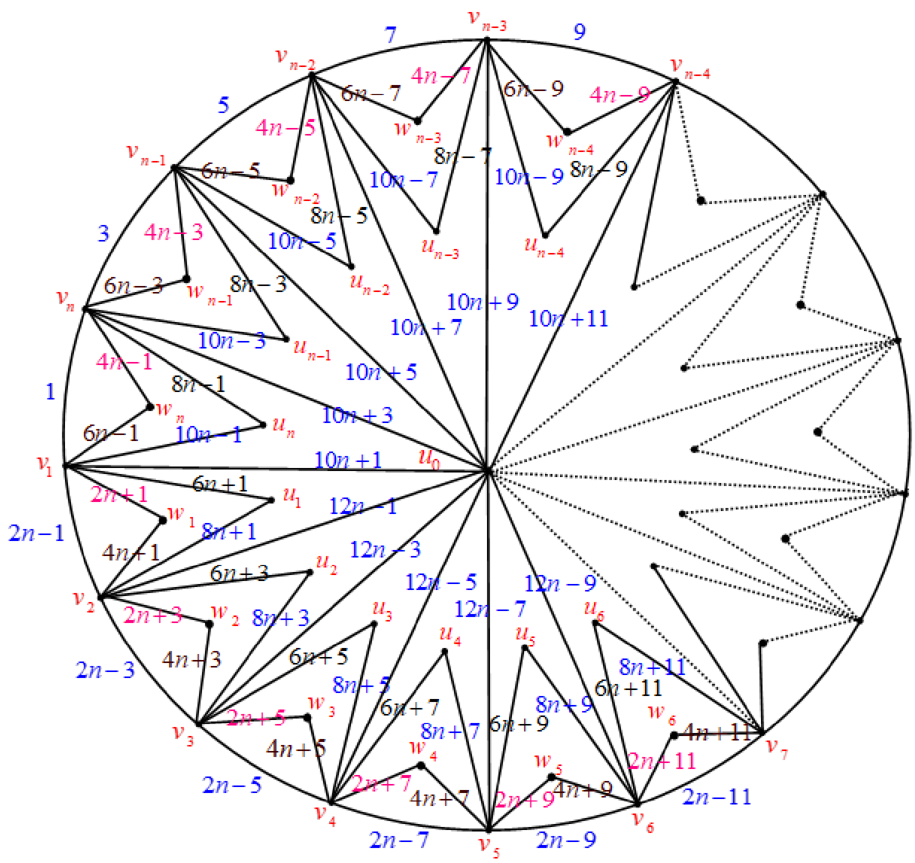

Figure 28. DCW_n, $n \equiv k(\bmod 12), k \ne 11$.

Define the map

$$\varphi : E(DCW_n) \to \{1, 3, \ldots, 12n-1\} \text{ by,}$$

$\varphi(v_1 v_n) = 1;$ $\quad\quad \varphi\left(v_{n-i} v_{n-(i+1)}\right) = 2i+3, 0 \le i \le n-2;$
$\varphi(w_i v_i) = 2n+2i-1, 1 \le i \le n;$ $\quad\quad \varphi(w_i v_{i+1}) = 4n+2i-1, 1 \le i \le n-1;$
$\varphi(w_n v_1) = 6n-1;$ $\quad\quad \varphi(u_i v_i) = 6n+2i-1, 1 \le i \le n;$
$\varphi(u_i v_{i+1}) = 8n+2i-1, 1 \le i \le n-1;$ $\quad\quad \varphi(u_n v_1) = 10n-1;$
$\varphi(u_0 v_1) = 10n+1;$ $\quad\quad \varphi(u_0 v_{n-i}) = 10n+2i+3, 0 \le i \le n-2.$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(v_i) = (2i-1) \bmod 12n; \qquad \varphi^*(u_i) = (2n+4i-2) \bmod 12n$$
$$\varphi^*(w_i) = (6n+4i-2) \bmod 12n, 1 \le i \le \ n. \text{ and}$$
$$\varphi^*(u_0) \equiv \frac{n}{2}(10n+1+12n-1) \equiv 11n^2 \bmod 12n \text{ Furthermore,.}$$

$$n \equiv 0 \bmod 12, \varphi^*(u_0) = 0; \qquad n \equiv 1 \bmod 12, \varphi^*(u_0) = 11n,$$
$$n \equiv 2 \bmod 12, \varphi^*(u_0) = 10n; \quad n \equiv 3 \bmod 12, \varphi^*(u_0) = 9n,$$
$$n \equiv 4 \bmod 12, \varphi^*(u_0) = 8n; \quad n \equiv 5 \bmod 12, \varphi^*(u_0) = 7n,$$
$$n \equiv 6 \bmod 12, \varphi^*(u_0) = 6n; \quad n \equiv 7 \bmod 12, \varphi^*(u_0) = 5n,$$
$$n \equiv 8 \bmod 12, \varphi^*(u_0) = 4n; \quad n \equiv 9 \bmod 12, \varphi^*(u_0) = 3n,$$
$$n \equiv 10 \bmod 12, \varphi^*(u_0) = 2n.$$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_{n-1}v_n, u_1, u_2, \ldots, u_{n-1}, u_n\}$ to $VU_{42} = \{1, 3, \ldots, 2n-3, 2n-1, 2n+2, 2n+6, \ldots, 6n-6, 6n-2\}$, respectively, is one to one. Additionally, it is a one to one map from $\{w_1, w_2, \ldots w_{n-1}, w_n\}$ to $W_{42} = \{6n+2, 6n+6, \ldots, 10n-6, 10n-2\}$, respectively. It is evident that VU_{42} and W_{42} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{04} = \{0, 2n, 3n, 4n, 5n, 6n, 7n, 8n, 9n, 10n, 11n\}$, and U_{04} has no elements in common with VU_{42} or W_{42}. Consequently, φ^* is a bijective function, and the double crown-wheel graph is edge odd graceful when $n \equiv k (\bmod 12), k \neq 11$. After considering case (1) and case (2), it can be concluded that every double crown-wheel graph possesses the property of being edge odd graceful for values of n greater than or equal to 3. □

Examples: The double crown-wheel graphs $DCW_{16} \cdots DCW_{19}$ and DCW_{23} and their explicit labeling are depicted in Figures 29–33, respectively.

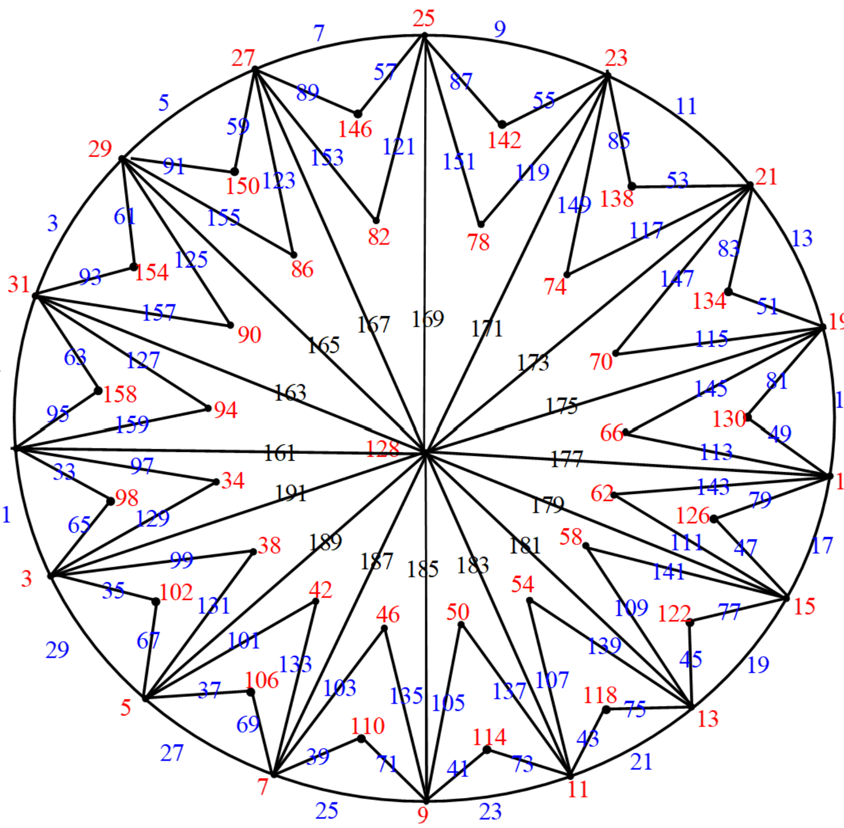

Figure 29. DCW_{16}.

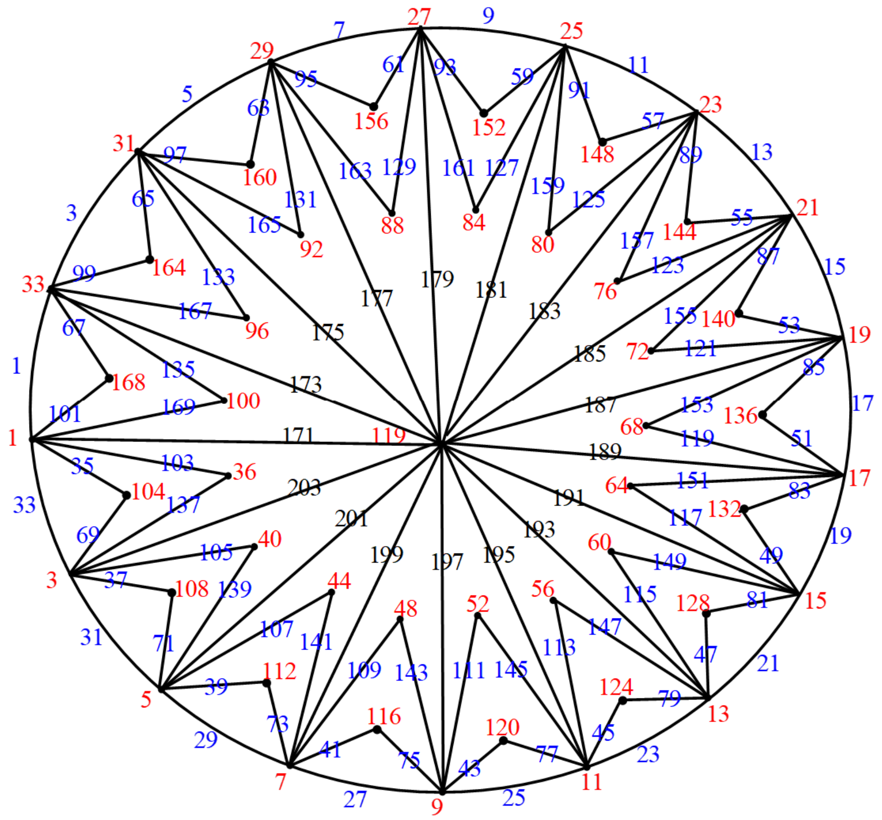

Figure 30. DCW_{17}.

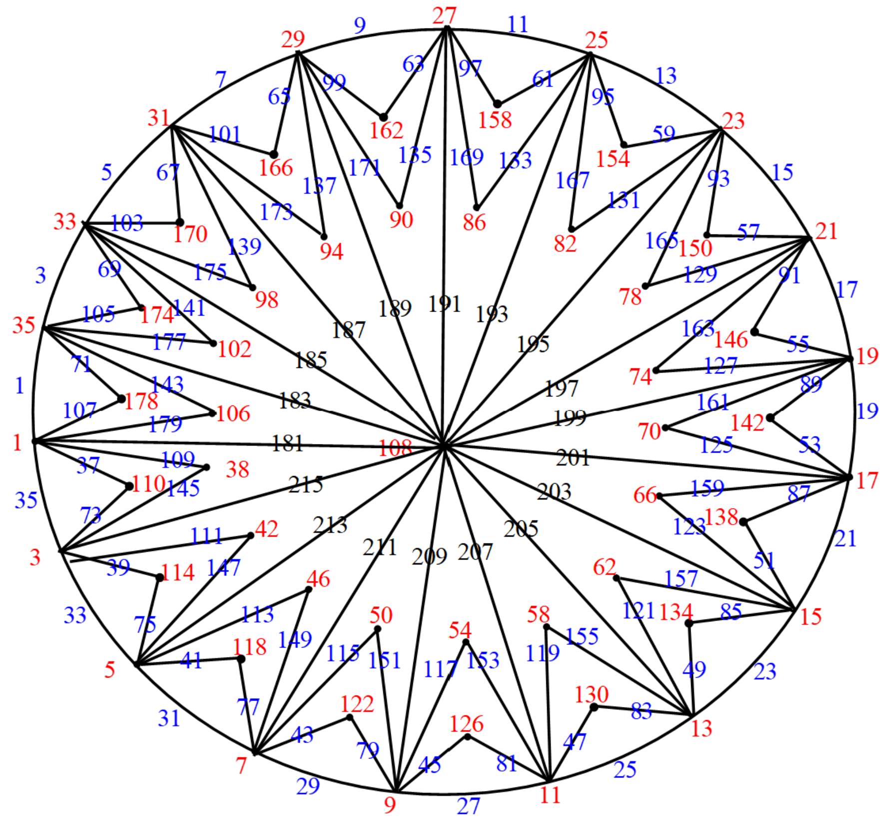

Figure 31. DCW_{18}.

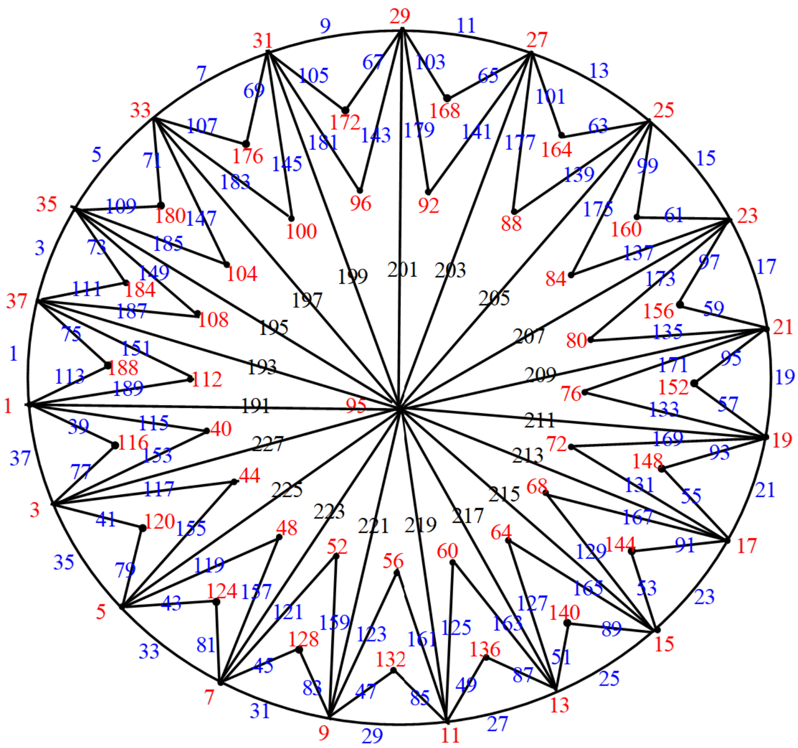

Figure 32. DCW_{19}.

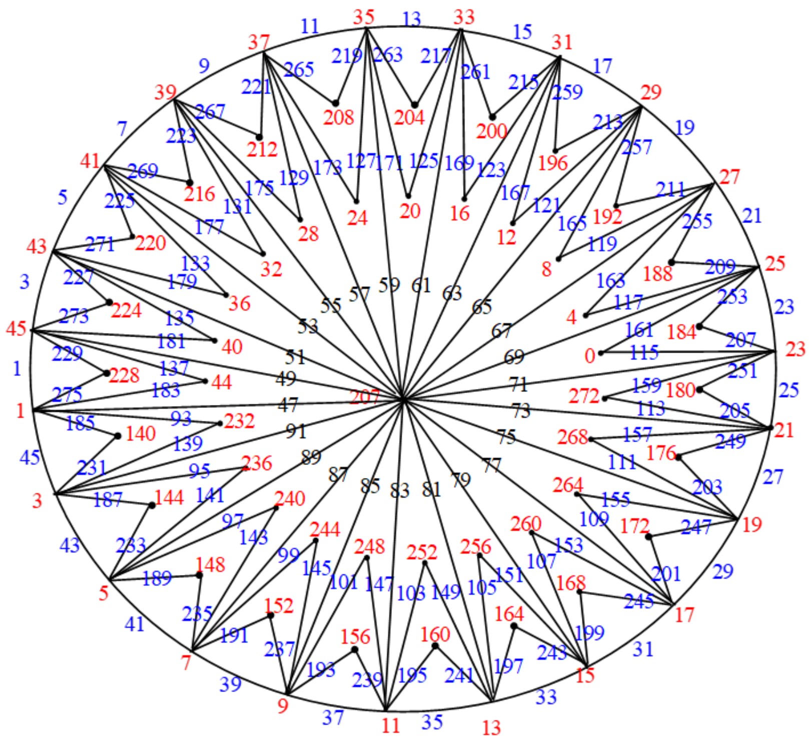

Figure 33. DCW_{23}.

2.5. Crown-Triangulated Wheel Graphs

The fifth graph is the ***crown-triangulated wheel graph***, CTW_n. It is obtained by combining the triangulated wheel graphs $TW_n = \{u_0, v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3, \ldots, u_n\}$ with the set $\{w_1, w_2, w_3, \ldots, w_n\}$, where each vertex w_i is adjacent to vertices v_i and v_{i+1}, for $1 \leq i \leq n-1$, and the vertex w_n is adjacent to vertices v_1 and v_n.

Theorem 5. *The crown-triangulated wheel graph, CTW_n is edge odd graceful for n greater than or equal to* 3.

Proof. Clearly, $|E(CTW_n)| = 7n = m$, the set of vertices of CTW_n is $\{u_0, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n, w_1, w_2, w_3, \ldots, w_n\}$.

There are two cases:

Case (1): Let CTW_n be as in Figure 34 and $n \equiv 5(\text{mod}\,14)$.

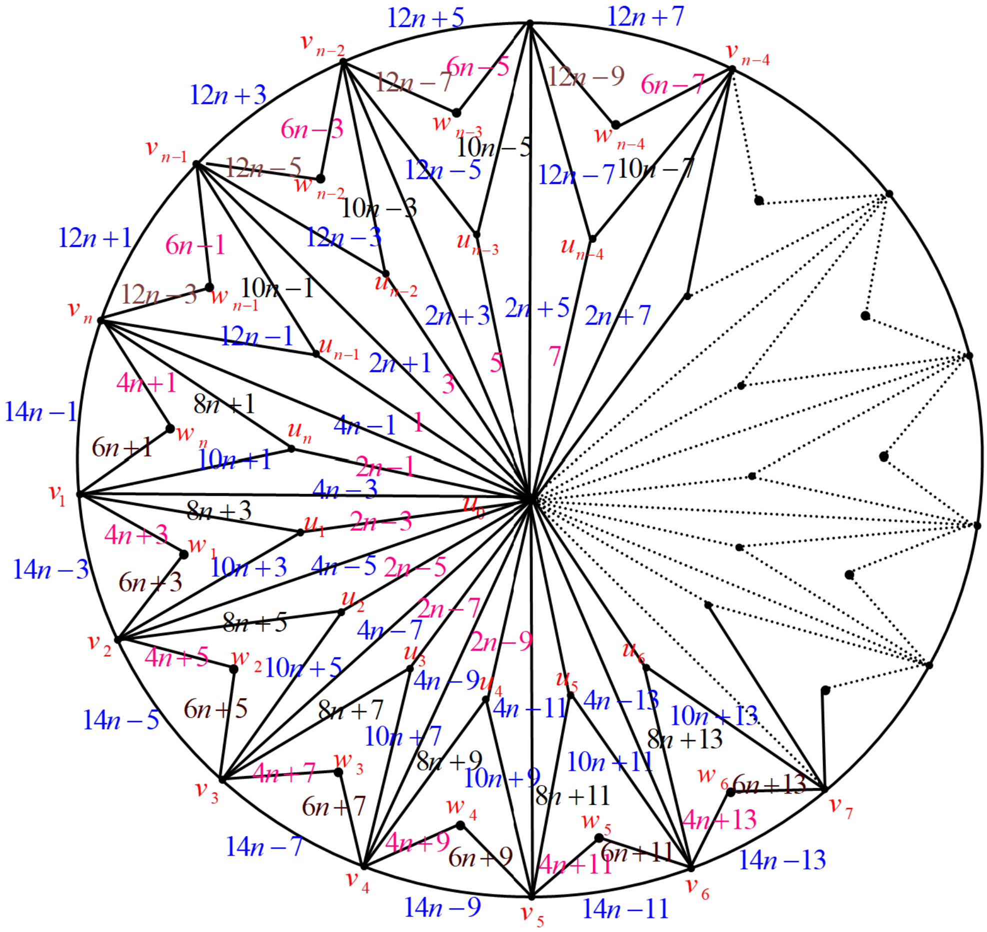

Figure 34. CTW_n, $n \equiv 5(\text{mod}\,14)$.

Define the map

$$\varphi : E(CTW_n) \to \{1, 3, \ldots, 14n - 1\} \text{ by}$$

$$\varphi(u_0 u_{n-i}) = 2i - 1, 1 \le i \le n - 1; \qquad \varphi(u_0 u_n) = 2n - 1;$$
$$\varphi(u_0 v_{n-i}) = 2n + 2i - 1, 1 \le i \le n - 1; \quad \varphi(u_0 v_n) = 4n - 1;$$
$$\varphi(w_i v_i) = 4n + 2i + 1, 1 \le i \le n - 1; \qquad \varphi(w_n v_n) = 4n + 1;$$
$$\varphi(w_i v_{i+1}) = 6n + 2i + 1, 1 \le i \le n - 1; \qquad \varphi(w_n v_1) = 6n + 1;$$
$$\varphi(u_i v_i) = 8n + 2i + 1, 1 \le i \le n - 1; \qquad \varphi(u_n v_n) = 8n + 1;$$
$$\varphi(u_i v_{i+1}) = 10n + 2i + 1, 1 \le i \le n - 1; \qquad \varphi(u_n v_1) = 10n + 1;$$
$$\varphi\left(v_{n-i} v_{n-(i+1)}\right) = 12n + 2i + 1, 0 \le i \le n - 2; \text{ and} \quad \varphi(v_1 v_n) = 14n - 1.$$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(v_i) = (4n + 2i - 1) \bmod 14n, 1 \leq i \leq n;$$
$$\varphi^*(u_n) = (6n + 1); \varphi^*(u_i) = (6n + 2i + 1) \bmod 14n, 1 \leq i \leq n - 1;$$
$$\varphi^*(w_n) = (10n + 2); \varphi^*(w_i) = (10n + 4i + 2) \bmod 14n, 1 \leq i \leq n - 1.$$

Furthermore,

$$\varphi^*(u_0) \equiv \frac{2n}{2}(1 + 4n - 1) \equiv 4n^2 \bmod 14n \equiv 6n.$$

In this case, the map φ^* from $\{v_1, v_2, \ldots, v_{n-1}, v_n, u_0, u_n, u_1, \ldots, u_{n-2}, u_{n-1}\}$ to $VU_{51} = \{4n+1, 4n+3, \ldots, 6n-5, 6n-3, 6n-1, 6n, 6n+1, 6n+3, \ldots, 8n-3, 8n-1\}$, respectively, is one to one. Additionally, it is one to one map from $\{w_n, w_1, w_2, \ldots, w_{n-1}\}$ to $W_{51} = \{10n+2, 10n+6, 10n+10, \ldots, 14n-2\}$, respectively. It is evident that VU_{51} and W_{51} do not share any common elements. Consequently, φ^* is a bijective function, and the crown-triangulated wheel graph is edge odd graceful when $n \equiv 5 (\bmod 14)$.
Case (2): Let CTW_n be as in Figure 35 and $n \equiv k (\bmod 14), k \neq 5$.

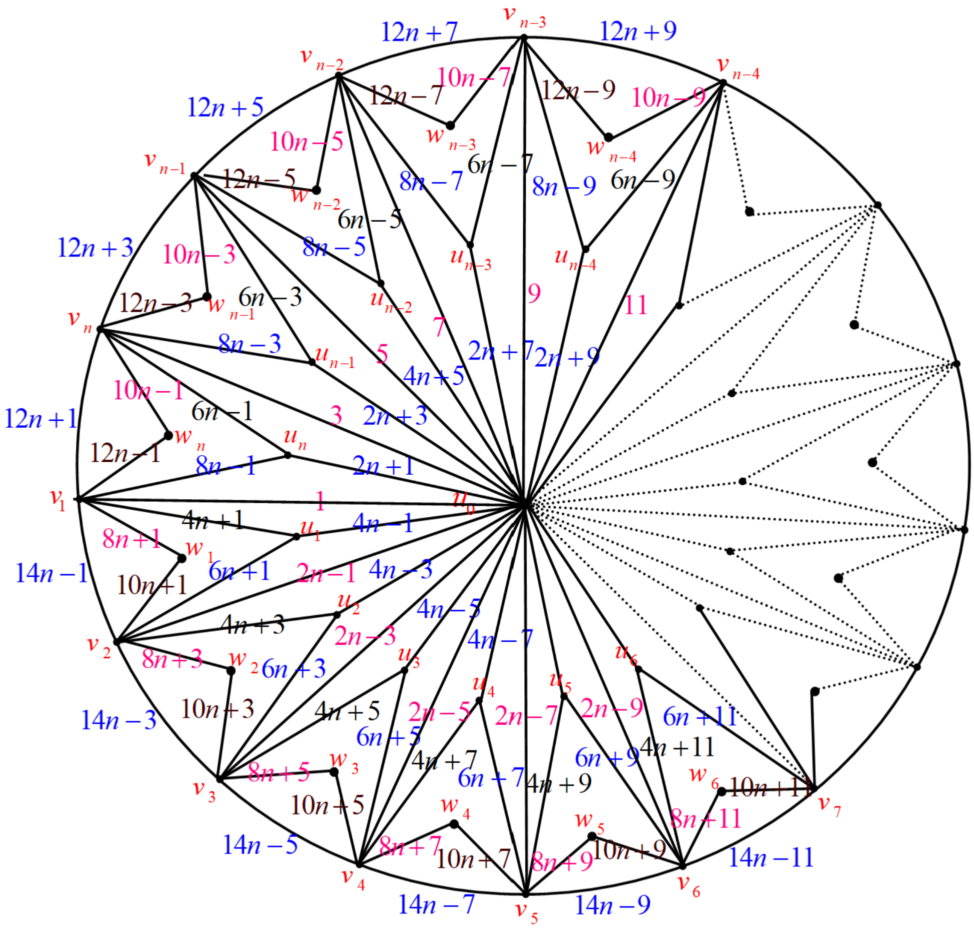

Figure 35. $CTW_n, n \equiv k (\bmod 14), k \neq 5$.

Define the map

$$\varphi : E(CTW_n) \to \{1, 3, \ldots, 14n - 1\} \text{ by}$$

$$\varphi(u_0v_{n-i}) = 2i+3, 0 \le i \le n-2;$$
$$\varphi(u_0v_1) = 1;$$
$$\varphi(u_0u_{n-i}) = 2n+2i+1, 0 \le i \le n-1;$$
$$\varphi(u_iv_i) = 4n+2i-1, 1 \le i \le n;$$
$$\varphi(u_iv_{i+1}) = 6n+2i-1, 1 \le i \le n-1;$$
$$\varphi(u_nv_1) = 8n-1;$$
$$\varphi(w_iv_i) = 8n+2i-1, 1 \le i \le n;$$
$$\varphi(w_nv_1) = 12n-1;$$
$$\varphi(w_iv_{i+1}) = 10n+2i-1, 1 \le i \le n-1;$$
$$\varphi(v_1v_n) = 12n+1;$$
$$\varphi\Big(v_{n-i}v_{n-(i+1)}\Big) = 12n+2i+3, 0 \le i \le n-2.$$

Therefore, the derived transformation φ^* on the set of vertices is given in the following manner:

$$\varphi^*(u_i) = (2i-1) \bmod 14n;$$
$$\varphi^*(w_i) = (4n+4i-2) \bmod 14n, 1 \le i \le n \text{ and}$$
$$\varphi^*(v_i) = (2n+2i-1) \bmod 14n, 1 \le i \le n.$$

Furthermore,

$$\varphi^*(u_0) \equiv \frac{2n}{2}(1+4n-1) \equiv 4n^2 \bmod 14n.$$

Furthermore,

$$n \equiv 0 \bmod 14, \varphi^*(u_0) = 0,$$
$$n \equiv 1 \bmod 14, \varphi^*(u_0) = 4n,$$
$$n \equiv 2 \bmod 14, \varphi^*(u_0) = 8n,$$
$$n \equiv 3 \bmod 14, \varphi^*(u_0) = 12n,$$
$$n \equiv 4 \bmod 14, \varphi^*(u_0) = 2n,$$
$$n \equiv 6 \bmod 14, \varphi^*(u_0) = 10n,$$
$$n \equiv 7 \bmod 14, \varphi^*(u_0) = 0,$$
$$n \equiv 8 \bmod 14, \varphi^*(u_0) = 4n,$$
$$n \equiv 9 \bmod 14, \varphi^*(u_0) = 8n,$$
$$n \equiv 10 \bmod 14, \varphi^*(u_0) = 12n,$$
$$n \equiv 11 \bmod 14, \varphi^*(u_0) = 2n,$$
$$n \equiv 12 \bmod 14, \varphi^*(u_0) = 6n,$$
$$n \equiv 13 \bmod 14, \varphi^*(u_0) = 10n.$$

In this case, the map φ^* from $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ to $VU_{52} = \{1, 3, \ldots, 2n-1, 2n+1, 2n+3, \ldots, 4n-1\}$, respectively, is one to one. Additionally, it is a one to one map from $\{w_1, w_2, \ldots, w_n\}$ to $W_{52} = \{4n+2, 4n+6, 4n+10, \ldots, 8n-2\}$, respectively. It is evident that VU_{52} and W_{52} do not share any common elements. Moreover, $\varphi^*(u_0)$ belongs to $U_{05} = \{0, 2n, 4n, 6n, 8n, 10n, 12n\}$, and U_{05} has no elements in common with VU_{52} or W_{52}. Consequently, φ^* is a bijective function, and the crown-triangulated wheel graph is edge odd graceful when $n \equiv k(\text{mod}14), k \ne 5$. After considering case (1) and case (2), it can be concluded that every crown-triangulated wheel graph possesses the property of being edge odd graceful for values of n greater than or equal to 3. □

Examples: The crown-triangulated wheel graphs $DCW_{16}, \cdots, DCW_{19}$ and their explicit labeling are depicted in Figures 36–39, respectively.

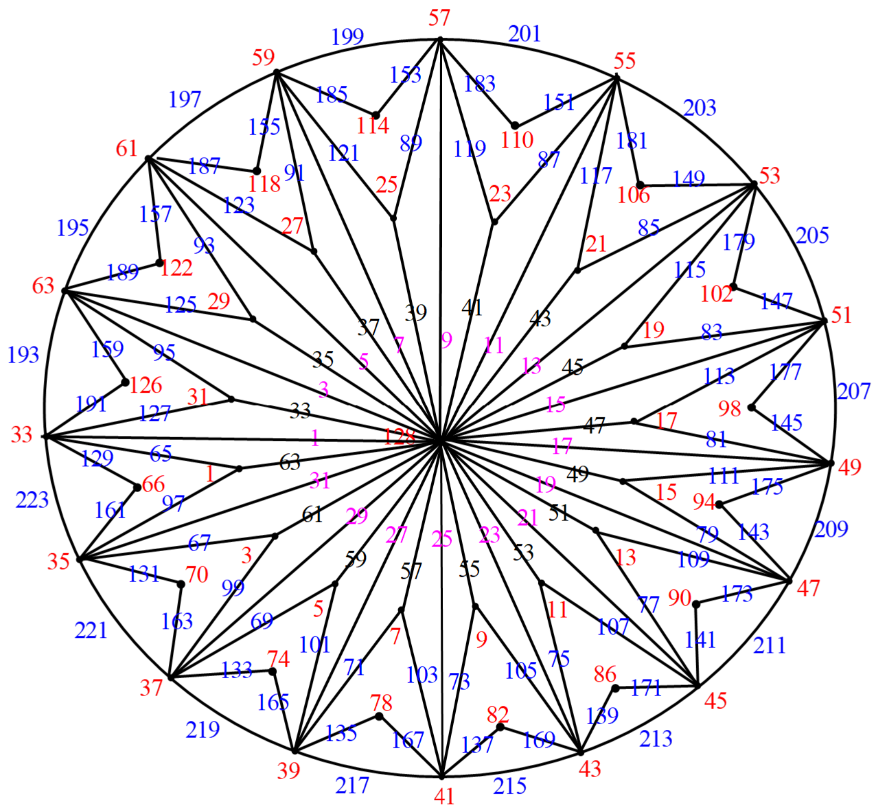

Figure 36. CTW_{16}.

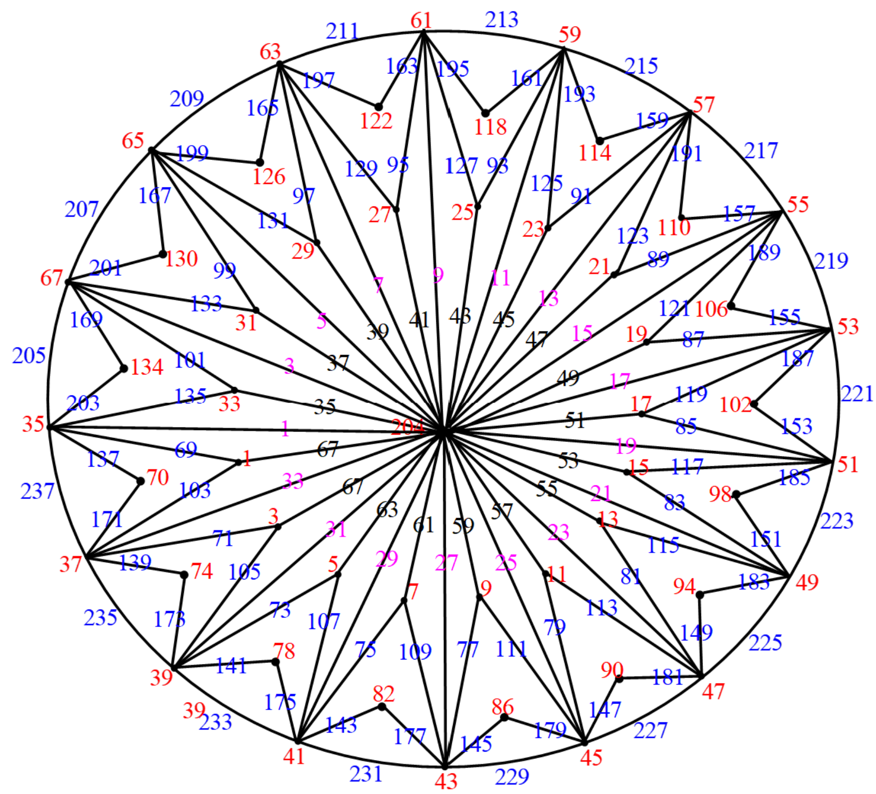

Figure 37. CTW_{17}.

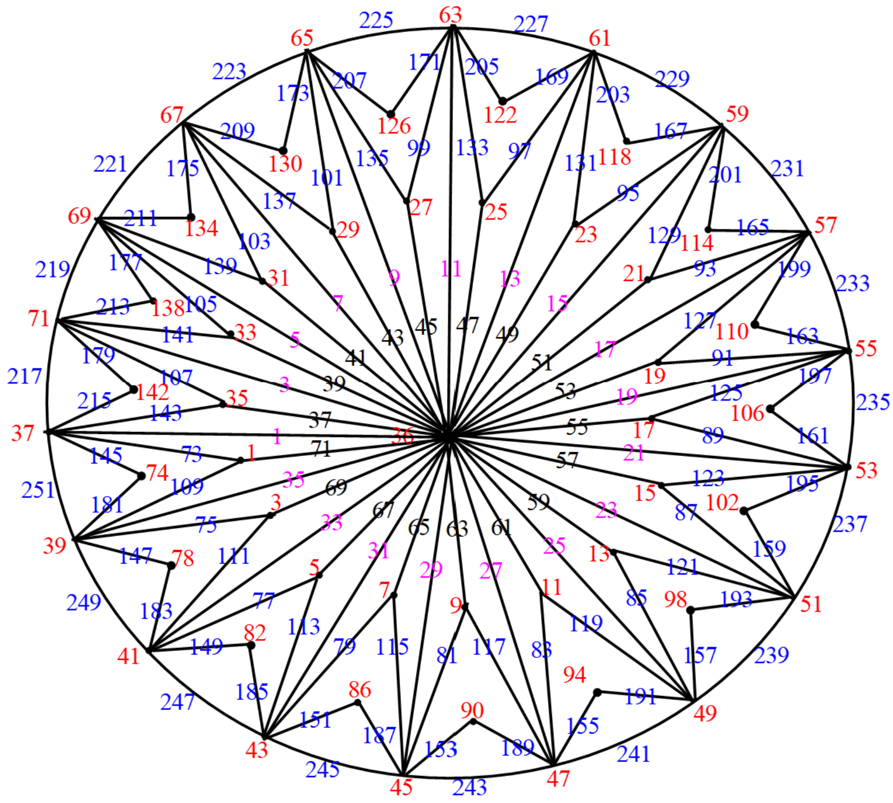

Figure 38. CTW_{18}.

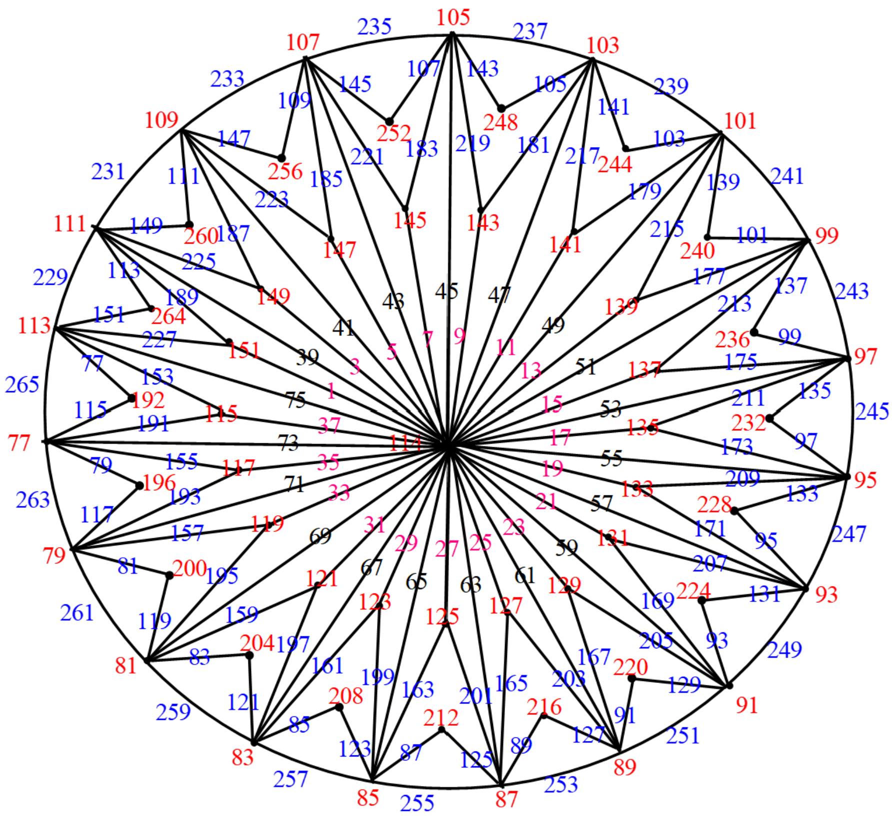

Figure 39. CTW_{19}.

3. Conclusions

The concepts of edge graceful labeling and edge odd graceful labeling have been recognized in the field of graph theory since 2009. Despite numerous studies on these topics, the edge-odd-gracefulness of many graphs remains unknown. One particular group of graphs, stars S_n, is proven to not be edge odd graceful if n is odd, as illustrated in Observation 1. In this research, we introduce five novel graph families that bear some resemblance to wheels. These families encompass closed flower graphs, cogwheel graphs, triangulated wheel graphs, double crown-wheel graphs, and crown-triangulated wheel graphs. By providing explicit labeling for each class, we establish the edge-odd-gracefulness of these graphs. Theorems 1–5 serve as the basis for demonstrating the edge-odd-gracefulness of their respective graph classes.

Author Contributions: Conceptualization, M.A. and S.N.D.; methodology, M.A. and S.N.D.; software, M.A. and S.N.D.; validation, M.A. and S.N.D.; formal analysis, M.A. and S.N.D.; investigation, M.A. and S.N.D.; resources, M.A. and S.N.D.; data curation, M.A. and S.N.D.; writing—original draft preparation, M.A. and S.N.D.; writing—review and editing, M.A. and S.N.D.; visualization, M.A. and S.N.D.; supervision, M.A. and S.N.D.; project administration, M.A. and S.N.D.; funding acquisition, M.A. and S.N.D. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number 445-9-935.

Data Availability Statement: Data are contained within the article.

Acknowledgments: We sincerely thank the referees for their invaluable advice and feedback, which greatly improved the quality of this paper. Their expertise and careful evaluation were instrumental in refining our research.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

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