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On α-Pseudo Spiralike Functions Associated with Exponential Pareto Distribution (EPD) and Libera Integral Operator

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Abstract: The present study aims at investigating some characterizations of a new subclass $G_\alpha(\mu,\tau)$ and obtaining the bounds on the first two Taylor–Maclaurin coefficients for functions belonging to the newly introduced subclass. In order to achieve this, a compound function $L^{\sigma}_{x,n}(z)$ is derived from the convolution of the analytic function $f(z)$ and a modified exponential Pareto distribution $G(x)$ in conjunction with the famous Libera integral operator $L(\zeta)$. With the aid of the derived function, the aforementioned subclass $G_\alpha(\mu,\tau)$ is introduced, while some properties of functions belonging to this subclass are considered in the open unit disk.

Keywords: analytic; univalent; starlike; spiralike; bounded turning; Libera integral

MSC: 30C45

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1. Introduction

Let Ω denote the class of analytic functions $f(\zeta)$ such that

$$f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k \tag{1}$$

and be normalized with $f'(0) = 0$ and $f'(0) = 1$ in the open unit disk $D = \{\zeta : |\zeta| < 1\}$. The single-valued function $f(\zeta)$ is said to be univalent in D if it never takes on the same value twice. In other words, if $f(\zeta_1) = f(\zeta_2)$ for $\zeta_1, \zeta_2 \in D$, then $\zeta_1 = \zeta_2$. Also, let A denote the class of all univalent functions that are in Ω. In 1973, Singh [1] examined a subclass of A (known as Bazilevic functions of type λ) denoted by $B_1(\lambda)(\lambda > 0)$ satisfying the condition

$$\Re\left\{\frac{f(\zeta)^{\lambda-1} f'(\zeta)}{\zeta^{\lambda-1}}\right\} > 0, \quad \lambda > 0, \quad \zeta \in D.$$

Letting $\lambda = 0$ and 1, respectively, the said class, $B_1(\lambda)$, yields the well-known subclasses of starlike and bounded turning functions satisfying the geometric conditions

$$\Re\left\{\frac{\zeta f'(\zeta)}{f(\zeta)}\right\} > 0, \quad \zeta \in D$$

and

$$\{f'(\zeta)\} > 0, \quad \zeta \in D.$$

In the present study, for function $f(\zeta) \in \Omega$, we say that $B_1(\lambda, \tau)$ is the class of Bazilevic functions of type λ, order τ provided that

$$\Re\left\{ \frac{f(\zeta)^{\lambda-1} f'(\zeta)}{\zeta^{\lambda-1}}, \right\} > \tau, \quad \lambda > 0, \quad 0 \leq \tau < 1, \quad \zeta \in D.$$

Furthermore, in [2], Al-kadim and Bashi defined the cumulative probability function of the exponential Pareto distribution (EPD) as

$$G(x) = 1 - e^{-\gamma(\frac{x}{n})^{\sigma}}; \;\; x > 0, \tag{2}$$

where $\gamma, \sigma > 0$ are shape parameters and $n > 0$ is a scale parameter.

The concept of the aforementioned distribution $G(x)$ was first initiated by Gupta et al. [3] in 1998, with its probability density function (pdf) expressed as

$$G(x, \gamma, \theta) = \gamma\theta(1 + x)^{-\gamma+1}[1 - (1 + x)^{-\gamma}]^{\theta-1},$$

where θ and γ are two shape parameters.

One major advantage of the exponential Pareto distribution (EPD) is the scaling parameter, which can be applied to different areas in real life situation. The distribution has the ability to capture the long-tailed nature of many real-world data sets, identify patterns and trends in data, and due to the importance of the exponential Pareto distribution in analyzing lifetime data FC, various applications of the distribution have been considered and studied in the literature. For example, Al-Kadim and Boshi in [2] discussed exponential and Pareto distributions and presented some properties which include the moment generated function, mean, mode, median, variance, the r-th moment about the mean, the r-th moment about the origin, reliability, hazard functions and coefficients of variation, of skweness and of kurtosis and estimated the parameter. Haj Ahmad et al. [4] used a unit exponential Pareto distribution to model the recovery rate of COVID-19; Idowu and Ajibode in [5] considered the use of the exponential Pareto distribution to improve raw material quality in cement production and for drawing control charts. See also [6,7] among others, for details.

It is, however, observed that there are no known applications of the exponential and Pareto distribution in geometric function theory in the literature now. Hence, the authors intend to investigate some relevant connections of this distribution (EPD) in geometric function theory in this study. Consequently, we let $\zeta = \gamma + \epsilon i$ for $\epsilon \geq 0$ such that $G(x)$ is now defined as

$$G(x) = 1 - e^{-\zeta(\frac{x}{n})^{\sigma}}; x, n, \sigma > 0, \zeta \in \mathbb{C}. \tag{3}$$

Remark 1. *If we set $\epsilon = 0$, then (3) reduces to the usual exponential Pareto distribution given by (2).*

Equation (3) can be normalized such that

$$(-1)^{-(k+1)} \left(\frac{x}{n}\right)^{-\sigma} G(x) = \zeta + \sum_{k=2}^{\infty} \left(\frac{x}{n}\right)^{\sigma(k-1)} \zeta^k. \tag{4}$$

In view of (1) and (4), we can say that

$$h(\zeta) = (-1)^{-(k+1)} . \left(\frac{x}{n}\right)^{-\sigma} G(x) * f(\zeta).$$

That is,

$$h(\zeta) = \zeta + \sum_{k=2}^{\infty} b^{(k-1)} a_k \zeta^k, \tag{5}$$

where for convenience we let $b = \left(\frac{x}{n}\right)^{\sigma}$.

Furthermore, in 1965, Libera [8] defined an integral operator $L(\zeta)$ as

$$L(\zeta) = \frac{2}{\zeta}\int_0^{\zeta} f(t)dt; \tag{6}$$

see also [9,10], among others.

It is worthy to note that the Libera integral operator defined in (6) maps each of the subclasses of starlike, convex and close-to-convex functions into itself, which makes the operator symmetric in nature. It converges uniformly, which makes it asymptotic in nature. Since (5) and (6) converge uniformly, in this study, we can replace $f(t)$ in (6) with the function $h(\zeta)$ derived in (5), such that

$$L_b(\zeta) = \frac{2}{\zeta}\int_0^{\zeta} h(t)dt.$$

It is trivial to see that

$$L_b(\zeta) = \zeta + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)\left(\frac{x}{n}\right)^{\sigma(k-1)} a_k\zeta^k. \tag{7}$$

It is imperative to note that the series derived in (7) is uniformly convergent for $|\zeta| < \frac{|n^{\delta}|}{p|x^{\delta}|}$. If $p \neq 0$, then the radius of convergence R is given by $R = \frac{|n^{\delta}|}{p|x^{\delta}|}$, while for $p = 0$, $R = \infty$.

Given the series

$$L_b(\zeta) = \zeta + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)\left(\frac{x}{n}\right)^{\sigma(k-1)} a_k\zeta^k$$

as a convolution of $f(\zeta)$ defined by (1) and $g(\zeta)$ defined as follows:

$$g(\zeta) = \zeta + \sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)\left(\frac{x}{n}\right)^{\sigma(k-1)} \zeta^k, \tag{8}$$

so that $L^{\sigma}_{x,n}(\zeta) = f(\zeta) * g(\zeta) = L_b(\zeta)$, Now let

$$b_k = \left(\frac{2}{k+1}\right)\left(\frac{x}{n}\right)^{\sigma(k-1)},$$

which implies that

$$b_{k+1} = \left(\frac{2}{k+2}\right)\left(\frac{x}{n}\right)^{k\sigma}.$$

By the ratio test of convergence,

$$\lim_{k\to\infty}\left|\frac{b_{k+1}}{b_k}\right| = \lim_{k\to\infty}\left|\frac{2x^{k\sigma}}{(k+2)n^{k\sigma}} \times \frac{(k+1)n^{k\sigma-\sigma}}{2x^{k\sigma-\sigma}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(k+1)x^{\sigma}}{(k+2)n^{\sigma}}\right| = \left|\frac{x^{\sigma}}{n^{\sigma}}\right| \lim_{k\to\infty}\left|\frac{k+1}{k+2}\right| = \frac{x^{\sigma}}{n^{\sigma}} < 1$$

for a convergence series. Hence, the series converges for $x^{\sigma} < n^{\sigma}$ and diverges for $n^{\sigma} < x^{\sigma}$. The test fails for $n^{\sigma} = x^{\sigma}$. For the purpose of this study, therefore, we assume $x^{\sigma} < n^{\sigma}$, since the series converges with the radius of convergence $\frac{x^{\sigma}}{n^{\sigma}}$ and the interval of convergence $-\frac{x^{\sigma}}{n^{\sigma}} < |\zeta| < \frac{x^{\sigma}}{n^{\sigma}}$. We remark that since x, n and σ are greater than zero or non-negative, the parameters $\frac{x^{\sigma}}{n^{\sigma}}$ is such that $\frac{x^{\sigma}}{n^{\sigma}} \in (0, 1)$, the unit disc.

Using (7), therefore, we give the following definition:

Definition 1. *Let the functions $f(\zeta)$ and $L_b(\zeta)$ be defined, respectively, by (1) and (7). Then, $L_b(\zeta)$ belongs to the class $G_\alpha(\tau,\mu)$ of α-pseudo-spiralike functions of order τ (associated with the exponential Pareto distribution and Libera integral operator), provided*

$$Re\left\{ e^{i\mu}\frac{\zeta\left(L_b'(\zeta)\right)^\alpha}{L_b(\zeta)}\right\} > \tau\cos\mu, \quad \zeta\in D, \tag{9}$$

where $\alpha>0,\ 0\le\tau<1,\ |\mu|<\frac{\pi}{2}, x, n, \sigma>0$.

Remark 2. *(i.) Firstly, we note that if we set $\tau=0$ in (9), then we write $G_\alpha(\mu)$ in place of $G_\alpha(0,\mu)$.*

(ii.) Setting $\alpha=1$ in (9), we obtain the class $G_1(\tau,\mu)$ of 1-pseudo-spiralike functions of order τ, or simply the class of spiralike functions of order τ.

(iii.) If we let $\alpha=2$ in (9), then we obtain the class $G_2(\tau,\mu)$ given by

$$Re\left\{ e^{i\mu}\frac{\zeta L_b'(\zeta)}{L_b(\zeta)}\left[L_b'(\zeta)\right]\right\} > \tau\cos\mu,$$

which is the product combination of geometric expressions for spiralike and bounded turning functions of order τ.

(iv.) Letting $\alpha=1$ and $\mu=0$ in (9), we obtain the class $G_1(\tau,0)$ of 1-pseudo-starlike functions of order τ, or simply the class of starlike functions of order τ.

(v.) If we let $\alpha=2$ and $\mu=0$ in (9), then we obtain the class $G_2(\tau,0)$ given by

$$Re\left\{ \frac{\zeta L_b'(\zeta)}{L_b(\zeta)}\left[L_b'(\zeta)\right]\right\} > \tau\cos\mu,$$

which, obviously, is the product combination of geometric expressions for starlike and bounded turning functions of order τ.

Next, we show that $G_\alpha(\tau,\mu)$ is a subclass of $B_1(\lambda,\tau)$ and, therefore, α-pseudo-spiralike functions are Bazilevic and univalent in D. Also, we obtain some characterizations as well as integral representations of these new classes of functions. Furthermore, we consider the coefficient estimates and Fekete functional for the functions belonging to the newly defined class $G_\alpha(\tau,\mu)$. At this juncture, it is noted that though for $\alpha>1$, the classes of α-pseudo-spiralike functions have similar traits as the analytic representation of spiralike functions, the likely inclusion relations between them is an open problem. The Libera integral operator as a tool for investigating the geometric function is used in this study because of its useful properties, such as preservation properties and conformal mapping properties, among others.

2. Some Characterizations for the Class $G_\alpha(\tau,\mu)$

Before proceeding to the results and their proofs, the following well-known Lemmas shall be considered.

Let P_τ denote the class of analytic functions $p(\zeta)$ in D, given by

$$p(\zeta)=1+p_1\zeta+p_2\zeta^2+\ldots=1+\sum_{k=1}^{\infty}p_k\zeta^k \qquad (\zeta\in D),$$

and satisfy the condition

$$Re\big[p(\zeta)\big] > \tau\cos\mu, \quad 0 \le \tau < 1, \quad |\mu| < \frac{\pi}{2}. \tag{10}$$

The class P_τ is the class of Caratheodory functions of order τ. For the case $\tau = 0$, we simply write P instead of P_τ.

Lemma 1 ([11]). *Let ζ be a complex number with positive real part. Then, for $t > 0$, $t \in [0,1]$, we have $Re\zeta^t \ge [Re\zeta]^t$.*

Lemma 2 ([11–13]). *Let the function $p \in P$ (class of Caratheodory functions) be given by*

$$p(\zeta) = 1 + \sum_{k=1}^{\infty} p_k \zeta^k \qquad (\zeta \in D); \tag{11}$$

then

$$|p_k| \le 2 \qquad\qquad (k \in \mathbb{N} = \{1,2,\ldots\}), \tag{12}$$

where

$$p(0) = 1 \qquad \text{and} \qquad Re(p(\zeta)) > 0 \qquad (\zeta \in D),\ 0 \le \tau < 1,\ |\mu| < \frac{\pi}{2}.$$

Lemma 3 ([11–13]). *Let $p \in P_\tau$. If*

$$\psi(\zeta) = [p(\zeta)]^t, \quad t \in [0,1],$$

then

$$\psi(0) = 1 \quad \text{and} \quad \Re\Big[\psi(\zeta)\Big] > (\tau\cos\mu)^t.$$

Proof. We observe that $\psi(0) = 1$. By applying (10) and Lemma 1, we can say that

$$Re\big[\psi(\zeta)\big] = Re\big[p(\zeta)\big]^t,$$

which implies that

$$Re\big[\psi(\zeta)\big] > \Big(Re\big[p(\zeta)\big]\Big)^t.$$

Therefore,

$$Re\big[\psi(\zeta)\big] > \Big(\tau\cos\mu\Big)^t$$

and this obviously completes the proof. □

Lemma 4 ([11–13]). *If $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \ldots$ is an analytic function with positive real part and v is a complex number, then*

$$|p_2 - vp_1^2| \le 2max\{1;\ |2v-1|\}.$$

The result is sharp for the functions given by

$$p(\zeta) = \frac{1+\zeta^2}{1-\zeta^2} \ \text{and} \ \ p(\zeta) = \frac{1+\zeta}{1-\zeta}.$$

Lemma 5 ([14]). *Let $q(\zeta)$ be analytic in D with $q(0) = 1$ and suppose that*

$$Re\left\{1 + \zeta\frac{q'(\zeta)}{q(\zeta)}\right\} > \frac{4\beta\rho + 8\beta - \rho - 1}{4\beta(\rho+1)}, \quad (\zeta \in D).$$

Then $Re\Big[q(\zeta)\Big] > \beta$ for $\frac{1}{4} \le \beta < 1$ and $0 < \rho \le 1$.

Theorem 1. *Let $L_b(\zeta)$ be of the form (7) and belongs to $G_\alpha(\mu)$. Then, for $\alpha \geq 1$, $|\mu| < \frac{\pi}{2}$, $0 \leq \tau < 1$ and $\zeta \in D$, $G_\alpha(\tau,\mu) \subset B_1(1-1/\alpha,\ \tau^{\frac{1}{\alpha}} \cos^{\frac{1}{\alpha}} \mu)$.*

Proof. Suppose that $L_b(\zeta) \in G_\alpha(\mu)$. Then for some $p \in P_\tau$, we set

$$p(\zeta) = e^{i\mu} \frac{\zeta \left(L_b'(\zeta)\right)^\alpha}{L_b(\zeta)}. \tag{13}$$

Obviously, (13) can be expressed as

$$p(\zeta) = e^{i\mu} \left\{ \frac{\zeta^{\frac{1}{\alpha}} L_b'(\zeta)}{\left(L_b(\zeta)\right)^{\frac{1}{\alpha}}} \right\}^\alpha. \tag{14}$$

From (14), we obtain

$$\left(p(\zeta)\right)^{\frac{1}{\alpha}} = \left(e^{i\mu}\right)^{\frac{1}{\alpha}} \frac{\zeta^{\frac{1}{\alpha}} \left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)^{\frac{1}{\alpha}}}. \tag{15}$$

In view of Lemma 3, we obtain

$$Re\left[e^{i\mu/\alpha} \frac{\zeta^{1/\alpha}\left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)^{1/\alpha}}\right] > \left(\tau \cos \mu\right)^{1/\alpha}$$

and, taking $\lambda = 1 - \frac{1}{\alpha}$, we obtain the required result. □

Theorem 2. *Let $L_b(\zeta)$ be of the form (7) and belongs to $G_\alpha(\mu)$. Then $L_b(\zeta)$ can be represented by the following integral:*

$$L_b^\sigma(z) = \left[\int_0^\zeta \left(\frac{\alpha-1}{\alpha}\right) e^{-i\frac{\mu}{\alpha}} \left(\frac{p(t)}{t}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{\alpha}{\alpha-1}}, \quad if \quad \alpha > 1.$$

Proof. Since $L_b(\zeta) \in G_\alpha(\tau,\mu)$, there exists p in P_τ such that

$$\left(p(\zeta)\right)^{\frac{1}{\alpha}} = \left(e^{i\mu}\right)^{\frac{1}{\alpha}} \frac{\zeta^{\frac{1}{\alpha}} \left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)^{\frac{1}{\alpha}}}. \tag{16}$$

Then, taking $\lambda = 1 - \frac{1}{\alpha}$, we have

$$\left(p(\zeta)\right)^{1-\lambda} = \left(e^{i\mu}\right)^{1-\lambda} \frac{\zeta^{\frac{1}{\alpha}} \left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)^{1-\lambda}}$$

such that

$$\left(e^{i\mu}\right)^{\lambda-1} \left(p(\zeta)\right)^{1-\lambda} \zeta^{\lambda-1} = \frac{\left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)^{1-\lambda}}. \tag{17}$$

Since

$$\frac{L_b'(\zeta)}{\left(L_b(\zeta)\right)^{1-\lambda}} = \left[L_b(\zeta)\right]^{\lambda-1}\left[L_b'(\zeta)\right] = \frac{\left[\left(L_b(\zeta)\right)^{\lambda}\right]'}{\lambda},$$

therefore,

$$\left[\left(L_b(\zeta)\right)^{\lambda}\right]' = \lambda\left(e^{i\mu}\right)^{\lambda-1}\left(p(\zeta)\right)^{1-\lambda}\zeta^{\lambda-1}. \tag{18}$$

Simple computation of (18) yields

$$L_b(\zeta) = \left[\left(e^{i\mu}\right)^{\lambda-1}\int_0^{\zeta}\lambda t^{\lambda-1}\left(p(t)\right)^{1-\lambda}dt\right]^{\frac{1}{\lambda}},$$

which yields the required result. □

Remark 3. *(i) If $\alpha = 1$ in (12), we have the integral representation of the spiralike function given*

$$L_b(\zeta) = exp\int_0^{\zeta} e^{-i\mu}\frac{p(t)}{t}dt.$$

(ii) If $\alpha = 1$ and $\mu = 0$ in (12), then the integral representation of the starlike function given by

$$L_b(\zeta) = exp\int_0^{\zeta}\frac{p(t)}{t}dt, \quad if \quad \alpha = 1$$

is well-known.

Theorem 3. *Let $L_b(\zeta)$ be of the form (7) and satisfy the condition that*

$$Re\left\{e^{i\mu}\left[\alpha\frac{\zeta\left(L_b''(\zeta)\right)}{\left(L_b'(\zeta)\right)} - \frac{\zeta\left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)}\right]\right\} > \frac{4\beta - (\rho+1)\left(1+4\beta\cos\mu\right)}{4\beta(\rho+1)}. \tag{19}$$

Then, $L_b(\zeta) \in G_\alpha(\mu)$ for $\frac{1}{4} \le \beta < 1$, $0 < \rho \le 1$, $\alpha \ge 1$, $|\mu| < \frac{\pi}{2}$ and σ, x, n are as earlier defined.

Proof. Define

$$p(\zeta) = e^{i\mu}\frac{\zeta\left[L_b'(\zeta)\right]^{\alpha}}{L_b(\zeta)}.$$

Taking logarithm differentiation of both sides, we obtain

$$\frac{p'(\zeta)}{p(\zeta)} = \alpha e^{i\mu}\frac{\left(L_b''(\zeta)\right)}{\left(L_b(\zeta)\right)'} - \frac{\left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)} + e^{i\mu}. \tag{20}$$

Now, using Lemma 5 in (20), we have

$$\begin{aligned} Re\left\{1+\zeta\frac{p'(\zeta)}{p(\zeta)}\right\} &= Re\left\{\left(1+e^{i\mu}\right) + e^{i\mu}\left[\alpha\zeta\frac{\left(L_b''(\zeta)\right)}{\left(L_b'(\zeta)\right)} - \zeta\frac{\left(L_b'(\zeta)\right)}{\left(L_b(\zeta)\right)}\right]\right\} \\ &> \frac{4\beta\rho + 8\beta - \rho - 1}{4\beta(\rho+1)} \end{aligned}$$

and this completes the proof. If we set $\beta = \frac{1}{4}$ in Theorem 3, we obtain the following corollary. □

Theorem 4. *Let $L_b(\zeta)$ be of the form (7). If $L_b(\zeta)$ belongs to the class $G_\alpha(\mu, \tau)$ of order τ, type α, then*

$$|a_2| \leq \frac{3\left(e^{i\mu} - \tau\cos\mu\right)}{(2\alpha_1 - 1)\left(\frac{x}{n}\right)^{\sigma}} \tag{21}$$

and

$$|a_3| \leq \frac{4\left(e^{i\mu} - \tau\cos\mu\right)}{(3\alpha_1 - 1)\left(\frac{x}{n}\right)^{2\sigma}}.max\left\{1; \left|1 + \frac{2(4\alpha_2 - 2\alpha_1 + 1)\left(e^{i\mu} - \tau\cos\mu\right)}{(2\alpha_1 - 1)^2}\right|\right\}. \tag{22}$$

Proof. From (9), we have

$$\Re\left\{e^{i\mu}\frac{\zeta\left(\left[L_b'(\zeta)\right]\right)^{\alpha}}{L_b(\zeta)}\right\} > \tau\cos\mu.$$

Now, set

$$\frac{e^{i\mu}\frac{\zeta\left(\left[L_b'(\zeta)\right]\right)^{\alpha}}{L_b(\zeta)} - \tau\cos\mu}{e^{i\mu} - \tau\cos\mu} = p(\zeta).$$

Then

$$e^{i\mu}\frac{\zeta\left(\left[L_b'(\zeta)\right]\right)^{\alpha}}{L_b(\zeta)} = \tau\cos\mu + \left(e^{i\mu} - \tau\cos\mu\right)p(\zeta). \tag{23}$$

This implies that

$$e^{i\mu}\zeta + \frac{4}{3}e^{i\mu}\alpha_1\left(\frac{x}{n}\right)^{\sigma}a_2\zeta^2 + \left[\frac{3}{2}\alpha_1 a_3 + e^{i\mu}\frac{16}{9}\alpha_2 a_2^2\right]\left(\frac{x}{n}\right)^{2\sigma}\zeta^3 + \dots$$

$$= e^{i\mu}\zeta + \left[\left(e^{i\mu} - \tau\cos\mu\right)p_1 + \frac{2}{3}e^{i\mu}\left(\frac{x}{n}\right)^{\sigma}a_2\right]\zeta^2$$

$$+\left[\left(e^{i\mu} - \tau\cos\mu\right)p_2 + \frac{2}{3}\left(e^{i\mu} - \tau\cos\mu\right)\left(\frac{x}{n}\right)^{\sigma}p_1 a_2 + \frac{1}{2}\left(\frac{x}{n}\right)^{2\sigma}e^{i\mu}a_3\right]\zeta^3 + \dots \quad .$$

By comparing the coefficients of both sides, we obtain

$$a_2 = \frac{3\left(e^{i\mu} - \tau\cos\mu\right)}{2e^{i\mu}(2\alpha_1 - 1)\left(\frac{x}{n}\right)^{\sigma}}p_1 \tag{24}$$

and

$$a_3 = \frac{2\left(e^{i\mu} - \tau\cos\mu\right)}{(3\alpha_1 - 1)\left(\frac{x}{n}\right)^{2\sigma}e^{i\mu}}\left\{p_2 - \frac{(4\alpha_2 - 2\alpha_1 + 1)\left(e^{i\mu} - \tau\cos\mu\right)}{(2\alpha_1 - 1)^2 e^{i\mu}}p_1^2\right\}. \tag{25}$$

Applying Lemma 2 in (24) and Lemma 4 in (25), we obtain the required results, as seen in (21) and (22), respectively. □

Theorem 5. *Let $L_b(\zeta)$ be of the form (7). If $L_b(\zeta)$ belongs to the class $G_\alpha(\mu,\tau)$ of order τ, type α, then*

$$\left|a_3 - va_2^2\right|$$
$$\leq \frac{4\left(e^{i\mu} - \tau\cos\mu\right)}{(3\alpha_1 - 1)\left(\frac{x}{n}\right)^{2\sigma}}.max\left\{1; \left|1 + \frac{\left(e^{i\mu} - \tau\cos\mu\right)}{4(2\alpha_1 - 1)^2}\left[8\left(4\alpha_2 - 2\alpha_1 + 1\right) + 9v(3\alpha_1 - 1)\right]\right|\right\}.$$

Proof. Using (24) and (25) for any complex number v, we have

$$a_3 - va_2^2 = \frac{2\left(e^{i\mu} - \tau\cos\mu\right)}{e^{i\mu}(3\alpha_1 - 1)\left(\frac{x}{n}\right)^{2\sigma}}\left\{p_2 - \frac{\left(e^{i\mu} - \tau\cos\mu\right)}{8e^{i\mu}(2\alpha_1 - 1)^2}\left[8\left(4\alpha_2 - 2\alpha_1 + 1\right) + 9v(3\alpha_1 - 1)\right]p_1^2\right\}.$$

In view of Lemma 4, we obtain the desired result. □

3. Conclusions

The present study is primarily concerned with a new function $L_b(\zeta)$, derived through Hadamard product/convolution, modified exponential Pareto distribution (EPD), Libera integral operator and differential calculus. Using the new function with the help of the subordination principle, a new subclass $G_\alpha(\mu,\tau)$, associated with modified EPD and the Libera integral operator is introduced. In view of the newly defined subclass, some characterizations as well as coefficient bounds for functions belonging to the aforementioned subclass are investigated using a succinct mathematical approach, while several other corollaries follow as simple consequences. Interestingly, it is worthy to note that the exponential Pareto distribution (EPD) is a great tool in analyzing many lifetime data and the symmetric properties are due to the involvement of the Libera integral operator and convolution transform. Finally, it is noted here that the bounds obtained in this work could be used in the future to study bi-univalent problems as well as Hankel determinants and these are left has open problems.

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References

1. Singh, R. On Bazilevic Functions. *Proc. Am. Math. Soc.* **1973**, *38*, 261–271.
2. Al-Kadim, K.A.; Boshi, M.A. Exponential Pareto Distribution. *Math. Theory Model.* **2013**, *3*, 135–146.
3. Gupta, R.C.; Gupta, R.D.; Gupta, P.L. Modelling failure time data by Lehman alternatives. *Commun. Stat. Theory Methods* **1998**, *27*, 887–904. [CrossRef]
4. Haj Ahmad, H.; Almetwally, E.M.; Elgarhy, M.; Ramadan, D.A. On Unit Exponential Pareto Distribution for Modeling the Recovery Rate of COVID-19. *Processes* **2023**, *11*, 232. [CrossRef]
5. Idowu, A.O.; Ajibode, I.A. An Exponential-Pareto Distribution Approach to improving raw material quality. *FEPI-JOPAS* **2023**, *5*, 33–39. Available online: https://fepi-jopas.federalpolyilaro.edu.ng (accessed on 17 April 2024).

6. Elbatal, I.; Zayed, M.; Rasekhi, M.; Butt, N.S. The Exponential Pareto Power Series Distribution: Theory and Application. *Pak. J. Stat. Oper. Res.* **2017**, *13* , 603–615. [CrossRef]
7. Hassan, A.S.; Hemeda, S.E.; Nassr, S.G. On the Extension of Exponentiated Pareto Distribution. *J. Mod. Appl. Stat. Methods* **2020**, *19*, 21. [CrossRef]
8. Libera, R.J. Some Classes of Regular Univalent Functions. *Proc. Math. Soc.* **1965**, *16*, 755–758. [CrossRef]
9. Hamzat, J.O.; Oladipo, A.T.; Oros, G.I. Application of a Multiplier Transformation to Libera Integral Operator Associated with Generalized Distribution. *Symmetry* **2022**, *14*, 1934. [CrossRef]
10. Oros, G.I. New Differential Subordination Obtained by Using a Differential-integral Ruscheweyh-Libera Operator. *Miskolc Math. Notes* **2020**, *21*, 303–317. [CrossRef]
11. Babalola, K.O. On λ-Pseudo-Starlike Functions. *J. Class. Anal.* **2013**, *3*, 137–147. [CrossRef]
12. Hamzat, J.O.; Sangoniyi, S.O. Fekete-Szego inequalities Associated with Certain Generalized Class of Non-Bazilevic Functions. *Int. J. Math. Sci. Opt. Theory Appl.* **2021**, *6*, 874–885.
13. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceeding of Conference on Complex Analytic*; Li, Z., Ren, F., Lang, L., Zhang, S., Eds.; International Press: New York, NY, USA, 1994; Volume 1, pp. 157–169.
14. Hamzat, J.O.; Oladipo, A.T. On a Comprehensive Class of Analytic p-Valent Functions Associated with Shell-like curve and Modified Sigmoid Function. *Malays. J. Comput.* **2022**, *7*, 995–1010. [CrossRef]