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Article

Novel Robust Estimation-Based Control of One-Sided Lipschitz Nonlinear Systems Subject to Output and Input Delays

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Abstract: This paper highlights the design of a controller established on estimated states for one-sided Lipschitz (OSL) nonlinear systems subject to output and input delays. The controller has been devised by involving Luenberger-like estimated states. The stability of the time-delayed nonlinear system is reckoned by assuming a Lyapunov functional for delayed dynamics and for which a delay-range dependent criterion is posed with a delay ranging between known upper and lower bounds. The time derivative of the functional is further exploited with linear matrix inequality (LMI) procedures, and employing Wirtinger's inequality for the integral terms instead of the traditional and more conservative Jensen's condition. Moreover, a sufficient and necessary solution is derived for the proposed design by involving the tedious decoupling technique to attain controller and observer gain simultaneously. The proposed methodology validates the observer error stability between observers and states asymptotically. The solution of matrix inequalities was obtained by employing cone-complementary linearization techniques to solve the tiresome constraints through simulation tools by convex optimization. Additionally, a novel scheme of an observer-based controller for the linear counterpart is also derived for one-sided Lipschitz nonlinear systems with multiple delays. Finally, the effectualness of the presented observer-based controller under input and output delays for one-sided Lipschitz nonlinear systems is validated by considering a numerical simulation of mobile systems in Cartesian coordinates.

Keywords: estimation-based control; delay-range-dependent (DRD); one-sided Lipschitz; Wirtinger's inequality; decoupling procedure; input and output delays

MSC: 37M05; 37M15; 37M25

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1. Introduction

State estimation has attracted significant attention in the last two decades for applications in various control systems, including the design of output feedback, fault detection, image processing incorporating high-level processing such as computer vision, cryptography, chemical processes, and complex systems such as biochemical reactors, grid-connected PV systems, aircraft, nuclear reactors, neural networks, and many more [1–8]. Physical systems are subjected to observers to retrieve knowledge of inputs and respective outputs as they constitute a more practical method, offering a unique theoretical perspective usually associated with linear system theories of controllability, stability, and observability [3,6]. Missing state variable information, either due to sensor unavailability or infeasible control techniques, can be suitably approximated by using an observer [9]. State estimation subjected to unknown inputs for linear and nonlinear, time-invariant and variant control systems with multiple variables, and full and reduced-order observers using Simulink (an estimator for slow and fast singularly perturbed systems) is presented in the literature [8–12]. In these studies, efforts have been made to derive a single classical generic observer to generate estimated internal states for the entire plant. M. Sławiński and T. Kaczorek presented a method for the full reduced-order observer for continuous-time singular linear systems [6]. A design of a positive linear observer system to approximate states of the system with greater accuracy is presented in [7]. The authors of [9] present linear matrix inequality (LMI)-based conditions for state and unknown input observers for nonlinear systems under delayed dynamics. A novel approach for disturbance observers for nonlinear systems is presented in [12] for input and output models.

Estimation-based control has been employed for both linear and non-linear control schemes to approximate unknown states and inputs that are not measured due to sensor unavailability or impractical control techniques [13–19]. Over recent years, observer-based systems for adaptive control, fuzzy control, backstepping control, non-linear fault control theory, sampled-data control, event-driven control, output-based control, Luenberger control, convex optimization control, time-delay based control, etc., have been adopted based on observer-based nonlinear control. In [13], observers were investigated for slow and fast singularly perturbed linear systems to address the numerical ill-conditioning problem of the original system. Disturbance observer-based control to estimate the states of non-linear systems and attenuation of disturbances is accomplished by designing a two-stage controller where both disturbance observer and controller operate separately [14]. In [15], the problem of decentralized fault tolerant control (FTC) for nonlinear systems in the feedback form is addressed by combining backstepping and non-linear FTC theory to develop a novel adaptive fuzzy decentralized FTC scheme. Robust observer-based fuzzy control for nonlinear dynamic systems subjected to uncertain parameters is accomplished in [19] using the Takagi–Sugeno fuzzy model system. The sufficient solution as linear matrix inequality (LMIs) is derived for robust stabilization.

Various practical systems, especially those influenced by transmission, transportation, or inertial phenomena, incorporate delayed differential equations. Conventional control methods described in the literature are not directly applicable to such delayed dynamics systems, and hence have received considerable attention from researchers for linear and nonlinear systems [20–29]. Time delay can be observed in input, output, or both. Estimation-based output feedback control was presented in [20] for linear systems with single/multiple output and input delays where the Truncated predictor feedback technique was used. Refs. [21,22] highlighted the stabilization of open loop systems with control and delay in state for linear and nonlinear systems. Delayed differential equations modeled in various nonlinear physical frameworks such as industrial processes, robotics, telecommunication systems, earth-controlled satellite devices, bio-medical engineering, etc., employed delays in input, output, and state, or all three. Adaptive neural network observer-based control for time-delayed stochastic SISO systems was studied by employing dynamic surface control to avoid the intricacy of the backstepping scheme [23]. In [24], a relatively new approach for a time-delayed dynamical scheme based on estimation-based feedback control was presented for nonlinear systems targeting the problem of infinite dimensioning in characteristic equations. This proposed method guaranteed all signals to be ultimately uniformly bounded. The authors of [25] employed active disturbance rejection control (ADRC) for systems that are non-linear, uncertain, and time-varying all at the same time (such as many industrial processes including combustion, wastewater treatment, etc.) to achieve disturbance rejection and maintenance of stability all at once. Sliding mode observer-based fuzzy output-feedback control has been employed for stochastic and non-linear systems incorporating multiple delays [26]. In [22], Luenberger-like estimators were designed for nonlinear structures having delays in state and output for real-time reconstruction of insulinemia in human beings from plasma glucose measurements. In [27], a new method was proposed which incorporated external disturbances and delayed dynamics both in output and state of the system. Adaptive consensus control based on the dynamic surface control technique for non-linear time-delayed and multi-agent systems was presented in [28]. In [29], ADRC was used for uncertain non-linear systems with input time-delay which was studied with a novel extended state observer as a predictor. The stability of time-delayed systems is exploited with delay-independent, delay-dependent, and delay-range-dependent approaches. Extensive research has been done by scholars on stability control and analysis of delay-independent non-linear systems [30–32], and delay-dependent non-linear systems for various types of systems ranging from robust filtering to adaptive and impulsive control for time-delayed non-linear systems [33–35]. Also, observers and controller designs have been investigated for delay-range-dependent non-linear systems with the help of various controlling techniques in a variety of systems such as biomedical systems, neural networks, process control, and safe and secure communication ([36–39], and references therein).

Control for estimation-based time-delayed Lipschitz and One-sided Lipschitz (OSL) systems with uncertain parameters and external turbulences and OSL systems with input saturation have been investigated recently by many researchers. Many techniques such as static and dynamic gain filter structures, and functions including Lyapunov-Krasovskii, Jensen's Inequality, LMIs, etc., have been employed successfully to stabilize the above-mentioned systems after parametric uncertainties and external or internal disturbances [30,37,38,40–42]. The conservatism of Lipschitzian nonlinear systems as OSL systems accompanies quadratic inner-boundedness constraints which leads to less generic results, which is highlighted in various studies [38,43].

Employing Wirtinger's condition for the feedback control of nonlinear and linear dynamic systems, which requires the abstraction of estimator and controller gain simultaneously, is an intriguing research problem ([43,44], and references therein). For nonlinear physical systems, information on all state vectors is required and in case of non-availability, estimation is carried out for specific states. For such systems, state-feedback estimation-based controllers have been devised which further engage LMI-based results. LMIs are obtained by integrating the triple integral terms for a delay-range-dependent approach. Previously, Jensen's inequality performed the task of solving the integral terms, yet conservatism still appears for that reason. Wirtinger's inequality fulfills the role of providing extractable observer and controller gain by involving the decoupling techniques.

Inspired by the revealed factors, this study highlights the estimation-based control of OSL nonlinear schemes exposed to output and input delays. The Lyapunov-Krasovskii (LK) functional is employed as a delay-range-dependent stability criterion to derive controller and observer gain. Time-derivation of the functional is further exploited with Wirtinger's condition instead of Jensen's inequality condition [43,44], due to conservatism added by the latter. Stability constraints for the LK functional are obtained by considering non-zero lower and upper bounds of time-varying delays. Furthermore, the OSL condition is incorporated instead of the Lipschitz condition which observes the local behavior of nonlinearity. A Luenberger-like observer scheme is presented to provide an estimation of states and to ensure the availability of all states for the feedback controller, from which the closed-loop feedback dynamics are prepared. In order to have the observer and controller gain simultaneously from the simulation tools, a decoupling technique ([40,43,45,46] and references therein) is engaged which provides sufficient and necessary conditions for nonlinear inequality. Nonlinear matrix inequality is then exploited with a convex cone complementary linearization algorithm to obtain LMI, which renders both observer and controller gains for a nonlinear system that has input and output delays. A simulation example is provided by considering a moving object in 2D Cartesian coordinates to exhibit the efficiency of the projected scheme.

The main contributions of the proposed methodology are as follows:

1. To the best of our knowledge, the delay-range-dependent stability approach to devise an observer-based controller for OSL nonlinearity under input and output delays is explored for the first time in the literature.

2. The Wirtinger inequality method is employed for the solution of multiple integrals in this study to reduce the conservatism of Jensen's inequality, in comparison to previous techniques reported in [43] to have a trivial solution.
3. Additionally, a necessary and sufficient solution is derived for the proposed design by involving the tedious decoupling technique to attain controller and observer gain simultaneously. The proposed methodology validates the observer error stability between observers and states asymptotically.

Further content in the paper is structured in five sections, which are as follows: Section 2 presents the system description of a one-sided nonlinear dynamical system with delay in input and output, and observer and controller structures, and important results to be utilized. Key results are delivered in Section 3 for the Delay-range-dependent (DRD) stability approach of estimation-based control with comprehensive proofs. Section 4 presents simulation outcomes by considering the motion of an object in a 2D Cartesian plane, and Section 5 is the conclusion.

2. System Explanation

Assuming a dynamical system with nonlinearity and delay encountered in the output and input of the system as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau_1(t)) + f(t,x), \\ y(t) &= Cx(t-\tau_2(t)), \end{aligned} \tag{1}$$

for which the state of the system is given by $x(t) \in \Re^n$, $y(t) \in \Re^p$ depicts the output vector, control input of the system is taken as $u(t) \in \Re^m$, and the nonlinearity is denoted by $f(t,x) \in \Re^n$. A, B and C are assumed as constant system matrices of appropriate dimensions for the linear dynamics. The continuous input and output function for delay terms are $\tau_1(t)$ and $\tau_2(t)$, respectively, which satisfies

$$\begin{aligned} &0 \le h_{j1} \le \tau_j \le h_{j2},\ \forall j = 1,2, \\ &\dot{\tau}_j \le \mu_j,\ \forall j = 1,2. \end{aligned}$$

Definition 1. *The nonlinear function given by $f(t,x)$, fulfills*

$$\langle f(t,x) - f(t,\hat{x}), \lambda_1 - \lambda_2 \rangle \le \rho \|\lambda_1 - \lambda_2\|^2, \tag{2}$$

with λ_1, $\lambda_2 \in R^n$ and $\rho \in \Re$ is assumed as a one-sided Lipschitz constant [40,41,43].

Definition 2. *The nonlinear function given by $f(t,x)$ is assumed to satisfy a quadratic inner-boundedness condition, if*

$$(f(t,\lambda_1) - f(t,\lambda_2))^T (f(t,\lambda_1) - f(t,\lambda_2)) \ \le \beta \|\lambda_1 - \lambda_2\|^2 + \alpha \langle \lambda_1 - \lambda_2, f(t,\lambda_1) - f(t,\lambda_2) \rangle, \tag{3}$$

The OSL condition provided in Definition 1 integrated with the quadratic inner-boundedness condition helps in extending the Lipschitz nonlinearity application and is easily adapted to a larger class of nonlinear systems. Since the OSL constant can have any zero or non-zero value, the extrapolation of Lipschitz nonlinearity emerged, where the constant can only be positive and for a specific range of nonlinear systems.

Assumption 1. *The nonlinear function assumed as $f(t,x)$ in (1) corroborates OSL nonlinearity and QIB conditions provided in relations (3) and (4).*

The control input for the estimation-based control strategy for OSL nonlinear systems is provided as

$$u(t) = K_1 \hat{x}(t) \tag{4}$$

where the matrix for controller gain is referred to as $K_1 \in \Re^{m\times n}$ and an estimate of $x(t)$ is given by $\hat{x}(t)$. A Luenberger-like observer for the nonlinear system in (1) is selected as

$$\begin{aligned} \dot{\hat{x}}(t) &= Bu(t-\tau_1(t)) + A\hat{x}(t) + f(t,\hat{x}(t)) + K_2(y(t)-\hat{y}(t)), \\ \hat{y}(t) &= C\hat{x}(t-\tau_2(t)), \end{aligned} \tag{5}$$

where observer gain is assumed as $K_2 \in \Re^{n\times p}$. Employing relations in Equations (1) and (5) renders

$$\dot{e}(t) = Ae(t) - K_2Ce(t-\tau_2(t)) + \Phi(x,\hat{x}), \tag{6}$$

where the error is provided as $e(t) = x(t) - \hat{x}(t)$ and the nonlinear terms are modeled as $\Phi(x,\hat{x}) = -f(t,\hat{x}) + f(t,x)$. Integrating (4) and (1) renders

$$\dot{x}(t) = f(t,x) + Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t)). \tag{7}$$

By combining (6) and (7) we have an augmented system as

$$\begin{aligned} &\dot{z}(t) = \overline{A}_1 z(t-\tau_1) + \overline{A}z(t) + \overline{A}_2 z(t-\tau_2) + Ig(t), \\ &\textit{where} \\ &z(t) = \begin{bmatrix} x(t)^T & e(t)^T \end{bmatrix}^T, \ z(t-\tau_1(t)) = \begin{bmatrix} x(t-\tau_1(t))^T & e(t-\tau_1(t))^T \end{bmatrix}^T, \\ &z(t-\tau_2(t)) = \begin{bmatrix} 0 & e(t-\tau_2(t))^T \end{bmatrix}^T, \\ &\overline{A} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} BK_1 & -BK_1 \\ 0 & 0 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} 0 \\ -K_2C \end{bmatrix}, \\ &g(t) = \begin{bmatrix} f(t,x) \\ \Phi(x,\hat{x}) \end{bmatrix}, I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{8}$$

If states of the systems, i.e., $x(t)$ and the estimation error $e(t)$, converge asymptotically to the vicinity of origin, then we say that the augmented system is asymptotically stable. For this purpose, Wirtinger's inequality is adopted in this study. Main results and inequality are defined in the following Lemmas, also provided in [43,44].

Lemma 1. *The following inequality holds, for a function M, that is continuously differentiable such that $[u,v] \to \Re^n$, such that there is a matrix $S > 0$,*

$$\begin{aligned} \int_w^v \dot{x}^T(s)M\dot{x}(s)ds \ &\geq \tfrac{1}{v-w}[x(v)-x(w)]^T M[x(v)-x(w)] \\ &+\tfrac{3}{v-w}\left[x(v)+x(w)-\tfrac{2}{v-w}\int_w^v x(s)ds\right]^T \\ &\times M\left[x(v)+x(w)-\tfrac{2}{v-w}\int_w^v x(s)ds\right]. \end{aligned}$$

Lemma 2. *For positive scalars, m, n, consider a function $\Omega(\delta, W)$ for which we have the following relation $\Omega(\delta,W) = \frac{1}{\delta}\Gamma^T R_1^T W\Gamma R_1 + \frac{1}{1-\delta}\Gamma^T R_2^T W\Gamma R_2$, in which $W \in \Re^{n\times n}$, $\Gamma \in \Re^n$, R_1, $R_2 \in \Re^{n\times m}$ and δ exists in interval $(0,1)$. If there exists $X \in \Re^{n\times n}$, that is $\begin{bmatrix} W & X \\ * & W \end{bmatrix} > 0$, then the inequality $min\Omega(\delta,W) = \begin{bmatrix} R_1\Gamma \\ R_2\Gamma \end{bmatrix}^T \begin{bmatrix} W & X \\ * & W \end{bmatrix} \begin{bmatrix} R_1\Gamma & R_2\Gamma \end{bmatrix}$, holds.*

The main task of the current research is to highlight an estimation-based control of OSL nonlinear systems which are subjected to multiple delays that are delayed in state and output by utilizing a DRD stability scheme.

3. Proposed Results

A synchronized approach for the attainment of observer and controller gains, simultaneously, is provided in the forthcoming section.

Theorem 1. *Consider a dynamical system (8), incurred by integrating OSL nonlinearity in Assumption 1, observer dynamics (5), closed-loop system (7), estimation error (6) and controller (4) and systems (1) satisfying the time-varying delay bounds in (2). The state vector $e(t)$ and the state observer error $e(t)$ converge asymptotically in the vicinity of origin for the mentioned symmetrical vectors $P_i \in \Re^{n\times n}$, $M_j \in \Re^{n\times n}$, $Q_{lj} \in \Re^{n\times n}$, $Z_{lk} \in \Re^{n\times n}$ and $Y_k \in \Re^{n\times n}$ endorsing $P_i > 0$, $Y_k > 0$, $Q_{lj} > 0$ and $Z_{lk} > 0$, for $i = k = l = 1$ and 2, and $j = 1, 2, 3$ and scalar $\rho > 0$, so that the following matrices*

$$\begin{gathered}\begin{bmatrix} \phi_1 & \phi_2 \\ * & -\operatorname{diag}(Y_1, Y_2, Z_{11}, Z_{12}, Z_{21}, Z_{22}) \end{bmatrix} \\ -\xi^T(t)T_1^T\vartheta_1T_1\xi(t) - \frac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_2^T\vartheta_2T_2\xi(t) - \xi^T(t)T_3^T\vartheta_3T_3\xi(t) \\ -\xi^T(t)T_4^T\vartheta_4T_4\xi(t) - \frac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_5T_5\xi(t) - \frac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_6^T\vartheta_6T_6\xi(t). \end{gathered} \tag{9}$$

where

$$\phi_1 = \begin{bmatrix} \phi_1^{(1)} & 0 \\ * & \phi_1^{(4)} \end{bmatrix},$$

$$\phi_1^{(1)} = \begin{bmatrix} Y_1 & P_1BK_1 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} \\ * & \Gamma_1^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_2^{(1)} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(1)} & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix}, \tag{10}$$

$$\phi_1^{(4)} = \begin{bmatrix} Y_2 & 0 & 0 & 0 & 0 & 0 & 0 & Y_3 & 0 & 0 & 0 & 0 & 0 & Y_4 \\ * & \Gamma_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_1^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_2^{(3)} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_3^{(3)} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix}$$

$$\phi_2 = \begin{bmatrix} A^T h_{11} Y_1 & A^T h_{1,12} Y_2 & 0 & 0 & 0 & 0 \\ K_1^T B^T h_{11} Y_1 & K_1^T B^T Y_2 h_{1,12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ h_{11} Y_1 & Y_2 h_{1,12} & 0 & 0 & 0 & 0 \\ 0 & 0 & A^T h_{11} Z_{11} & A^T Z_{12} h_{1,12} & A^T Z_{21} h_{21} & A^T Z_{22} h_{2,12} \\ -K_1^T B^T h_{11} Y_1 & -K_1^T B^T Y_2 h_{1,12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C^T K_2^T h_{11} Z_{11} & -C^T K_2^T h_{1,12} Z_{12} & -C^T K_2^T h_{21} Z_{21} & -C^T K_2^T h_{2,12} Z_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{11} Z_{11} & h_{1,12} Z_{12} & h_{21} Z_{21} & h_{2,12} Z_{22} \end{bmatrix},$$

are satisfied, where

$$\begin{aligned}
&\Upsilon_1 = P_1 A + A^T P_1 + \sum_{i=1}^{3} M_i + \rho \varepsilon_1 I + \beta \varepsilon_2 I, \\
&\Gamma_1^{(1)} = -(1-\mu_1) M_3, \\
&\Gamma_2^{(1)} = -M_1, \\
&\Gamma_3^{(1)} = -M_2, \\
&\Upsilon_2 = A^T P_2 + P_2 A + \sum_{i=1}^{2} \sum_{j=1}^{3} Q_{ij}, \\
&\Upsilon_3 = -P_2 K_2 C, \\
&\Upsilon_4 = P_2 - \tfrac{\varepsilon_1 I}{2} + \tfrac{\alpha \varepsilon_2 I}{2}, \\
&\Gamma_1^{(i+1)} = -Q_{i3}(1-\mu_i), \text{ for } i = 1 \text{ and } 2, \\
&\Gamma_2^{(i+1)} = -Q_{i1}, \text{ for } i = 1 \text{ and } 2, \\
&\Gamma_3^{(i+1)} = -Q_{i2}, \text{ for } 1 \text{ and } 2, \\
&h_{i,12} = h_{i2} - h_{i1}, \text{ for } 1 \text{ and } 2, \\
&\widetilde{W} = W^{-1}, \\
&T_1 = \begin{bmatrix} G_3^T & G_4^T & G_5^T & G_6^T \end{bmatrix}^T, T_2 = \begin{bmatrix} G_5^T & G_6^T & G_7^T & G_8^T \end{bmatrix}^T, \\
&T_3 = \begin{bmatrix} G_9^T & G_{11}^T & G_{13}^T & G_{15}^T \end{bmatrix}^T, T_4 = \begin{bmatrix} G_{10}^T & G_{12}^T & G_{14}^T & G_{16}^T \end{bmatrix}^T, \\
&T_5 = \begin{bmatrix} G_{13}^T & G_{15}^T & G_{17}^T & G_{19}^T \end{bmatrix}^T, T_6 = \begin{bmatrix} G_{14}^T & G_{16}^T & G_{18}^T & G_{20}^T \end{bmatrix}^T, \\
&G_3 = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_4 = \begin{bmatrix} I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_5 = \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_6 = \begin{bmatrix} 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_7 = \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_8 = \begin{bmatrix} 0 & I & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
&G_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{11}$$

$$
\begin{aligned}
G_{11} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{12} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 \end{bmatrix}, \\
G_{13} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{14} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{15} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{16} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 \end{bmatrix}, \\
G_{17} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{18} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{19} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
G_{20} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & -2I & 0 \end{bmatrix}, \\
\vartheta_1 &= \begin{bmatrix} \widetilde{Y}_1 & 0 \\ 0 & \widetilde{Y}_1 \end{bmatrix}, \vartheta_2 = \begin{bmatrix} \widetilde{Y}_2 & 0 \\ 0 & \widetilde{Y}_2 \end{bmatrix}, \vartheta_3 = \begin{bmatrix} \widetilde{Z}_{11} & 0 \\ 0 & \widetilde{Z}_{11} \end{bmatrix}, \vartheta_4 = \begin{bmatrix} \widetilde{Z}_{21} & 0 \\ 0 & \widetilde{Z}_{21} \end{bmatrix}, \vartheta_5 = \begin{bmatrix} \widetilde{Z}_{11} & 0 \\ 0 & \widetilde{Z}_{11} \end{bmatrix}, \vartheta_6 = \begin{bmatrix} \widetilde{Z}_{12} & 0 \\ 0 & \widetilde{Z}_{22} \end{bmatrix}, \\
\widetilde{Y}_1 &= \begin{bmatrix} Y_1 & 0 \\ 0 & 3Y_1 \end{bmatrix}, \widetilde{Y}_2 = \begin{bmatrix} Y_2 & 0 \\ 0 & 3Y_2 \end{bmatrix}, \widetilde{Z}_{i1} = \begin{bmatrix} Z_{i1} & 0 \\ 0 & 3Z_{i1} \end{bmatrix}, \widetilde{Z}_{i2} = \begin{bmatrix} Z_{i2} & 0 \\ 0 & 3Z_{i2} \end{bmatrix}, \text{ for } i = 1, 2.
\end{aligned}
\tag{11}
$$

Proof. Selecting a Lyapunov Krasovskii functional for the delayed system from the literature [43,45] provided by

$$
\begin{aligned}
V(e,x,t) = {} & e^T(t)P_2 e(t) + x^T(t)P_1 x(t) + \sum_{i=1}^{2} \int_{t-h_{1i}}^{t} x^T(\alpha) M_i x(\alpha) d\alpha \\
& + \int_{t-\tau_1(t)}^{t} x^T(\alpha) M_3 x(\alpha) d\alpha + \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{t-h_{ij}}^{t} e^T(\alpha) Q_{ij} e(\alpha) d\alpha \\
& + \sum_{i=1}^{2} \int_{t-\tau_i(t)}^{t} e^T(\alpha) Q_{i3} e(\alpha) d\alpha + \int_{-h_{11}}^{0} \int_{t+s}^{t} h_{11} \dot{x}^T(\alpha) Y_1 \dot{x}(\alpha) d\alpha ds \\
& + \int_{-h_{12}}^{-h_{11}} \int_{t+s}^{t} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha ds + \sum_{i=1}^{2} \int_{-h_{i1}}^{0} \int_{t+s}^{t} h_{i1} \dot{e}^T(\alpha) Z_{i1} \dot{e}(\alpha) d\alpha ds \\
& + \sum_{i=1}^{2} \int_{-h_{i2}}^{-h_{i1}} \int_{t+s}^{t} h_{i,12} \dot{e}^T(\alpha) Z_{i2} \dot{e}(\alpha) d\alpha ds.
\end{aligned}
\tag{12}
$$

Time-derivative of function in (12) and involving the derivative of delay $\dot{\tau}_j \leq \mu_j$ for $j = 1, 2$, we have

$$
\begin{aligned}
\dot{V}(e,x,t) \leq {} & 2e^T(t)P_2\dot{e}(t) + 2x^T(t)P_1\dot{x}(t) \\
& + \sum_{i=1}^{2} \left\{ -x^T(t-h_{1i}) M_i x(t-h_{1i}) + x^T(t) M_i x(t) \right\} \\
& -(1-\mu_1) x^T(t-\tau_1(t)) M_3 x(t-\tau_1(t)) + x^T(t) M_3 x(t) \\
& + \sum_{i=1}^{2} \sum_{j=1}^{2} \left\{ -e^T(t-h_{ij}) Q_{ij} e(t-h_{ij}) + e^T(t) Q_{ij} e(t) \right\} \\
& + \sum_{i=1}^{2} \left\{ e^T(t) Q_{i3} e(t) - (1-\mu_i) e^T(t-\tau_i(t)) Q_{i3} e(t-\tau_i(t)) \right\} \\
& + \dot{x}^T(t) \left(h_{11}^2 Y_1 + h_{1,12}^2 Y_2 \right) \dot{x}(t) - \int_{t-h_{11}}^{t} h_{11} \dot{x}^T(\alpha) Y_1 \dot{x}(\alpha) d\alpha \\
& + \int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha + \sum_{i=1}^{2} \dot{e}^T(t) \left(h_{i1}^2 Z_{i1} + h_{i,12}^2 Z_{i2} \right) \dot{e}(t) \\
& - \sum_{i=1}^{2} \int_{t-h_{i1}}^{t} h_{i1} \dot{e}^T(\alpha) Z_{i1} \dot{e}(\alpha) d\alpha + \sum_{i=1}^{2} \int_{t-h_{i2}}^{t-h_{i1}} h_{i,12} \dot{e}^T(\alpha) Z_{i2} \dot{e}(\alpha) d\alpha.
\end{aligned}
\tag{13}
$$

Using Assumption 1, select $U = V = W^{-1}$, such that it renders

$$
\Phi^T(x, \hat{x}) Q^{-1} e(t) \leq \rho e^T(t) W^{-1} e(t) \, . \tag{14}
$$

Integrating (6)–(7) and (13)–(14), renders

$$
\begin{aligned}
\dot{V}(e,x,t) \leq\ & 2e^T(t)P_2(\Phi(x,\hat{x}) + Ae(t) - K_2Ce(t-\tau_2(t))) \\
& +2x^T(t)P_1(BK_1x(t-\tau_1(t)) + Ax(t) - BK_1e(t-\tau_1(t)) + f(t,x)) \\
& -\sum_{i=1}^{2} x^T(t-h_{1i})M_ix(t-h_{1i}) - (1-\mu_1)x^T(t-\tau_1(t))M_3x(t-\tau_1(t)) \\
& +\sum_{i=1}^{2}\sum_{j=1}^{3} e^T(t)Q_{ij}e(t) - \sum_{i=1}^{2}\sum_{j=1}^{2} e^T\left(t-h_{ij}\right)Q_{ij}e\left(t-h_{ij}\right) + \sum_{j=1}^{3} x^T(t)M_jx(t) \\
& -\sum_{i=1}^{2} (1-\mu_i)e^T(t-\tau_i(t))Q_{i3}e(t-\tau_i(t)) - \int_{t-h_{11}}^{t} h_{11}\dot{x}^T(\alpha)Y_1\dot{x}(\alpha)d\alpha \\
& +(f(t,x) + Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t)))^T \left(h_{11}^2Y_1 + h_{1,12}^2Y_2\right) \\
& \times(f(t,x) + Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t))) \\
& -\int_{t-h_{12}}^{t-h_{11}} h_{1,12}\dot{x}^T(\alpha)Y_2\dot{x}(\alpha)d\alpha - \sum_{i=1}^{2}\int_{t-h_{i1}}^{t} h_{i1}\dot{e}(\alpha)^TZ_{i1}\dot{e}(\alpha)d\alpha \\
& +\sum_{i=1}^{2}\left[(\Phi(x,\hat{x}) + Ae(t) - K_2Ce(t-\tau_2(t)))^T\left(h_{i1}^2Z_{i1} + h_{i,12}^2Z_{i2}\right)\right. \\
& \left.\times(\Phi(x,\hat{x}) + Ae(t) - K_2Ce(t-\tau_2(t)))\right] \\
& -\sum_{i=1}^{2}\int_{t-h_{i2}}^{t-h_{i1}} h_{i,12}\dot{e}(\alpha)^TZ_{i2}\dot{e}(\alpha)d\alpha.
\end{aligned}
\tag{15}
$$

Engaging the integral terms in (15) as

$$
-\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq -\int_{t-\tau_1}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds - \int_{t-h_{11}}^{t-\tau_1} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds. \tag{16}
$$

Involving Lemma 1 for Wirtinger's inequality, the integral term renders

$$
\begin{aligned}
-\int_{t-\tau_1}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \quad & -\int_{t-h_{11}}^{t-\tau_1} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq -\tfrac{h_{11}}{\tau_1}[x(t) - x(t-\tau_1)]^TY_1[x(t) - x(t-\tau_1)] \\
& -\tfrac{3h_{11}}{\tau_1}\left[x(t) - x(t-\tau_1) - \tfrac{2}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right]^T Y_1\left[x(t) - x(t-\tau_1) - \tfrac{2}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds\right] \\
& -\tfrac{h_{11}}{h_{11}-\tau_1}[x(t-\tau_1) - x(t-h_{11})]^TY_1[x(t-\tau_1) - x(t-h_{11})] \\
& -\tfrac{3h_{11}}{h_{11}-\tau_1}\left[x(t-\tau_1) - x(t-h_{11}) - \tfrac{2}{h_{11}-\tau_1}\int_{t-h_{11}}^{t-\tau_1} x(s)ds\right]^T \\
& \times Y_1\left[x(t-\tau_1) - x(t-h_{11}) - \tfrac{2}{h_{11}-\tau_1}\int_{t-h_{11}}^{t-\tau_1} x(s)ds\right].
\end{aligned}
\tag{17}
$$

Defining

$$
\begin{aligned}
\xi(t) = \Bigg[\ & x^T(t) \quad x^T(t-\tau_1(t)) \quad x^T(t-h_{11}) \quad x^T(t-h_{12}) \quad \tfrac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^{t} x^T(s)ds \\
& \tfrac{1}{h_{11}-\tau_1(t)}\int_{t-h_{11}}^{t-\tau_1(t)} x^T(s)ds \quad \tfrac{1}{h_{12}-\tau_1(t)}\int_{t-h_{12}}^{t-\tau_1(t)} x^T(s)ds \quad f^T(t,x) \quad e^T(t) \quad e^T(t-\tau_1(t)) \\
& e^T(t-h_{11}) \quad e^T(t-h_{12}) \quad \tfrac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^{t} e^T(s)ds \quad \tfrac{1}{h_{11}-\tau_1(t)}\int_{t-h_{11}}^{t-\tau_1(t)} e^T(s)ds \quad \tfrac{1}{h_{12}-\tau_1(t)}\int_{t-h_{12}}^{t-\tau_1(t)} x^T(s)ds \\
& e^T(t-\tau_2(t)) \quad e^T(t-h_{21}) \quad e^T(t-h_{22}) \quad \tfrac{1}{\tau_2(t)}\int_{t-\tau_2(t)}^{t} e^T(s)ds \quad \tfrac{1}{h_{21}-\tau_2(t)}\int_{t-h_{21}}^{t-\tau_2(t)} e^T(s)ds \\
& \tfrac{1}{h_{22}-\tau_2(t)}\int_{t-h_{22}}^{t-\tau_2(t)} e^T(s)ds \quad \Phi^T(x,\hat{x})\ \Bigg],
\end{aligned}
\tag{18}
$$

since we have a relation as

$$
\begin{aligned}
&\begin{bmatrix} x(t) & x(t-\tau_1) \end{bmatrix} \\
&\qquad = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\qquad = G_3\xi(t), \\
&\begin{bmatrix} x(t) & x(t-\tau) & \frac{2}{\tau_1(t)} \int\limits_{t-\tau_1(t)}^{t} x^T(s)ds \end{bmatrix} \\
&\qquad = \begin{bmatrix} I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\qquad = G_4\xi(t), \\
&\begin{bmatrix} x(t-\tau_1) & x(t-h_{11}) \end{bmatrix} \\
&\qquad = \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\qquad = G_5\xi(t),
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
&\begin{bmatrix} x(t-\tau_1) & x(t-h_{11}) & \frac{2}{h_{11}-\tau_1(t)} \int\limits_{t-h_{11}}^{t-\tau_1(t)} x^T(s)ds \end{bmatrix} \\
&\qquad = \begin{bmatrix} 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\qquad = G_6\xi(t).
\end{aligned}
$$

Furthermore, inputting values from (19), the relation in (17) renders

$$
\begin{aligned}
-\textstyle\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq & -\tfrac{h_{11}}{\tau_1}\xi^T(t)G_3^T Y_1 G_3\xi(t) - \tfrac{3h_{11}}{\tau_1}\xi^T(t)G_4^T Y_1 G_4\xi(t) \\
& -\tfrac{h_{11}}{h_{11}-\tau_1}\xi^T(t)G_5^T Y_1 G_5\xi(t) - \tfrac{3h_{11}}{h_{11}-\tau_1}\xi^T(t)G_6^T Y_1 G_6\xi(t),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
-\textstyle\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq & -\tfrac{h_{11}}{\tau_1}\xi^T(t)\begin{bmatrix} G_3^T & G_4^T \end{bmatrix}\begin{bmatrix} Y_1 & 0 \\ 0 & 3Y_1 \end{bmatrix}\begin{bmatrix} G_3 \\ G_4 \end{bmatrix}\xi(t) \\
& -\tfrac{h_{11}}{h_{11}-\tau_1}\xi^T(t)\begin{bmatrix} G_5^T & G_6^T \end{bmatrix}\begin{bmatrix} Y_1 & 0 \\ 0 & 3Y_1 \end{bmatrix}\begin{bmatrix} G_5 \\ G_6 \end{bmatrix}\xi(t).
\end{aligned}
\tag{21}
$$

Defining $\kappa_1 = \frac{\tau_1}{h_{11}}$ and $\widetilde{Y}_1 = \begin{bmatrix} Y_1 & 0 \\ 0 & 3Y_1 \end{bmatrix}$, incorporating terms gives

$$
\begin{aligned}
-\textstyle\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq & -\tfrac{1}{\kappa_1}\xi^T(t)\begin{bmatrix} G_3^T & G_4^T \end{bmatrix}\widetilde{Y}_1\begin{bmatrix} G_3 \\ G_4 \end{bmatrix}\xi(t) \\
& -\tfrac{1}{1-\kappa_1}\xi^T(t)\begin{bmatrix} G_5^T & G_6^T \end{bmatrix}\widetilde{Y}_1\begin{bmatrix} G_5 \\ G_6 \end{bmatrix}\xi(t).
\end{aligned}
\tag{22}
$$

Using Lemma 2 and defining $\vartheta_1 = \begin{bmatrix} \widetilde{Y}_1 & 0 \\ 0 & \widetilde{Y}_1 \end{bmatrix}$, we have

$$
-\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq -\xi^T(t)\begin{bmatrix} G_3^T & G_4^T & G_5^T & G_6^T \end{bmatrix}\vartheta_1\begin{bmatrix} G_3 \\ G_4 \\ G_5 \\ G_6 \end{bmatrix}\xi(t).
\tag{23}
$$

Taking $T_1^T = \begin{bmatrix} G_3^T & G_4^T & G_5^T & G_6^T \end{bmatrix}$ which makes

$$
-\int_{t-h_{11}}^{t} h_{11}\dot{x}^T(s)Y_1\dot{x}(s)ds \leq -\xi^T(t)T_1^T\vartheta_1 T_1\xi(t).
\tag{24}
$$

Other integrals in the Lyapunov functional in (15) result into

$$
-\int_{t-h_{12}}^{t-h_{11}} h_{1,12}\dot{x}^T(\alpha)Y_2\dot{x}(\alpha)d\alpha = -\int_{t-\tau_1}^{t-h_{11}} h_{1,12}\dot{x}^T(\alpha)Y_2\dot{x}(\alpha)d\alpha - \int_{t-h_{12}}^{t-\tau_1} h_{1,12}\dot{x}^T(\alpha)Y_2\dot{x}(\alpha)d\alpha.
\tag{25}
$$

Note that for this, terms are

$$\begin{aligned}
&\begin{bmatrix} x(t-\tau_1) & x(t-h_{12}) \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\quad = G_7 \xi(t), \\
&\begin{bmatrix} x(t-\tau_1) & x(t-h_{12}) & \frac{2}{h_{12}-\tau_1(t)} \int\limits_{t-h_{12}}^{t-\tau_1(t)} x^T(s) ds \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & I & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
&\quad = G_8 \xi(t).
\end{aligned} \tag{26}$$

Utilizing G_5 and G_6 defined in (19) and G_7 and G_8 defined in (26), the relation in (25) becomes

$$\begin{aligned}
-\int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha &= -\tfrac{h_{1,12}}{h_{11}-\tau_1} \xi^T(t) G_5^T Y_2 G_5 \xi(t) - \tfrac{3h_{1,12}}{h_{11}-\tau_1} \xi^T(t) G_6^T Y_2 G_6 \xi(t) \\
&\quad - \tfrac{h_{1,12}}{h_{12}-\tau_1} \xi^T(t) G_7^T Y_2 G_7 \xi(t) - \tfrac{3h_{1,12}}{h_{12}-\tau_1} \xi^T(t) G_8^T Y_2 G_8 \xi(t), \\
-\int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha &\leq -\tfrac{h_{1,12}}{h_{11}-\tau_1} \xi^T(t) \begin{bmatrix} G_5^T & G_6^T \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ 0 & 3Y_2 \end{bmatrix} \begin{bmatrix} G_5 \\ G_6 \end{bmatrix} \xi(t) \\
&\quad - \tfrac{h_{1,12}}{h_{12}-\tau_1} \xi^T(t) \begin{bmatrix} G_7^T & G_8^T \end{bmatrix} \begin{bmatrix} Y_2 & 0 \\ 0 & 3Y_2 \end{bmatrix} \begin{bmatrix} G_7 \\ G_8 \end{bmatrix} \xi(t).
\end{aligned} \tag{27}$$

Utilizing $\kappa_2 = \frac{\tau_1 - h_{11}}{h_{12}-h_{11}}$ and $\widetilde{Y}_2 = \begin{bmatrix} Y_2 & 0 \\ 0 & 3Y_2 \end{bmatrix}$ in (27) results into

$$\begin{aligned}
-\int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha &\leq -\tfrac{h_{1,12}}{h_{12}-h_{11}} \left[\tfrac{1}{\kappa_2} \xi^T(t) \begin{bmatrix} G_5^T & G_6^T \end{bmatrix} \widetilde{Y}_2 \begin{bmatrix} G_5 \\ G_6 \end{bmatrix} \xi(t) \right] \\
&\quad - \tfrac{h_{1,12}}{h_{12}-h_{11}} \left[\tfrac{1}{1-\kappa_2} \xi^T(t) \begin{bmatrix} G_7^T & G_8^T \end{bmatrix} \widetilde{Y}_2 \begin{bmatrix} G_7 \\ G_8 \end{bmatrix} \xi(t) \right].
\end{aligned} \tag{28}$$

Using Lemma 2 and defining $\vartheta_2 = \begin{bmatrix} \widetilde{Y}_2 & 0 \\ 0 & \widetilde{Y}_2 \end{bmatrix}$, the relation in (28) renders

$$-\int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha \leq -\frac{h_{1,12}}{h_2 - h_1} \xi^T(t) \begin{bmatrix} G_5^T & G_6^T & G_7^T & G_8^T \end{bmatrix} \vartheta_2 \begin{bmatrix} G_5 \\ G_6 \\ G_7 \\ G_8 \end{bmatrix} \xi(t). \tag{29}$$

Utilizing $T_2 = \begin{bmatrix} G_5^T & G_6^T & G_7^T & G_8^T \end{bmatrix}$ in (29) gives

$$-\int_{t-h_{12}}^{t-h_{11}} h_{1,12} \dot{x}^T(\alpha) Y_2 \dot{x}(\alpha) d\alpha \leq -\frac{h_{1,12}}{h_{12} - h_{11}} \xi^T(t) T_2^T \vartheta_2 T_2 \xi(t). \tag{30}$$

The integral terms for error in (15) will be rendered by using

$$-\sum_{i=1}^{2} \int_{t-h_{i1}}^{t} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha = -\sum_{i=1}^{2} \left[\int_{t-\tau_i}^{t} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha - \int_{t-h_{i1}}^{t-\tau_i} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha \right]. \tag{31}$$

For this, terms are represented as

$$
\begin{aligned}
&\begin{bmatrix} e(t) & e(t-\tau_1) \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_9\xi(t),\\
&\begin{bmatrix} e(t) & e(t-\tau_2) \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{10}\xi(t),\\
&\begin{bmatrix} e(t) & e(t-\tau_1) & \frac{1}{\tau_1(t)}\int\limits_{t-\tau_1}^{t} e(s)ds \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{11}\xi(t).\\
&\begin{bmatrix} e(t) & e(t-\tau_2) & \frac{1}{\tau_2(t)}\int\limits_{t-\tau_2}^{t} e(s)ds \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{12}\xi(t).\\
&\begin{bmatrix} e(t-\tau_1) & e(t-h_{11}) \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{13}\xi(t).\\
&\begin{bmatrix} e(t-\tau_2) & e(t-h_{21}) \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & -I & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{14}\xi(t).\\
&\begin{bmatrix} e(t-\tau_1) & e(t-h_{11}) & \frac{1}{h_{11}-\tau_1(t)}\int\limits_{t-h_{11}}^{t-\tau_1(t)} e^T(s)ds \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{15}\xi(t).\\
&\begin{bmatrix} e(t-\tau_2) & e(t-h_{21}) & \frac{1}{h_{21}-\tau_2(t)}\int\limits_{t-h_{21}}^{t-\tau_2(t)} e^T(s)ds \end{bmatrix}\\
&\qquad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & -2I & 0 & 0 \end{bmatrix}\xi(t)\\
&\qquad = G_{16}\xi(t).
\end{aligned}
\tag{32}
$$

Utilizing relations defined in (32), the integral term in (31) becomes

$$
\begin{aligned}
-\sum_{i=1}^{2}\int_{t-h\ i1}^{t} h_{i1}\dot{e}(\alpha)^T Z_{i1}\dot{e}(\alpha)d\alpha = &-\tfrac{h_{11}}{\tau_1}\xi^T(t)G_9^T Z_{11}G_9\xi(t) - \tfrac{3h_{11}}{\tau_1}\xi^T(t)G_{11}^T Z_{11}G_{11}\xi(t)\\
&-\tfrac{h_{11}}{h_{11}-\tau_1}\xi^T(t)G_{13}^T Z_{11}G_{13}\xi(t) - \tfrac{3h_{11}}{h_{11}-\tau_1}\xi^T(t)G_{15}^T Z_{11}G_{15}\xi(t)\\
&-\tfrac{h_{21}}{\tau_1}\xi^T(t)G_{10}^T Z_{21}G_{10}\xi(t) - \tfrac{3h_{21}}{\tau_1}\xi^T(t)G_{12}^T Z_{21}G_{12}\xi(t)\\
&-\tfrac{h_{11}}{h_{11}-\tau_1}\xi^T(t)G_{14}^T Z_{21}G_{14}\xi(t) - \tfrac{3h_{11}}{h_{11}-\tau_1}\xi^T(t)G_{16}^T Z_{21}G_{16}\xi(t),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
-\sum_{i=1}^{2}\int_{t-h\ i1}^{t} h_{i1}\dot{e}(\alpha)^T Z_{i1}\dot{e}(\alpha)d\alpha \le &-\tfrac{h_{11}}{\tau_1}\xi^T(t)\begin{bmatrix} G_9^T & G_{11}^T \end{bmatrix}\begin{bmatrix} Z_{11} & 0 \\ 0 & 3Z_{11} \end{bmatrix}\begin{bmatrix} G_9 \\ G_{11} \end{bmatrix}\xi(t)\\
&-\tfrac{h_{11}}{h_{11}-\tau_1}\xi^T(t)\begin{bmatrix} G_{13}^T & G_{15}^T \end{bmatrix}\begin{bmatrix} Z_{11} & 0 \\ 0 & 3Z_{11} \end{bmatrix}\begin{bmatrix} G_{13} \\ G_{15} \end{bmatrix}\xi(t)\\
&-\tfrac{h_{21}}{\tau_2}\xi^T(t)\begin{bmatrix} G_{10}^T & G_{12}^T \end{bmatrix}\begin{bmatrix} Z_{21} & 0 \\ 0 & 3Z_{21} \end{bmatrix}\begin{bmatrix} G_{10} \\ G_{12} \end{bmatrix}\xi(t)\\
&-\tfrac{h_{21}}{h_{21}-\tau_2}\xi^T(t)\begin{bmatrix} G_{14}^T & G_{16}^T \end{bmatrix}\begin{bmatrix} Z_{21} & 0 \\ 0 & 3Z_{21} \end{bmatrix}\begin{bmatrix} G_{14} \\ G_{16} \end{bmatrix}\xi(t).
\end{aligned}
\tag{34}
$$

Utilizing $\kappa_3 = \frac{\tau_1}{h_{11}}$, $\kappa_4 = \frac{\tau_2}{h_{21}}$, $\widetilde{Z}_{i1} = \begin{bmatrix} Z_{i1} & 0 \\ 0 & 3Z_{i1} \end{bmatrix}$, for $i = 1, 2$, in (34) results into

$$
\begin{aligned}
-\sum_{i=1}^{2} \int_{t-h_{i1}}^{t} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha \leq & -\tfrac{h_{11}}{\tau_1} \left[\tfrac{1}{\kappa_3} \xi^T(t) \begin{bmatrix} G_9^T & G_{11}^T \end{bmatrix} \widetilde{Z}_{11} \begin{bmatrix} G_9 \\ G_{11} \end{bmatrix} \xi(t) \right] \\
& -\tfrac{h_{11}}{h_{11}-\tau_1} \left[\tfrac{1}{1-\kappa_3} \xi^T(t) \begin{bmatrix} G_{13}^T & G_{15}^T \end{bmatrix} \widetilde{Z}_{11} \begin{bmatrix} G_{13} \\ G_{15} \end{bmatrix} \xi(t) \right] \\
& -\tfrac{h_{21}}{\tau_2} \left[\tfrac{1}{\kappa_4} \xi^T(t) \begin{bmatrix} G_{10}^T & G_{12}^T \end{bmatrix} \widetilde{Z}_{21} \begin{bmatrix} G_{10} \\ G_{12} \end{bmatrix} \xi(t) \right] \\
& -\tfrac{h_{21}}{h_{21}-\tau_2} \left[\tfrac{1}{1-\kappa_4} \xi^T(t) \begin{bmatrix} G_{14}^T & G_{16}^T \end{bmatrix} \widetilde{Z}_{21} \begin{bmatrix} G_{14} \\ G_{16} \end{bmatrix} \xi(t) \right].
\end{aligned} \tag{35}
$$

Using Lemma 2 and define

$$
\vartheta_3 = \begin{bmatrix} \widetilde{Z}_{11} & 0 \\ 0 & \widetilde{Z}_{11} \end{bmatrix}, \ \vartheta_4 = \begin{bmatrix} \widetilde{Z}_{21} & 0 \\ 0 & \widetilde{Z}_{21} \end{bmatrix},
$$

the relation in (35) renders

$$
\begin{aligned}
-\sum_{i=1}^{2} \int_{t-h_{i1}}^{t} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha \quad & \leq -\xi^T(t) \begin{bmatrix} G_9^T & G_{11}^T & G_{13}^T & G_{15}^T \end{bmatrix} \vartheta_3 \begin{bmatrix} G_9 \\ G_{11} \\ G_{13} \\ G_{15} \end{bmatrix} \xi(t) \\
& -\xi^T(t) \begin{bmatrix} G_{10}^T & G_{12}^T & G_{14}^T & G_{16}^T \end{bmatrix} \vartheta_4 \begin{bmatrix} G_{10} \\ G_{12} \\ G_{14} \\ G_{16} \end{bmatrix} \xi(t).
\end{aligned} \tag{36}
$$

Utilizing $T_3 = \begin{bmatrix} G_9^T & G_{11}^T & G_{13}^T & G_{15}^T \end{bmatrix}$ and $T_4 = \begin{bmatrix} G_{10}^T & G_{12}^T & G_{14}^T & G_{16}^T \end{bmatrix}$, in (36) gives

$$
-\sum_{i=1}^{2} \int_{t-h_{i1}}^{t} h_{i1} \dot{e}(\alpha)^T Z_{i1} \dot{e}(\alpha) d\alpha \leq -\xi^T(t) T_3^T \vartheta_3 T_3 \xi(t) - \xi^T(t) T_4^T \vartheta_4 T_4 \xi(t). \tag{37}
$$

A similar approach for the second term of error in (15) results in

$$
-\sum_{i=1}^{2} \int_{t-h_{i2}}^{t-h_{i1}} h_{i,12} \dot{e}(\alpha)^T Z_{i2} \dot{e}(\alpha) d\alpha = -\frac{h_{1,12}}{h_{12}-h_{11}} \xi^T(t) T_5^T \vartheta_5 T_5 \xi(t) - \frac{h_{2,12}}{h_{21}-h_{22}} \xi^T(t) T_6^T \vartheta_6 T_6 \xi(t), \tag{38}
$$

in which

$$
\begin{aligned}
& \begin{bmatrix} e(t-\tau_1) & e(t-h_{12}) \end{bmatrix} \\
& \quad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
& \quad = G_{17} \xi(t). \\
& \begin{bmatrix} e(t-\tau_2) & e(t-h_{22}) \end{bmatrix} \\
& \quad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
& \quad = G_{18} \xi(t). \\
& \begin{bmatrix} e(t-\tau_1) & e(t-h_{12}) & \frac{2}{h_{11}-\tau_1(t)} \int\limits_{t-h_{11}}^{t-\tau_1(t)} e^T(s) ds \end{bmatrix} \\
& \quad = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & I & 0 & 0 & -2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xi(t) \\
& \quad = G_{19} \xi(t),
\end{aligned}
$$

which renders (15) as

$$
\begin{aligned}
\dot{V}(e,x,t) \leq\ & 2e^T(t)P_2(\Phi(x,\hat{x}) + Ae(t) - K_2Ce(t-\tau_2(t))) \\
& + 2x^T(t)P_1(Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t)) + f(t,x)) \\
& - \sum_{i=1}^{2} x^T(t-h_{1i})M_ix(t-h_{1i}) - (1-\mu_1)x^T(t-\tau_1(t))M_3x(t-\tau_1(t)) \\
& + \sum_{i=1}^{2}\sum_{j=1}^{3} e^T(t)Q_{ij}e(t) - \sum_{i=1}^{2}(1-\mu_i)e^T(t-\tau_i(t))Q_{i3}e(t-\tau_i(t)) \\
& - \sum_{i=1}^{2}\sum_{j=1}^{2} e^T(t-h_{ij})Q_{ij}e(t-h_{ij}) + \sum_{j=1}^{3} x^T(t)M_jx(t) \\
& + (f(t,x) + Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t)))^T\left(h_{11}^2Y_1 + h_{1,12}^2Y_2\right) \\
& \times (f(t,x) + Ax(t) + BK_1x(t-\tau_1(t)) - BK_1e(t-\tau_1(t))) \\
& + \sum_{i=1}^{2}\Big[(Ae(t) + \Phi(x,\hat{x}) - K_2Ce(t-\tau_2(t)))^T\left(h_{i1}^2Z_{i1} + h_{i,12}^2Z_{i2}\right) \\
& \times(Ae(t) + \Phi(x,\hat{x}) - K_2Ce(t-\tau_2(t)))\Big] - \xi^T(t)T_1^T\vartheta_1T_1\xi(t) \\
& - \frac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_2^T\vartheta_2T_2\xi(t) - \xi^T(t)T_3^T\vartheta_3T_3\xi(t) - \xi^T(t)T_4^T\vartheta_4T_4\xi(t) \\
& - \frac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_5T_5\xi(t) - \frac{h_{2,12}}{h_{21}-h_{22}}\xi^T(t)T_6^T\vartheta_6T_6\xi(t).
\end{aligned} \tag{39}
$$

Involving the Assumption 1 relation for OSL and QIB for scalars $\varepsilon_i > 0$ for $i = 1,2,3,4$, provides an equivalent form that can be seen in Appendix A as Equations (A1)–(A4). Integrating all the relations of OSL and QIB with (39)

$$
\dot{V}(x,e,t) \leq \xi^T(t)\begin{bmatrix} \phi_1 & \phi_2 \\ * & -diag(Y_1,Y_2,Z_{11},Z_{12},Z_{21},Z_{22}) \end{bmatrix}\xi(t) \tag{40}
$$

To show $\dot{V}(x,e,t) < 0$, $\phi_1 < 0$ and $\phi_2 < 0$ are mandatory. Further engaging the Schur complement to $\phi_1 < 0$ and $\phi_2 < 0$ renders (9), which delivers Theorem 1. □

Remark 1. *The methodology highlighted in Theorem 1 incorporates the novel observer-based control for the OSL nonlinear dynamical systems subjected to input and output delays which vary between upper and lower bounds. They are also applicable to time-delayed systems with both larger and smaller delays in comparison to conventional techniques for Lipschitz and OSL nonlinear systems [37,40,43]. The proposed method integrates the OSL nonlinearity condition for an observer-based scheme along with delayed dynamics. In contrast to [43], the present work reduces the conservatism of the quadratic inner-boundedness condition by including this GOSL inequality which extends the applicability of estimation techniques for nonlinear systems.*

Remark 2. *To extend the scope and provide better results for time-delayed systems, the delay-independent stability criterion utilized in older studies is extended to the delay-range-dependent criterion for input and output delayed systems in this study [45]. Delay-range-dependent stabilization employed in this strategy is further exploited with the Wirtinger inequality to have less conservative results in contrast to the conventional Jensen's inequality procedure. Previously, estimation techniques with delayed dynamics were treated with Jensen's inequality to provide the solution of integral terms in the Lyapunov functional of time-delayed systems [37,40,45]. In this study, the Wirtinger inequality presents the results for the systems which were previously treated with Jensen's inequality. To the best of the authors' knowledge, an estimation-based scheme of control for OSL nonlinear schemes with multiple delays processed with a delay-range-dependent stability approach using the Wirtinger inequality is being covered here for the first time. The present study includes the solution for various convex routines to obtain the controller and observer gains simultaneously through simulation tools.*

Remark 3. *A distinctive Lyapunov-Krasovskii functional (provided in [43,45]) is nominated in Theorem 1 to have an observer-based control scheme that may address multiple time delays as compared to the orthodox techniques [36,39]. Physical systems integrate both input and output delays; the proposed technique may also render the solution of any one of the delays by just assuming other terms as zero. If input delay $\tau_1(t)$ is required to be zero, a special case of Theorem 1 can be incurred by the selection of corresponding matrices M_i for $i = 1,2,3$, Z_{1i}, for $i = 1,2,3$, Q_{1i} , for $i = 1,2,3$, Y_i, for $i = 1,2$, taken as zero. Likewise, results for systems with no delay can be obtained by assuming $\tau_2(t) = 0$, $Z_{21} = Z_{22} = Z_{23} = 0$ and $Q_{21} = Q_{22} = Q_{23} = 0$.*

The proposed methodology in Theorem 1 ensures the convergence of states in (1) and estimation error by obtaining the controller and estimation gain assumed as K_1 and K_2. Whereas simultaneous tuning of K_1 and K_2 is not possible in the present form of Theorem1, therefore decoupling techniques provided in [40,43,46] are employed to have both gain matrices independently.

Theorem 2. *The solution of constraints in Theorem 1 deduced from a necessary and sufficient condition is such that there should be matrices $\overline{P}_1 \in \Re^{n\times n}$, $\widetilde{P}_2 \in \Re^{n\times n}$, $\overline{M}_j \in \Re^{n\times n}$, $V_k \in \Re^{n\times n}$, $\widetilde{Q}_{lj} \in \Re^{n\times n}$ and $\widetilde{Z}_{lk} \in \Re^{n\times n}$ endorsing $\overline{P}_1 > 0$, $\widetilde{P}_2 > 0$, $\widetilde{Q}_{lj} > 0$, $\overline{M}_j > 0$, $\widetilde{Z}_{lk} > 0$, $V_k > 0$ for $j = 1,2,3$, and $k = l = 1,2$ so that the given matrices*

$$\Omega_1 = \begin{bmatrix} \tilde{Y}_1 & BX_1 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & \overline{P}_1 A^T h_{11} & \overline{P}_1 A^T h_{1,12} \\ * & \tilde{\Gamma}_1^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & X_1 B^T h_{11} & X_1 B^T h_{1,12} \\ * & * & \Gamma_2^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I & h_{11} I & h_{1,12} I \\ * & * & * & * & * & * & * & * & -V_1 & 0 \\ * & * & * & * & * & * & * & * & * & -V_2 \end{bmatrix} \tag{41}$$
$$-\xi^T(t) T_1^T \vartheta_{1C} T_1 \xi(t) - \frac{h_{1,12}}{h_{12}-h_{11}} \xi^T(t) T_2^T \vartheta_{2C} T_2 \xi(t)\,.$$

$$\Omega_2 = \begin{bmatrix} \phi_1^{(4)} & \widetilde{\phi}_2 \\ * & -diag(\widetilde{Z}_{11}, \widetilde{Z}_{12}, \widetilde{Z}_{21}, \widetilde{Z}_{22}) \end{bmatrix} - \xi^T(t) T_3^T \vartheta_{3o} T_3 \xi(t) \tag{42}$$
$$-\xi^T(t) T_4^T \vartheta_{4o} T_4 \xi(t) - \frac{h_{1,12}}{h_{12}-h_{11}} \xi^T(t) T_5^T \vartheta_{5o} T_5 \xi(t) - \frac{h_{2,12}}{h_{21}-h_{22}} \xi^T(t) T_6^T \vartheta_{6o} T_6 \xi(t),$$

where

$$\phi_1^{(4)} = \begin{bmatrix} \tilde{Y}_2 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{Y}_3 & 0 & 0 & 0 & 0 & 0 & Y_4 \\ * & \Gamma_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_1^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_2^{(3)} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_3^{(3)} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix}$$

$$\widetilde{\phi}_2 = \begin{bmatrix} A^T h_{11}\widetilde{P}_2 & A^T\widetilde{P}_2 h_{1,12} & A^T h_{21}\widetilde{P}_2 h_{1,12} & A^T h_{2,12}\widetilde{P}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C^T X_2^T h_{11} & -C^T X_2^T h_{1,12} & -C^T X_2^T h_{21} & -C^T X_2^T h_{2,12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{11}\widetilde{P}_2 & h_{1,12}\widetilde{P}_2 & h_{21}\widetilde{P}_2 & h_{2,12}\widetilde{P}_2 \end{bmatrix},$$

$$\begin{aligned}
&\widetilde{\Upsilon}_1 = \overline{P}_1 A + A^T\overline{P}_1 + \sum_{i=1}^{3} M_i + \rho\varepsilon_1 I + \beta\varepsilon_2 I, \\
&\Gamma_1^{(1)} = -(1-\mu_1)M_3, \\
&\Gamma_2^{(1)} = -M_1, \\
&\Gamma_3^{(1)} = -M_2, \\
&\widetilde{\Upsilon}_2 = \widetilde{P}_2 A + A^T\widetilde{P}_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij} + \rho\varepsilon_3 I + \beta\varepsilon_4 I, \\
&\widetilde{\Upsilon}_3 = -X_2 C, \\
&\Upsilon_4 = I - \tfrac{\varepsilon_3\widetilde{P}_2}{2} + \tfrac{\alpha\varepsilon_4\widetilde{P}_2}{2}, \\
&\Gamma_1^{(i+1)} = -(1-\mu_i)Q_{i3}, \text{ for } i = 1 \text{ and } 2, \\
&\Gamma_2^{(i+1)} = -Q_{i1}, \text{ for } i = 1 \text{ and } 2, \\
&\Gamma_3^{(i+1)} = -Q_{i2}, \text{ for } i = 1 \text{ and } 2, \\
&h_{i,12} = h_{i2} - h_{i1}, \text{ for } i = 1 \text{ and } 2.
\end{aligned}$$

Proof. *Sufficiency:* By applying the Schur complement procedure to inequalities (41) and (42), treated further by the congruence transformation, $diag(\overline{P}_1, \overline{P}_1, I, I, I, I, I, \widetilde{Y}_1^{-1}, \widetilde{Y}_2^{-1})$, in which $\overline{P}_1 = \widetilde{P}_1^{-1}$, $X_1 = K_1\overline{P}_1$, $\overline{Y}_1 = \overline{P}_1\widetilde{Y}_1\overline{P}_1$, $\overline{Y}_2 = \overline{P}_1\widetilde{Y}_2\overline{P}_1$, $T_1 = \overline{P}_1\overline{Y}_1^{-1}\,\overline{P}_1$ and $T_2 = \overline{P}_1\overline{Y}_2^{-1}\,\overline{P}_1$, we have

$$\begin{aligned}
&\begin{bmatrix} \hat{\phi}_1 & 0 \\ * & \hat{\phi}_2 \end{bmatrix} - \xi^T(t)T_1^T\vartheta_1 T_1\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_2^T\vartheta_2 T_2\xi(t) - \xi^T(t)T_3^T\vartheta_3 T_3\xi(t) \\
&-\xi^T(t)T_4^T\vartheta_4 T_4\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_5 T_5\xi(t) - \tfrac{h_{2,12}}{h_{21}-h_{22}}\xi^T(t)T_6^T\vartheta_6 T_6\xi(t).
\end{aligned} \tag{43}$$

$$\hat{\phi}_1 = \begin{bmatrix} \widetilde{\Upsilon}_1 & \widetilde{P}_1 BK_1 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & A^T h_{11}\widetilde{Y}_1 & A^T h_{1,12}\widetilde{Y}_2 \\ * & \widetilde{\Gamma}_1^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & K_1^T B^T h_{11}\widetilde{Y}_1 & K_1^T B^T h_{1,12}\widetilde{Y}_2 \\ * & * & \Gamma_2^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I & h_{11}\widetilde{Y}_1 & h_{1,12}\widetilde{Y}_2 \\ * & * & * & * & * & * & * & * & -\widetilde{Y}_1 & 0 \\ * & * & * & * & * & * & * & * & * & -\widetilde{Y}_2 \end{bmatrix}, \tag{44}$$

$$
\begin{aligned}
&-\xi^T(t)T_1^T\vartheta_{1c}T_1\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_2^T\vartheta_{2c}T_2\xi(t)\\
&\vartheta_{1c} = \begin{bmatrix} Y_1 & 0 & 0 & 0 \\ 0 & 3Y_1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 3Y_1 \end{bmatrix}, \ \vartheta_{2c} = \begin{bmatrix} Y_2 & 0 & 0 & 0 \\ 0 & 3Y_2 & 0 & 0 \\ 0 & 0 & Y_2 & 0 \\ 0 & 0 & 0 & 3Y_2 \end{bmatrix},
\end{aligned} \tag{44}
$$

where the resultant is obtained by employing the Schur compliment. Similarly for the remaining, apply congruence transformation by involving a block-diagonal matrix notation $diag\Big(I, I, I, I, I, I, I, I, I, I, I, I, I, \widetilde{P}_2\widetilde{Z}_{11}^{-1}, \widetilde{P}_2\widetilde{Z}_{12}^{-1}, \widetilde{P}_2\widetilde{Z}_{21}^{-1}, \widetilde{P}_2\widetilde{Z}_{22}^{-1}\Big)$ and by substituting $X_2 = \widetilde{P}_2K_2$, $V_{11} = \widetilde{P}_2\widetilde{Z}_{11}^{-1}\widetilde{P}_2$, $V_{12} = \widetilde{P}_2\widetilde{Z}_{12}^{-1}\widetilde{P}_2$, $V_{21} = \widetilde{P}_2\widetilde{Z}_{21}^{-1}\widetilde{P}_2$ and $V_{22} = \widetilde{P}_2\widetilde{Z}_{22}^{-1}\widetilde{P}_2$.

$$
\begin{aligned}
&\hat{\phi}_2 = \begin{bmatrix} \phi_1^{(4)} & \widetilde{\phi}_2 \\ * & -diag(\widetilde{Z}_{11}, \widetilde{Z}_{12}, \widetilde{Z}_{21}, \widetilde{Z}_{22}) \end{bmatrix} - \xi^T(t)T_3^T\vartheta_{3o}T_3\xi(t)\\
&-\xi^T(t)T_4^T\vartheta_{4o}T_4\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_{5o}T_5\xi(t) - \tfrac{h_{2,12}}{h_{21}-h_{22}}\xi^T(t)T_6^T\vartheta_{6o}T_6\xi(t),\\
&\vartheta_{3o} = \begin{bmatrix} Z_{11} & 0 & 0 & 0 \\ 0 & 3Z_{11} & 0 & 0 \\ 0 & 0 & Z_{11} & 0 \\ 0 & 0 & 0 & 3Z_{11} \end{bmatrix}, \ \vartheta_{4o} = \begin{bmatrix} Z_{21} & 0 & 0 & 0 \\ 0 & 3Z_{21} & 0 & 0 \\ 0 & 0 & Z_{21} & 0 \\ 0 & 0 & 0 & 3Z_{21} \end{bmatrix},\\
&\vartheta_{5o} = \begin{bmatrix} Z_{12} & 0 & 0 & 0 \\ 0 & 3Z_{12} & 0 & 0 \\ 0 & 0 & Z_{12} & 0 \\ 0 & 0 & 0 & 3Z_{12} \end{bmatrix}, \ \vartheta_{6o} = \begin{bmatrix} Z_{22} & 0 & 0 & 0 \\ 0 & 3Z_{22} & 0 & 0 \\ 0 & 0 & Z_{22} & 0 \\ 0 & 0 & 0 & 3Z_{22} \end{bmatrix},
\end{aligned} \tag{45}
$$

$$
\widetilde{\phi}_2 = \begin{bmatrix}
A^Th_{11}\widetilde{Z}_{11} & A^Th_{1,12}\widetilde{Z}_{12} & A^Th_{21}\widetilde{Z}_{21} & A^Th_{2,12}\widetilde{Z}_{22} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-C^TK_2^Th_{11}\widetilde{Z}_{11} & -C^TK_2^T\widetilde{Z}_{12}h_{1,12} & -h_{21}C^TK_2^T\widetilde{Z}_{21} & -C^TK_2^T\widetilde{Z}_{22}h_{2,12} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
h_{11}\widetilde{Z}_{11} & h_{1,12}\widetilde{Z}_{12} & h_{21}\widetilde{Z}_{21} & h_{2,12}\widetilde{Z}_{22}
\end{bmatrix}, \tag{46}
$$

$$
\begin{aligned}
\widetilde{\Upsilon}_1 &= \widetilde{P}_1A + A^T\widetilde{P}_1 + \sum_{i=1}^{3}\widetilde{M}_i + \beta\varepsilon_2 I + \rho\varepsilon_1 I,\\
\widetilde{\Gamma}_1^{(1)} &= -(1-\mu_1)\widetilde{M}_3,
\end{aligned}
$$

are comparable to the matrix inequalities (43)–(46) when treated with the Schur complement. Further, inequalities (44) and (45) imply

$$
\begin{bmatrix} \hat{\phi}_1 & \Pi^T \\ \Pi & \lambda\hat{\phi}_2 \end{bmatrix} < 0, \text{ for } i = 1, 2, \tag{47}
$$

in which we have λ which is an adequately large number greater than zero and

$$\Pi^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\tilde{Y}_1 h_{11} BK_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\tilde{Y}_2 h_{1,12} BK_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

Suppose I_i signifies a block of $n \times 19n$ order. The identity matrix I_i includes the ith block and all remaining blocks are considered as zero. Integrating (43), (44), and (48) into (47), and further by the pre- and post-multiplication of $\left[I_1^T, I_2^T, I_3^T, I_4^T, I_5^T, I_{14}^T, I_{15}^T, I_6^T, I_7^T, I_8^T, I_9^T, I_{10}^T, I_{11}^T, I_{12}^T, I_{13}^T, I_{16}^T, I_{17}^T, I_{18}^T, I_{19}^T\right]^T$ and its transposed matrix in the subsequent matrices, correspondingly, ends up into the inequalities in (9)–(10) for provided $P_1 = \widetilde{P}_1$, $P_2 = \lambda\widetilde{P}_2$, $Y_1 = \widetilde{Y}_1$, $Y_2 = \widetilde{Y}_2$, $Z_{l1} = \lambda\widetilde{Z}_{l1}$, $Z_{l2} = \lambda\widetilde{Z}_{l2}$, $M_i = \widetilde{M}_i$, $Q_i = \lambda\widetilde{Q}_i$, $\alpha_1 = \lambda\widetilde{\alpha}_1$ and $\alpha_2 = \lambda\widetilde{\alpha}_2$ (for $i = 1,2,3$ and $l = 1,2,3$), which provides relation as in (9).

Necessity: Considering the form of a positive definite matrix as

$$P = \begin{bmatrix} \breve{P}_1 & v \\ * & v \end{bmatrix} \text{ and } P = \begin{bmatrix} v & v \\ * & \breve{P}_2 \end{bmatrix},$$

which satisfies the conditions of Theorem 1, and also v is an entry that did not affect the derivation steps. Employing the relations of P in (9)

$$\begin{bmatrix} \psi_1 & \psi_2 \\ * & \psi_3 \end{bmatrix} - \xi^T(t)T_1^T\vartheta_1 T_1\xi(t) - \frac{h_{1,12}}{h_2 - h_1}\xi^T(t)T_2^T\vartheta_2 T_2\xi(t) < 0, \tag{49}$$

in which ψ_1, ψ_2 and ψ_3 can be observed in Appendix A as Equation (A5) and similarly for other

$$\begin{aligned} \varsigma = \begin{bmatrix} \tau_1 & \tau_2 \\ * & \tau_3 \end{bmatrix} - \xi^T(t)T_3^T\vartheta_{3o}T_3\xi(t) - \xi^T(t)T_4^T\vartheta_{4o}T_4\xi(t) \\ -\tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_{5o}T_5\xi(t) - \tfrac{h_{2,12}}{h_{21}-h_{22}}\xi^T(t)T_6^T\vartheta_{6o}T_6\xi(t) < 0, \end{aligned} \tag{50}$$

where τ_1, τ_2 and τ_3 are shown in Appendix A as (A6)–(A8). Further, (49) and (50) are pre- and post-multiplied with

$$\begin{bmatrix} \breve{P}_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \breve{P}_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{Y}_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{Y}_1^{-1} & 0 \end{bmatrix},$$

$$\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0
\end{bmatrix},$$

$$\text{and}\begin{bmatrix}
\breve{P}_2\widetilde{Z}_{11}^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \breve{P}_2\widetilde{Z}_{12}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \breve{P}_2\widetilde{Z}_{21}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \breve{P}_2\widetilde{Z}_{22}^{-1}
\end{bmatrix},$$

respectively, and further engaging $\breve{P}_1^{-1} = \overline{P}_1$ and $\breve{P}_2^{-1} = \hat{P}_2$, renders inequalities (41) and (42) which ends the proof. □

Remark 4. *The proposed results in Theorem 2 add exquisiteness to obtain the gain of the controller. The estimator assumed as K_1 and K_2 individually for the augmented system for the delay-range-dependent estimation-based control of OSL nonlinear time-varying delayed system (1). Compared to the proposed Theorem 1, the solution in Theorem 2 is manageable and easy to solve due to the eradication of complex relations in the matrix Π. The novel solution provided in Theorem 2 is more generic and renders necessary and sufficient stipulation for the constraints presented in Theorem 1. Decoupling techniques furnish observer and controller gains simultaneously through simulation tools.*

By taking $f(t,x) = 0$, the proposed scheme is modeled for a linear system with multiple delays in the succeeding corollary.

Corollary 1. *Consider an augmented system (8), incurred by integrating observer dynamics (5), closed-loop system (7), estimation error (6), and controller (4) and systems (1) underlying time-varying delay limits in (2) and assuming nonlinear terms as $f(t,x) = 0$. The state vector $e(t)$ and the state estimation error $e(t)$ converge asymptotically in the vicinity of origin for given matrices $\overline{P}_1 \in \Re^{n\times n}$, $\widetilde{P}_2 \in \Re^{n\times n}$, $\overline{M}_j \in \Re^{n\times n}$, $\widetilde{Q}_{lj} \in \Re^{n\times n}$, $V_k \in \Re^{n\times n}$ and $\widetilde{Z}_{lk} \in \Re^{n\times n}$ validating $\overline{P}_1 > 0$, $\widetilde{P}_2 > 0$, $\widetilde{Q}_{lj} > 0$, $\overline{M}_j > 0$ and $V_k > 0$, for $j = 1,2,3$, $k = l = 1,2$, and there exists the solution of Theorem 1 as it satisfies the following matrix inequalities*

$$\begin{aligned}\Omega_1 = &\begin{bmatrix} \kappa_1 & \kappa_2 \\ * & -diag(V_1, V_2) \end{bmatrix}, \\ &-\xi^T(t)T_1^T\vartheta_{1C}T_1\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_2^T\vartheta_{2C}T_2\xi(t)\,.\end{aligned} \tag{51}$$

$$\begin{aligned}\Omega_2 = &\begin{bmatrix} \phi_1^{(4)} & \widetilde{\phi}_2 \\ * & -diag(\widetilde{Z}_{11}, \widetilde{Z}_{12}, \widetilde{Z}_{21}, \widetilde{Z}_{22}) \end{bmatrix} - \xi^T(t)T_3^T\vartheta_{3o}T_3\xi(t) \\ &-\xi^T(t)T_4^T\vartheta_{4o}T_4\xi(t) - \tfrac{h_{1,12}}{h_{12}-h_{11}}\xi^T(t)T_5^T\vartheta_{5o}T_5\xi(t) - \tfrac{h_{2,12}}{h_{21}-h_{22}}\xi^T(t)T_6^T\vartheta_{6o}T_6\xi(t),\end{aligned} \tag{52}$$

where

$$\kappa_1 = \begin{bmatrix} \widetilde{\Upsilon}_1 & BX_1 & 0 & 0 & 0 & 0 & -\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} \\ * & \widetilde{\Gamma}_1^{(1)} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_2^{(1)} & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(1)} & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix},$$

$$\kappa_2 = \begin{bmatrix} \overline{P}_1 A^T h_{11} & \overline{P}_1 A^T h_{1,12} \\ X_1 B^T h_{11} & X_1 B^T h_{1,12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ h_{11} I & h_{1,12} I \end{bmatrix},$$

$$\phi_1^{(4)} = \begin{bmatrix} \hat{\Upsilon}_2 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Upsilon}_3 & 0 & 0 & 0 & 0 & 0 & \hat{\Upsilon}_4 \\ * & \Gamma_1^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Gamma_2^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Gamma_3^{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Gamma_1^{(3)} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Gamma_2^{(3)} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Gamma_3^{(3)} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix}$$

$$\widetilde{\phi}_2 = \begin{bmatrix} A^T h_{11} \widetilde{P}_2 & A^T h_{1,12} \widetilde{P}_2 & A^T h_{21} \widetilde{P}_2 & A^T h_{2,12} \widetilde{P}_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C^T X_2^T h_{11} & -C^T X_2^T h_{1,12} & -C^T X_2^T h_{21} & -C^T X_2^T h_{2,12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{11} \widetilde{P}_2 & h_{1,12} \widetilde{P}_2 & h_{21} \widetilde{P}_2 & h_{2,12} \widetilde{P}_2 \end{bmatrix},$$

$$
\begin{aligned}
&\widetilde{\Upsilon}_1 = \overline{P}_1 A + A^T \overline{P}_1 + \sum_{i=1}^{3} M_i + \overline{P}_1(\rho\varepsilon_1 I + \beta\varepsilon_2 I)\overline{P}_1,\\
&\Gamma_1^{(1)} = -(1-\mu_1)M_3,\\
&\Gamma_2^{(1)} = -M_1,\\
&\Gamma_3^{(1)} = -M_2,\\
&\hat{\Upsilon}_2 = \widetilde{P}_2 A + A^T \widetilde{P}_2 + \sum_{i=1}^{2}\sum_{j=1}^{3} Q_{ij},\\
&\widetilde{\Upsilon}_3 = -X_2 C,\\
&\hat{\Upsilon}_4 = I - \tfrac{\varepsilon_1 \widetilde{P}_2}{2} + \tfrac{\alpha\varepsilon_2 \widetilde{P}_2}{2},\\
&\Gamma_1^{(i+1)} = -(1-\mu_i)Q_{i3}, \text{ for } i = 1 \text{ and } 2,\\
&\Gamma_2^{(i+1)} = -Q_{i1}, \text{ for } i = 1 \text{ and } 2,\\
&\Gamma_3^{(i+1)} = -Q_{i2}, \text{ for } i = 1 \text{ and } 2,\\
&h_{i,12} = h_{i2} - h_{i1}, \text{ for } i = 1 \text{ and } 2.
\end{aligned}
$$

The gain for the controller assumed as K_1 and the estimator assumed as K_2 is obtained by assessing $K_1 = X_1 \overline{P}_1^{-1}$ and $K_2 = \widetilde{P}_2^{-1} X_2$, respectively.

Remark 5. *Corollary 1 delivers the solution of an estimation-based control technique for a linear counterpart exposed to interval time-varying multiple lags. Delay-range-dependent stability approaches for a linear counterpart are not exploited to that extent in various studies, for instance [47,48]. Furthermore, Corollary 1 presents the estimation-based control technique for linear systems subjected to interval output and input delays, which additionally elucidates the innovation of nonlinear outcomes in Theorems 1 and 2. This result also contributes to the simultaneous obtainment of controller and observer gains for linear systems using the decoupling technique.*

The constraints in Theorem 2 (or in derived results of Corollary 1) incorporate bilinear constraints which are required to be treated by engaging convex optimization through a cone complementary linearization algorithm approach [36,37,43,45]. The constraints (41) and (42) are solvable through convex optimization

$$
\begin{cases}
\min \mathit{Trace}\left(\sum_{k=1}^{2}\left(V_k\overline{V}_k + Y_k\overline{Y}_k + S_{ik}\overline{S}_{ik} + \overline{Y}_k S_k + P_1 Y_k P_1 \overline{S}_{ik} + P_1 S_k P_1 \overline{V}_k\right) + \widetilde{P}_1\overline{P}_1\right),\\
\text{subject to}\\
\begin{bmatrix} \widetilde{P}_1 & I \\ * & \overline{P}_1 \end{bmatrix} \geq 0\,, \begin{bmatrix} V_k & I \\ * & \overline{V}_k \end{bmatrix} \geq 0, \begin{bmatrix} Y_k & I \\ * & \overline{Y}_k \end{bmatrix} \geq 0, \begin{bmatrix} Y_k & I \\ * & \overline{S}_k \end{bmatrix} \geq 0, \begin{bmatrix} V_k & P_1 \\ * & \overline{S}_k \end{bmatrix} \geq 0, \begin{bmatrix} Y_k & P_1 \\ * & \overline{S}_{ik} \end{bmatrix} \geq 0,\\
k = 1 \text{ and } 2 \text{ and inequality (41)}\,.
\end{cases}
\tag{53}
$$

Similarly, for (42), matrix inequality constraints can be determined by

$$
\begin{cases}
\min \mathit{Trace}\left(\sum_{j=1}^{2}\sum_{i=1}^{2}\left(\widetilde{Z}_{ij}S_{ij} + P_2 S_{ij} P_2 \overline{V}_{ij} + V_{ij}\overline{V}_{ij}\right) + \widetilde{P}_2\overline{P}_2\right),\\
\text{subject to}\\
\begin{bmatrix} \widetilde{P}_2 & I \\ * & \overline{P}_2 \end{bmatrix} \geq 0\,, \begin{bmatrix} \widetilde{Z}_{ij} & I \\ * & \overline{S}_{ij} \end{bmatrix} \geq 0, \begin{bmatrix} V_{ij} & I \\ * & \overline{V}_{ij} \end{bmatrix} \geq 0, \begin{bmatrix} S_{ij} & \overline{P}_2 \\ * & \overline{V}_{ij} \end{bmatrix} \geq 0,\\
i = 1, 2, \quad j = 1, 2 \text{ and inequality (42) in Theorem 2.}
\end{cases}
\tag{54}
$$

Remark 6. *Results presented in proposed Theorem 2 and respectively derived Corollary 1, in comparison to Theorem 1, render the controller and the estimator gain vectors directly by the utilization of LMI tools and a cone complementary convex linearization procedure. The present study proposed an LMI-based strategy in (53)–(54) as a noteworthy elaboration to the approach in [45] for input and output delays, for the gain of the controller K_1 and the gain of the observer K_2. Notably, the presented technique of estimation-based controller involving the DRD stability* *approach for GOSL nonlinear systems solved through convex optimization techniques can be adopted for systems under actuation and measurement delays.*

4. Simulation Results

Assuming dynamics of motion of an object system in 2D Cartesian coordinates subjected to output and input delay to show the effectualness of the presented approach, and assuming the dynamical nonlinear system (1) stated in [36,43] as

$$\begin{aligned} f(x) &= -(x_1^2 + x_2^2)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ A &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T. \end{aligned} \tag{55}$$

Input and output vectors for delay are assumed as $0.01 \sin 2t + 0.03$ and $0.013 \sin 3t + 0.065$, correspondingly, taking the OSL nonlinearity constant as $\rho_c = 0.00008$ and $\rho_o = 0.009$ and delay-derivative bound for both controller and observer as $\mu_1 = 2$ and $\mu_2 = 0.04$, respectively. Furthermore, working out the optimization problems from (53) and (54) and employing a cone complementary algorithm, the viable solution obtained for the observer-based controller is accomplished for delay bounds $h_{11} = 0.001$ and $h_{21} = 0.01$ for the controller and $h_{12} = 0.065$ and $h_{22} = 0.065$ for the observer. The controller gain referred to K_1 and the estimation gain K_2 are calculated as

$$K_1 = \begin{bmatrix} -1.1112 & -0.1074 \\ 0.1047 & -1.1053 \end{bmatrix} \tag{56}$$

$$K_2 = \begin{bmatrix} 2.698 \\ 3.348 \end{bmatrix} \tag{57}$$

Both the measured states and estimated states convergence are shown in Figures 1 and 2 whereas the estimation error converging to origin is presented in Figure 3, which is obtained by improvising the proposed estimation-based control technique. Further, it confirms the potential of this novel scheme for implementation in industrial applications.

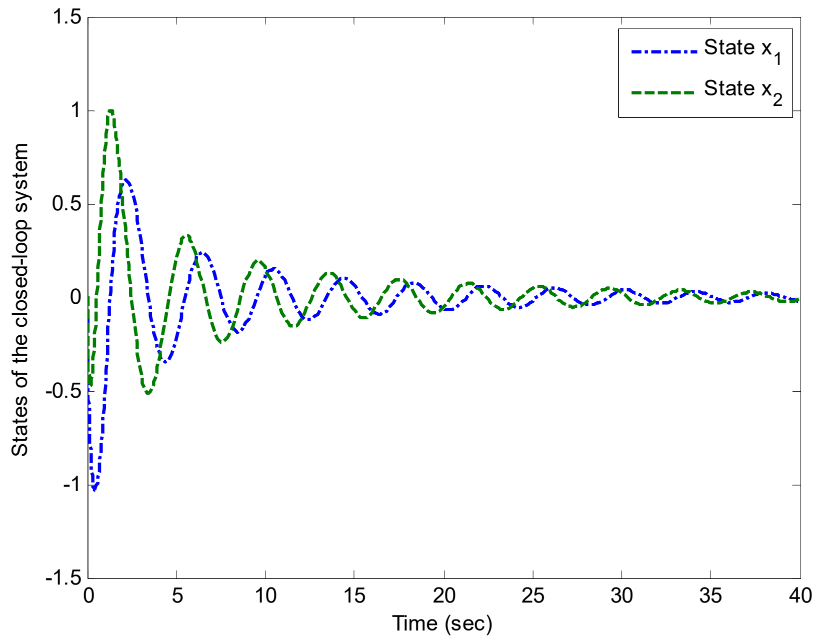

Figure 1. System states subject to input and output delays via the proposed estimation-based control strategy.

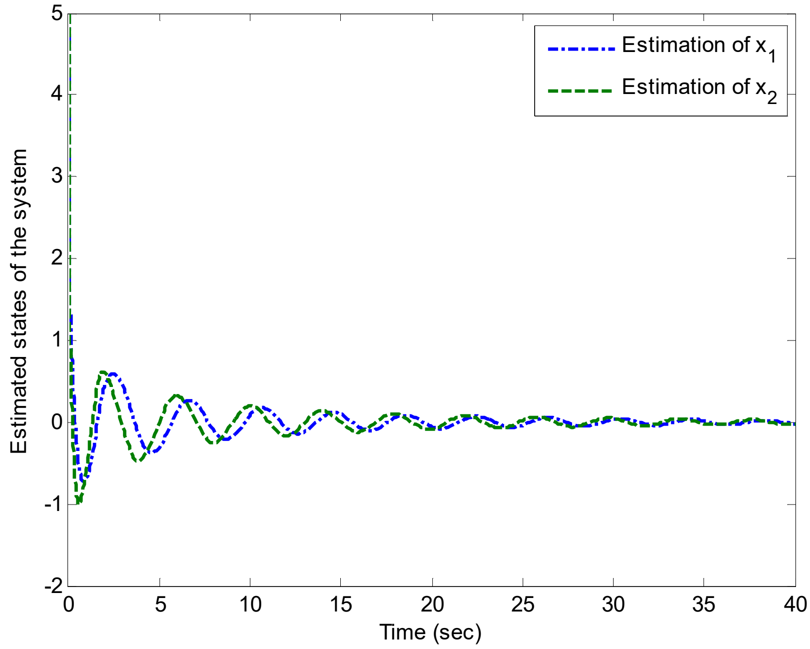

Figure 2. Observer states subject to input and output delays via the proposed estimation-based control strategy.

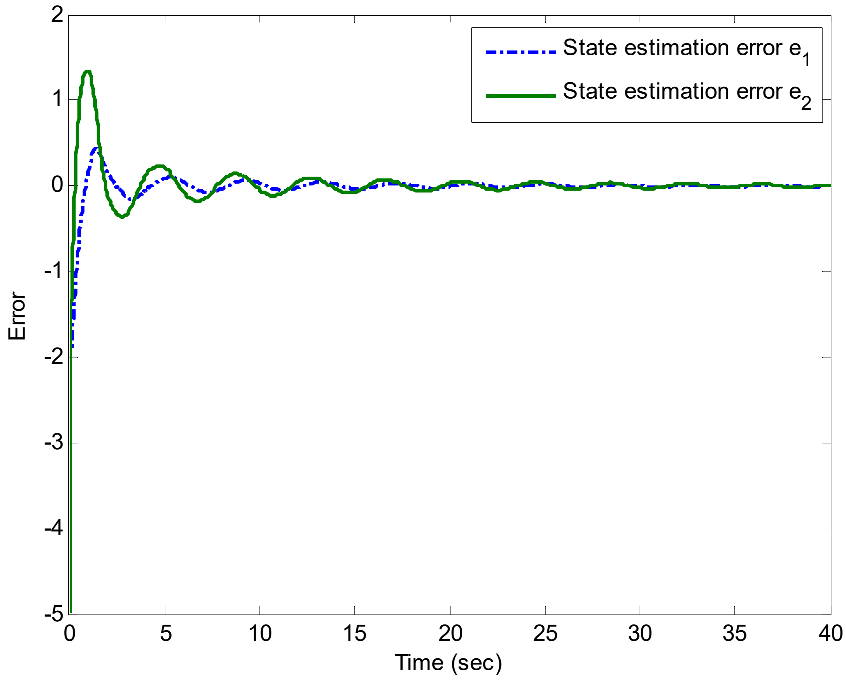

Figure 3. Estimation error for plant and estimated states subject to input and output delays.

An analysis of the upper bounds of delayed dynamics for the stability of multiple delayed systems is provided in Table 1 for $h_{12} = h_{22}$, for $h_{11} = h_{21} = 10$. This comparison study shows that the methodology proposed in this work is viable for the broader ranges of delays as well as for multiple delayed systems, in contrast to [45], as the Lipschitz nonlinearity counterpart leads to an infeasible solution. Furthermore, the technique applies to systems with delay in the output of state as in [36,43]. Moreover, the presented scheme can be deliberated for the range of delays when the lower bound is not equal to zero, i.e., $h_{11} = h_{21} \neq 0$.

Table 1. Upper allowed limit of $h_{12} = h_{22}$, assuming $h_{11} = h_{21} = 10s$.

Methods	Estimation Technique in [36,43,45]	Proposed Methodology
$h_{12} = h_{22}$ (for observer)	Inapplicable	$\approx 10^{11}$ s
$h_{12} = h_{22}$ (for controller)		$\approx 10^{08}$ s

5. Conclusions

In this work, an observer-based controller design approach for one-sided Lipschitz nonlinear systems in the presence of multiple time-varying delays was presented. Estimation-based control techniques for multiple delayed dynamics and nonlinear systems of a broader range are the main scope of the study. This technique integrates an observer-based controller technique, one-sided Lipschitz nonlinearity, multiple time-delayed dynamics, and nonlinear matrix inequalities to ensure the asymptotic conjunction of estimation error to origin. The availability of observer and controller gains is guaranteed by using a Lyapunov-Krasovskii functional for systems with delays. This functional leverages the time-derivative of delayed dynamics along with a delay-range-dependent criterion, replacing the conservative Jensen's inequality with Wirtinger's inequality condition. Additionally, the process involves the OSL condition, convex optimization techniques, and finally, the decoupling method for systems with multiple delays. Decoupling techniques aid the simultaneous extraction of controller and observer gains using the cone-complementary linearization technique by convex optimization. Furthermore, a sufficient and necessary solution for a novel main design is provided for one-sided Lipschitz nonlinear multi-delayed systems. Numerical simulation for the motion of an object in the Cartesian plane is furnished to show the usefulness of the proposed results, showing the pragmatic impact on a broader class of nonlinear systems subject to multiple delays. The proposed methodology may be considered for systems with consensus control or multi-agent systems. Moreover, the method for estimation-based control under multiple delays can also be explored for event-triggered systems.

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Appendix A

$$\xi^T(t)\begin{bmatrix}
\rho\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\varepsilon_1 I}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0
\end{bmatrix}\xi(t) > 0. \tag{A1}$$

$$\xi^T(t)\begin{bmatrix}
\beta\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\varepsilon_2 I}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0
\end{bmatrix}\xi(t) > 0. \tag{A2}$$

$$
\xi^T(t)\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \rho\varepsilon_3 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3 \\
* & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
* & * & * & * & * & * & * & * & * & * & * & & & & & & & \\
* & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0
\end{bmatrix}\xi(t) > 0. \tag{A3}
$$

and

$$
\xi^T(t)\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & \beta\varepsilon_4 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\alpha\varepsilon_4 I}{2} \\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_4 I
\end{bmatrix}\xi(t) > 0. \tag{A4}
$$

$$
\psi_1 = \begin{bmatrix}
\Upsilon_1 & v & P_1BK_1 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v \\
* & v & v & v & v & v & v & v & v & v & v & v & v & v \\
* & * & \Gamma_1^{(1)} & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v \\
* & * & * & v & v & v & v & v & v & v & v & v & v & v \\
* & * & * & * & \Gamma_2^{(1)} & v & 0 & v & 0 & v & 0 & v & 0 & v \\
* & * & * & * & * & v & v & v & v & v & v & v & v & v \\
* & * & * & * & * & * & \Gamma_3^{(1)} & v & 0 & v & 0 & v & 0 & v \\
* & * & * & * & * & * & * & v & v & v & v & v & v & v \\
* & * & * & * & * & * & * & * & 0 & v & 0 & v & 0 & v \\
* & * & * & * & * & * & * & * & * & v & v & v & v & v \\
* & * & * & * & * & * & * & * & * & * & 0 & v & 0 & v \\
* & * & * & * & * & * & * & * & * & * & * & v & v & v \\
* & * & * & * & * & * & * & * & * & * & * & * & 0 & v \\
* & * & * & * & * & * & * & * & * & * & * & * & * & v
\end{bmatrix},
$$
$$
\psi_2 = \begin{bmatrix}
-\frac{\varepsilon_1 I}{2} + \frac{\alpha\varepsilon_2 I}{2} & v & A^T h_{11}\tilde{Y}_1 & v & A^T h_{1,12}\tilde{Y}_2 & v \\
v & v & K_1^T B^T h_{11}\tilde{Y}_1 & v & K_1^T B^T h_{1,12}\tilde{Y}_2 & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v \\
0 & v & 0 & v & 0 & v \\
v & v & v & v & v & v
\end{bmatrix},
\tag{A5}
$$
$$
\psi_3 = \begin{bmatrix}
-\varepsilon_2 I & v & h_{11}\tilde{Y}_1 & v & h_{1,12}\tilde{Y}_2 & v \\
* & v & v & v & v & v \\
* & * & -\tilde{Y}_1 & v & 0 & v \\
* & * & * & v & v & v \\
* & * & * & * & -\tilde{Y}_2 & v \\
* & * & * & * & * & v
\end{bmatrix},
$$

$$\tau_1=\begin{bmatrix}
v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \tilde{\Upsilon}_2 & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & v & \tilde{\Upsilon}_3 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & \Upsilon_4 \\
\ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \Gamma_1^{(2)} & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \Gamma_2^{(2)} & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_3^{(1)} & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_2^{(3)} & v & 0 & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \Gamma_3^{(3)} & v & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & 0 & v & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & v & v \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & -\varepsilon_4 I
\end{bmatrix} \tag{A6}$$

$$\tau_2=\begin{bmatrix}
v & v & v & v & v & v & v & v \\
v & A^T h_{11}\widetilde{Z}_{11} & v & A^T h_{1,12}\widetilde{Z}_{12} & v & A^T h_{21}\widetilde{Z}_{21} & v & A^T h_{2,12}\widetilde{Z}_{22} \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & -C^T K_2^T h_{11}\widetilde{Z}_{11} & v & -C^T K_2^T \widetilde{Z}_{12} h_{1,12} & v & -C^T K_2^T \widetilde{Z}_{21} h_{21} & v & -C^T K_2^T \widetilde{Z}_{22} h_{2,12} \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & 0 & v & 0 & v & 0 & v & 0 \\
v & v & v & v & v & v & v & v \\
v & h_{11}\widetilde{Z}_{11} & v & h_{1,12}\widetilde{Z}_{12} & v & \widetilde{Z}_{21} h_{21} & v & h_{2,12}\widetilde{Z}_{22}
\end{bmatrix}, \tag{A7}$$

$$\tau_3 = \begin{bmatrix} v & v & v & v & v & v & v & v \\ v & \widetilde{Z}_{11} & v & v & v & v & v & v \\ v & v & v & v & v & v & v & v \\ v & v & v & \widetilde{Z}_{12} & v & v & v & v \\ v & v & v & v & v & v & v & v \\ v & v & v & v & v & \widetilde{Z}_{21} & v & v \\ v & v & v & v & v & v & v & v \\ v & v & v & v & v & v & v & \widetilde{Z}_{22} \end{bmatrix}. \tag{A8}$$

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