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Analyzing Curvature Properties and Geometric Solitons of the Twisted Sasaki Metric on the Tangent Bundle over a Statistical Manifold

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Abstract: Let (M, ∇, g) be a statistical manifold and TM be its tangent bundle endowed with a twisted Sasaki metric G. This paper serves two primary objectives. The first objective is to investigate the curvature properties of the tangent bundle TM. The second objective is to explore conformal vector fields and Ricci, Yamabe, and gradient Ricci–Yamabe solitons on the tangent bundle TM according to the twisted Sasaki metric G.

Keywords: conformal vector field; Ricci and Yamabe solitons; statistical manifold; twisted Sasaki metric; tangent bundle

MSC: 53C07; 53B05; 53B12; 53A45

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1. Introduction

Geometric flows are a mathematical concept that explore the evolution of metric structures on Riemannian manifolds. These flows enable the analysis of how the metric structure of a manifold changes over time and its consequent impact on the geometry of the manifold. One significant type of geometric flow is the Ricci flow, developed by Richard S. Hamilton [1], famously used by Grigori Perelman to prove the Poincaré Conjecture. The Ricci flow serves as a fundamental mathematical tool for studying the changes in metric structures on Riemannian manifolds over time. This flow systematically modifies the metric structure of a Riemannian manifold with the objective of achieving a constant Ricci curvature. The Ricci curvature is defined as the trace of the Ricci tensor, quantifying the curvature of the manifold at a specific point. Ricci flow is instrumental in comprehending the evolution of the metric structure of a manifold and deriving topological results.

Yamabe flow, introduced by Hidehiko Yamabe, is another type of geometric flow. It seeks to maintain the Yamabe curvature of a Riemannian manifold at a constant level while simultaneously altering the metric structure. The Yamabe curvature represents a specific aspect of the metric structure, and this flow aims to highlight these characteristics. Yamabe flow plays a crucial role in geometric analysis and understanding the evolution of metric structures on manifolds.

Ricci solitons are closely related to the Ricci flow and soliton solutions defined by a specific vector field on a Riemannian manifold. They represent particular solutions of the Ricci flow, describing how the manifold evolves in a specific manner. There are two main categories of Ricci solitons: gradient solitons, which correspond to metric structures with certain properties, and Ricci flow solitons, which are solutions of the Ricci flow that exhibit specific behavior.

Yamabe solitons are mathematical concepts used to characterize and study metric structures on Riemannian manifolds. These solitons are a special type that characterize the metric structure of a Riemannian manifold (M, g) through a specific vector field X. This vector field X characterizes a soliton that alters the Yamabe curvature of the manifold, a curvature measure reflecting the properties of the metric structure. A negative Yamabe curvature indicates specific curvature and contraction properties on the manifold.

The significance of Ricci solitons arises from their correspondence to self-similar solutions of the Ricci flow [1] and their natural generalization of Einstein metrics. Recent years have witnessed growing interest in the study of Ricci solitons and their various generalizations in the realm of Riemannian geometry.

Yamabe flow was introduced by Hamilton [1], coinciding with the development of Ricci flow. Both the Ricci soliton and Yamabe soliton are special solitons associated with Ricci flow and Yamabe flow, respectively. In soliton structures, if the potential vector field is the gradient of a function, it is termed a gradient soliton (such as gradient Ricci solitons or gradient Yamabe solitons).

These geometric flows and solitons play pivotal roles in addressing a wide range of problems related to the topology, geometry, and differential equations of manifolds. Regarding different geometric flows, some researchers have obtained different estimates and inequalities, such as, Li–Yau-type estimates, Perelman-type differential Harnack inequalities, etc. [2–5]. The studies on gradient estimates and differential Harnack inequalities were presented in [6–11]. We can find some papers regarding different manifolds of different connections in a tangent bundle [12–18]. All of these papers provide powerful mathematical tools for comprehending manifold evolution and changes, contributing to the derivation of numerous mathematical theorems and results.

Several authors have explored soliton structures in different contexts. For instance, Abbassi and Amri [19] investigated natural Ricci soliton structures on the tangent and unit tangent bundles of Riemannian manifolds. Chen and Deshmukh [20] introduced the concept of quasi-Yamabe solitons on Riemannian manifolds. Güler and Crasmareanu [21] introduced the notion of Ricci–Yamabe flow by considering a scalar combination of the Ricci flow and Yamabe flow. Dey and Majhi [22] studied generalized gradient Ricci–Yamabe solitons on complete Sasakian three-manifolds. Nurowski and Randall [23] introduced generalized Ricci solitons, and Kumara et al. [24] demonstrated that a Riemannian concurrent–recurrent manifold is Einstein when its metric is a generalized Ricci-type soliton. Gezer, Bilen, and De [25] explored almost Ricci and almost Yamabe soliton structures on the tangent bundle using the ciconia metric. Recently, Li and Khan et al. studied solitons, inequalities, and submanifolds using soliton theory, submanifold theory, and other related theories [26–31]. They obtained a number of interesting results and inspired the idea of this paper. De and Gezer [32] studied k-almost Yamabe solitons for the perfect fluid spacetime of general relativity. In particular, they constructed two examples to prove the existence of k-almost Yamabe solitons.

In the 1980s, the concept of a statistical structure emerged, playing a pivotal role in the development of an effective branch known as information geometry, which combines differential geometry and statistics. This field found applications across various scientific domains, including image processing, data analysis, physics, computer science, and machine learning (see [33–38]). A comprehensive survey of information geometry is available in [39]. Researchers have conducted numerous studies related to statistical manifolds. For example, Gezer, Peyghan, and Nourmohammadifar explored Kähler–Norden structures on statistical manifolds in [40]. Matsuzoe provided an overview of the geometry of statistical manifolds and discussed the connections between information geometry and affine differential geometry [41]. In [42], Peyghan, Seifipour, and Gezer investigated statistical structures on the tangent bundle TM equipped with two Riemannian g-natural metrics and lift connections.

In our present study, we introduce a novel natural metric for the tangent bundle TM over a statistical manifold, known as the twisted Sasaki metric. In this paper, firstly, we delve into the geometry of twisted Sasaki metrics, scrutinizing their properties and inherent geometrical aspects in depth. Our exploration aims to provide a comprehensive understanding of the behavior of these metrics, setting the stage for further investigations. Secondly, we shift our focus to the study of soliton structures associated with the twisted Sasaki metric when it is employed on the tangent bundle TM over a statistical manifold. Solitons, as mathematical constructs, reveal specific patterns of evolution within a given metric space. In our analysis, we aimed to elucidate and characterize the soliton structures that emerge from the utilization of the twisted Sasaki metric on TM within the context of a statistical manifold. This investigation sheds light on the dynamic interplay between the metric structure and the statistical properties of the underlying manifold.

Our research endeavors to enhance our comprehension of the geometry and curvature characteristics of the twisted Sasaki metric applied to TM, while also uncovering the intriguing soliton structures that manifest in this scenario. By doing so, we contribute to the broader fields of differential geometry and metric analysis, introducing a novel metric that challenges traditional rigidity when applied to tangent bundles and statistical manifolds.

Throughout this paper, we consistently assume that all manifolds, tensor fields, and connections are differentiable of class C^∞.

2. The Twisted Sasaki Metric on the Tangent Bundle over a Statistical Manifold

Statistical manifolds have found applications in various domains, including information science, information theory, neural networks, and statistical mechanics (as demonstrated in references [33–35]). Essentially, a statistical manifold is a mathematical space where the points represent probability distributions, providing a geometric model for understanding these distributions. A statistical structure on a differentiable manifold M is defined by a pair (∇, g), where g represents a (pseudo-)Riemannian metric, and ∇ denotes a torsion-free linear connection with the property that ∇g is totally symmetric. When a manifold possesses such a statistical structure, it is referred to as a statistical manifold. It is worth noting that a typical example of a statistical manifold is a (pseudo-)Riemannian manifold (M, g) paired with a Levi-Civita connection ∇ for g. In essence, statistical manifolds serve as generalizations of (pseudo-)Riemannian manifolds, providing a broader geometric framework for probabilistic modeling.

Definition 1. *Consider an arbitrary linear connection ∇ defined on a (pseudo-)Riemannian manifold (M, g). With the given pair (∇, g), we construct the $(0,3)$-tensor field denoted as F through the expression*

$$F(X, Y, Z) := (\nabla_Z g)(X, Y).$$

Clearly, $F(X, Y, Z) = F(Y, X, Z)$, due to the symmetry of g. This tensor field F is occasionally referred to as the cubic form associated with the pair (∇, g) [43]. Now, when we have a symmetric bilinear form ρ defined on a manifold M, we designate (∇, ρ) as a Codazzi pair if the covariant derivative $(\nabla\rho)$ is (totally) symmetric concerning vector fields X, Y, and Z [44]:

$$(\nabla_Z \rho)(X, Y) = (\nabla_X \rho)(Z, Y) = (\nabla_Y \rho)(Z, X).$$

Expressed in terms of the cubic form F, this condition can be rephrased as

$$F(X, Y, Z) = F(Z, Y, X) = F(Z, X, Y),$$

which means that the condition for (∇, g) to form a Codazzi pair is equivalent to F being entirely symmetric with respect to all of its indices.

Now, let us consider a torsion-free linear connection ∇ defined on a (pseudo-)Riemannian manifold (M, g). In the case where the pair (∇, g) forms a Codazzi pair, a concept well known to information geometers as characterizing statistical structures, the manifold M, when coupled with this statistical structure (∇, g), is termed a statistical manifold. It is important to note that the notion of a statistical manifold was originally introduced by Lauritzen [45]. These statistical manifolds are extensively explored in the realm of affine differential geometry, as evidenced in [45,46], and they play a pivotal role in the field of information geometry.

Consider an n-dimensional statistical manifold denoted as (M_n, ∇, g). In this article, we employ the C^∞-category to comprehensively elucidate various concepts. We focus on connected manifolds with dimension of $n > 1$. To facilitate our analysis, we introduce the tangent bundle of M_n, denoted as TM. We define the natural projection as $\pi : TM \to M_n$. When we employ a system of local coordinates (U, x^i) in M_n, it induces a corresponding system of local coordinates on TM, denoted as $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, where $\bar{i}$ ranges from $n+1$ to $2n$. Here, (u^i) represents the cartesian coordinates within each tangent space T_pM for all $p \in U$. It is important to note that p is any arbitrary point within U.

Consider the linear connection ∇ on the statistical manifold (M_n, ∇, g). We can decompose the tangent space of the tangent bundle TM into two distributions: the horizontal distribution determined by ∇ and the vertical distribution defined by $\ker \pi_*$. In this context, the local frame is given by

$$E_i = \frac{\partial}{\partial x^i} - u^s \Gamma^h_{is} \frac{\partial}{\partial u^h}; \; i = 1, ..., n,$$

and

$$E_{\bar{i}} = \frac{\partial}{\partial u^i}; \; \bar{i} = n+1, ..., 2n.$$

Here, Γ^h_{is} represents the Christoffel symbols of the linear connection ∇. The local frame $\{E_\beta\} = (E_i, E_{\bar{i}})$ is commonly referred to as the adapted frame. Let $A = A^i \frac{\partial}{\partial x^i}$ be a vector field. We can obtain the horizontal and vertical lifts of A with respect to the adapted frame as follows [47]:

$$\begin{aligned} {}^H A &= A^i E_i, \\ {}^V A &= A^i E_{\bar{i}}. \end{aligned}$$

Within TM, the local 1−form system $(dx^i, \delta u^i)$ forms the dual frame of the adapted frame $\{E_\beta\}$, where

$$\delta u^i = {}^H\left(dx^i\right) = du^i + u^s \Gamma^i_{hs} dx^h.$$

From the Riemannian manifold (M_n, g) to its tangent bundle TM, a variety of Riemannian or pseudo-Riemannian metrics have been devised. These metrics are constructed by naturally lifting the Riemannian metric g to the tangent bundle TM and are commonly referred to as g-natural metrics. In [48], the authors systematically derived a family of Riemannian g-natural metrics, which depend on six arbitrary functions characterizing the norm of a vector $u \in TM$. The study of natural metrics on tangent bundles arises from the need to understand the geometric and physical properties of objects moving on a Riemannian manifold. These metrics provide a way to extend the geometry of the base manifold to the tangent bundle, which is essential in various fields, including physics, differential geometry, and mechanics. Now, let us introduce the twisted Sasaki metric on the tangent bundle of a statistical manifold.

Let (M_n, ∇, g) be a statistical manifold and $a, b \in \mathbb{R}$. On the tangent bundle TM, the twisted Sasaki metric G is defined by

$$\begin{aligned} (i) \;\; G({}^H X, {}^H Y) &= a g(X, Y), \\ (ii) \;\; G({}^V X, {}^H Y) &= 0, \\ (iii) \;\; G({}^V X, {}^V Y) &= b g(X, Y) \end{aligned}$$

for all vector fields X and Y.

In the adapted frame $\{E_\beta\}$, we can express the twisted Sasaki metric and its inverse as follows:

$$(G) = \begin{pmatrix} ag_{ij} & 0 \\ 0 & bg_{ij} \end{pmatrix}$$

and

$$\left(G^{-1}\right) = \begin{pmatrix} \frac{1}{a}g^{ij} & 0 \\ 0 & \frac{1}{b}g^{ij} \end{pmatrix}.$$

The horizontal lift ${}^H\nabla$ of a linear connection ∇ on the manifold M_n to the tangent bundle TM is a unique linear connection defined on the TM and is characterized by the following conditions:

$$\begin{aligned} {}^H\nabla_{{}^HX}{}^HY &= {}^H(\nabla_X Y),\ {}^H\nabla_{{}^HX}{}^VY = {}^V(\nabla_X Y), \\ {}^H\nabla_{{}^VX}{}^HY &= 0,\ {}^H\nabla_{{}^VX}{}^VY = 0 \end{aligned}$$

for all vector fields X and Y on M_n. The torsion tensor ${}^HT$ of ${}^H\nabla$ satisfies the conditions

$$\begin{aligned} {}^HT({}^VX,{}^VY) &= 0,\ {}^HT({}^VX,{}^HY) = {}^V(T(X,Y)), \\ {}^HT({}^HX,{}^HY) &= {}^H(T(X,Y)) - \gamma R(X,Y). \end{aligned}$$

In these expressions, T and R represent the torsion and curvature tensor fields, respectively, of the linear connection ∇ on the manifold M_n. Importantly, the last set of identities implies that the connection ${}^H\nabla$ can have a non-zero torsion tensor even if ∇ is selected as a torsion-free linear connection. For a more detailed explanation, we can refer to [47]. Now, our objective is to determine the conditions under which the pair $({}^H\nabla, G)$ forms a Codazzi pair on the tangent bundle TM. To facilitate future use, we provide the following.

$$\begin{aligned} & ({}^H\nabla_{E_i}G)(E_j, E_k) \\ =\ & E_iG(E_j, E_k) - G({}^H\nabla_{E_i}E_j, E_k) - G(E_j, {}^H\nabla_{E_i}E_k) \\ =\ & E_i(ag_{jk}) - G({}^H\nabla_{{}^H(\partial_i)}{}^H(\partial_j), {}^H(\partial_k)) - G({}^H(\partial_j), {}^H\nabla_{{}^H(\partial_i)}{}^H(\partial_k)) \\ =\ & a(\partial_i g_{jk}) - G({}^H(\nabla_{\partial_i}\partial_j), {}^H(\partial_k)) - G({}^H(\partial_j), {}^H(\nabla_{\partial_i}\partial_k)) \\ =\ & a(\partial_i g_{jk}) - ag(\nabla_{\partial_i}\partial_j, \partial_k) - ag(\partial_j, \nabla_{\partial_i}\partial_k) \\ =\ & a(\partial_i g_{jk}) - ag(\Gamma^h_{ij}\partial_h, \partial_k) - ag(\partial_j, \Gamma^h_{ik}\partial_h) \\ =\ & a(\partial_i g_{jk} - \Gamma^h_{ij}g_{hk} - \Gamma^h_{ik}g_{jh}) \\ =\ & a\nabla_i g_{jk}, \end{aligned} \tag{1}$$

$$\begin{aligned} & ({}^H\nabla_{E_i}G)(E_j, E_{\bar{k}}) \\ =\ & E_iG(E_j, E_{\bar{k}}) - G({}^H\nabla_{E_i}E_j, E_{\bar{k}}) - G(E_j, {}^H\nabla_{E_i}E_{\bar{k}}) \\ =\ & E_iG({}^H(\partial_j), {}^V(\partial_k)) - G({}^H\nabla_{{}^H(\partial_i)}{}^H(\partial_j), {}^V(\partial_k)) - G({}^H(\partial_j), {}^H\nabla_{{}^H(\partial_i)}{}^V\partial_k) \\ =\ & -G({}^H(\nabla_{\partial_i}\partial_j), {}^V(\partial_k)) - G({}^H(\partial_j), {}^V(\nabla_{\partial_i}\partial_k)) \\ =\ & 0, \end{aligned} \tag{2}$$

$$\begin{aligned} ({}^H\nabla_{E_i}G)(E_{\bar{j}}, E_{\bar{k}}) &= E_iG(E_{\bar{j}}, E_{\bar{k}}) - G({}^H\nabla_{E_i}E_{\bar{j}}, E_{\bar{k}}) - G(E_{\bar{j}}, {}^H\nabla_{E_i}E_{\bar{k}}) \\ &= E_i bg(\partial_j, \partial_k) - G({}^V(\nabla_{\partial_i}\partial_j), {}^V(\partial_k)) - G({}^V(\partial_j), {}^V(\nabla_{\partial_i}\partial_k)) \\ &= b(\partial_i g_{jk} - \Gamma^h_{ij}g_{hk} - \Gamma^h_{jk}g_{jh}) \\ &= b\nabla_i g_{jk}, \end{aligned} \tag{3}$$

$$\begin{aligned}({}^H\nabla_{E_{\bar{i}}}G)(E_j,E_k) &= E_{\bar{i}}G(E_j,E_k)-G({}^H\nabla_{E_{\bar{i}}}E_j,E_k)-G(E_j,{}^H\nabla_{E_{\bar{i}}}E_k)\\ &= E_{\bar{i}}G({}^H(\partial_j),{}^H(\partial_k))-G({}^H\nabla_{V(\partial_i)}{}^H(\partial_j),{}^H(\partial_k))\\ &\quad -G({}^H(\partial_j),{}^H\nabla_{V(\partial_i)}{}^H(\partial_k))\\ &= 0, \end{aligned} \tag{4}$$

$$({}^H\nabla_{E_{\bar{i}}}G)(E_j,E_{\bar{k}}) = E_{\bar{i}}G(E_j,E_{\bar{k}})-G({}^H\nabla_{E_{\bar{i}}}E_j,E_{\bar{k}})-G(E_j,{}^H\nabla_{E_{\bar{i}}}E_{\bar{k}})=0, \tag{5}$$

$$({}^H\nabla_{E_{\bar{i}}}G)(E_{\bar{j}},E_{\bar{k}}) = E_{\bar{i}}G(E_{\bar{j}},E_{\bar{k}})-G({}^H\nabla_{E_{\bar{i}}}E_{\bar{j}},E_{\bar{k}})-G(E_{\bar{j}},{}^H\nabla_{E_{\bar{i}}}E_{\bar{k}})=0.$$

Theorem 1. *Given a statistical manifold (M_n, g, ∇), and the tangent bundle TM equipped with the twisted Sasaki metric G, the pair $({}^H\nabla, G)$ can be classified as a Codazzi pair if and only if ∇ is a metric connection with respect to g, where ${}^H\nabla$ represents the horizontal lift of the linear connection ∇.*

Proof. Consider a statistical manifold (M_n, g, ∇) with a (pseudo-)Riemannian metric g and a linear connection ∇. Then, the following relationships hold:

$$\nabla_i g_{jk} = \nabla_j g_{ik} = \nabla_k g_{ij}.$$

Now, let $(TM, G, {}^H\nabla)$ be a statistical manifold derived from (M_n, g, ∇), where G is the twisted Sasaki metric, and ${}^H\nabla$ is the associated horizontal lift connection. From the above relationships (1), we obtain

$$\begin{aligned}({}^H\nabla_{E_i}G)(E_j,E_k) &= ({}^H\nabla_{E_j}G)(E_i,E_k)=({}^H\nabla_{E_k}G)(E_i,E_j)\\ a\nabla_i g_{jk} &= a\nabla_j g_{ik} = a\nabla_k g_{ij}.\end{aligned}$$

Additionally, from Equations (2) and (4), we find that

$$({}^H\nabla_{E_i}G)(E_j,E_{\bar{k}}) = ({}^H\nabla_{E_{\bar{k}}}G)(E_j,E_i) = ({}^H\nabla_{E_j}G)(E_{\bar{k}},E_i)=0.$$

Furthermore, from Equations (3) and (5), we have

$$\begin{aligned}({}^H\nabla_{E_i}G)(E_{\bar{j}},E_{\bar{k}}) &= ({}^H\nabla_{E_{\bar{j}}}G)(E_i,E_{\bar{k}})=({}^H\nabla_{E_{\bar{k}}}G)(E_i,E_{\bar{j}})\\ b\nabla_i g_{jk} &= 0,\end{aligned}$$

which means that ∇ is a metric connection with respect to g. If the above equations are performed in reverse, the sufficient condition of the theorem is easily reached. □

To calculate the Levi-Civita connection $\tilde{\nabla}$ of the twisted Sasaki metric G, we need to find the Christoffel symbols $\tilde{\Gamma}^\alpha_{\gamma\beta}$ associated with this connection. The Levi-Civita connection ensures that the metric is compatible with the connection, meaning that the metric is parallel with respect to $\tilde{\nabla}$. To find the connection coefficients, you can use the following formula for the Christoffel symbols:

$$\tilde{\Gamma}^\alpha_{\gamma\beta} = \frac{1}{2}G^{\alpha\varepsilon}(E_\gamma G_{\varepsilon\beta}+E_\beta G_{\gamma\varepsilon}-E_\varepsilon G_{\gamma\beta})+\frac{1}{2}(\Omega_{\gamma\beta}{}^{\alpha}+\Omega^\alpha_{\gamma\beta}+\Omega^\alpha_{\beta\gamma}),$$

where

$$\begin{cases}\Omega^\alpha_{\gamma\beta} = G^{\alpha\varepsilon}G_{\delta\beta}\Omega_{\varepsilon\gamma}{}^{\delta},\\ \Omega_{ji}{}^{\bar{h}} = -\Omega_{ij}{}^{\bar{h}} = -R_{jis}{}^{h}y^s\ ,\\ \Omega_{\bar{j}i}{}^{\bar{h}} = -\Omega_{i\bar{j}}{}^{\bar{h}} = \Gamma^h_{ji}\end{cases}$$

and it will be used as $\gamma = j;\bar{j} \quad \beta = i;\bar{i} \quad \alpha = h;\bar{h} \quad \varepsilon = k;\bar{k} \quad \delta = m;\bar{m}$.

We present the following proposition regarding the Levi-Civita connection $\tilde{\nabla}$ associated with the twisted Sasaki metric G.

Proposition 1. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the local expression for the Levi-Civita connection $\tilde{\nabla}$ associated with the twisted Sasaki metric G on TM can be stated as follows:*

$$
\begin{aligned}
\tilde{\nabla}_{E_i} E_j &= \left(\Gamma^k_{ij}\right) E_k + \left(\frac{1}{2} y^s R_{jis}{}^{k}\right) E_{\bar{k}}, \\
\tilde{\nabla}_{E_i} E_{\bar{j}} &= \left(\frac{b}{2\alpha} y^s R_{sji}{}^{k}\right) E_k + \left(\Gamma^k_{ij} + \frac{1}{2} g^{km} (\nabla_i g_{mj})\right) E_{\bar{k}}, \\
\tilde{\nabla}_{E_{\bar{i}}} E_j &= \left(\frac{b}{2\alpha} y^s R_{sij}{}^{k}\right) E_k + \left(\frac{1}{2} g^{km} (\nabla_j g_{mi})\right) E_{\bar{k}}, \\
\tilde{\nabla}_{E_{\bar{i}}} E_{\bar{j}} &= \left(-\frac{b}{2a} g^{kh} (\nabla_h g_{ij})\right) E_k,
\end{aligned}
$$

where R is the curvature tensor of the linear connection ∇.

3. Curvature Properties of the Twisted Sasaki Metric

The components of the Riemannian curvature tensor for the twisted Sasaki metric G are calculated using the following expression:

$$
\tilde{R}_{\delta\gamma\beta}{}^{\alpha} = E_\delta \tilde{\Gamma}^\alpha_{\gamma\beta} - E_\gamma \tilde{\Gamma}^\alpha_{\delta\beta} + \tilde{\Gamma}^\alpha_{\delta\varepsilon} \tilde{\Gamma}^\varepsilon_{\gamma\beta} - \tilde{\Gamma}^\alpha_{\gamma\varepsilon} \tilde{\Gamma}^\varepsilon_{\delta\beta} - \Omega_{\delta\gamma}{}^{\varepsilon} \tilde{\Gamma}^\alpha_{\varepsilon\beta}.
$$

In this context, the indices are denoted as follows: $\gamma = i; \bar{i} \quad \beta = j; \bar{j} \quad \alpha = k; \bar{k} \quad \varepsilon = h; \bar{h}$ $\delta = m; \overline{m}$. In the forthcoming proposition, we provide the components of the Riemannian curvature tensor associated with the twisted Sasaki metric G.

Proposition 2. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the local expression for the corresponding Riemannian curvature tensor $\tilde{R}$ can be stated as follows:*

$$
\begin{aligned}
\tilde{R}_{mij}{}^{k} &= R_{mij}{}^{k} + \frac{b}{4a} y^s y^p \left[R_{shm}{}^{k} R_{jip}{}^{h} - R_{shi}{}^{k} R_{jmp}{}^{h} + 2R_{mis}{}^{h} R_{phj}{}^{k}\right], \\
\tilde{R}_{mij}{}^{\bar{k}} &= -\frac{1}{4} y^s \left[R_{jis}{}^{h} A^k_{mh} - R_{jms}{}^{h} A^k_{ih} + 2R_{mis}{}^{h} A^k_{jh}\right] \\
&\quad + \frac{1}{2} y^s \left[\nabla_m R_{jis}{}^{k} - \nabla_i R_{jms}{}^{k}\right], \\
\tilde{R}_{\bar{m}ij}{}^{k} &= \frac{b}{2a} y^s \nabla_i R_{msj}{}^{k} - \frac{b}{4a} y^s \left[R_{jis}{}^{h} A^k_{mh} + R_{shi}{}^{k} A^h_{jm}\right], \\
\tilde{R}_{\bar{m}ij}{}^{\bar{k}} &= \frac{1}{2} R_{jim}{}^{k} - \frac{b}{4a} y^s y^p R_{his}{}^{k} R_{pmj}{}^{h} - \frac{1}{2} \nabla_i A^k_{jm} - \frac{1}{4} A^k_{ih} A^h_{jm}, \\
\tilde{R}_{m\bar{i}j}{}^{k} &= \frac{b}{2a} y^s \nabla_m R_{sij}{}^{k} + \frac{b}{4a} y^s \left[R_{shm}{}^{k} A^h_{ji} + R_{jms}{}^{h} A^k_{ih}\right], \\
\tilde{R}_{m\bar{i}j}{}^{\bar{k}} &= \frac{1}{2} R_{mji}{}^{h} + \frac{b}{4a} y^s y^p R_{hms}{}^{k} R_{pij}{}^{h} + \frac{1}{2} \nabla_m A^k_{ji} + \frac{1}{4} A^k_{mh} A^h_{ji}, \\
\tilde{R}_{mi\bar{j}}{}^{k} &= \frac{b}{4a} y^s \left[R_{shm}{}^{k} A^h_{ij} - R_{shi}{}^{k} A^h_{mj} - 2R_{mis}{}^{h} A^k_{hj}\right] \\
&\quad + \frac{b}{2a} y^s \left[\nabla_m R_{sji}{}^{k} - \nabla_i R_{sjm}{}^{k}\right], \\
\tilde{R}_{mi\bar{j}}{}^{\bar{k}} &= R_{mij}{}^{k} + \frac{1}{2} \left[\nabla_m A^k_{ij} - \nabla_i A^k_{mj}\right] + \frac{1}{4} \left[A^k_{mh} A^h_{ij} - A^k_{ih} A^h_{mj}\right] \\
&\quad + \frac{b}{4a} y^s y^p \left[R_{hms}{}^{k} R_{pji}{}^{h} - R_{his}{}^{k} R_{pjm}{}^{h}\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}_{\bar{m}\bar{i}j}{}^{k} &= \frac{b}{4a}\left[A^k_{ih}A^h_{jm} - A^k_{mh}A^h_{ji}\right] + \frac{b^2}{4a^2}y^s y^p\left[R_{smh}{}^{k}R_{pij}{}^{h} - R_{sih}{}^{k}R_{pmj}{}^{h}\right] \\
&\quad + \frac{b}{a}R_{mij}{}^{k}, \\
\tilde{R}_{\bar{m}\bar{i}j}{}^{\bar{k}} &= \frac{b}{4a}y^s\left[R_{sij}{}^{h}A^k_{hm} - R_{smj}{}^{h}A^k_{hi}\right], \\
\tilde{R}_{\bar{m}i\bar{j}}{}^{k} &= \frac{b}{2a}R_{mji}{}^{k} + \frac{b}{2a}\left(\nabla_i A^k_{mj}\right) - \frac{b}{4a}A^k_{mh}A^h_{ij} + \frac{b^2}{4a^2}y^s y^p R_{smh}{}^{k}R_{pji}{}^{h}, \\
\tilde{R}_{\bar{m}i\bar{j}}{}^{\bar{k}} &= \frac{b}{4a}y^s\left[R_{sji}{}^{h}A^k_{hm} + R_{his}{}^{k}A^h_{mj}\right], \\
\tilde{R}_{m\bar{i}\bar{j}}{}^{k} &= \frac{b}{2a}R_{jim}{}^{k} - \frac{b}{2a}\left(\nabla_m A^k_{ij}\right) + \frac{b}{4a}A^k_{ih}A^h_{mj} - \frac{b^2}{4a^2}y^s y^p R_{sih}{}^{k}R_{pjm}{}^{h}, \\
\tilde{R}_{m\bar{i}\bar{j}}{}^{\bar{k}} &= \frac{b}{4a}y^s\left[R_{mhs}{}^{k}A^h_{ij} + R_{jsm}{}^{h}A^k_{hi}\right], \\
\tilde{R}_{\bar{m}\bar{i}\bar{j}}{}^{k} &= \frac{b^2}{4a^2}y^s\left[R_{sih}{}^{k}A^h_{mj} - R_{smh}{}^{k}A^h_{ij}\right], \\
\tilde{R}_{\bar{m}\bar{i}\bar{j}}{}^{\bar{k}} &= \frac{b}{4a}\left[A^k_{hi}A^h_{mj} - A^k_{hm}A^h_{ij}\right],
\end{aligned}
$$

where R is the curvature tensor of the linear connection ∇ and $A^k_{ij} = g^{kl}\left(\nabla_i g_{jl}\right)$.

Continuing with our analysis, we now turn our attention to the Ricci and scalar curvature tensors. Utilizing the results from Proposition 2 and performing standard calculations, we obtain the following outcomes.

Proposition 3. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the local expression for the corresponding Ricci curvature tensor $\tilde{R}_{IJ} = \tilde{R}_{MIJ}{}^{M}$ can be stated as follows:*

$$
\begin{aligned}
\tilde{R}_{ij} &= R_{ij} + \frac{b}{4a}y^s y^p\left[R_{mis}{}^{h}R_{phj}{}^{m} + R_{msi}{}^{h}R_{jhp}{}^{m}\right] - \frac{1}{4}\left(\nabla_i g^{ml}\right)\left(\nabla_j g_{ml}\right) \\
&\quad - \frac{1}{2}g^{ml}\left(\nabla_i \nabla_j g_{ml}\right), \\
\tilde{R}_{\bar{i}j} &= \frac{b}{2a}y^s \nabla_m R_{sij}{}^{m} + \frac{b}{4a}y^s\left[R_{sij}{}^{h}A^m_{hm} + R_{jsm}{}^{h}A^m_{ih}\right], \\
\tilde{R}_{i\bar{j}} &= \frac{b}{2a}y^s \nabla_m R_{sji}{}^{m} + \frac{b}{4a}y^s\left[R_{sji}{}^{h}A^m_{hm} + R_{ism}{}^{h}A^m_{jh}\right], \\
\tilde{R}_{\bar{i}\bar{j}} &= \frac{b}{4a}\left[2A^m_{hi}A^h_{mj} - A^m_{hm}A^h_{ij}\right] - \frac{b}{2a}\left(\nabla_m A^m_{ij}\right) - \frac{b^2}{4a^2}y^s y^p R_{sih}{}^{m}R_{pjm}{}^{h},
\end{aligned}
$$

where R is the curvature tensor of the linear connection ∇, and $A^k_{ij} = g^{kl}\left(\nabla_i g_{jl}\right)$.

Proposition 4. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the local expression for the corresponding scalar curvature tensor $\tilde{r}$ can be stated as follows:*

$$
\tilde{r} = \frac{1}{a}r + \frac{b}{4a^2}\|R\| + \frac{1}{4a}g^{ij}\left[A^m_{hi}A^h_{mj} - A^m_{hm}A^h_{ij}\right] - \frac{1}{2a}g^{ij}\left[\nabla_i A^m_{jm} + \nabla_m A^m_{ij}\right],
$$

where $A^k_{ij} = g^{kl}\left(\nabla_i g_{jl}\right)$, and r and R are the scalar curvature and curvature tensor of the linear connection ∇, respectively.

4. Conformal Vector Fields according to the Twisted Sasaki Metric

Let $L_{\tilde{X}}$ be the Lie derivation with respect to the vector field $\tilde{X}$. A vector field $\tilde{X}$ with components $(v^h, v^{\bar{h}})$ is fiber-preserving if and only if v^h depends only on the variables (x^h). Therefore, every fiber-preserving vector field $\tilde{X}$ on TM induces a vector field $V = v^h \frac{\partial}{\partial x^h}$ on M_n. We shall first state the following lemmas, which are needed later on.

Lemma 1. *Consider a statistical manifold (M_n, g, ∇) and its tangent bundle TM. Let $\tilde{X}$ be a fiber-preserving vector field on TM with the components $(v^h, v^{\bar{h}})$. Then, the Lie derivates of the adapted frame and the dual basis are given as follows (see also [49]):*

$$\begin{aligned}
(i)\quad L_{\tilde{X}} E_i &= -(E_i v^h) E_h + \left[y^b v^c R_{icb}{}^h - v^{\bar{b}} \Gamma^h_{bi} - (E_i v^{\bar{h}}) \right] E_{\bar{h}}, \\
(ii)\quad L_{\tilde{X}} E_{\bar{i}} &= \left[v^b \Gamma^h_{bi} - (E_{\bar{i}} v^{\bar{h}}) \right] E_{\bar{h}}, \\
(iii)\quad L_{\tilde{X}} dx^i &= (E_h v^i) dx^h, \\
(iv)\quad L_{\tilde{X}} \delta y^i &= -\left[y^c v^b R_{bhc}{}^i + v^{\bar{b}} \Gamma^i_{bh} + (E_h v^{\bar{i}}) \right] dx^h - \left[v^b \Gamma^i_{bh} - (E_{\bar{h}} v^{\bar{i}}) \right] \delta y^h.
\end{aligned}$$

Here, R denotes the curvature tensor of the linear connection ∇, and Γ^h_{is} represents the Christoffel symbols of the linear connection ∇.

Through the Lemma given above, we provide the following lemma that we will use later.

Lemma 2. *In the context of a statistical manifold (M_n, g, ∇) and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the Lie derivative of twisted Sasaki metric G with respect to the fiber-preserving vector field $\tilde{X}$ is given as follows:*

$$\begin{aligned}
L_{\tilde{X}} G = \; & a \left[L_V g_{ij} + 2 \left(E_i v^h \right) g_{hj} \right] dx^i dx^j \\
& -2b g_{hj} \left[y^s v^b R_{bis}{}^h + v^{\bar{b}} \Gamma^h_{bi} + \left(E_i v^{\bar{h}} \right) \right] dx^i \delta y^j \\
& +b \left[L_V g_{ij} - 2 v^b \Gamma^h_{bi} g_{hj} + 2 g_{hj} \left(E_{\bar{i}} v^{\bar{h}} \right) \right] \delta y^i \delta y^j,
\end{aligned}$$

where $L_V g_{ij}$ denotes the components of the Lie derivative of $L_V g$, and $\tilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$. $V = v^h \frac{\partial}{\partial x^h}$ is a vector field on M_n.

If we have the tangent bundle TM over a statistical manifold (M_n, g, ∇) equipped with the twisted Sasaki metric G, and there exists a scalar function Ω such that the equation

$$L_{\tilde{X}} G = 2\Omega G \tag{6}$$

is satisfied, then we refer to $\tilde{X}$ as an infinitesimal fiber-preserving conformal vector field on (TM, G).

Theorem 2. *In the context of a statistical manifold (M_n, g, ∇) of dimension $n \geq 2$ and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, the fiber-preserving vector field $\tilde{X}$ is an infinitesimal fiber-preserving conformal vector field on (TM, G) if and only if the scalar function Ω on TM depends only on the variables $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, x^{\bar{h}}\right)$ and the following conditions are satisfied:*

$$
\begin{aligned}
&(i) && \tilde{X} = \left(v^h, v^{\bar{h}}\right) = \left(v^h, y^s A_s^h + B^h\right) = {}^H V + {}^V B + \gamma A, \\
&(ii) && \Omega = \frac{1}{2n} g^{ij}(L_V g_{ij}) + \frac{1}{n}\left(E_i v^i\right), \\
&(iii) && v^l R_{lis}{}^h + \nabla_i \nabla_s v^h = 0, \\
&(iv) && \nabla_i v^h = A_i^h, \\
&(v) && \nabla_i B^h = 0,
\end{aligned}
$$

where $\tilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ *is a fiber-preserving vector field on* TM*,* $V = v^h \frac{\partial}{\partial x^h}$ *is a vector field on* M_n*,* Ω *is a scalar function,* ${}^H V$ *is the horizontal lift of* V*,* ${}^V B$ *is the vertical lift of* B*, and* $A = \left(A_i^h\right)$ *and* $B = \left(B^h\right)$ *are* $(1,1)$ *and* $(1,0)$ *tensor fields on* M_n*, respectively.*

Proof. Starting from the expressions given in Lemma 2 and Lemma 6, we obtain the following equations:

$$
\begin{gathered}
L_{\tilde{X}} G_{ij} = 2\Omega G_{ij} \\
L_V g_{ij} + 2(E_i v^h) g_{hj} = 2\Omega g_{ij}, \qquad (7) \\
L_{\tilde{X}} G_{\bar{i}j} = 2\Omega G_{\bar{i}j} \\
2b g_{hj}\left[y^s v^l R_{lis}{}^h + v^{\bar{l}} \Gamma_{li}^h + \left(E_i v^{\bar{h}}\right)\right] = 0 \qquad (8)
\end{gathered}
$$

and

$$
\begin{gathered}
L_{\tilde{X}} G_{\bar{i}\bar{j}} = 2\Omega G_{\bar{i}\bar{j}} \\
L_V g_{ij} - 2v^l \Gamma_{li}^h g_{hj} + 2g_{hj}\left(E_{\bar{i}} v^{\bar{h}}\right) = 2\Omega g_{ij}. \qquad (9)
\end{gathered}
$$

Now, let us apply $E_{\bar{k}}$ to both sides of Equation (9):

$$
g_{hj} E_{\bar{k}}\left(E_{\bar{i}} v^{\bar{h}}\right) = \left(E_{\bar{k}} \Omega\right) g_{ij}. \qquad (10)
$$

By interchanging i with k in the last equation, we obtain

$$
g_{hj} E_{\bar{i}}\left(E_{\bar{k}} v^{\bar{h}}\right) = \left(E_{\bar{i}} \Omega\right) g_{kj}. \qquad (11)
$$

From the equalities of (10) and (11), we can write

$$
\left(E_{\bar{k}} \Omega\right) g_{ij} = \left(E_{\bar{i}} \Omega\right) g_{kj}.
$$

By contracting with g^{ij} in the above equation, we obtain

$$
\begin{aligned}
n\left(E_{\bar{k}} \Omega\right) &= \left(E_{\bar{k}} \Omega\right) \\
\left(E_{\bar{k}} \Omega\right) &= 0. \qquad (12)
\end{aligned}
$$

This means that the scalar function Ω depends only on the variables $\left(x^h\right)$. Thus, we can see Ω as a function on M_n. Substituting Equation (12) into Equation (11), we can write

$$
E_{\bar{i}}\left(E_{\bar{k}} v^{\bar{h}}\right) = 0
$$

$$
v^{\bar{h}} = y^s A_s^h + B^s, \qquad (13)
$$

where $A = \left(A_i^h\right)$ and $B = \left(B^h\right)$ are $(1,1)$ and $(1,0)$ tensor fields on M_n, respectively. Through substituting Equation (13) into Equation (8), we have

$$v^l R_{lis}{}^h + \nabla_i A_s^h = 0$$

and

$$\nabla_i B^h = 0.$$

By substituting Equation (13) into Equation (9), we have

$$L_V g_{ij} - 2v^l \Gamma_{li}^h g_{hj} + 2g_{hj} A_i^h = 2\Omega g_{ij}. \tag{14}$$

When Equations (7) and (14) are evaluated together, we can write

$$\nabla_i v^h = A_i^h.$$

If the last equation is used in equality (14), the following result is obtained

$$v^l R_{lis}{}^h + \nabla_i \nabla_s v^h = 0.$$

By contracting Equation (7) with $\frac{1}{2n} g^{ij}$, we have

$$\Omega = \frac{1}{2n} g^{ij} \left(L_V g_{ij}\right) + \frac{1}{n}\left(E_i v^i\right).$$

This completes the proof. □

5. The Ricci Solitons according to the Twisted Sasaki Metric

A Ricci soliton on a smooth manifold M_n (where $n \geq 2$) is defined as a triple (g, X, λ), where g represents a pseudo-Riemannian metric on M_n, Ric is the associated Ricci tensor, X is a vector field, and λ is a real constant. This triple can satisfy the following equation:

$$Ric + \frac{1}{2} L_X g = \lambda g,$$

where L_X represents the Lie derivative with respect to the vector field X. A Ricci soliton can be classified into three categories, shrinking, steady, or expanding, depending on the sign of the constant λ. It is called “shrinking” when λ is positive, “steady” when λ is zero, and “expanding” when λ is negative. These classifications are important because Ricci solitons correspond to self-similar solutions of the Ricci flow [1] and serve as natural generalizations of Einstein metrics.

In the case of a gradient Ricci soliton, the soliton vector field X can be expressed as the gradient of a smooth function f on M_n. For gradient Ricci solitons, the following equation holds:

$$\left(\nabla^2 f\right) + Ric = \lambda g,$$

where ∇^2 denotes the Hessian operator.

A Ricci soliton defined on the tangent bundle TM equipped with the twisted Sasaki metric G, which is defined over a statistical manifold (M_n, g), is characterized by the equation

$$\widetilde{Ric} + \frac{1}{2} L_{\tilde{X}} G = \lambda G, \tag{15}$$

where $\widetilde{Ric}$ is the Ricci tensor of G, $\tilde{X}$ is a vector field on TM, and λ is a smooth function on TM.

Theorem 3. *In the context of a statistical manifold (M_n, g, ∇) of dimension $n \geq 2$ and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, (TM, G, λ) constitutes a Ricci soliton if and only if the base manifold is flat, and the following conditions are satisfied:*

$$
\begin{aligned}
&(i) \quad \tilde{X} = (v^h, v^{\overline{h}}) = (v^h, y^s M_s^h + N^h), \\
&(ii) \quad \lambda = \frac{1}{n}\left[\frac{1}{2} g^{ij}(L_V g_{ij}) + (E_i v^i) - \frac{1}{2a}\nabla^j \nabla^m g_{jm} - \frac{1}{4a} g^{ij}(\nabla^m g_{ih})\left(\nabla^h g_{jm}\right)\right], \\
&(iii) \quad \nabla M = \nabla N = 0,
\end{aligned}
$$

where $\tilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ is a vector field on TM, $V = v^h \frac{\partial}{\partial x^h}$ is a vector field on M_n, $\lambda \in \mathbb{R}$, and $M = (M_s^h)$ and $N = (N^h)$ are $(1,1)$ and $(1,0)$ tensor fields on M_n, respectively.

Proof. With the help of Equation (15), we have

$$\tilde{R}_{ij} + \frac{1}{2} L_{\tilde{X}} G_{ij} = \lambda G_{ij},$$

$$
\begin{aligned}
\lambda a g_{ij} &= R_{ij} + \frac{b}{4a} y^s y^p [R_{mis}{}^h R_{phj}{}^m + R_{msi}{}^h R_{jhp}{}^m] - \frac{1}{4} A_{ih}^m A_{jm}^h \\
&\quad -\frac{1}{2}\nabla_i A_{jm}^m + \frac{a}{2}[L_V g_{ij} + 2(E_i v^h) g_{hj}],
\end{aligned} \tag{16}
$$

$$\tilde{R}_{\bar{i}j} + \frac{1}{2} L_{\tilde{X}} G_{\bar{i}j} = \lambda G_{\bar{i}j},$$

$$\frac{1}{2a}\nabla_m R_{sij}{}^m + \frac{1}{4a}(R_{sij}{}^h A_{hm}^m + R_{jsm}{}^h A_{ih}^m) - g_{hj} v^l R_{lis}{}^h = 0$$

and

$$v^{\bar{l}} \Gamma_{li}^h + (E_i v^{\bar{h}}) = 0. \tag{17}$$

Also,

$$\tilde{R}_{\bar{i}\bar{j}} + \frac{1}{2} L_{\tilde{X}} G_{\bar{i}\bar{j}} = \lambda G_{\bar{i}\bar{j}},$$

$$
\begin{aligned}
\lambda g_{ij} &= \frac{1}{4a}[2A_{hi}^m A_{mj}^h - A_{hm}^m A_{ij}^h] - \frac{1}{2a}(\nabla_m A_{ij}^m) + \frac{1}{2} L_V g_{ij} \\
&\quad + g_{hj}(E_{\bar{i}} v^{\overline{h}}) - v^l \Gamma_{li}^h g_{hj}
\end{aligned} \tag{18}
$$

and

$$\frac{b^2}{4a^2} y^s y^p R_{sih}{}^m R_{pjm}{}^h = 0 \Rightarrow \frac{b^2}{4a^2} y^s y^p \|R\| = 0.$$

This means that the base manifold is flat. By contracting with g^{ij} in Equation (18), we obtain

$$
\begin{aligned}
\lambda n &= \frac{1}{4a} g^{ij}[2A_{hi}^m A_{mj}^h - A_{hm}^m A_{ij}^h] - \frac{1}{2a} g^{ij}(\nabla_m A_{ij}^m) + \frac{1}{2} g^{ij} L_V g_{ij} \\
&\quad + (E_{\bar{i}} v^{\bar{i}}) - v^l \Gamma_{li}^i.
\end{aligned}
$$

Applying $E_{\bar{k}}$ to both sides of the last equation, we have

$$E_{\bar{k}}\left(E_{\bar{i}} v^{\bar{i}}\right) = 0$$

$$v^{\bar{i}} = y^s M_s^i + N^i, \tag{19}$$

where $M = (M^i_s)$ and $N = (N^i)$ are $(1,1)$ and $(1,0)$ tensor fields on M_n. Substituting Equation (19) in Equation (17), we obtain

$$y^s \nabla_i M^h_s + \nabla_i N^h = 0.$$

By contracting with $\frac{1}{na} g^{ij}$ in Equation (16) and considering that the base manifold is flat, we have the following equation

$$\lambda = \frac{1}{n}\left[\frac{1}{2} g^{ij}(L_V g_{ij}) + (E_i v^i) - \frac{1}{2a}\nabla^j \nabla^m g_{jm} - \frac{1}{4a} g^{ij}(\nabla^m g_{ih})\left(\nabla^h g_{jm}\right)\right],$$

which completes the proof. □

6. The Yamabe Solitons according to the Twisted Sasaki Metric

In the context of a complete Riemannian manifold (M_n, g), where $n \geq 2$, the Riemannian metric g is considered to have a Yamabe soliton if it satisfies the following equation:

$$\frac{1}{2} L_X g = (r - \lambda) g,$$

where λ is a scalar constant, X is a differentiable vector field known as the soliton vector field, and r represents the scalar curvature of the Riemannian manifold (M_n, g). If λ is a smooth function, then the Riemannian metric g is said to admit an almost Yamabe soliton, which is a natural generalization of Yamabe solitons [50]. Additionally, almost Yamabe solitons are classified as steady, expanding, or shrinking, depending on the value of λ. Specifically, they are categorized as follows: (1) steady if $\lambda = 0$, (2) expanding if $\lambda < 0$, and (3) shrinking if $\lambda > 0$. It is important to note that Einstein manifolds are a special case of almost Yamabe solitons.

Now, when considering a Yamabe soliton defined on the tangent bundle TM with the twisted Sasaki metric G over a statistical manifold (M_n, g, ∇), we can describe it with the following equation:

$$\frac{1}{2} L_{\tilde{X}} G = (\tilde{r} - \lambda) G, \tag{20}$$

where $\tilde{r}$ represents the scalar curvature of (TM, G), $\tilde{X}$ is a vector field on TM, and λ is a smooth function on TM.

In the case of a gradient Yamabe soliton, the soliton vector field X can be expressed as the gradient of a C^∞ function f on M_n, and this leads to the following equation:

$$\left(\nabla^2 f\right) = (r - \lambda) g.$$

Theorem 4. *In the context of a statistical manifold (M_n, g, ∇) of dimension $n \geq 2$ and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, (TM, G, λ) is a Yamabe soliton if and only if the following conditions are satisfied:*

$$\begin{aligned}
&(i) && \tilde{X} = (v^h, v^{\bar{h}}) = (v^h, y^s M^h_s + N^h),\\
&(ii) && \lambda = \tilde{r} - \frac{1}{2n} g^{ij}(L_V g_{ij}) - \frac{1}{n}(E_i v^i),\\
&(iii) && \nabla_i M^h_s = v^l R_{lis}{}^h,\\
&(iv) && \nabla_i N^h = 0,
\end{aligned}$$

where $\tilde{X} = v^h E_h + v^{\bar{h}} E_{\bar{h}}$ is a vector field on TM, $V = v^h \frac{\partial}{\partial x^h}$ is a vector field on M_n, $\lambda \in \mathbb{R}$, and $M = (M^h_s)$ and $N = (N^h)$ are $(1,1)$ and $(1,0)$ tensor fields on M_n, respectively.

Proof. Starting from Equation (20), we have

$$\frac{1}{2}L_{\tilde{X}}G_{ij} = (\tilde{r} - \lambda)G_{ij},$$

$$\lambda g_{ij} = \tilde{r}g_{ij} - \frac{1}{2}L_V g_{ij} - (E_i v^h)g_{hj}.$$

By contracting the last equation with $\frac{1}{n}g^{ij}$, we obtain

$$\lambda = \tilde{r} - \frac{1}{2n}(L_V g_{ij}) - \frac{1}{n}(E_i v^i). \tag{21}$$

Now, from Equation (20), we also have

$$\frac{1}{2}L_{\tilde{X}}G_{\bar{i}j} = (\tilde{r} - \lambda)G_{\bar{i}j} = 0,$$

which implies that

$$y^s v^l R_{lis}{}^h + \Gamma^h_{li}v^{\bar{l}} + E_i v^{\bar{h}} = 0 \tag{22}$$

$$\frac{1}{2}L_{\tilde{X}}G_{\bar{i}\bar{j}} = (\tilde{r} - \lambda)G_{\bar{i}\bar{j}}$$

$$\frac{1}{2}[L_V g_{ij} - 2v^l\Gamma^h_{li}g_{hj} + 2g_{hj}(E_{\bar{i}}v^{\bar{h}})] = (\tilde{r} - \lambda)g_{ij}.$$

By contracting the last equation with $\frac{1}{n}g^{ij}$, we find that

$$\lambda = \tilde{r} - \frac{1}{2n}g^{ij}(L_V g_{ij}) + \frac{1}{n}v^l\Gamma^i_{li} - \frac{1}{n}(E_{\bar{i}}v^{\bar{i}}). \tag{23}$$

When Equations (21) and (23) are evaluated together, we obtain

$$-(E_i v^i) = v^l\Gamma^i_{li} - (E_{\bar{i}}v^{\bar{i}}).$$

Applying $E_{\bar{k}}$ to both sides of the last equation, we have

$$E_{\bar{k}}\left(E_{\bar{i}}v^{\bar{i}}\right) = 0,$$

$$v^{\bar{i}} = y^s M^i_s + N^i. \tag{24}$$

Substituting Equation (24) into Equation (22), we obtain

$$y^s v^l R_{lis}{}^h + \Gamma^h_{li}(y^s M^l_s + N^l) + E_i(y^s M^h_s + N^h) = 0$$

and

$$y^s(v^l R_{lis}{}^h + \nabla_i M^h_s) + \nabla_i N^h = 0.$$

Form the above equations, we obtain

$$\nabla_i M^h_s = -v^l R_{ils}{}^h$$

and

$$\nabla_i N^h = 0.$$

Thus, the proof is complete. □

7. The Generalized Ricci–Yamabe Solitons according to the Twisted Sasaki Metric

Güler and Crasmareanu [21] introduced the concept of a Ricci–Yamabe flow on a Riemannian manifold (M_n, g), where $n \geq 2$, by combining the Ricci flow and Yamabe flow into a scalar equation:

$$\frac{\partial g}{\partial t}(t) + 2\alpha R(t) + \beta r(t) g(t) = 0, \tag{25}$$

where g is a Riemannian metric, R is the Ricci tensor, r is the scalar curvature tensor, and $\alpha, \beta \in \mathbb{R}$.

A Riemannian manifold (M_n, g) with $n > 2$ is said to have a generalized Ricci–Yamabe soliton $(g, X, \lambda, \alpha, \beta, \gamma)$ if the following equation holds:

$$L_X g + 2\alpha R = (2\lambda - \beta r) g + 2\gamma X^{\#} \otimes X^{\#}, \tag{26}$$

where $\lambda, \alpha, \beta, \gamma \in \mathbb{R}$, and $X^{\#}$ is the associated $1-$form with X.

If X is the gradient of a smooth function f on M_n, then this concept is referred to as a generalized gradient Ricci–Yamabe soliton, and Equation (26) simplifies to

$$\nabla^2 f + \alpha R = \left(\lambda - \frac{1}{2}\beta r\right) g + \gamma df \otimes df.$$

The generalized (gradient) Ricci–Yamabe soliton is categorized as expanding if $\lambda < 0$, steady if $\lambda = 0$, or shrinking if $\lambda > 0$.

Theorem 5. *In the context of a statistical manifold (M_n, g, ∇) of dimension $n \geq 2$ and its tangent bundle (TM, G) equipped with the twisted Sasaki metric G, $(TM, G, {}^C V, \lambda)$ is a generalised Ricci–Yamabe soliton if and only if the following conditions are satisfied:*

$$(i) \quad \lambda = \frac{1}{2n}\left[g^{ij}(L_V g_{ij}) + 2(E_i v^i) - g^{ij}\partial_t g_{ij} - 2\gamma a g_{kt} v^k v^t\right],$$
$$(ii) \quad L_V g_{ij} = 2\lambda g_{ij} + 2 g_{hj}(\nabla_i v^h) + \partial_t g_{ij},$$
$$(iii) \quad g_{kt}(\partial_s v^k)(\partial_p v^t) = 0,$$

where the potential vector field ${}^C V$ is the complete lift of a vector field V on M_n to the tangent bundle TM. This lift is given by ${}^C V = (v^m, v^{\overline{m}}) = (v^m, y^s \partial_s v^m)$, where $V = v^h \frac{\partial}{\partial x^h}$ is a vector field on M_n, and $\lambda \in \mathbb{R}$.

Proof. We will demonstrate the existence of the scalar λ. Starting from Equation (25), we obtain

$$2\alpha R + \beta r G = -\partial_t G. \tag{27}$$

When we substitute the expressions for $L_{{}^C X} G$ and Equation (27) into Equation (26), we obtain

$$L_{{}^C X} G_{ij} + 2\alpha \tilde{R}_{ij} = (2\lambda - \beta \tilde{r}) G_{ij} + 2\gamma\, {}^C X_i^{\#} \otimes {}^C X_j^{\#}, \tag{28}$$

$$L_{{}^C X} G_{\bar{i}j} + 2\alpha \tilde{R}_{\bar{i}j} = (2\lambda - \beta \tilde{r}) G_{\bar{i}j} + 2\gamma\, {}^C X_{\bar{i}}^{\#} \otimes {}^C X_j^{\#}$$

and

$$L_{{}^C X} G_{\bar{i}\bar{j}} + 2\alpha \tilde{R}_{\bar{i}\bar{j}} = (2\lambda - \beta \tilde{r}) G_{\bar{i}\bar{j}} + 2\gamma\, {}^C X_{\bar{i}}^{\#} \otimes {}^C X_{\bar{j}}^{\#}. \tag{29}$$

Here, the potential vector field ${}^C X$ and its associated $1-$form ${}^C X^{\#}$ are defined as follows:

$${}^C X = \begin{pmatrix} v^m \\ v^{\overline{m}} \end{pmatrix} = \begin{pmatrix} v^m \\ y^s \partial_s v^m \end{pmatrix}$$

and

$$\begin{aligned} {}^{C}X^{\#}_{j} &= {}^{C}X^{I}G_{Ij} = ag_{ij}v^{i}, \\ {}^{C}X^{\#}_{\bar{j}} &= {}^{C}X^{I}G_{I\bar{j}} = bg_{ij}y^{s}\partial_{s}v^{i}. \end{aligned}$$

From (28), we deduce that

$$L_V g_{ij} + 2(E_i v^h)g_{hj} - \partial_t g_{ij} = 2\lambda g_{ij} + 2\gamma a g_{ki} v^k g_{tj} v^t.$$

By contracting this equation with $\frac{1}{2n}g^{ij}$, we obtain

$$\lambda = \frac{1}{2n}\Big[g^{ij}(L_V g_{ij}) + 2(E_i v^i) - g^{ij}\partial_t g_{ij} - 2\gamma a g_{kt} v^k v^t\Big].$$

From Equation (29), we have

$$b\Big[L_V g_{ij} - 2v^l\Gamma^h_{li}g_{hj} + 2h_{hj}(E_{\bar{i}}v^{\bar{h}})\Big] + 2\alpha\tilde{R}_{\bar{i}\bar{j}} + \beta\tilde{r}G_{\bar{i}\bar{j}} = 2\lambda G_{\bar{i}\bar{j}} + 2\gamma\, {}^{C}X^{\#}_{\bar{i}} \otimes {}^{C}X^{\#}_{\bar{j}}.$$

Using Equation (27) in the last equation, we can rewrite it as

$$\begin{aligned} 0 &= L_V g_{ij} - 2\lambda g_{ij} - 2g_{hj}v^l\Gamma^h_{li} + 2g_{hj}(\partial_i v^h) - \partial_t g_{ij} \\ L_V g_{ij} &= 2\lambda g_{ij} + 2g_{hj}(\nabla_i v^h) + \partial_t g_{ij}. \end{aligned}$$

Finally, from the contracted equation

$$2\gamma b y^s y^p g_{ki}(\partial_s v^k) g_{tj}(\partial_p v^t) = 0,$$

we obtain

$$g_{kt}(\partial_s v^k)(\partial_p v^t) = 0.$$

Therefore, the proof is complete. □

8. Conclusions

This paper endeavors to examine the geometric characteristics of the tangent bundles of statistical manifolds by employing the twisted Sasaki metric on their tangent bundles. Our inquiry was structured into two principal phases, each delineating specific yet interconnected objectives.

Firstly, we delved into the intricacies of the twisted Sasaki metrics, meticulously scrutinizing their properties and inherent geometrical aspects. Our comprehensive analysis has provided a deeper understanding of these metrics, elucidating their behavior and implications for differential geometry within the context of statistical manifolds.

Secondly, we shifted our focus toward the study of soliton structures arising from the application of the twisted Sasaki metric to the tangent bundle. Solitons, as mathematical constructs, offer invaluable insights into the dynamic interplay between metric structures and statistical properties. By characterizing these soliton structures, we uncovered novel patterns of evolution within the manifold, further enriching our comprehension of its geometry.

Our research makes a significant contribution to the fields of differential geometry and metric analysis by introducing a novel metric that challenges traditional rigidity when applied to tangent bundles and statistical manifolds. By elucidating the curvature properties of the tangent bundle and uncovering intriguing soliton structures, we have expanded the horizons of geometric understanding in this domain. In the future, our findings will facilitate further investigations into the manifold implications of the twisted Sasaki metric and its role in shaping the geometric and statistical landscapes of diverse mathematical structures. It is our hope that this paper will inspire future endeavors aimed at unraveling the profound mysteries of geometric structures in statistical manifolds.

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