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Characterization of Nonlinear Mixed Bi-Skew Lie Triple Derivations on ∗-Algebras

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Abstract: This paper concentrates on examining the characterization of nonlinear mixed bi-skew Lie triple $*$- derivations within an $*$-algebra denoted by $\mathcal{A}$ which contains a nontrivial projection with a unit I. Additionally, we expand this investigation to applications by describing these derivations within prime $*$-algebras, von Neumann algebras, and standard operator algebras.

Keywords: mixed bi-skew Lie triple derivation; $*$-derivation; $*$-algebra

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1. Introduction

Consider an algebra $\mathcal{A}$ defined over the complex field $\mathbb{C}$. A map $* : \mathcal{A} \to \mathcal{A}$ is called an involution if the following conditions hold for all $\mathcal{J}, \mathcal{K} \in \mathcal{A}$ and $\alpha \in \mathbb{C}$: (i) $(\mathcal{J} + \mathcal{K})^* = \mathcal{J}^* + \mathcal{K}^*$; (ii) $(\alpha\mathcal{J})^* = \bar{\alpha}\mathcal{J}^*$; and (iii) $(\mathcal{J}\mathcal{K})^* = (\mathcal{K})^*(\mathcal{J})^*$ and $(\mathcal{J}^*)^* = \mathcal{J}$. An algebra $\mathcal{A}$ with involution $*$ is called a $*$-algebra. Let $\mathcal{J}$ and $\mathcal{K}$ be elements of $\mathcal{A}$. The notation $[\mathcal{J}, \mathcal{K}]_\bullet$ represents the bi-skew Lie product defined as $[\mathcal{J}, \mathcal{K}]_\bullet = \mathcal{J}\mathcal{K}^* - \mathcal{K}\mathcal{J}^*$, while $[\mathcal{J}, \mathcal{K}]$ denotes the Lie product of $\mathcal{J}$ and $\mathcal{K}$, defined as $[\mathcal{J}, \mathcal{K}] = \mathcal{J}\mathcal{K} - \mathcal{K}\mathcal{J}$. Lie and bi-skew Lie products are gaining importance across a number of research fields, and many authors have been interested in investigating them (see [1–6]). An additive mapping $\Pi : \mathcal{A} \to \mathcal{A}$ is termed an additive derivation if it satisfies the condition $\Pi(\mathcal{J}\mathcal{K}) = \Pi(\mathcal{J})\mathcal{K} + \mathcal{J}\Pi(\mathcal{K})$ for all $\mathcal{J}, \mathcal{K} \in \mathcal{A}$. If, in addition, $\Pi(\mathcal{J}^*) = \Pi(\mathcal{J})^*$ holds for all $\mathcal{J} \in \mathcal{A}$, then Π is an additive $*$-derivation.

The investigation of the additive properties of mappings on rings and algebras, particularly in relation to their structure, has been a captivating area of research for the past sixty years. Martindale, in his work [7], addressed the question, "When is a multiplicative mapping additive?". He presented a significant technique along with a set of conditions on a ring that compel a multiplicative isomorphism to be additive. Notably, he demonstrated that every multiplicative isomorphism from a prime ring containing a nontrivial idempotent to any ring is necessarily additive.

Building upon Martindale's work, Daif [8] extended the concept to multiplicative derivations of rings, establishing their additivity and introducing the notion of multiplicative derivations. Subsequently, various results have been derived in both associative and alternative rings and algebras. In 2009, Wang [9] delved into the additivity of n-multiplicative isomorphisms and n-multiplicative derivations of rings. Recently, Rehman et al. [10] mixed the concept of Jordan and Jordan $*$-products and proved that every nonlinear mixed Jordan triple derivation on an $*$-algebra is an additive $*$-derivation. Motivated by the above works, in this paper, we mixed the concept of Lie and bi-skew Lie products and accordingly defined nonlinear mixed bi-skew Lie triple derivation as follows: let $\Pi : \mathcal{A} \to \mathcal{A}$ be a map (without additivity). If

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$, then Π is called a nonlinear mixed bi-skew Lie triple derivation, proving that every nonlinear mixed bi-skew Lie triple derivation is an additive $*$-derivation under some conditions.

Before presenting the main result, it is essential to provide an example that satisfies the condition

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$ for which the mapping Π is nontrivial.

Example 1. *Consider $\mathcal{A} = M_2(\mathbb{C})$, and let the algebra of all square matrices of the order 2 over the field of complex numbers $\mathbb{C}$, and let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be a unity of $M_2(\mathbb{C})$. The map $* : \mathcal{A} \to \mathcal{A}$ given by $*(A) = A^\theta$, in which A^θ denotes the conjugate transpose of the matrix A, is an involution. Hence, $\mathcal{A}$ is a unital $*$-algebra with a unity I. Now, define a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that $\Pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} o & ib \\ -ic & o \end{pmatrix}$. Note that $\Pi(A)$ is a derivation on $\mathcal{A}$. So, it also satisfies*

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$. Moreover, $\mathcal{A}$ contains a nontrivial projection $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and Π is also nontrivial.

2. Main Result

In this section, we will prove the following theorem.

Theorem 1. *Let $\mathcal{A}$ be a unital $*$-algebra with a unity I containing a nontrivial projection P which satisfies*

$$X\mathcal{A}P = 0 \implies X = 0 \tag{1}$$

and

$$X\mathcal{A}(I-P) = 0 \implies X = 0. \tag{2}$$

Define a map $\Pi : \mathcal{A} \to \mathcal{A}$ such that if

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$, then Π is additive. Moreover, if $\Pi(iI) = i\Pi(I)$, then Π is also an additive $*$-derivation.

Let $\mathcal{P} = \mathcal{P}_1$ be a nontrivial projection in $\mathcal{A}$ and $\mathcal{P}_2 = I - \mathcal{P}_1$, where I is the unity of this algebra $\mathcal{A}$. Then, by the Peirce decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{P}_1\mathcal{A}\mathcal{P}_1 \oplus \mathcal{P}_1\mathcal{A}\mathcal{P}_2 \oplus \mathcal{P}_2\mathcal{A}\mathcal{P}_1 \oplus \mathcal{P}_2\mathcal{A}\mathcal{P}_2$, and denote $\mathcal{A}_{11} = \mathcal{P}_1\mathcal{A}\mathcal{P}_1$, $\mathcal{A}_{12} = \mathcal{P}_1\mathcal{A}\mathcal{P}_2$, $\mathcal{A}_{21} = \mathcal{P}_2\mathcal{A}\mathcal{P}_1$ and $\mathcal{A}_{22} = \mathcal{P}_2\mathcal{A}\mathcal{P}_2$. Note that any $\mathcal{J} \in \mathcal{A}$ can be written as $\mathcal{J} = \mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}$, where $\mathcal{J}_{ij} \in \mathcal{A}_{ij}$ and $\mathcal{J}_{ij}^* \in \mathcal{A}_{ji}$ for $i,j = 1,2$.

We use various lemmas in order to prove Theorem 1.

Lemma 1. $\Pi(0) = 0$.

Proof. It is easy to check that

$$\begin{aligned} \Pi(0) &= \Pi([[0,0]_\bullet, 0]) \\ &= [[\Pi(0), 0]_\bullet, 0] + [[0, \Pi(0)]_\bullet, 0] + [[0,0]_\bullet, \Pi(0)] \\ &= 0. \end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. *For any* $\mathcal{J}_{11} \in \mathcal{A}_{11}, \mathcal{J}_{12} \in \mathcal{A}_{12}, \mathcal{J}_{21} \in \mathcal{A}_{21}$, *and* $\mathcal{J}_{22} \in \mathcal{A}_{22}$, *we have*

$$\Pi(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}) = \Pi(\mathcal{J}_{11}) + \Pi(\mathcal{J}_{12}) + \Pi(\mathcal{J}_{21}) + \Pi(\mathcal{J}_{22}).$$

Proof. Let $\mathcal{M} = \Pi(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}) - \Pi(\mathcal{J}_{11}) - \Pi(\mathcal{J}_{12}) - \Pi(\mathcal{J}_{21}) - \Pi(\mathcal{J}_{22})$. It is easy to check that $[[\mathcal{J}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2] = [[\mathcal{J}_{12}, \mathcal{P}_1]_\bullet, \mathcal{P}_2] = [[\mathcal{J}_{22}, \mathcal{P}_1]_\bullet, \mathcal{P}_2] = 0$, and, using Lemma 1, we obtain

$$\begin{aligned} \Pi([[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_2]) &= \Pi([[\mathcal{J}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) + \Pi([[\mathcal{J}_{12}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) \\ &\quad + \Pi([[\mathcal{J}_{21}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) + \Pi([[\mathcal{J}_{22}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) \\ &= [[\Pi(\mathcal{J}_{11}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{11}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{11}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)] + [[\Pi(\mathcal{J}_{12}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{12}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{12}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)] \\ &\quad + [[\Pi(\mathcal{J}_{21}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{21}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{21}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)] + [[\Pi(\mathcal{J}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{22}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{22}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)]). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \Pi([[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_2]) &= [[\Pi(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] \\ &\quad + [[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] \\ &\quad + [[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)]. \end{aligned}$$

By using the last two expressions, we obtain $[[\mathcal{M}, \mathcal{P}_1]_\bullet, \mathcal{P}_2] = 0$. This means that $-\mathcal{P}_1\mathcal{M}^*\mathcal{P}_2 - \mathcal{P}_2\mathcal{M}\mathcal{P}_1 = 0$. Multiplying both sides by $\mathcal{P}_1$ from the right, we obtain $\mathcal{P}_2\mathcal{M}\mathcal{P}_1 = 0$. Similarly, we can show that $\mathcal{P}_1\mathcal{M}\mathcal{P}_2 = 0$. Now, for any $X_{12} \in \mathcal{A}_{12}$, we find

$$\begin{aligned} \Pi([[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), X_{12}]_\bullet, \mathcal{P}_1]) &= [[\Pi(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), X_{12}]_\bullet, \mathcal{P}_1] \\ &\quad + [[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), \Pi(X_{12})]_\bullet, \mathcal{P}_1] \\ &\quad + [[(\mathcal{J}_{11} + \mathcal{J}_{12} + \mathcal{J}_{21} + \mathcal{J}_{22}), X_{12}]_\bullet, \Pi(\mathcal{P}_1)]. \end{aligned}$$

It follows from $[[\mathcal{J}_{11}, X_{12}]_\bullet, \mathcal{P}_1] = [[\mathcal{J}_{12}, X_{12}]_\bullet, \mathcal{P}_1] = [[\mathcal{J}_{22}, X_{12}]_\bullet, \mathcal{P}_1] = 0$ that

$$\begin{aligned}\Pi([[(\mathcal{J}_{11}+\mathcal{J}_{12}+\mathcal{J}_{21}+\mathcal{J}_{22}),X_{12}]_\bullet,\mathcal{P}_1]) &= \Pi([[\mathcal{J}_{11},X_{12}]_\bullet,\mathcal{P}_1])+\Pi([[\mathcal{J}_{12},X_{12}]_\bullet,\mathcal{P}_1])\\ &\quad +\Pi([[\mathcal{J}_{21},X_{12}]_\bullet,\mathcal{P}_1])+\Pi([[\mathcal{J}_{22},X_{12}]_\bullet,\mathcal{P}_1])\\ &= [[\Pi(\mathcal{J}_{11}),X_{12}]_\bullet,\mathcal{P}_1]+[[\mathcal{J}_{11},\Pi(X_{12})]_\bullet,\mathcal{P}_1]\\ &\quad +[[\mathcal{J}_{11},X_{12}]_\bullet,\Pi(\mathcal{P}_1)]+[[\Pi(\mathcal{J}_{12}),X_{12}]_\bullet,\mathcal{P}_1]\\ &\quad +[[\mathcal{J}_{12},\Pi(X_{12})]_\bullet,\mathcal{P}_1]+[[\mathcal{J}_{12},X_{12}]_\bullet,\Pi(\mathcal{P}_1)]\\ &\quad +[[\Pi(\mathcal{J}_{21}),X_{12}]_\bullet,\mathcal{P}_1]+[[\mathcal{J}_{21},\Pi(X_{12})]_\bullet,\mathcal{P}_1]\\ &\quad +[[\mathcal{J}_{21},X_{12}]_\bullet,\Pi(\mathcal{P}_1)]+[[\Pi(\mathcal{J}_{22}),X_{12}]_\bullet,\mathcal{P}_1]\\ &\quad +[[\mathcal{J}_{22},\Pi(X_{12})]_\bullet,\mathcal{P}_1]+[[\mathcal{J}_{22},X_{12}]_\bullet,\Pi(\mathcal{P}_1)]).\end{aligned}$$

By comparing the last two equations, we find that $[[\mathcal{M},X_{12}]_\bullet,\mathcal{P}_1]=0$. This means that $\mathcal{M}X_{12}^*-X_{12}\mathcal{M}^*\mathcal{P}_1-\mathcal{P}_1\mathcal{M}X_{12}^*-X_{12}\mathcal{M}^*=0$. By multiplying $\mathcal{P}_2$ from the right, we obtain $X_{12}\mathcal{M}^*\mathcal{P}_2=0$. By using (1) and (2), we obtain $\mathcal{P}_2\mathcal{M}\mathcal{P}_2=0$. In a similar way, we can show that $\mathcal{P}_1\mathcal{M}\mathcal{P}_1=0$. Hence, $\mathcal{M}=0$. This completes the proof. □

Lemma 3. *For any $\mathcal{J}_{ij},\mathcal{K}_{ij}\in\mathcal{A}_{ij}$ with $i\neq j$ and $i,j=1,2$, we have*

$$\Pi(\mathcal{J}_{ij}+\mathcal{K}_{ij})=\Pi(\mathcal{J}_{ij})+\Pi(\mathcal{K}_{ij}).$$

Proof. First, we prove for $i=1$ and $j=2$, i.e., we have to show that

$$\Pi(\mathcal{J}_{12}+\mathcal{K}_{12})=\Pi(\mathcal{J}_{12})+\Pi(\mathcal{K}_{12}).$$

Let $\mathcal{M}=\Pi(\mathcal{J}_{12}+\mathcal{K}_{12})-\Pi(\mathcal{J}_{12})-\Pi(\mathcal{K}_{12})$. It follows that

$$\begin{aligned}\Pi([[\mathcal{J}_{12}+\mathcal{K}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]) &= [[\Pi(\mathcal{J}_{12}+\mathcal{K}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]+[[\mathcal{J}_{12}+\mathcal{K}_{12},\Pi(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]\\ &\quad +[[\mathcal{J}_{12}+\mathcal{K}_{12},\mathcal{P}_1]_\bullet,\Pi(\mathcal{P}_2)].\end{aligned}$$

From the other side, using Lemma 2, we obtain

$$\begin{aligned}\Pi([[\mathcal{J}_{12}+\mathcal{K}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2]) &= \Pi([[\mathcal{J}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2])+\Pi([[\mathcal{K}_{12},\mathcal{P}_1]_\bullet,\mathcal{P}_2])\\ &= [[\Pi(\mathcal{J}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]+[[\mathcal{J}_{12},\Pi(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]\\ &\quad +[[\mathcal{J}_{12},\mathcal{P}_1]_\bullet,\Pi(\mathcal{P}_2)]+[[\Pi(\mathcal{K}_{12}),\mathcal{P}_1]_\bullet,\mathcal{P}_2]\\ &\quad +[[\mathcal{K}_{12},\Pi(\mathcal{P}_1)]_\bullet,\mathcal{P}_2]+[[\mathcal{K}_{12},\mathcal{P}_1]_\bullet,\Pi(\mathcal{P}_2)].\end{aligned}$$

By using the above two equations, we obtain $[[\mathcal{M},\mathcal{P}_1]_\bullet,\mathcal{P}_2]=0$. Thus, $-\mathcal{P}_1\mathcal{M}^*\mathcal{P}_2-\mathcal{P}_2\mathcal{M}\mathcal{P}_1=0$. Multiplying both sides by $\mathcal{P}_2$ from the left, we obtain $\mathcal{P}_2\mathcal{M}\mathcal{P}_1=0$. Similarly, we can show that $\mathcal{P}_1\mathcal{M}\mathcal{P}_2=0$. Now, for any $X_{21}\in\mathcal{A}_{21}$, we have

$$\begin{aligned}\Pi([[X_{12},(\mathcal{J}_{12}+\mathcal{K}_{12})]_\bullet,\mathcal{P}_1]) &= [[\Pi(X_{12}),(\mathcal{J}_{12}+\mathcal{K}_{12})]_\bullet,\mathcal{P}_1]+[[X_{12},\Pi(\mathcal{J}_{12}+\mathcal{K}_{12})]_\bullet,\mathcal{P}_1]\\ &\quad +[[X_{12},(\mathcal{J}_{12}+\mathcal{K}_{12})]_\bullet,\Pi(\mathcal{P}_1)].\end{aligned}$$

On the other side, it follows from $[[X_{12},\mathcal{J}_{12}]_\bullet,\mathcal{P}_1]=0$ that

$$\begin{aligned}\Pi([[X_{12},(\mathcal{J}_{12}+\mathcal{K}_{12})]_\bullet,\mathcal{P}_1]) &= \Pi([[X_{12},\mathcal{J}_{12}]_\bullet,\mathcal{P}_1])+\Pi([[X_{12},\mathcal{K}_{12}]_\bullet,\mathcal{P}_1])\\ &= [[\Pi(X_{12}),\mathcal{J}_{12}]_\bullet,\mathcal{P}_1]+[[X_{12},\Pi(\mathcal{J}_{12})]_\bullet,\mathcal{P}_1]\\ &\quad +[[X_{12},\mathcal{J}_{12}]_\bullet,\Pi(\mathcal{P}_1)]+[[\Pi(X_{12}),\mathcal{K}_{12}]_\bullet,\mathcal{P}_1]\\ &\quad +[[X_{12},\Pi(\mathcal{K}_{12})]_\bullet,\mathcal{P}_1]+[[X_{12},\mathcal{K}_{12}]_\bullet,\Pi(\mathcal{P}_1)].\end{aligned}$$

By obtaining the above two equations, we find that $[[X_{12},\mathcal{M}]_\bullet,\mathcal{P}_1]=0$. Thus, $X_{12}\mathcal{M}^*\mathcal{P}_1-\mathcal{M}X_{12}^*-X_{12}\mathcal{M}^*+\mathcal{P}_1\mathcal{M}X_{12}^*=0$. By multiplying $\mathcal{P}_1$ from the right, we obtain $(\mathcal{P}_1-I)\mathcal{M}X_{12}^*=0$. Thus, by using (1) and (2), we find that $\mathcal{P}_2\mathcal{M}\mathcal{P}_2=0$. Similarly, we can show that $\mathcal{P}_1\mathcal{M}\mathcal{P}_1 = 0$. Hence, $\mathcal{M} = 0$. In a similar way, we can prove for $i = 2, j = 1$. $\square$

Lemma 4. *For any $\mathcal{J}_{ii}, \mathcal{K}_{ii} \in \mathcal{A}_{ii}, 1 \leq i \leq 2$, we have*

1. $\Pi(\mathcal{J}_{11} + \mathcal{K}_{11}) = \Pi(\mathcal{J}_{11}) + \Pi(\mathcal{K}_{11})$.
2. $\Pi(\mathcal{J}_{22} + \mathcal{K}_{22}) = \Pi(\mathcal{J}_{22}) + \Pi(\mathcal{K}_{22})$.

Proof. 1. Let $\mathcal{M} = \Pi(\mathcal{J}_{11} + \mathcal{K}_{11}) - \Pi(\mathcal{J}_{11}) - \Pi(\mathcal{K}_{11})$. We have

$$\begin{aligned}\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) &= [[\Pi(\mathcal{J}_{11} + \mathcal{K}_{11}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{11} + \mathcal{K}_{11}, \Pi(X_{12})]_\bullet, \mathcal{P}_1] \\ &\quad + [[\mathcal{J}_{11} + \mathcal{K}_{11}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)]\end{aligned}$$

Since $[[\mathcal{J}_{11}, X_{12}]_\bullet, \mathcal{P}_2] = 0]$, and using Lemma 1, we have

$$\begin{aligned}\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) &= \Pi([[\mathcal{J}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) + \Pi([[\mathcal{K}_{11}, \mathcal{P}_1]_\bullet, \mathcal{P}_2]) \\ &= [[\Pi(\mathcal{J}_{11}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{11}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{11}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)] + [[\Pi(\mathcal{K}_{11}), \mathcal{P}_1]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{K}_{11}, \Pi(\mathcal{P}_1)]_\bullet, \mathcal{P}_2] + [[\mathcal{K}_{11}, \mathcal{P}_1]_\bullet, \Pi(\mathcal{P}_2)].\end{aligned}$$

From the above two equations, we have $[[\mathcal{M}, \mathcal{P}_1]_\bullet, \mathcal{P}_2] = 0$. This yields $-\mathcal{P}_1\mathcal{M}^*\mathcal{P}_2 - \mathcal{P}_2\mathcal{M}\mathcal{P}_1 = 0$. Hence, $\mathcal{P}_2\mathcal{M}\mathcal{P}_1 = 0$. Similarly, we can show that $\mathcal{P}_1\mathcal{M}\mathcal{P}_2 = 0$.

Now, for any $X_{12} \in \mathcal{A}_{12}$ and using Lemma 1, we have

$$\begin{aligned}\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, X_{12}]_\bullet, \mathcal{P}_2]) &= [[\Pi(\mathcal{J}_{11} + \mathcal{K}_{11}), X_{12}]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{11} + \mathcal{K}_{11}, \Pi(X_{12})]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{11} + \mathcal{K}_{11}, X_{12}]_\bullet, \Pi(\mathcal{P}_2)].\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, X_{12}]_\bullet, \mathcal{P}_2]) &= \Pi([[\mathcal{J}_{11}, X_{12}]_\bullet, \mathcal{P}_2]) + \Pi([[\mathcal{K}_{11}, X_{12}]_\bullet, \mathcal{P}_2]) \\ &= [[\Pi(\mathcal{J}_{11}), X_{12}]_\bullet, \mathcal{P}_2] + [[\mathcal{J}_{11}, \Pi(X_{12})]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{J}_{11}, X_{12}]_\bullet, \Pi(\mathcal{P}_2)] + [[\Pi(\mathcal{K}_{11}), X_{12}]_\bullet, \mathcal{P}_2] \\ &\quad + [[\mathcal{K}_{11}, \Pi(X_{12})]_\bullet, \mathcal{P}_2] + [[\mathcal{K}_{11}, X_{12}]_\bullet, \Pi(\mathcal{P}_2)].\end{aligned}$$

From the above two equations, we obtain $[[\mathcal{M}, X_{12}]_\bullet, \mathcal{P}_2] = 0$. This means that $-X_{12}\mathcal{M}^*\mathcal{P}_2 - \mathcal{P}_2\mathcal{M}X_{12}^* = 0$. Multiplying $\mathcal{P}_1$ from the left, we obtain $X_{12}\mathcal{M}^*\mathcal{P}_2 = 0$, i.e., $\mathcal{P}_1X\mathcal{P}_2\mathcal{M}^*\mathcal{P}_2 = 0$ for all $X \in \mathcal{A}$. It follows from (1) and (2) that $\mathcal{P}_2\mathcal{M}\mathcal{P}_2 = 0$.

Also, for any $X_{21} \in \mathcal{A}_{21}$, we have

$$\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, X_{21}], \mathcal{P}_1]) = \Pi([[\mathcal{J}_{11}, X_{21}], \mathcal{P}_1]) + \Pi([[\mathcal{K}_{11}, X_{21}], \mathcal{P}_1]).$$

On the other hand, it follows from Lemmas 2 and 3 that

$$\begin{aligned}\Pi([[\mathcal{J}_{11} + \mathcal{K}_{11}, X_{21}], \mathcal{P}_1]) &= \Pi(-X_{21}\mathcal{J}_{11}^* - \mathcal{J}_{11}X_{21}^* - X_{21}\mathcal{K}_{11}^* - \mathcal{K}_{11}X_{21}^*) \\ &= \Pi(-X_{21}\mathcal{J}_{11}^*) + \Pi(-\mathcal{J}_{11}X_{21}^*) + \Pi(-X_{21}^*\mathcal{K}_{11}) + \Pi(-\mathcal{K}_{11}X_{21}^*) \\ &= \Pi(-X_{21}\mathcal{J}_{11}^* - \mathcal{J}_{11}X_{21}^*) + \Pi(-X_{21}\mathcal{K}_{11}^* - \mathcal{K}_{11}X_{21}^*) \\ &= \Pi([[\mathcal{J}_{11}, X_{21}]_\bullet, \mathcal{P}_1]) + \Pi([[\mathcal{K}_{11}, X_{21}]_\bullet, \mathcal{P}_1]) \\ &= [[\Pi(\mathcal{J}_{11}), X_{21}]_\bullet, \mathcal{P}_1] + [[\mathcal{J}_{11}, \Pi(X_{21})]_\bullet, \mathcal{P}_1] \\ &\quad + [[\mathcal{J}_{11}, X_{21}]_\bullet, \Pi(\mathcal{P}_1)] + [[\Pi(\mathcal{K}_{11}), X_{21}]_\bullet, \mathcal{P}_1] \\ &\quad + [[\mathcal{K}_{11}, \Pi(X_{21})]_\bullet, \mathcal{P}_1] + [[\mathcal{K}_{11}, X_{21}]_\bullet, \Pi(\mathcal{P}_1)].\end{aligned}$$

From the last two expressions, we find $[[\mathcal{M}, X_{21}]_\bullet, \mathcal{P}_1] = 0$. This means $-X_{21}\mathcal{M}^*\mathcal{P}_1 - \mathcal{P}_1\mathcal{M}X_{21}^* = 0$. By pre-multiplying this by $\mathcal{P}_1$, we get $\mathcal{P}_1\mathcal{M}X_{21}^* = 0$. It follows from (1) and

(2) that $\mathcal{P}_1\mathcal{M}\mathcal{P}_1 = 0$. Hence, $\mathcal{M} = 0$.

2. By using the same technique that was used in the proof of Lemma 4 (1), we can show that
$$\Pi(\mathcal{J}_{22} + \mathcal{K}_{22}) = \Pi(\mathcal{J}_{22}) + \Pi(\mathcal{K}_{22}).$$

This completes the proof. □

Lemma 5. Π *is an additive map.*

Proof. For any $\mathcal{J}, \mathcal{K} \in \mathcal{A}$, we write $\mathcal{J} = \sum_{i,j=1}^{2} \mathcal{J}_{ij}$ and $\mathcal{K} = \sum_{i,j=1}^{2} \mathcal{K}_{ij}$. By using Lemmas 2–4, we obtain

$$\begin{aligned}
\Pi(\mathcal{J} + \mathcal{K}) &= \Pi(\sum_{i,j=1}^{2} \mathcal{J}_{ij} + \sum_{i,j=1}^{2} \mathcal{K}_{ij}) \\
&= \Pi(\sum_{i,j=1}^{2} (\mathcal{J}_{ij} + \mathcal{K}_{ij})) \\
&= \sum_{i,j=1}^{2} \Pi(\mathcal{J}_{ij} + \mathcal{K}_{ij}) \\
&= \sum_{i,j=1}^{2} \Pi(\mathcal{J}_{ij}) + \Pi(\mathcal{K}_{ij}) \\
&= \Pi(\sum_{i,j=1}^{2} \mathcal{J}_{ij}) + \Pi(\sum_{i,j=1}^{2} \mathcal{K}_{ij}) \\
&= \Pi(\mathcal{J}) + \Pi(\mathcal{K}).
\end{aligned}$$

Hence, Π is an additive map. □

Now in the rest of the paper, we prove that Π is an additive $*$-derivation.

Lemma 6. *If* $\Pi(iI) = i\Pi(I)$, *then*

1. $\Pi(I)$ *and* $\Pi(I)^*$ *are central elements of* $\mathcal{A}$.
2. $\Pi(\mathcal{J}^*) = \Pi(\mathcal{J})^*$.

Proof. 1. For any $\mathcal{K} = \mathcal{K}^*$ and since $\Pi(iI) = i\Pi(I)$, we have
$$0 = \Pi([[I, \mathcal{K}]_\bullet, \mathcal{L}]) = [[\Pi(I), \mathcal{K}]_\bullet, \mathcal{L}]$$
for all $\mathcal{L} \in \mathcal{A}$. This implies that
$$[\Pi(I), \mathcal{K}]_\bullet \in \mathcal{Z}(\mathcal{A}).$$

Taking $\mathcal{K} = I$ in the above equation, we obtain
$$\Pi(I) - \Pi(I)^* \in \mathcal{Z}(\mathcal{A}). \tag{3}$$

Similarly, taking $\mathcal{K} = -I$, we obtain
$$\Pi(I) + \Pi(I)^* \in \mathcal{Z}(\mathcal{A}). \tag{4}$$

It follows from Equations (3) and (4) that Π(I) and $\Pi(I)^*$ are central elements of $\mathcal{A}$.

2. It is clear from Lemma 6 (1) that
$$0 = \Pi([[\mathcal{J}, I]_\bullet, \mathcal{K}]) = [[\Pi(\mathcal{J}), I]_\bullet, \mathcal{K}] \tag{5}$$

for all $\mathcal{J}, \mathcal{K} \in \mathcal{A}$ with $\mathcal{J} = \mathcal{J}^*$. This gives that $[\Pi(\mathcal{J}), I]_\bullet \in \mathcal{Z}(\mathcal{A})$. Therefore, $\Pi(\mathcal{J}) - \Pi(\mathcal{J})^* \in \mathcal{Z}(\mathcal{A})$ if $\mathcal{J} = \mathcal{J}^*$. It follows that there exists an additive map $f : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ such that $f(\mathcal{J}) = \Pi(\mathcal{J})^* - \Pi(\mathcal{J}^*)$. Also, from the other side,

$$\Pi([[\mathcal{J}, \mathcal{K}]_\bullet, \mathcal{L}]) = [[\Pi(\mathcal{J}), \mathcal{K}]_\bullet, \mathcal{L}] + [[\mathcal{J}, \Pi(\mathcal{K})]_\bullet, \mathcal{L}] + [[\mathcal{J}, \mathcal{K}]_\bullet, \Pi(\mathcal{L})]$$

and

$$\begin{aligned} i\Pi([\mathcal{J}\mathcal{K}^* + \mathcal{K}\mathcal{J}^*, \mathcal{L}]) &= \Pi([[i\mathcal{J}, \mathcal{K}]_\bullet, \mathcal{L}]) \\ &= [[\Pi(i\mathcal{J}), \mathcal{K}]_\bullet, \mathcal{L}] + [[i\mathcal{J}, \Pi(\mathcal{K})]_\bullet, \mathcal{L}] + [[i\mathcal{J}, \mathcal{K}]_\bullet, \Pi(\mathcal{L})] \\ &= i[\Pi(\mathcal{J})\mathcal{K}^* + \mathcal{K}\Pi(\mathcal{J})^*, \mathcal{L}] + i[\mathcal{J}\Pi(\mathcal{K})^* + \Pi(\mathcal{K})\mathcal{J}^*, \mathcal{L}] \\ &\quad + i[\mathcal{J}\mathcal{K}^* + \mathcal{K}\mathcal{J}^*, \Pi(\mathcal{L})]. \end{aligned}$$

This implies that

$$\Pi([\mathcal{J}\mathcal{K}^*, \mathcal{L}]) = [\Pi(\mathcal{J})\mathcal{K}^*, \mathcal{L}] + [\mathcal{J}\Pi(\mathcal{K})^*, \mathcal{L}] + [\mathcal{J}\mathcal{K}^*, \Pi(\mathcal{L})] \tag{6}$$

and

$$\Pi([\mathcal{J}\mathcal{K}^*, \mathcal{L}]) = [\Pi(\mathcal{J})\mathcal{K}^*, \mathcal{L}] + [\mathcal{J}\Pi(\mathcal{K}^*), \mathcal{L}] + [\mathcal{J}\mathcal{K}^*, \Pi(\mathcal{L})]. \tag{7}$$

Replacing $\mathcal{K}$ with $\mathcal{K}^*$ in Equation (7), we obtain

$$\Pi([\mathcal{J}\mathcal{K}, \mathcal{L}]) = [\Pi(\mathcal{J})\mathcal{K}, \mathcal{L}] + [\mathcal{J}\Pi(\mathcal{K}), \mathcal{L}] + [\mathcal{J}\mathcal{K}, \Pi(\mathcal{L})]. \tag{8}$$

From Equations (6) and (7), we obtain

$$f(\mathcal{K})[\mathcal{J}, \mathcal{L}] = 0$$

for all $\mathcal{J}, \mathcal{K}, \mathcal{L} \in \mathcal{A}$. Thus, $f(\mathcal{K}) = 0$ for all $\mathcal{K} \in \mathcal{A}$. Hence, $\Pi(\mathcal{J}^*) = \Pi(\mathcal{J})^*$ for all $\mathcal{J} \in \mathcal{A}$. □

Proof of Theorem 1. By using Lemmas 5 and 6, we can say that Π is additive and $\Pi(\mathcal{J}^*) = \Pi(\mathcal{J})^*$ for all $\mathcal{J} \in \mathcal{A}$. Now, we only have to show that Π is also a derivation. Taking $\mathcal{J} = I$ in Equation (8), we obtain

$$\eta([\mathcal{K}, \mathcal{L}]) = [\eta(\mathcal{K}), \mathcal{L}] + [\mathcal{K}, \eta(\mathcal{L})],$$

where $\eta(X) = \Pi(X) + \Pi(I)X$ for all $X \in \mathcal{A}$. Hence, η is an additive Lie derivation. It follows from [11] (Theorem 2) that $\eta(X) = \mu(X) + \zeta(X)$, where $\mu : \mathcal{A} \to \mathcal{A}$ is an additive derivation and $\zeta : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is an additive map that vanishes at the commutator. It follows that $\Pi(X) = \mu(X) + \zeta(X) - \Pi(I)X$ for all $X \in \mathcal{A}$. Now, from Equation (8), we obtain

$$\zeta(\mathcal{J})[\mathcal{K}, \mathcal{L}] + \zeta(\mathcal{K})[\mathcal{J}, \mathcal{L}] - 2\Pi(I)[\mathcal{J}\mathcal{K}, \mathcal{L}] = 0. \tag{9}$$

Taking $\mathcal{J} = P_1, \mathcal{K} = X_{12} = [X_{12}, P_2]$, and $\mathcal{L} = P_2$ in Equation (9), we find $\zeta(P_1)X_{12} - 2\Pi(I)X_{12} = 0$ for all $X_{12} \in \mathcal{A}_{12}$. This means that

$$(\zeta(P_1) - 2\Pi(I))P_1 = 0. \tag{10}$$

On the other hand, putting $\mathcal{J} = X_{21}, \mathcal{K} = P_1$, and $\mathcal{L} = P_2$ in Equation (9), we obtain $\zeta(P_1)X_{21} - 2\Pi(I)X_{21} = 0$ for all $X_{21} \in \mathcal{A}_{21}$. This means that

$$(\zeta(P_1) - 2\Pi(I))P_2 = 0. \tag{11}$$

Using Equations (10) and (11), we have $\zeta(P_1) = 2\Pi(I)$. Similarly, we can show that $\zeta(P_2) = 2\Pi(I)$. Hence, $\zeta(I) = 4\Pi(I)$. It follows from $\Pi(X) = \mu(X) + \zeta(X) - \Pi(I)X$ that $\zeta(I) = 2\Pi(I)$. Thus, $\Pi(I) = 0$. It follows from Equation (9) that $\zeta = 0$. Hence, Π is an additive $*$-derivation. □

3. Applications

As a direct result of Theorem 1, we have the corollaries described below.

Let $\mathcal{H}$ be a Hilbert space over a field $\mathbb{F}$ of real or complex numbers, and $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$. The rank of an operator is the dimension of its range. Thus, an operator of a finite rank is one which has a finite dimensional range. We denote $\mathcal{F}(\mathcal{H})$, the subalgebra of all bounded linear operators on $\mathcal{H}$ of a finite rank.

Let $\mathcal{H}$ be a Banach space over a field $\mathbb{F}$ of real or complex numbers. A subalgebra $\mathrm{S}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ is called a standard operator algebra if $\mathcal{F}(\mathcal{H}) \subseteq \mathrm{S}(\mathcal{H})$.

Corollary 1. *Let $\mathcal{A}$ be a standard operator algebra on an infinite, dimensional, complex Hilbert space $\mathcal{H}$ containing an identity operator I. Suppose that $\mathcal{A}$ is closed under adjoint operation. Define $\Pi : \mathcal{A} \to \mathcal{A}$ such that if*

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$, then Π is additive. Moreover, if $\Pi(iI) = i\Pi(I)$, then Π is also an additive $*$-derivation.

Proof. It is a fact that every standard operator algebra $\mathcal{A}$ is a prime algebra, which is a consequence of the Hahn–Banach theorem. Then, by the definition of primeness, $\mathcal{A}$ also satisfies (1) and (2). Hence, by Theorem 1, Π is an additive $*$-derivation. □

A von Neumann algebra $\mathcal{M}$ is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator. Equivalently, a von Neumann algebra $\mathcal{M}$ is a self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ which satisfies the double commutant property, i.e., $\mathcal{M}'' = \mathcal{M}$.

A von Neumann algebra $\mathcal{M}$ is called a factor von Neumann algebra if its center is trivial, i.e., $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$. If $\mathcal{Z}(\mathcal{M}) = \mathcal{M}$, then $\mathcal{M}$ is called abelian.

Corollary 2. *Let $\mathcal{M}$ ba a factor von Neumann algebra with $dim\mathcal{M} \geq 2$. Define $\Pi : \mathcal{M} \to \mathcal{M}$ such that if*

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$, then Π is additive. Moreover, if $\Pi(iI) = i\Pi(I)$, then Π is also an additive $*$-derivation.

Proof. It follows from [12] (Lemma 2.2) that every factor von Neumann algebra $\mathcal{M}$ satisfies (1) and (2). Hence, using Theorem 1, Π is an additive $*$-derivation. □

An algebra $\mathcal{A}$ is called prime if $\mathcal{J}\mathcal{A}\mathcal{K} = \{0\}$ for $\mathcal{J},\mathcal{K} \in \mathcal{A}$ implies either $\mathcal{J} = 0$ or $\mathcal{K} = 0$.

Corollary 3. *Let $\mathcal{A}$ be a prime $*$-algebra with a unit I containing a nontrivial projection P. If a map $\Pi : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\Pi([[\mathcal{J},\mathcal{K}]_\bullet,\mathcal{L}]) = [[\Pi(\mathcal{J}),\mathcal{K}]_\bullet,\mathcal{L}] + [[\mathcal{J},\Pi(\mathcal{K})]_\bullet,\mathcal{L}] + [[\mathcal{J},\mathcal{K}]_\bullet,\Pi(\mathcal{L})]$$

for all $\mathcal{J},\mathcal{K},\mathcal{L} \in \mathcal{A}$, then Π is additive. Moreover, if $\Pi(iI) = i\Pi(I)$, then Π is also an additive $*$-derivation.

Proof. By the definition of the primeness of $\mathcal{A}$, it is easy to see that $\mathcal{A}$ also satisfies (1) and (2); hence, by Theorem 1, Π is an additive $*$-derivation. $\square$

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