mathematics

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A Collocation Approach for the Nonlinear Fifth-Order KdV Equations Using Certain Shifted Horadam Polynomials

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Abstract: This paper proposes a numerical algorithm for the nonlinear fifth-order Korteweg–de Vries equations. This class of equations is known for its significance in modeling various complex wave phenomena in physics and engineering. The approximate solutions are expressed in terms of certain shifted Horadam polynomials. A theoretical background for these polynomials is first introduced. The derivatives of these polynomials and their operational metrics of derivatives are established to tackle the problem using the typical collocation method to transform the nonlinear fifth-order Korteweg–de Vries equation governed by its underlying conditions into a system of nonlinear algebraic equations, thereby obtaining the approximate solutions. This paper also includes a rigorous convergence analysis of the proposed shifted Horadam expansion. To validate the proposed method, we present several numerical examples illustrating its accuracy and effectiveness.

Keywords: generalized Fibonacci polynomials; Korteweg–de Vries equations; operational matrices; convergence analysis; collocation method

MSC: 33C45; 65M70; 35Q53

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Academic Editor: Patricia J. Y. Wong

Received: 9 December 2024
Revised: 12 January 2025
Accepted: 14 January 2025
Published: 17 January 2025

Citation: Abd-Elhameed, W.M.; Alqubori, O.M.; Atta, A.G. A Collocation Approach for the Nonlinear Fifth-Order KdV Equations Using Certain Shifted Horadam Polynomials. *Mathematics* **2025**, *13*, 300. https://doi.org/10.3390/math13020300

1. Introduction

Nonlinear differential equations (DEs) are vital since they can represent complicated real-world phenomena in numerous fields of science and engineering branches. Complex systems can be better understood by studying nonlinear DEs, which, in contrast to linear ones, can display a wide range of behaviors. These behaviors include multistability, chaos, and bifurcations. Many models in different disciplines, such as electrodynamics, neuroscience, epidemiology, mechanical engineering, fluid dynamics, and economics, can be modeled using nonlinear DEs; see, for example, [1–3]. Since most of these models lack exact solutions, numerical methods are crucial in treating these nonlinear DEs. For example, the authors of [4] followed a numerical approach for treating the Black–Scholes model. The authors of [5] used a collocation approach to solve the Fitzhugh–Nagumo nonlinear DEs in neuroscience. Another numerical approach was followed in [6] to solve the nonlinear equations of Emden–Fowler models. Nonlinear thermal diffusion problems were handled in [7]. In [8], a numerical scheme for solving a stochastic nonlinear advection-diffusion dynamical model was handled. In [9], the authors employed Petrov-Galerkin methods for treating some linear and nonlinear partial DEs. The authors of [10] used a specific difference scheme for some nonlinear fractional DEs. The authors of [11] a collocation procedure for treating the nonlinear fractional FitzHugh–Nagumo equation.

An important nonlinear partial differential equation that describes the motion of solitons (individual waves) in shallow water and other systems is the Korteweg–de Vries (KdV) equation. Over the years, the KdV equation has undergone several revisions to account for non-local interactions, dissipation effects, higher-order terms, and various physical phenomena. Many scientific fields have used these modifications; nonlinear optics, fluid dynamics, and plasma physics are just a few examples. The following are a few notable variations of the KdV equation and the several scientific domains that have used them: the standard KdV equations, the modified KdV equation, the generalized KdV equation, the KdV–Burgers equation, and the KdV–Kawahara equation. Furthermore, regarding some applications of some specific problems of the KdV-type equations, we mention three of these problems and their applications.

- The Caudrey–Dodd–Gibbon problem. This problem has applications in shallow water waves, nonlinear optics, and plasma physics; see [12].
- The Sawada–Kotera problem has applications in hydrodynamics, elasticity, plasma physics, and soliton theory; see [13].
- The Kaup-Kuperschmidt problem has applications in fluid mechanics, biological wave propagation, plasma physics, and quantum field theory; see [14].

Numerous contributions have focused on their handling due to the significance of the various KdV-type equations. For example, in [15], analytical and numerical solutions for the fifth-order KdV equation were presented. In [16], some hyperelliptic solutions of certain modified KdV equations were proposed. A numerical study for the stochastic KdV equation was presented in [17]. To treat the generalized Kawahara equation, an operational matrix approach was proposed in [18]. The authors of [19] followed a finite difference approach to handle the fractional KdV equation. Two algorithms were presented in [20] to treat the nonlinear time-fractional Lax's KdV equation. Another numerical approach was given in [21] for approximating the modified KdV equation. A computational approach was used to handle a higher-order KdV equation in [22]. The time-fractional KdV equation was investigated numerically in [23]. In [24], the method of lines was proposed to solve the KdV equation. A Bernstein polynomial basis was employed in [25] to treat the KdV-type equations.

Special functions are fundamental in the scientific, mathematical, and engineering fields. For examples of the usage of these polynomials in signal processing, quantum mechanics, and physics, one can consult [26,27]. These functions have the potential to solve several types of DEs. For example, the authors of [28] numerically treated the fractional Rayleigh-Stokes problem using certain orthogonal combinations of Chebyshev polynomials. The shifted Fibonacci polynomials were utilized in [29] to treat the fractional Burgers equation. In [30], the authors used Vieta–Fibonacci polynomials to treat certain two-dimensional problems. Other two-dimensional FDEs were handled using Vieta Lucas polynomials in [31]. The authors of [32] used Changhee polynomials to treat a high-dimensional chaotic Lorenz system. In [33], some FDEs were treated using shifted Chebyshev polynomials.

Horadam sequences, named after the mathematician Alwyn Horadam, who initially developed them in the 1960s, generalize several well-known polynomials, such as Fibonacci, Lucas, Pell, and Pell Lucas polynomials. Many authors investigated Horadam sequences of polynomials. For example, the authors in [34] investigated some generalized Horadam polynomials and numbers. Some identities regarding Horadam sequences were developed in [35]. Some subclasses of bi-univalent functions associated with the Horadam polynomials were given in [36]. In [37], some characterizations of periodic generalized Horadam sequences were introduced. An application to specific Horadam sequences in coding theory was presented in [38].

Spectral methods are becoming essential in the applied sciences; see, for instance [39,40], for some of their applications in fields like engineering and fluid dynamics. These methods involve approximating differential and integral equation solutions by expansions of various special functions. The three spectral techniques most frequently employed are the collocation, tau, and Galerkin methods. The type of differential equation and the boundary conditions it governs determine which spectral method is suitable. The three spectral approaches use different trial and test functions. The Galerkin approach selects all basis function members to satisfy the underlying conditions imposed by a specific differential equation, treating the test and trial functions as equivalent. (For a few references, see [41–43].) The tau method is easier than the Galerkin method in application since there are no restrictions on selecting the trial and test functions; see, for example, [44–46]. The collocation method is the most popular spectral method because it works well with nonlinear DEs and can be used with all kinds of DEs, no matter what the underlying conditions are; see, for example, [47–50].

We comment here that the motivations for our work are as follows:

- KdV-type equations are among the most important problems encountered in applied sciences, which motivates us to investigate them using a new approach.
- Several spectral approaches were followed to solve KdV-type equations with various orthogonal polynomials as basis functions. The basis functions used in this article are a family of polynomials that are not orthogonal. This article will motivate us to apply these polynomials to other problems in the applied sciences.
- To the best of our knowledge, the specific Horadam sequence of polynomials used in this paper was not previously used in numerical analysis, which provides a compelling reason to introduce and utilize them.

Furthermore, the work's novelty is due to the following points:

- We have developed novel simplified formulas for the new sequence of polynomials, including their high-order derivatives and operational matrices of derivatives.
- This paper presents a new comprehensive study on the convergence analysis of the used Horadam expansion.

The main objectives of this paper can be listed in the following items:

(a) Introducing a class of shifted Horadam polynomials and developing new essential formulas concerned with them.
(b) Developing operational matrices of derivatives of the introduced shifted polynomials.
(c) Analyzing a collocation procedure for solving the nonlinear fifth-order KdV equations.
(d) Investigating the convergence analysis of the proposed Horadam expansion.
(e) Verifying our numerical algorithm by presenting some illustrative examples.

This paper is structured as follows: Section 2 gives an overview of Horadam polynomials, their representation, and some particular polynomials of them. Section 3 introduces certain shifted Horadam polynomials and develops some theoretical formulas that will be used to design our numerical algorithm. Section 4 presents a collocation approach for treating the nonlinear fifth-order KdV-type equations. Section 5 discusses the convergence and error analysis of the proposed expansion in more detail. Section 6 presents some illustrative examples and comparisons. Finally, some discussions are given in Section 7.

2. An Overview of Horadam Polynomials and Some Particular Polynomials

Horadam presented a set of generalized polynomials in his seminal work [51]. These polynomials may be generated using the following recursive formula:

$$W_j(x) = p(x)\,W_{j-1}(x) + q(x)\,W_{j-2}(x), \quad W_0(x) = 0,\, W_1(x) = 1. \tag{1}$$

The polynomials $W_j(x)$ can be written in the following Binet's form:

$$W_j(x) = \frac{\left[p(x) + \sqrt{p^2(x) + 4\,q(x)}\right]^j - \left[p(x) - \sqrt{p^2(x) + 4\,q(x)}\right]^j}{2^j\sqrt{p^2(x) + 4\,q(x)}}, \quad j \geq 0. \tag{2}$$

The above sequence of polynomials generalizes some well-known polynomials, such as Fibonacci, Pell, Lucas, and Pell–Lucas polynomials.

The standard Fibonacci polynomials can be generated with the following recursive formula:

$$F_j(x) = x\,F_{j-1}(x) + F_{j-2}(x), \quad j \geq 2, \quad F_0(x) = 0,\, F_1(x) = 1. \tag{3}$$

The standard Fibonacci polynomials, which are special ones of Horadam polynomials, have several extensions. The generalized Fibonacci polynomials are one example of such a generalization; they are derived using the following recursive formula:

$$F_k^{a,b}(x) = a\,x\,F_{k-1}^{a,b}(x) + b\,F_{k-2}^{a,b}(x), \quad F_0^{a,b}(x) = 1,\, F_1^{a,b}(x) = a\,x, \quad k \geq 2. \tag{4}$$

It is worth noting here that for every k, $F_k^{a,b}(x)$ is of degree k. These polynomials involve many celebrated sequences, such as Fibonacci, Pell, Fermat, and Chebyshev polynomials of the second kind. More precisely, we have the following expressions:

$$F_{i+1}(x) = F_i^{(1,1)}(x), \qquad P_{i+1}(x) = F_i^{(2,1)}(x), \tag{5}$$

$$\mathcal{F}_{i+1}(x) = F_i^{(3,-2)}(x), \qquad U_i(x) = F_i^{(2,-1)}(x). \tag{6}$$

Recently, the authors of [29] have developed some new formulas for the shifted Fibonacci polynomials, defined as

$$F_i^*(x) = F_i(2x-1).$$

In addition; they used these polynomials to solve the fractional Burgers' equation. This paper will introduce specific polynomials of the shifted generalized Fibonacci polynomials, defined as

$$\theta_m(x) = F_{m+1}^{-2,-1}(2x-1). \tag{7}$$

Note that for every $m \geq 0$, $\theta_m(x)$ is of degree m.
The following formula is used to generate these polynomials:

$$\theta_k(x) = -2\,x\,\theta_{k-1}(x) - \theta_{k-2}(x), \quad \theta_0(x) = 1, \quad \theta_1(x) = -2(2x-1), \quad k \geq 2. \tag{8}$$

The following section introduces fundamental formulas concerning the introduced polynomials $\theta_m(x)$.

3. Some New Formulas Concerned with the Introduced Shifted Polynomials

We will develop new formulas for the specific shifted Horadam polynomials defined in (7). The following two lemmas will present the power form representation and inversion formula for these polynomials, which are pivotal in this paper. Next, we will establish new derivative expressions for these polynomials and their operational matrices of derivatives.

3.1. Analytic Form and Its Inversion Formula

Theorem 1. *Let m be a non-negative integer. The power form representation of $\theta_m(x)$ is given by*

$$\theta_m(x) = \frac{-1}{2\sqrt{\pi}} \sum_{r=0}^{m} \frac{(1+2m-r)!\,\Gamma\left(-\frac{1}{2}-m+r\right)}{(m-r)!\,r!} x^{m-r}. \tag{9}$$

Proof. We will proceed by induction. Assume the validity of (9) for every j with $j < m$; that is, we have

$$\theta_j(x) = \sum_{r=0}^{j} A_{r,j} x^{j-r}, \quad \forall\, j < m, \tag{10}$$

where

$$A_{r,j} = \frac{-(1+2j-r)!\,\Gamma\left(-\frac{1}{2}-j+r\right)}{2\sqrt{\pi}(j-r)!\,r!}.$$

To complete the proof; we have to show the validity of (9).
Starting from the recurrence relation of $\theta_m(x)$, we have

$$\theta_m(x) = -2\,(2x-1)\theta_{m-1}(x) - \theta_{m-2}(x). \tag{11}$$

Making use of (10), we can write

$$\theta_m(x) = -2(2x-1) \sum_{r=0}^{m-1} A_{r,m-1}\, x^{m-r-1} - \sum_{r=0}^{m-2} A_{r,m-2}\, x^{m-r-2}. \tag{12}$$

The last formula can be written as

$$\theta_m(x) = -4 \sum_{r=0}^{m-1} A_{r,m-1}\, x^{m-r} + 2 \sum_{r=0}^{m-1} A_{r,m-1}\, x^{m-r-1} - \sum_{r=0}^{m-2} A_{r,m-2}\, x^{m-r-2}, \tag{13}$$

which has the form

$$\begin{aligned} \theta_m(x) &= -4 \sum_{r=0}^{m-1} A_{r,m-1}\, x^{m-r} + 2 \sum_{r=0}^{m} A_{r-1,m-1}\, x^{m-r} - \sum_{r=0}^{m} A_{r-2,m-2}\, x^{m-r} \\ &= \sum_{r=0}^{m} (-4A_{r,m-1} + 2\,A_{r-1,m-1} - A_{r-2,m-2})\, x^{m-r}. \end{aligned}$$

If we note the identity:

$$-4A_{r,m-1} + 2\,A_{r-1,m-1} - A_{r-2,m-2} = A_{r,m},$$

then, Formula (9) can be proved. □

Theorem 2. *Consider a non-negative integer m. The following inversion formula is valid:*

$$x^m = \sum_{r=0}^{m} \frac{(-1)^{r-m} 2^{1-2m} (1+m-r)(1+2m)!}{(2+2m-r)!r!} \theta_{m-r}(x). \tag{14}$$

Proof. We prove Formula (14) by induction. The formula holds for $m = 0$. Assume the validity of (14), and we have to show the following formula:

$$x^{m+1} = \sum_{r=0}^{m+1} \frac{(-1)^{r-m+1}\, 2^{-1-2m}\, (m-r+2)\, (2m+3)!}{(2m-r+4)!\, r!}\, \theta_{m-r+1}(x). \tag{15}$$

Now, if we multiply Formula (14) by x, and make use of the recursive formula (8) in the following form:

$$x\, \theta_m(x) = \frac{1}{4}(2\, \theta_m(x) - \theta_{m+1}(x) - \theta_{m-1}(x)), \tag{16}$$

then the following formula can be obtained:

$$x^{m+1} = \sum_{r=0}^{m} \frac{(-1)^{r-m} 2^{-1-2m} (m-r+1)(2m+1)!}{(2m-r+2)! r!} (2\theta_{m-r}(x) - \theta_{m-r+1}(x) - \theta_{m-r-1}(x)), \tag{17}$$

which can be turned after some algebraic computations into the following form:

$$x^{m+1} = \sum_{r=0}^{m+1} \frac{(-1)^{r-m+1}\, 2^{-1-2m}\, (m-r+2)\, (2m+3)!}{(2m-r+4)!\, r!}\, \theta_{m-r+1}(x). \tag{18}$$

This completes the proof. □

3.2. Derivative Expressions and Operational Matrices of Derivatives of $\theta_m(x)$

Based on Theorems 1 and 2, an expression for the high-order derivatives of $\theta_m(x)$ in terms of their original polynomials can be deduced. The following theorem exhibits this expression.

Theorem 3. *Consider two positive integers m, p with $m \geq p$. We have*

$$D^p \theta_m(x) = \sum_{k=0}^{m-p} U_{k,m,p}\, \theta_k(x), \tag{19}$$

where

$$U_{k,m,p} = \frac{v_{k,m,p}\, 2^{2p}\, (-1)^{m-k} (k+1) \left(\frac{1}{2}(4+m+k-p)\right)_{p-1} (p)_{\frac{1}{2}(m-k-p)}}{\left(\frac{1}{2}(m-k-p)\right)!},$$

with

$$v_{k,m,p} = \begin{cases} 1, & (m-k-p) \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The analytic form of $\theta_m(x)$ in (9) enables one to write $D^p \theta_m(x)$ as

$$D^p \theta_m(x) = -\frac{1}{2\sqrt{\pi}} \sum_{r=0}^{m-p} \frac{(1+2m-r)!\, \Gamma\left(-\frac{1}{2}-m+r\right) (1+m-p-r)_p}{(m-r)!\, r!}\, x^{m-r-p}. \tag{20}$$

The inversion formula in (14) converts the above formula into the following one:

$$D^p\theta_m(x) = \frac{-2}{\pi}\sum_{r=0}^{m-p}\frac{(2m-r+1)!\,\Gamma\left(\frac{3}{2}+m-p-r\right)\Gamma\left(-\frac{1}{2}-m+r\right)}{r!}\times \sum_{n=0}^{m-r-p}\frac{(-1)^{-m+p+r+n}(1+m-p-r-n)}{n!\,(2m-2p-2r-n+2)!}\,\theta_{m-r-p-n}(x). \tag{21}$$

The last formula can be rearranged to be written in a more convenient form:

$$D^p\theta_m(x) = \frac{2}{\pi}\sum_{\ell=0}^{m-p}(-1)^{-m+\ell+p}(-1-m+\ell+p)\times \sum_{r=0}^{\ell}\frac{(2m-r+1)!\,\Gamma\left(\frac{3}{2}+m-p-r\right)\Gamma\left(-\frac{1}{2}-m+r\right)}{r!\,(\ell-r)!\,(2m-\ell-2p-r+2)!}\,\theta_{m-p-\ell}(x). \tag{22}$$

Now, to obtain a simplified formula for the derivatives $D^p\theta_m(x)$ $m \geq p$, we use symbolic algebra to find a closed form for the second sum that appears in the right-hand side of (22). For this purpose, we set

$$S_{\ell,m,p} = \sum_{r=0}^{\ell}\frac{(2m-r+1)!\,\Gamma\left(\frac{3}{2}+m-p-r\right)\Gamma\left(-\frac{1}{2}-m+r\right)}{r!\,(\ell-r)!\,(2m-\ell-2p-r+2)!},$$

and use Zeilberger's algorithm [52] to show that the following recursive formula is satisfied by $S_{\ell,m,p}$:

$$(2-\ell-2p)(-4+\ell-2m+2p)S_{\ell-2,m,p} + \ell(-2+\ell-2m)\,S_{\ell,m,p} = 0, \tag{23}$$

with the following initial values:

$$S_{0,m,p} = \frac{4^{-1-m+p}\sqrt{\pi}\,\Gamma\left(-\frac{1}{2}-m\right)(2m+1)!}{(m-p+1)!},\quad S_{1,m,p} = 0,$$

which can be solved to obtain

$$S_{\ell,m} = \begin{cases} \dfrac{(-1)^{\ell/2}\,2^{-2-2m+\ell+2p}\sqrt{\pi}\,(2m-\ell+1)!\,\Gamma\left(\frac{1}{2}(-1-2m+\ell)\right)(p)_{\frac{\ell}{2}}}{\left(\frac{\ell}{2}\right)!\left(m-\frac{\ell}{2}-p+1\right)!}, & \ell \text{ even},\\ 0, & \ell \text{ odd}, \end{cases}$$

and accordingly, Formula (22) reduces into the following one:

$$D^p\theta_m(x) = 2^{2p}\sum_{k=0}^{\lfloor\frac{m-p}{2}\rfloor}\frac{(-1)^p(1-2k+m-p)\left(\frac{1}{2}(4-2k+2m-2p)\right)_{p-1}(p)_k}{k!}\,\theta_{m-p-2k}(x). \tag{24}$$

The last formula can be written in the following alternative form:

$$D^p\theta_m(x) = 2^{2p}\sum_{k=0}^{m-p}v(m,k,p)\frac{(-1)^{m-k}(k+1)\left(\frac{1}{2}(4+m+k-p)\right)_{p-1}(p)_{\frac{1}{2}(m-k-p)}}{\left(\frac{1}{2}(m-k-p)\right)!}\theta_k(x), \tag{25}$$

with

$$v_{k,m,p} = \begin{cases} 1, & (m-k-p) \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

This finalizes the proof of Theorem 3. □

The following four corollaries exhibit the first-, second-, third-, and fifth-order derivatives of the polynomials $\theta_m(x)$. They are all consequences of Theorem 3.

Corollary 1. *The first derivative of $\theta_m(x)$ can be expressed in the following form:*

$$\frac{d\,\theta_m(x)}{d\,x} = \sum_{k=0}^{m-1} \lambda^1_{k,m}\,\theta_k(x), \quad m \geq 1, \tag{26}$$

where

$$\lambda^1_{k,m} = 4\,(k+1)(-1)^{m-k}\,v_{m,k,1}. \tag{27}$$

Corollary 2. *The second derivative of $\theta_m(x)$ can be expressed in the following form:*

$$\frac{d^2\,\theta_m(x)}{d\,x^2} = \sum_{k=0}^{m-2} \lambda^2_{k,m}\,\theta_k(x), \quad m \geq 2, \tag{28}$$

where

$$\lambda^2_{k,m} = 4\,(k+1)\,(-1)^{m-k}\,(m-k)\,(k+m+2)\,v_{m,k,2}. \tag{29}$$

Corollary 3. *The third derivative of $\theta_m(x)$ can be expressed in the following form:*

$$\frac{d^3\,\theta_m(x)}{d\,x^3} = \sum_{k=0}^{m-3} \lambda^3_{k,m}\,\theta_k(x), \quad m \geq 3, \tag{30}$$

where

$$\lambda^3_{k,m} = 2\,(k+1)\,(-1)^{m-k}\,(k-m-1)\,(k-m+1)\,(k+m+1)\,(k+m+3)\,v_{m,k,3}. \tag{31}$$

Corollary 4. *The fifth derivative of $\theta_m(x)$ can be expressed in the following form:*

$$\frac{d^5\,\theta_m(x)}{d\,x^5} = \sum_{k=0}^{m-5} \lambda^4_{k,m}\,\theta_k(x), \quad m \geq 5, \tag{32}$$

where

$$\lambda^4_{k,m} = \frac{8\,(k+1)\,(-1)^{m-k}\,(k+m-1)\,(k+m+1)\,(k+m+3)\,(k+m+5)\Gamma\Big(\frac{1}{2}(-k+m+5)\Big)\,v_{m,k,5}}{3\,\Gamma\Big(\frac{1}{2}(-k+m-3)\Big)}. \tag{33}$$

Proof. The proof of Corollaries 1–4 can be easily obtained after putting $p = 1, 2, 3, 5$ respectively in Theorem 3. □

The following corollary presents the operational matrices of the integer derivatives of the polynomials $\theta_m(x)$, which can be deduced from the above four corollaries.

Corollary 5. *If we consider the following vector:*

$$\boldsymbol{\theta}(x) = [\theta_0(x), \theta_1(x), \ldots, \theta_N(x)]^T, \tag{34}$$

then, the first-, second-, third-, and fifth-order derivatives of the vector $\boldsymbol{\theta}(x)$ can be written in the following matrix forms:

$$\begin{aligned} \frac{d\,\boldsymbol{\theta}(x)}{d\,x} &= \boldsymbol{A}\,\boldsymbol{\theta}(x), \\ \frac{d^2\,\boldsymbol{\theta}(x)}{d\,x^2} &= \boldsymbol{B}\,\boldsymbol{\theta}(x), \\ \frac{d^3\,\boldsymbol{\theta}(x)}{d\,x^3} &= \boldsymbol{F}\,\boldsymbol{\theta}(x), \\ \frac{d^5\,\boldsymbol{\theta}(x)}{d\,x^5} &= \boldsymbol{G}\,\boldsymbol{\theta}(x), \end{aligned} \tag{35}$$

where $\boldsymbol{A} = (\lambda^1_{k,m})$, $\boldsymbol{B} = (\lambda^2_{k,m})$, $\boldsymbol{F} = (\lambda^3_{k,m})$, and $\boldsymbol{G} = (\lambda^4_{k,m})$ are the operational matrices of derivatives of order $(N+1)^2$.

Proof. The expressions in (35) are direct consequences of Corollaries 1–4. □

4. A Collocation Approach for the Nonlinear Fifth-Order KdV-Type Partial DEs

Consider the following nonlinear fifth-order KdV-type partial differential equation [53,54]:

$$\begin{aligned} &\frac{\partial\,\eta(x,t)}{\partial\,t} + a\,\eta(x,t)^2\left(\frac{\partial\,\eta(x,t)}{\partial\,x}\right) + b\left(\frac{\partial\,\eta(x,t)}{\partial\,x}\right)\left(\frac{\partial^2\,\eta(x,t)}{\partial\,x^2}\right) \\ &+ d\,\eta(x,t)\left(\frac{\partial^3\,\eta(x,t)}{\partial\,x^3}\right) + \frac{\partial^5\,\eta(x,t)}{\partial\,x^5} = 0, \quad 0 \le x,t \le 1, \end{aligned} \tag{36}$$

governed by the following initial and boundary conditions:

$$\eta(x,0) = f(x), \tag{37}$$

$$\eta(0,t) = g_0(t), \quad \eta(1,t) = g_1(t), \tag{38}$$

$$\frac{\partial\,\eta(0,t)}{\partial\,x} = g_2(t), \quad \frac{\partial\,\eta(1,t)}{\partial\,x} = g_3(t), \quad \frac{\partial^2\,\eta(0,t)}{\partial\,x^2} = g_4(t), \tag{39}$$

where a, b, d are arbitrary constants.

Now, consider the following space:

$$\mathcal{Z}^N = \text{span}\{\theta_m(x)\,\theta_n(t) : 0 \le m,n \le N\}. \tag{40}$$

Consequently, it can be assumed that any function $\eta_N = \eta_N(x,t) \in \mathcal{Z}^N$ can be represented as

$$\eta_N = \sum_{m=0}^{N}\sum_{n=0}^{N} \hat{\eta}_{mn}\,\theta_m(x)\,\theta_n(t) = \boldsymbol{\theta}(x)^T\,\hat{\boldsymbol{\eta}}\,\boldsymbol{\theta}(t), \tag{41}$$

where $\boldsymbol{\theta}(x)$ is the vector defined in (34), and $\hat{\boldsymbol{\eta}} = (\hat{\eta}_{mn})_{0\le m,n\le N}$ is the matrix of unknowns, whose order is $(N+1)^2$.

Now, we can write the residual $\mathcal{R}_N(x,t)$ of Equation (36) as

$$\mathcal{R}_N(x,t) = \frac{\partial\,\eta_N}{\partial\,t} + a\,{\eta_N}^2\left(\frac{\partial\,\eta_N}{\partial\,x}\right) + b\left(\frac{\partial\,\eta_N}{\partial\,x}\right)\left(\frac{\partial^2\,\eta_N}{\partial\,x^2}\right) + d\,\eta_N\left(\frac{\partial^3\,\eta_N}{\partial\,x^3}\right) + \frac{\partial^5\,\eta_N}{\partial\,x^5}. \tag{42}$$

Thanks to Corollary 5 along with the expansion (41), the following expressions for the terms $\frac{\partial \eta_N}{\partial t}, \eta_N^2 \left(\frac{\partial \eta_N}{\partial x}\right), \left(\frac{\partial^2 \eta_N}{\partial x^2}\right), \eta_N \left(\frac{\partial^3 \eta_N}{\partial x^3}\right)$ and $\frac{\partial^5 \eta_N}{\partial x^5}$, can be obtained:

$$\frac{\partial \eta_N}{\partial t} = \boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, (\boldsymbol{A}\,\boldsymbol{\theta}(t)), \tag{43}$$

$$\eta_N^2 \left(\frac{\partial \eta_N}{\partial x}\right) = [\boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)]^2 \, [(\boldsymbol{A}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)], \tag{44}$$

$$\left(\frac{\partial \eta_N}{\partial x}\right) \left(\frac{\partial^2 \eta_N}{\partial x^2}\right) = [(\boldsymbol{A}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \, [(\boldsymbol{B}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)], \tag{45}$$

$$\eta_N \left(\frac{\partial^3 \eta_N}{\partial x^3}\right) = [\boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \, [(\boldsymbol{F}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)], \tag{46}$$

$$\frac{\partial^5 \eta_N}{\partial x^5} = (\boldsymbol{G}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t). \tag{47}$$

By virtue of the expressions (43)–(47), the residual $\mathcal{R}_N(x,t)$ can be written in the following form:

$$\begin{aligned} \mathcal{R}_N(x,t) =& \boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, (\boldsymbol{A}\,\boldsymbol{\theta}(t)) + a\,[\boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)]^2 \, [(\boldsymbol{A}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \\ &+ b\,[(\boldsymbol{A}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \, [(\boldsymbol{B}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \\ &+ d\,[\boldsymbol{\theta}(x)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] \, [(\boldsymbol{F}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t)] + (\boldsymbol{G}\,\boldsymbol{\theta}(x))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(t). \end{aligned} \tag{48}$$

Now, to obtain the expansion coefficients c_{mn}, we apply the spectral collocation method by forcing the residual $\mathcal{R}_N(x,t)$ to be zero at some collocation points $\left(\frac{m+1}{N+2}, \frac{n+1}{N+2}\right)$, as follows:

$$\mathcal{R}_N\left(\frac{m+1}{N+2}, \frac{n+1}{N+2}\right) = 0, \quad 0 \le m \le N-5, \quad 0 \le n \le N-1. \tag{49}$$

Moreover, the initial and boundary conditions (37)–(39) imply the following equations:

$$\boldsymbol{\theta}\left(\frac{m+1}{N+2}\right)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}(0) = f\left(\frac{m+1}{N+2}\right), \quad 0 \le m \le N, \tag{50}$$

$$\boldsymbol{\theta}(0)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}\left(\frac{n+1}{N+2}\right) = g_0\left(\frac{n+1}{N+2}\right), \quad 0 \le n \le N-1, \tag{51}$$

$$\boldsymbol{\theta}(1)^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}\left(\frac{n+1}{N+2}\right) = g_1\left(\frac{n+1}{N+2}\right), \quad 0 \le n \le N-1, \tag{52}$$

$$(\boldsymbol{A}\,\boldsymbol{\theta}(0))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}\left(\frac{n+1}{N+2}\right) = g_2\left(\frac{n+1}{N+2}\right), \quad 0 \le n \le N-1, \tag{53}$$

$$(\boldsymbol{A}\,\boldsymbol{\theta}(1))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}\left(\frac{n+1}{N+2}\right) = g_3\left(\frac{n+1}{N+2}\right), \quad 0 \le n \le N-1, \tag{54}$$

$$(\boldsymbol{B}\,\boldsymbol{\theta}(0))^T \,\hat{\boldsymbol{\eta}}\, \boldsymbol{\theta}\left(\frac{n+1}{N+2}\right) = g_4\left(\frac{n+1}{N+2}\right), \quad 0 \le n \le N-1. \tag{55}$$

The $(N+1)^2$ nonlinear system of equations formed by the equations in (50)–(55) and (49) may be solved with the use of a numerical solver, such as Newton's iterative technique, and thus the approximate solution given by (41) can be found.

5. The Convergence and Error Analysis

This section provides a detailed convergence analysis of the proposed approximate expansion. To begin this study, specific inequalities are necessary.

Lemma 1. *The following inequality holds [55]:*

$$|I_n(x)| \leq \frac{x^n \cosh(x)}{2^n\,\Gamma(n+1)}, \quad x > 0, \tag{56}$$

where $I_n(x)$ is the modified Bessel function of order n of the first kind.

Lemma 2. *Consider the infinitely differentiable function $g(x)$ at the origin. $g(x)$ can be expanded as*

$$g(x) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{4\,(-1)^n\, g^{(s)}(0)\,(n+1)\,\Gamma(s+\frac{3}{2})}{\sqrt{\pi}\,(s-n)!\,(n+s+2)!}\,\theta_n(x). \tag{57}$$

Proof. Consider the following expansion for $g(x)$:

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}\,x^n. \tag{58}$$

As a result of the inversion formula (14), the previous expansion transforms into the following form:

$$g(x) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(-1)^r\, g^{(n)}(0)\, 2^{1-2\,n}\,(2\,n+1)!\,(r+1)}{n!\,(n-r)!\,(2+n+r)!}\theta_r(x). \tag{59}$$

Now, expanding the right-hand side of the last equation and rearranging the similar terms, the following expansion can be obtained:

$$g(x) = \sum_{n=0}^{\infty}\sum_{s=n}^{\infty} \frac{4\,(-1)^n\, g^{(s)}(0)\,(n+1)\,\Gamma(s+\frac{3}{2})}{\sqrt{\pi}\,(n-s)!\,(n+s+2)!}\,\theta_n(x). \tag{60}$$

This completes the proof of this lemma. □

Lemma 3. *Consider any non-negative integer m. The following inequality holds for $\theta_m(x)$:*

$$|\theta_m(x)| \leq m+1, \quad x \in (0,1). \tag{61}$$

Proof. Using the analytic form for $\theta_m(x)$ in (9), we can write

$$|\theta_m(x)| = \frac{1}{2\sqrt{\pi}\,m!}\left|\sum_{r=0}^{m}\binom{m}{m-r}\Gamma\left(-\frac{1}{2}-r\right)\right|, \quad x \in (0,1). \tag{62}$$

Now, we will use the symbolic algebra to find a closed formula for the summation in (62). Set

$$S_m = \sum_{r=0}^{m}\binom{m}{m-r}\Gamma\left(-\frac{1}{2}-r\right).$$

Zeilberger's algorithm [52] aids in demonstrating that the following first-order recursive formula is satisfied by S_m:

$$S_{m+1} + (m+2)S_m = 0, \quad S_0 = -2\sqrt{\pi},$$

which can be immediately solved to give

$$S_m = 2(-1)^{m+1}\sqrt{\pi}(m+1)!,$$

and this implies the following inequality:

$$|S_m| \leq 2\sqrt{\pi}(m+1)!. \tag{63}$$

Now, it is easy to see from (62) along with the inequality (63), that

$$|\theta_m(x)| \leq m+1. \tag{64}$$

This proves Lemma 3. □

Theorem 4. *If $g(x)$ is defined on $[0,1]$ and $|g^{(i)}(0)| \leq \mu^i, \quad i > 0$, where μ is a positive constant and $g(x) = \sum_{n=0}^{\infty} \hat{u}_n\,\theta_n(x)$, then we obtain*

$$|\hat{u}_n| \leq \frac{(e^\mu+1)2^{-2n-1}\mu^n}{n!}. \tag{65}$$

Moreover, the series converges absolutely.

Proof. Based on Lemma 2 and the assumptions of the theorem, we can write

$$\begin{aligned}|\hat{u}_n| &= \left|\sum_{s=n}^{\infty} \frac{4\,(-1)^n\,g^{(s)}(0)\,(n+1)\,\Gamma\left(s+\frac{3}{2}\right)}{\sqrt{\pi}\,(s-n)!\,(n+s+2)!}\right| \\ &\leq \sum_{s=n}^{\infty} \frac{4\,\mu^s\,(n+1)\,\Gamma\left(s+\frac{3}{2}\right)}{\sqrt{\pi}\,(s-n)!\,(n+s+2)!} \\ &= \frac{4\,e^{\mu/2}\,(n+1)\,I_{n+1}\left(\frac{\mu}{2}\right)}{\mu}.\end{aligned} \tag{66}$$

The application of Lemma 1 enables us to write the previous inequality as

$$|\hat{u}_n| \leq \frac{4\,e^{\mu/2}\,(n+1)\,\cosh\left(\frac{\mu}{2}\right)\left(\frac{\mu}{2}\right)^{n+1}}{\mu\,2^{n+1}\,(n+1)!}, \tag{67}$$

which can be rewritten after simplifying the right-hand side of the last inequality as

$$|\hat{u}_n| \leq \frac{(e^\mu+1)\,2^{-2n-1}\,\mu^n}{n!}. \tag{68}$$

We now show the second part of the theorem. Since we have

$$\sum_{n=0}^{\infty} |\hat{u}_n\,\theta_n(x)| \leq \sum_{i=0}^{\infty} \frac{2^{-2n-1}(e^\mu+1)\mu^n\,(n+1)}{n!} = \frac{1}{8}\,e^{\mu/4}\,(e^\mu+1)\,(\mu+4), \tag{69}$$

so the series converges absolutely. □

Theorem 5. *If $f(x)$ satisfies the hypothesis of Theorem 4, and $e_N(x) = \sum_{n=N+1}^{\infty} \hat{u}_n\,\theta_n(x)$, then the following error estimation is satisfied:*

$$|e_N(x)| < \frac{(e^\mu+1)\left(e^{\mu/4}(\mu+4)+4\right)2^{-2N-5}\mu^{N+1}}{N!}. \tag{70}$$

Proof. The definition of $e_N(x)$ enables us to write

$$
\begin{aligned}
|e_N(x)| &= \sum_{n=N+1}^{\infty} |c_n|\, |\theta_n(x)| \\
&\le \sum_{n=N+1}^{\infty} \frac{2^{-2n-1}(e^{\mu}+1)\mu^n\,(n+1)}{n!} \\
&= \frac{(e^{\mu}+1)2^{-2N-3}\Big(\mu^{N+1} + e^{\mu/4}(\mu+4)\,4^N\,\big(N! - \Gamma\big(N+1,\tfrac{\mu}{4}\big)\big)\Big)}{N!} \\
&< \frac{(e^{\mu}+1)\Big(e^{\mu/4}(\mu+4)+4\Big)2^{-2N-5}\mu^{N+1}}{N!},
\end{aligned} \tag{71}
$$

where $\Gamma(.,.)$ denotes upper incomplete gamma functions [56]. □

Theorem 6. *Let* $\eta(x,t) = \chi_1(x)\,\chi_2(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{\eta}_{ij}\,\theta_i(x)\,\theta_j(t)$, *with* $|\chi_1^{(i)}(0)| \le \ell_1^i$ *and* $|\chi_2^{(i)}(0)| \le \ell_2^i$, *where* ℓ_1 *and* ℓ_2 *are positive constants. One has*

$$
|\hat{\eta}_{ij}| \le \frac{\Big(e^{\ell_1}+1\Big)\,\Big(e^{\ell_2}+1\Big)\,2^{-2\,(i+j+1)}\,\ell_1^i\,\ell_2^j}{i!\,j!}. \tag{72}
$$

Moreover, the series converges absolutely.

Proof. If we apply Lemma 2 and use the assumption $\eta(x,t) = \chi_1(x)\,\chi_2(t)$, then we can write

$$
\hat{\eta}_{ij} = \sum_{p=i}^{\infty}\sum_{q=j}^{\infty} \frac{16\,(-1)^{i+j}\,\chi_1^{(p)}(0)\;\chi_2^{(q)}(0)\,(i+1)\,(j+1)\,\Gamma\big(q+\frac{3}{2}\big)\,\Gamma\big(p+\frac{3}{2}\big)}{\pi\,(p-i)!\,(i+p+2)!\,(q-j)!\,(j+q+2)!}. \tag{73}
$$

If we use the assumptions: $|\chi_1^{(i)}(0)| \le {\ell_1}^i$, and $|\chi_2^{(i)}(0)| \le {\ell_2}^i$, then we obtain

$$
|\hat{\eta}_{ij}| \le \sum_{p=i}^{\infty} \frac{4\,(-1)^i\,\chi_1^{(p)}(0)\,(i+1)\,\Gamma\big(p+\frac{3}{2}\big)}{\sqrt{\pi}\,(p-i)!\,(i+p+2)!} \times \sum_{q=j}^{\infty} \frac{4\,(-1)^j\,\chi_2^{(q)}(0)\,(j+1)\,\Gamma\big(q+\frac{3}{2}\big)}{\sqrt{\pi}\,(q-j)!\,(j+q+2)!}. \tag{74}
$$

We obtain the desired result by performing similar steps as in the proof of Theorem 4. □

Theorem 7. *If* $\eta = \eta(x,t)$ *satisfies the hypothesis of Theorem 6, then we have the following upper estimate on the truncation error:*

$$
\begin{aligned}
|E_N| = |\eta - \eta_N| &< \frac{e^{\ell_1/4}\Big(e^{\ell_1}+1\Big)(\ell_1+4)\,\Big(e^{\ell_2}+1\Big)\Big(e^{\ell_2/4}(\ell_2+4)+4\Big)2^{-2N-5}\ell_2^{N+1}}{N!} \\
&+ \frac{e^{\ell_2/4}\Big(e^{\ell_2}+1\Big)(\ell_2+4)\,\Big(e^{\ell_1}+1\Big)\Big(e^{\ell_1/4}(\ell_1+4)+4\Big)2^{-2N-5}\ell_1^{N+1}}{N!}.
\end{aligned} \tag{75}
$$

Proof. From definitions of η and η_N, we obtain

$$
\begin{aligned}
|E_N| = |\eta - \eta_N| &= \left| \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \hat{\eta}_{ij}\,\theta_i(x)\,\theta_j(t) - \sum_{i=0}^{N}\sum_{j=0}^{N} \hat{\eta}_{ij}\,\theta_i(x)\,\theta_j(t) \right| \\
&\le \left| \sum_{i=0}^{N}\sum_{j=N+1}^{\infty} \hat{\eta}_{ij}\,\theta_i(x)\,\theta_j(t) \right| + \left| \sum_{i=N+1}^{\infty}\sum_{j=0}^{\infty} \hat{\eta}_{ij}\,\theta_i(x)\,\theta_j(t) \right|.
\end{aligned} \tag{76}
$$

If Theorem 6 and Lemma 3 are applied, then the following inequalities can be obtained:

$$\sum_{i=0}^{N} \frac{2^{-2i-1}(i+1)\left(e^{\ell_1}+1\right)\ell_1^i}{i!} = \frac{(e^{\lambda}+1)2^{-2N-5}(N+1)\ell_1^{N+1}\left(e^{\ell_1/4}(\ell_1+4)E_{-N}\left(\frac{\ell_1}{4}\right)-4\right)}{(N+1)!} < e^{\ell_1/4}\left(e^{\ell_1}+1\right)(\ell_1+4), \tag{77}$$

$$\sum_{i=N+1}^{\infty} \frac{2^{-2i-1}(i+1)\left(e^{\ell_1}+1\right)\ell_1^i}{i!} = \frac{\left(e^{\ell_1}+1\right)2^{-2N-3}}{N!}\Bigg(\ell_1^{N+1}+e^{\ell_1/4}(\ell_1+4)4^N \times\left(N!-\Gamma\left(N+1,\frac{\ell_1}{4}\right)\right)\Bigg) < \frac{\left(e^{\ell_1}+1\right)\left(e^{\ell_1/4}(\ell_1+4)+4\right)2^{-2N-5}\ell_1^{N+1}}{N!}, \tag{78}$$

$$\sum_{i=0}^{\infty} \frac{2^{-2i-1}(i+1)\left(e^{\ell_1}+1\right)\ell_1^i}{i!} = \frac{1}{8}e^{\ell_1/4}\left(e^{\ell_1}+1\right)(\ell_1+4) < e^{\ell_1/4}\left(e^{\ell_1}+1\right)(\ell_1+4), \tag{79}$$

and accordingly, we obtain

$$|\eta-\eta_N| < \frac{e^{\ell_1/4}\left(e^{\ell_1}+1\right)(\ell_1+4)\left(e^{\ell_2}+1\right)\left(e^{\ell_2/4}(\ell_2+4)+4\right)2^{-2N-5}\ell_2^{N+1}}{N!} + \frac{e^{\ell_2/4}\left(e^{\ell_2}+1\right)(\ell_2+4)\left(e^{\ell_1}+1\right)\left(e^{\ell_1/4}(\ell_1+4)+4\right)2^{-2N-5}\ell_1^{N+1}}{N!}. \tag{80}$$

This completes the proof of this theorem. □

6. Illustrative Examples

In this section, we present numerical examples to validate and demonstrate the applicability and accuracy of our proposed numerical algorithm. We also present comparisons with some other methods. Now, if we consider the successive errors E_N and E_{N+1}, then the order of convergence for the given method can be calculated as [57]

$$\text{Order} = \frac{\log \frac{E_{N+1}}{E_N}}{\log \frac{N+1}{N}}. \tag{81}$$

Example 1 ([53,54])**.** *Consider the following Lax equation of order five:*

$$\frac{\partial \eta}{\partial t} + 30\,\eta^2\left(\frac{\partial \eta}{\partial x}\right) + 20\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 10\,\eta\left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x, t \le 1, \tag{82}$$

governed by

$$\eta(x,0) = 2k^2(3\tanh(k\,x_0 - kx) + 2), \tag{83}$$

$$\eta(0,t) = 2k^2\left(3\tanh\left(k\,x_0 + 56\,k^5\,t\right) + 2\right), \tag{84}$$

$$\eta(1,t) = 2k^2\left(2 - 3\tanh\left(-k\,x_0 - 56\,k^5\,t + k\right)\right), \tag{85}$$

$$\frac{\partial \eta(0,t)}{\partial x} = -\,6k^3 sech^2\left(k\,x_0 + 56\,k^5\,t\right), \tag{86}$$

$$\frac{\partial \eta(1,t)}{\partial x} = -\,6k^3 sech^2\left(-k\,x_0 - 56\,k^5\,t + k\right), \tag{87}$$

$$\frac{\partial^2 \eta(0,t)}{\partial x^2} = -\,12k^4\tanh\left(k\,x_0 + 56\,k^5\,t\right)sech^2\left(k\,x_0 + 56\,k^5 t\right), \tag{88}$$

where the analytic solution of this problem is

$$\eta(x,t) = 2k^2\Big(2 - 3\tanh\Big(k\,x - 56\,k^5\,t - k\,x_0\Big)\Big).$$

Table 1 presents a comparison of L^∞ error between our method at $N = 12$ and the method in [53] when $k = 0.01$ and $x_0 = 0$. Table 2 shows the CPU time used in seconds of the results in Table 1. Moreover, Table 3 shows the absolute errors (AEs) at different values of t when $k = 0.01$ and $x_0 = 0$. Table 4 shows the maximum AEs and the order of convergence, which is calculated by (81) at different values of N. Figure 1 illustrates the AEs (left) and approximate solution (right) at $N = 12$ when $k = 0.01$ and $x_0 = 0$.

Table 1. The L^∞ error of Example 1.

	Method in [53]			**Our Method at $N = 12$**
$2\hat{M}$	**$\Delta t = \frac{1}{10}$**	**$\Delta t = \frac{1}{100}$**	**$\Delta t = \frac{1}{1000}$**	
2	1.0635×10^{-7}	8.5821×10^{-8}	1.0503×10^{-7}	1.06572×10^{-15}
4	2.8529×10^{-7}	2.0122×10^{-8}	6.3916×10^{-9}	
8	2.9616×10^{-7}	3.0049×10^{-8}	3.4426×10^{-9}	
16	3.0042×10^{-7}	3.0238×10^{-8}	3.2241×10^{-9}	
32	3.0009×10^{-7}	3.0057×10^{-8}	3.0645×10^{-9}	
64	3.0045×10^{-7}	3.0055×10^{-8}	3.0198×10^{-9}	

Table 2. CPU time used in seconds of Table 1.

	Method in [53]			**Time of Our Method at $N = 12$**
$2\hat{M}$	**Time at $\Delta t = \frac{1}{10}$**	**Time at $\Delta t = \frac{1}{100}$**	**Time at $\Delta t = \frac{1}{1000}$**	
2	0.148614	0.117720	1.000305	1105.16
4	0.058584	0.117451	0.968710	
8	0.081574	0.41234	1.078842	
16	0.163095	0.182400	1.269367	
32	0.035235	0.190048	1.602233	
64	0.040135	0.232144	1.893176	

Table 3. The AEs of Example 1.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	1.0842×10^{-19}	2.71051×10^{-19}	1.6263×10^{-19}	9.21572×10^{-19}
0.2	1.79209×10^{-25}	2.71051×10^{-19}	2.71051×10^{-19}	7.04731×10^{-19}
0.3	2.71051×10^{-19}	2.1684×10^{-19}	4.33681×10^{-19}	4.11997×10^{-18}
0.4	5.96311×10^{-19}	1.6263×10^{-19}	7.58942×10^{-19}	9.59519×10^{-18}
0.5	1.0842×10^{-18}	2.71051×10^{-19}	1.13841×10^{-18}	1.62088×10^{-17}
0.6	1.51788×10^{-18}	1.6263×10^{-19}	1.46367×10^{-18}	2.22261×10^{-17}
0.7	1.68051×10^{-18}	1.0842×10^{-19}	1.57209×10^{-18}	2.54788×10^{-17}
0.8	1.51788×10^{-18}	1.6263×10^{-19}	1.40946×10^{-18}	2.29309×10^{-17}
0.9	9.21572×10^{-19}	2.71051×10^{-19}	6.50521×10^{-19}	1.21431×10^{-17}

Table 4. The maximum AEs and order of convergence for Example 1.

N	Error	Order
4	3.48827×10^{-7}	-
5	5.97282×10^{-8}	0.96358
6	1.05960×10^{-8}	0.99163
7	8.55756×10^{-10}	1.04696
8	9.73067×10^{-11}	1.03323
9	5.90808×10^{-12}	1.06141
10	4.43248×10^{-12}	0.96484
11	7.07445×10^{-12}	0.94307
12	2.18521×10^{-15}	1.26877

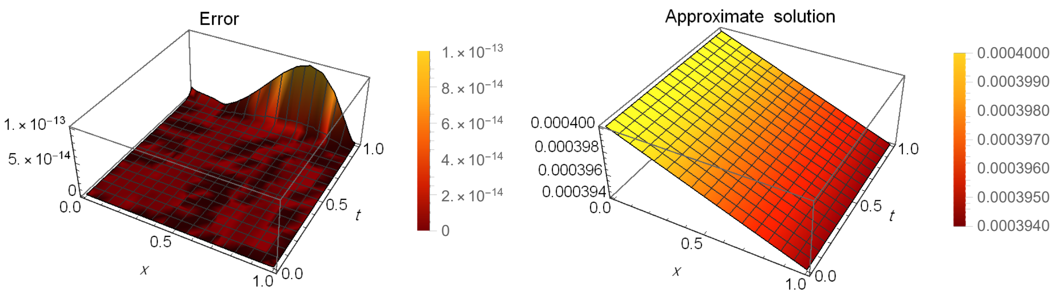

Figure 1. The AEs and the approximate solution for Example 1.

Remark 1 (Stability)**.** *We comment that our proposed method is stable in the sense that* $|\eta_{N+1} - \eta_N|$ *is sufficiently small for sufficiently large values of N, see [58]. To confirm this regarding Problem (82), we plot Figure 2 that shows that our method remains stable for* $x = t$.

Remark 2 (Consistency)**.** *To show the consistency of our numerical method in the sense that* $|\mathcal{R}_N(x,t)|$ *is small for sufficiently large values of N, we plot Figure 3 that gives the absolute residual* $|\mathcal{R}_N(x,t)|$ *at* $t = 0.5$ *for Problem (82). This figure shows the absolute residual* $|\mathcal{R}_N(x,t)|$ *at* $t = 0.5$ *is sufficiently small for sufficiently large values of N.*

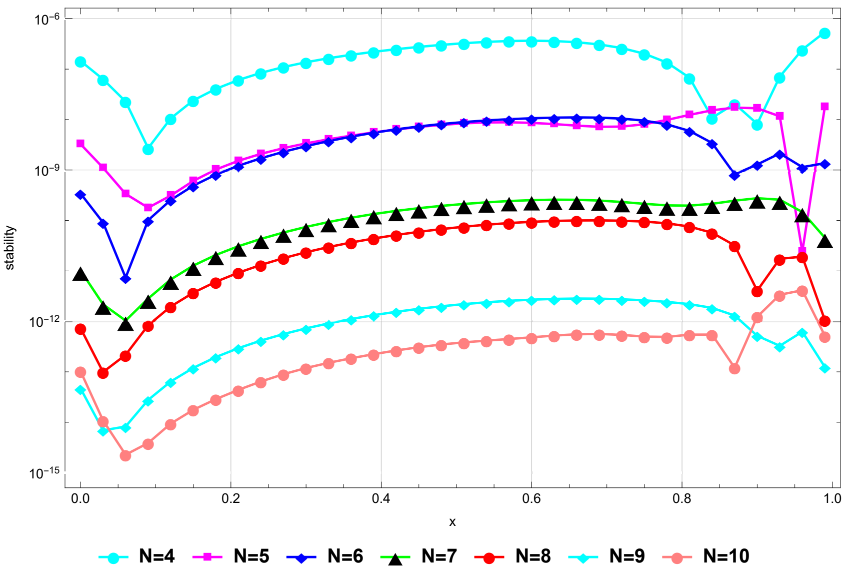

Figure 2. Stability $|\eta_{N+1} - \eta_N|$ for Example 1.

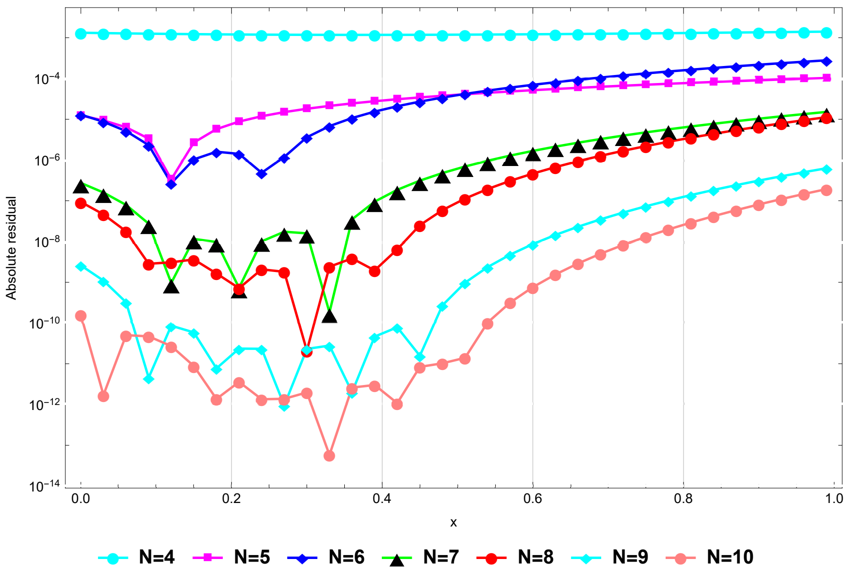

Figure 3. The absolute residual $|\mathcal{R}_N(x,t)|$ at $t = 0.5$ for Example 1.

Example 2 ([53,54])**.** *Consider the following Sawada–Kotera equation of order five:*

$$\frac{\partial \eta}{\partial t} + 45\,\eta^2 \left(\frac{\partial \eta}{\partial x}\right) + 15\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 15\,\eta\left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x, t \le 1, \tag{89}$$

governed by

$$\eta(x,0) = 2k^2 sech^2(k\,x_0 - kx), \tag{90}$$

$$\eta(0,t) = 2k^2 sech^2\left(k\,x_0 + 16k^5 t\right), \tag{91}$$

$$\eta(1,t) = 2k^2 sech^2\left(-k\,x_0 - 16k^5 t + k\right), \tag{92}$$

$$\frac{\partial \eta(0,t)}{\partial x} = 4k^3 \tanh\left(k\left(x_0 + 16k^4 t\right)\right) sech^2\left(k\left(x_0 + 16k^4 t\right)\right), \tag{93}$$

$$\frac{\partial \eta(1,t)}{\partial x} = -4k^3 \tanh\left(-k\,x_0 - 16k^5 t + k\right) sech^2\left(k\left(x_0 + 16k^4 t - 1\right)\right), \tag{94}$$

$$\frac{\partial^2 \eta(0,t)}{\partial x^2} = 4k^4\left(\cosh\left(2k\left(x_0 + 16k^4 t\right)\right) - 2\right) sech^4\left(k\left(x_0 + 16k^4 t\right)\right), \tag{95}$$

where the analytic solution of this problem is

$$\eta(x,t) = 2\,k^2 sech^2\left(k\,x - 16\,k^5\,t - k\,x_0\right).$$

Table 5 presents a comparison of L^∞ error between our method at $N = 12$ and the method in [53] when $k = 0.2$ and $x_0 = 0$. Table 6 shows the CPU time used in seconds of the results in Table 5. Moreover, Table 7 shows the AEs at different values of t when $k = 0.2$ and $x_0 = 0$. Figure 4 illustrates the AEs (left) and approximate solution (right) at $N = 12$ when $k = 0.2$ and $x_0 = 0$.

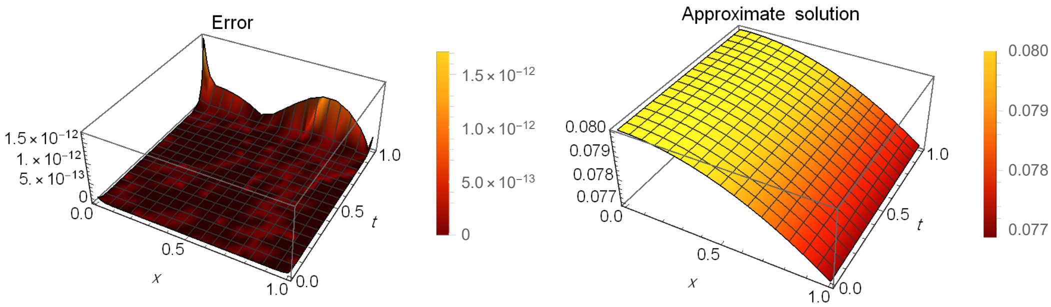

Figure 4. The AEs and the approximate solution for Example 2.

Table 5. The L^∞ error of Example 2.

	Method in [53]			Our Method at $N = 12$
$2\hat{M}$	$\Delta t = \frac{1}{10}$	$\Delta t = \frac{1}{100}$	$\Delta t = \frac{1}{1000}$	
2	1.1108×10^{-7}	1.1618×10^{-7}	1.1669×10^{-7}	1.11577×10^{-14}
4	1.6982×10^{-8}	2.3948×10^{-8}	2.4639×10^{-8}	
8	2.6394×10^{-9}	4.3489×10^{-9}	5.0420×10^{-9}	
16	6.6177×10^{-9}	4.8615×10^{-10}	1.1714×10^{-9}	
32	7.5484×10^{-9}	4.7326×10^{-10}	2.3286×10^{-10}	
64	7.7933×10^{-9}	7.0392×10^{-10}	5.9370×10^{-12}	

Table 6. CPU time used in seconds of Table 5.

	Method in [53]			Time of Our Method at $N = 12$
$2\hat{M}$	Time at $\Delta t = \frac{1}{10}$	Time at $\Delta t = \frac{1}{100}$	Time at $\Delta t = \frac{1}{1000}$	
2	0.086222	0.178449	1.073422	1111.03
4	0.090816	0.122136	1.106796	
8	0.038014	0.157665	1.095270	
16	0.038951	0.143982	1.266360	
32	0.041361	0.179976	1.616171	
64	0.047167	0.227436	1.924651	

Table 7. The AEs of Example 2.

x	$t = 0.1$	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.1	2.77564×10^{-17}	1.11023×10^{-16}	1.11023×10^{-16}	9.312×10^{-15}
0.2	8.04913×10^{-16}	8.46546×10^{-16}	8.18791×10^{-16}	8.28504×10^{-15}
0.3	2.70617×10^{-15}	2.47025×10^{-15}	2.45637×10^{-15}	5.41234×10^{-15}
0.4	5.49561×10^{-15}	4.96826×10^{-15}	4.96826×10^{-15}	1.06861×10^{-15}
0.5	8.86792×10^{-15}	7.93811×10^{-15}	7.95199×10^{-15}	4.28825×10^{-15}
0.6	1.19488×10^{-14}	1.06443×10^{-14}	1.0672×10^{-14}	9.68671×10^{-15}
0.7	1.34476×10^{-14}	1.1921×10^{-14}	1.19904×10^{-14}	1.37529×10^{-14}
0.8	1.18655×10^{-14}	1.05333×10^{-14}	1.06026×10^{-14}	1.47382×10^{-14}
0.9	5.99521×10^{-15}	5.32908×10^{-15}	5.37071×10^{-15}	1.11577×10^{-14}

Example 3 ([53,54])**.** *Consider the following Caudrey–Dodd–Gibbon equation of order five:*

$$\frac{\partial \eta}{\partial t} + 180\,\eta^2 \left(\frac{\partial \eta}{\partial x}\right) + 30\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 30\,\eta\left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x, t \le 1, \tag{96}$$

governed by

$$\eta(x,0) = \frac{k^2 e^{kx}}{(e^{kx}+1)^2}, \tag{97}$$

$$\eta(0,t) = \frac{k^2 e^{k^5 t}}{\left(e^{k^5 t}+1\right)^2}, \tag{98}$$

$$\eta(1,t) = \frac{k^2 e^{k^5 t + k}}{\left(e^{k^5 t}+e^{k}\right)^2}, \tag{99}$$

$$\frac{\partial\,\eta(0,t)}{\partial\,x} = \frac{k^3 e^{k^5 t}\left(e^{k^5 t} - 1\right)}{\left(e^{k^5 t} + 1\right)^3}, \tag{100}$$

$$\frac{\partial\,\eta(1,t)}{\partial\,x} = \frac{k^3 e^{k^5 t + k}\left(e^{k^5 t} - e^{k}\right)}{\left(e^{k^5 t} + e^{k}\right)^3}, \tag{101}$$

$$\frac{\partial^2\,\eta(0,t)}{\partial\,x^2} = \frac{k^4 e^{k^5 t}\left(-4e^{k^5 t} + e^{2k^5 t} + 1\right)}{\left(e^{k^5 t} + 1\right)^4}, \tag{102}$$

where the analytic solution of this problem is

$$\eta(x,t) = \frac{k^2 e^{k\left(x - k^4 t\right)}}{\left(e^{k\left(x - k^4 t\right)} + 1\right)^2}.$$

Table 8 presents a comparison of L^∞ error between our method at $N = 12$ and the method in [53] when $k = 0.01$. Table 9 shows the AEs at different values of t when $k = 0.01$. Figure 5 illustrates the AEs (left) and approximate solution (right) at $N = 12$ when $k = 0.01$.

Table 8. The L^∞ error of Example 3.

	Method in [53]			**Our Method at $N = 12$**
$2\hat{M}$	$\Delta t = \frac{1}{10}$	$\Delta t = \frac{1}{100}$	$\Delta t = \frac{1}{1000}$	
2	1.0354×10^{-10}	1.0418×10^{-10}	1.0424×10^{-10}	5.20417×10^{-18}
4	2.2802×10^{-11}	2.3648×10^{-11}	2.3732×10^{-11}	
8	3.1943×10^{-12}	4.0357×10^{-12}	4.1198×10^{-12}	
16	4.7979×10^{-14}	8.2657×10^{-13}	9.1186×10^{-13}	
32	7.2377×10^{-13}	1.2889×10^{-13}	2.1392×10^{-13}	
64	8.9292×10^{-13}	3.9850×10^{-14}	4.5942×10^{-14}	

Table 9. The AEs of Example 3.

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.1	1.69407×10^{-20}	3.38813×10^{-21}	1.35525×10^{-20}	6.77626×10^{-21}
0.2	6.77626×10^{-21}	6.77626×10^{-21}	3.0091×10^{-25}	6.77626×10^{-21}
0.3	6.77626×10^{-21}	1.35525×10^{-20}	8.7983×10^{-25}	1.01644×10^{-20}
0.4	1.69407×10^{-20}	3.38813×10^{-21}	1.35525×10^{-20}	6.77626×10^{-21}
0.5	2.03288×10^{-20}	6.77626×10^{-21}	3.38813×10^{-21}	2.03288×10^{-20}
0.6	1.35525×10^{-20}	1.35525×10^{-20}	1.35525×10^{-20}	2.37169×10^{-20}
0.7	2.37169×10^{-20}	6.77626×10^{-21}	1.01644×10^{-20}	1.01644×10^{-20}
0.8	1.69407×10^{-20}	3.7324×10^{-24}	1.69407×10^{-20}	1.69407×10^{-20}
0.9	2.71051×10^{-20}	6.77626×10^{-21}	2.03288×10^{-20}	1.69407×10^{-20}

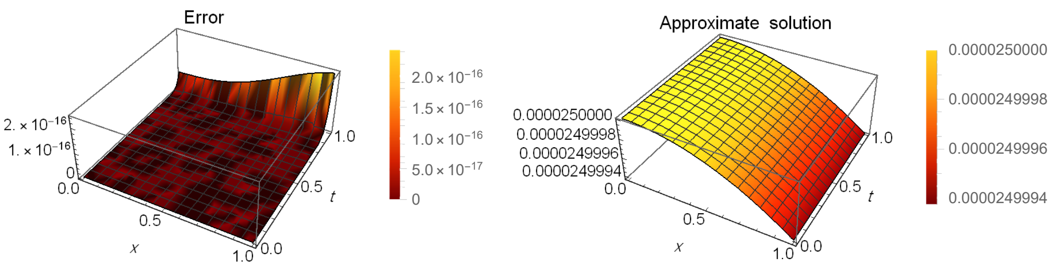

Figure 5. The AEs and the approximate solution for Example 3.

Example 4 ([53,54]). *Consider the following fifth-order Kaup–Kuperschmidt equation:*

$$\frac{\partial \eta}{\partial t} + 20\,\eta^2 \left(\frac{\partial \eta}{\partial x}\right) + 25 \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 10\,\eta \left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x, t \le 1, \quad (103)$$

governed by

$$\eta(x,0) = \frac{24k^2 e^{kx}\left(4e^{kx} + e^{2kx} + 16\right)}{\left(16e^{kx} + e^{2kx} + 16\right)^2}, \quad (104)$$

$$\eta(0,t) = \frac{24k^2 e^{k^5 t}\left(4e^{k^5 t} + 16e^{2k^5 t} + 1\right)}{\left(16e^{k^5 t} + 16e^{2k^5 t} + 1\right)^2}, \quad (105)$$

$$\frac{\partial \eta(0,t)}{\partial x} = \frac{24k^2 e^{k^5 t + k}\left(16e^{2k^5 t} + 4e^{k^5 t + k} + e^{2k}\right)}{\left(16e^{2k^5 t} + 16e^{k^5 t + k} + e^{2k}\right)^2}, \quad (106)$$

$$\frac{\partial \eta(1,t)}{\partial x} = \frac{24k^3 e^{k^5 t}\left(4e^{k^5 t} - 1\right)^3\left(4e^{k^5 t} + 1\right)}{\left(16e^{k^5 t} + 16e^{2k^5 t} + 1\right)^3}, \quad (107)$$

$$\frac{\partial^2 \eta(0,t)}{\partial x^2} = -\frac{24k^3 e^{k^5 t + k}\left(e^k - 4e^{k^5 t}\right)^3\left(4e^{k^5 t} + e^k\right)}{\left(16e^{2k^5 t} + 16e^{k^5 t + k} + e^{2k}\right)^3}, \quad (108)$$

where the analytic solution of this problem is

$$\eta(x,t) = \frac{24\,k^2 e^{k\left(x - k^4 t\right)}\left(4\,e^{k\left(x-k^4 t\right)} + e^{2k\left(x-k^4 t\right)} + 16\right)}{\left(16\,e^{k\left(x-k^4 t\right)} + e^{2k\left(x-k^4 t\right)} + 16\right)^2}.$$

Table 10 presents a comparison of L^∞ error between our method at $N = 12$, and the method in [53] when $k = 0.01$. Table 11 shows the AEs at different values of t when $k = 0.01$. Figure 6 illustrates the AEs (left) and approximate solution (right) at $N = 12$ when $k = 0.01$.

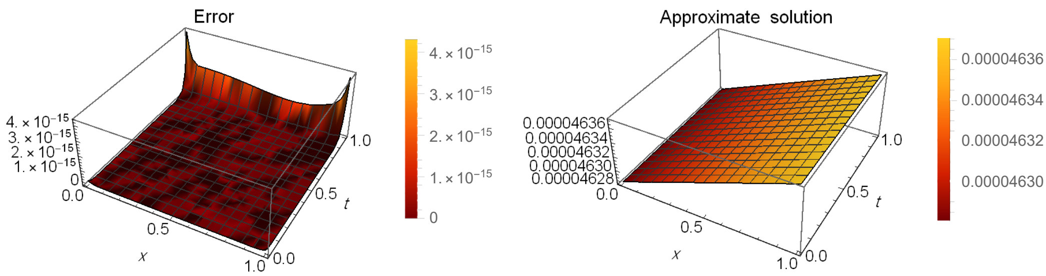

Figure 6. The AEs and the approximate solution for Example 4.

Table 10. The L^∞ error of Example 4.

Method in [53]		**Our Method at $N = 12$**
$2\,\hat{M}$	$\Delta t = \frac{1}{1000}$	
2	2.9045×10^{-10}	3.90177×10^{-17}
4	4.5383×10^{-10}	
8	5.0035×10^{-10}	
16	5.1233×10^{-10}	
32	5.1535×10^{-10}	
64	5.1611×10^{-10}	

Table 11. The AEs of Example 4.

x	$t = 0.15$	$t = 0.4$	$t = 0.65$	$t = 0.9$
0.1	4.40951×10^{-26}	4.74338×10^{-20}	1.35525×10^{-20}	4.29412×10^{-17}
0.2	1.35525×10^{-20}	4.74338×10^{-20}	3.38813×10^{-20}	4.23652×10^{-17}
0.3	2.03288×10^{-20}	6.77626×10^{-21}	3.38813×10^{-20}	3.99867×10^{-17}
0.4	2.71051×10^{-20}	6.77626×10^{-21}	3.38813×10^{-20}	3.6436×10^{-17}
0.5	3.38813×10^{-20}	1.35525×10^{-20}	6.09864×10^{-20}	3.24109×10^{-17}
0.6	4.74338×10^{-20}	1.35525×10^{-20}	6.77626×10^{-20}	2.89279×10^{-17}
0.7	4.74338×10^{-20}	4.74338×10^{-20}	9.48677×10^{-20}	2.77081×10^{-17}
0.8	2.71051×10^{-20}	2.71051×10^{-20}	7.45389×10^{-20}	3.07168×10^{-17}
0.9	6.77626×10^{-21}	1.35525×10^{-20}	4.06576×10^{-20}	3.90177×10^{-17}

Example 5 ([53,54])**.** *Consider the following fifth-order Ito equation:*

$$\frac{\partial \eta}{\partial t} + 2\eta^2 \left(\frac{\partial \eta}{\partial x}\right) + 6\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 3\eta\left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x, t \le 1, \tag{109}$$

governed by

$$\eta(x,0) = 10k^2\Big(2 - 3\tanh^2(k(x_0 + x))\Big), \tag{110}$$

$$\eta(0,t) = 10k^2\Big(2 - 3\tanh^2\Big(k\Big(x_0 - 96k^4 t\Big)\Big)\Big), \tag{111}$$

$$\eta(1,t) = 10k^2\Big(2 - 3\tanh^2\Big(k\,x_0 - 96k^5 t + k\Big)\Big), \tag{112}$$

$$\frac{\partial \eta(0,t)}{\partial x} = -60k^3 \tanh\Big(k\Big(x_0 - 96k^4 t\Big)\Big) sech^2\Big(k\Big(x_0 - 96k^4 t\Big)\Big), \tag{113}$$

$$\frac{\partial \eta(1,t)}{\partial x} = -60k^3 \tanh\Big(k\,x_0 - 96k^5 t + k\Big) sech^2\Big(k\,x_0 - 96k^5 t + k\Big), \tag{114}$$

$$\frac{\partial^2 \eta(0,t)}{\partial x^2} = 60k^4\Big(\cosh\Big(2k\Big(x_0 - 96k^4 t\Big)\Big) - 2\Big) sech^4\Big(k\Big(x_0 - 96k^4 t\Big)\Big), \tag{115}$$

where the analytic solution of this problem is

$$\eta(x,t) = 20\,k^2 - 30\,k^2 \tanh^2\Big(k\,x - 96\,k^5\,t + k\,x_0\Big).$$

Table 12 presents a comparison of L^∞ error between our method at $N = 12$ and the method in [53] when $k = 0.12$ and $x_0 = 0$. Table 13 shows the AEs at different values of t when $k = 0.12$ and $x_0 = 0$. Figure 7 illustrates the AEs (left) and approximate solution (right) at $N = 10$ when $k = 0.12$ and $x_0 = 0$. Table 14 shows the maximum AEs and order of convergence (81) at different values of N.

Remark 3. *Figure 8 confirms that the method remains stable when $x = t$ for higher values of N. Finally, Figure 9 verifies that the $|\mathcal{R}_N(x,t)|$ at $x = t$ is sufficiently small for sufficiently large values of N, and this proves the consistency of the presented method.*

Table 12. The L^∞ error of Example 5.

	Method in [53]			**Our method at $N = 12$**
$2\hat{M}$	$\Delta t = \frac{1}{10}$	$\Delta t = \frac{1}{100}$	$\Delta t = \frac{1}{1000}$	
2	3.8776×10^{-6}	1.1080×10^{-6}	8.3693×10^{-7}	1.55598×10^{-13}
4	4.6387×10^{-6}	7.1764×10^{-7}	3.3455×10^{-7}	
8	4.4590×10^{-6}	5.1088×10^{-7}	1.2552×10^{-7}	
16	4.4709×10^{-6}	4.5665×10^{-7}	6.4702×10^{-8}	
32	4.4544×10^{-6}	4.4071×10^{-7}	4.8748×10^{-8}	
64	4.4562×10^{-6}	4.3726×10^{-7}	4.4826×10^{-8}	

Table 13. The AEs of Example 5.

x	$t = 0.1$	$t = 0.3$	$t = 0.6$	$t = 0.9$
0.1	4.31486×10^{-19}	5.55128×10^{-17}	4.28135×10^{-19}	4.87388×10^{-14}
0.2	7.21651×10^{-16}	4.9961×10^{-16}	3.8859×10^{-16}	4.82947×10^{-15}
0.3	1.99842×10^{-15}	1.60985×10^{-15}	1.16577×10^{-15}	9.17599×10^{-14}
0.4	4.10786×10^{-15}	3.38623×10^{-15}	2.38705×10^{-15}	2.37477×10^{-13}
0.5	6.71691×10^{-15}	5.49568×10^{-15}	3.83037×10^{-15}	4.07507×10^{-13}
0.6	9.10391×10^{-15}	7.38308×10^{-15}	5.16267×10^{-15}	5.5611×10^{-13}
0.7	1.03252×10^{-14}	8.21576×10^{-15}	5.77331×10^{-15}	6.16396×10^{-13}
0.8	9.43697×10^{-15}	7.27206×10^{-15}	5.10716×10^{-15}	5.04319×10^{-13}
0.9	5.55115×10^{-15}	3.83032×10^{-15}	2.72011×10^{-15}	1.55598×10^{-13}

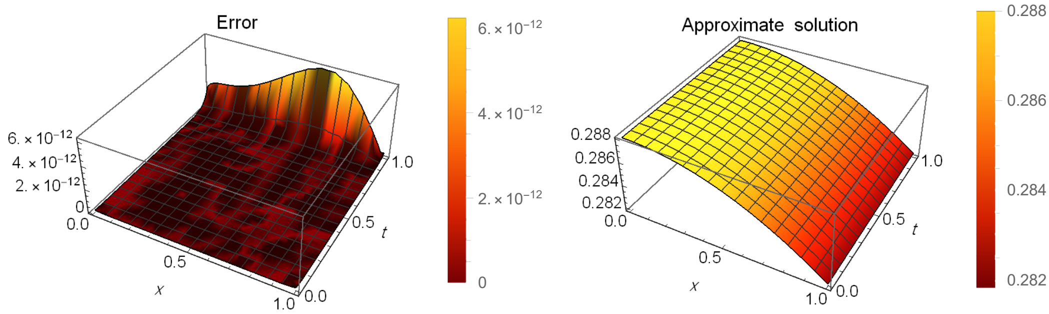

Figure 7. The AEs and the approximate solution for Example 5.

Table 14. The maximum AEs and order of convergence for Example 5.

N	Error	Order
4	4.81759×10^{-8}	-
5	8.14415×10^{-8}	0.834512
6	6.08548×10^{-10}	1.16769
7	4.26518×10^{-10}	0.936205
8	3.37397×10^{-12}	1.14569
9	1.31867×10^{-12}	0.980054
10	3.48999×10^{-13}	1.00061

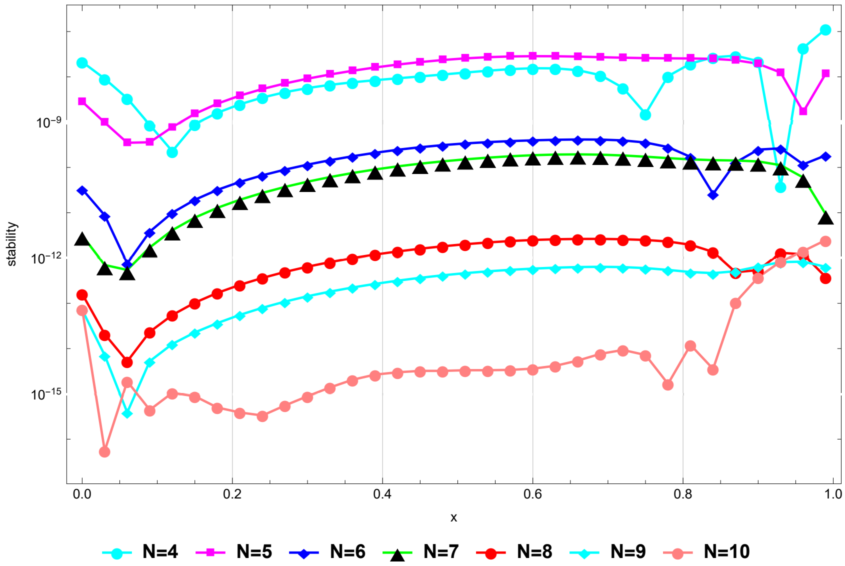

Figure 8. Stability $|\eta_{N+1} - \eta_N|$ for Example 5.

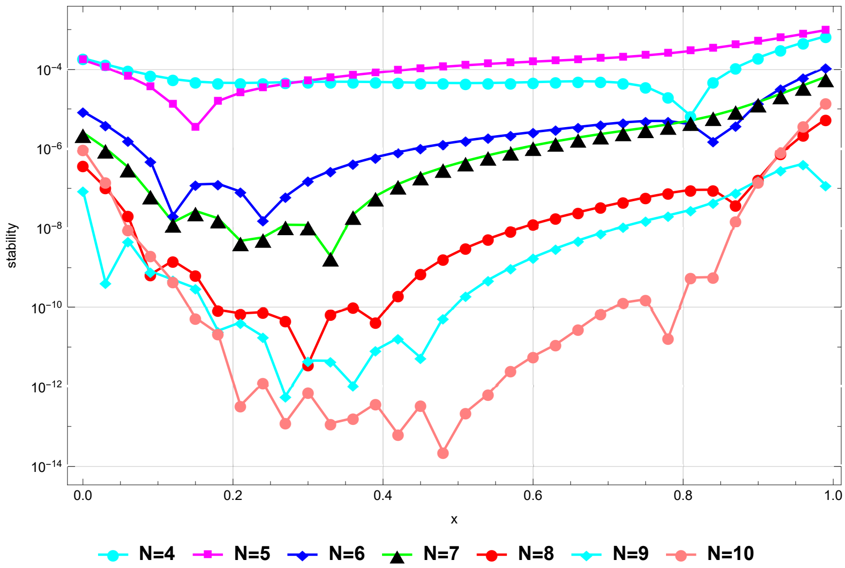

Figure 9. The absolute residual $|\mathcal{R}_N(x,t)|$ at $x = t$ for Example 5.

Example 6. *Consider the following Sawada–Kotera equation of order five:*

$$\frac{\partial \eta}{\partial t} + 45\,\eta^2 \left(\frac{\partial \eta}{\partial x}\right) + 15 \left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial^2 \eta}{\partial x^2}\right) + 15\,\eta \left(\frac{\partial^3 \eta}{\partial x^3}\right) + \frac{\partial^5 \eta}{\partial x^5} = 0, \quad 0 \le x,t \le 1, \quad (116)$$

governed by

$$\eta(x,0) = \eta(0,t) = \eta(1,t) = 0, \quad (117)$$

$$\frac{\partial \eta(0,t)}{\partial x} = \frac{\partial \eta(1,t)}{\partial x} = \frac{\partial^2 \eta(0,t)}{\partial x^2} = 0, \quad (118)$$

Since the exact solution is not available, so we define the following absolute residual error norm:

$$RE = \max_{(x,t)\in[0,1]^2} \left| \frac{\partial \eta_N}{\partial t} + 45\,\eta_N^2 \left(\frac{\partial \eta_N}{\partial x}\right) + 15 \left(\frac{\partial \eta_N}{\partial x}\right)\left(\frac{\partial^2 \eta_N}{\partial x^2}\right) + 15\,\eta_N \left(\frac{\partial^3 \eta_N}{\partial x^3}\right) + \frac{\partial^5 \eta_N}{\partial x^5} \right|, \quad (119)$$

and applying the presented method at $N = 5$ to obtain Table 15, which illustrates the RE.

Table 15. The RE of Example 6.

x	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.8$
0.1	1.32384×10^{-20}	2.52242×10^{-21}	1.0856×10^{-21}	1.54641×10^{-20}
0.2	1.32389×10^{-20}	2.52131×10^{-21}	1.08586×10^{-21}	1.54682×10^{-20}
0.3	1.32394×10^{-20}	2.51944×10^{-21}	1.08649×10^{-21}	1.54706×10^{-20}
0.4	1.32404×10^{-20}	2.51749×10^{-21}	1.0876×10^{-21}	1.54651×10^{-20}
0.5	1.32421×10^{-20}	2.51648×10^{-21}	1.08921×10^{-21}	1.54441×10^{-20}
0.6	1.32444×10^{-20}	2.51736×10^{-21}	1.09124×10^{-21}	1.54018×10^{-20}
0.7	1.3247×10^{-20}	2.52061×10^{-21}	1.09352×10^{-21}	1.53379×10^{-20}
0.8	1.32496×10^{-20}	2.52588×10^{-21}	1.09575×10^{-21}	1.52601×10^{-20}
0.9	1.32516×10^{-20}	2.53158×10^{-21}	1.09749×10^{-21}	1.51877×10^{-20}

7. Conclusions

This study successfully developed and analyzed a numerical algorithm to treat the fifth-order KdV-type equations, producing highly accurate results. The main idea was to introduce new shifted Horadam polynomials to act as basis functions. We established many basic formulas for these polynomials to design the proposed numerical method. In addition, some specific formulas and inequalities helped to investigate the convergence of the shifted Horadam approximate solutions in depth. We also offered numerical examples to confirm the method's applicability and usefulness in tackling complicated nonlinear problems in mathematical physics and related domains. To the best of our knowledge, this is the first time these polynomials have been used in the scope of numerical solutions of DEs. We plan to employ these polynomials to treat other types of DEs in the applied sciences.

Author Contributions: Conceptualization, W.M.A.-E. and A.G.A.; Methodology, W.M.A.-E., O.M.A. and A.G.A.; Software, A.G.A.; Validation, W.M.A.-E., O.M.A. and A.G.A.; Formal analysis, W.M.A.-E. and A.G.A.; Investigation, W.M.A.-E., O.M.A. and A.G.A.; Writing—original draft, A.G.A. and W.M.A.-E.; Writing—review and editing, W.M.A.-E., A.G.A. and O.M.A.; Visualization, A.G.A. and W.M.A.-E.; Supervision, W.M.A.-E.; Funding acquisition, W.M.A.-E. and O.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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