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Polynomial Identities for Binomial Sums of Harmonic Numbers of Higher Order

Takao Komatsu [1,*] and B. Sury [2]

1 Faculty of Education, Nagasaki University, Nagasaki 852-8521, Japan
2 Stat-Math Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560059, India; sury@isibang.ac.in
* Correspondence: komatsu@nagasaki-u.ac.jp or komatsu@zstu.edu.cn

Abstract: We study the formulas for binomial sums of harmonic numbers of higher order $\sum_{k=0}^{n} H_k^{(r)}\binom{n}{k}(1-q)^k q^{n-k} = H_n^{(r)} - \sum_{j=1}^{n} \mathcal{D}_r(n,j)\frac{q^j}{j}$. Recently, Mneimneh proved that $\mathcal{D}_1(n,j) = 1$. In this paper, we find several different expressions of $\mathcal{D}_r(n,j)$ for $r \geq 1$.

Keywords: polynomial identities; harmonic numbers; determinant; Bell polynomials

MSC: 11B65; 11A07; 05A10; 11B50; 11B73

1. Introduction

For a positive integer r, define the n-th harmonic number of order r by

$$H_n^{(r)} := \sum_{i=1}^{n} \frac{1}{i^r}.$$

When $r = 1$, $H_n = H_n^{(1)}$ is the original harmonic number. In this paper, we study the formula

$$\sum_{k=0}^{n} H_k^{(r)} \binom{n}{k} (1-q)^k q^{n-k} = H_n^{(r)} - \sum_{j=1}^{n} \mathcal{D}_r(n,j) \frac{q^j}{j}. \tag{1}$$

In [1], for a positive integer n and $0 \leq q \leq 1$, it is shown that $\mathcal{D}_1(n,j) = 1$. Namely,

$$\sum_{k=0}^{n} H_k \binom{n}{k} (1-q)^k q^{n-k} = H_n - \sum_{j=1}^{n} \frac{q^j}{j}. \tag{2}$$

This relation is derived by the author from an interesting probabilistic analysis. The identity (2) is a generalization of

$$\sum_{k=0}^{n} H_k \binom{n}{k} = 2^n \left(H_n - \sum_{j=1}^{n} \frac{1}{j 2^j} \right),$$

which has been proved in [2] in the field of symbolic computation and in [3] in finite differences. The identity (2) is a special case of a general result of Boyadzhiev [4]:

$$\sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k} H_k = (a+b)^n H_n - \left(b(a+b)^{n-1} + \frac{b^2}{2}(a+b)^{n-2} + \cdots + \frac{b^n}{n} \right). \tag{3}$$

In addition, Boyadzhiev's main result (3) has been generalized in [5] to multiple harmonic-like numbers.

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The main aim of this paper is to show several different expressions of $\mathcal{D}_r(n,j)$ as no simple form has been found.

In fact, more different generalizations of (1) or (2) can be considered. For example, recently, in [6], the so-called hyperharmonic number generalizes the harmonic number of the order r in the formula. In [7], more generalized sums and their application to multiple polylogarithms are given. In [8], some expressions of Mneimneh-type binomial sums are established, involving multiple harmonic-type sums in terms of finite sums of Stirling numbers, Bell numbers, and some related variables, and a conjecture is proposed. Then, in [9], this conjecture is resolved and generalized, and the transformation of generalized Mneimneh-like sums is presented. When we generalize too much, a lot of interesting and essential properties may be lost. Therefore, we do not consider further generalized harmonic numbers in this paper.

2. Observation

By using the harmonic numbers to express $\mathcal{D}_r(n,j)$, for $1 \le r \le 7$, we can manually obtain the following. Some initial trials for $r = 1, 2, 3$ are given below.

$$
\begin{aligned}
\mathfrak{D}_1(n,j) &= 1\,,\\
\mathfrak{D}_2(n,j) &= H_n - H_{n-j}\,,\\
\mathfrak{D}_3(n,j) &= \frac{(H_n - H_{n-j})^2}{2} + \frac{H_n^{(2)} - H_{n-j}^{(2)}}{2}\,,\\
\mathfrak{D}_4(n,j) &= \frac{(H_n - H_{n-j})^3}{6} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})}{2} + \frac{H_n^{(3)} - H_{n-j}^{(3)}}{3}\,,\\
\mathcal{D}_5(n,j) &= \frac{(H_n - H_{n-j})^4}{4!} + \frac{(H_n - H_{n-j})^2(H_n^{(2)} - H_{n-j}^{(2)})}{4}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(3)} - H_{n-j}^{(3)})}{3} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})^2}{8} + \frac{H_n^{(4)} - H_{n-j}^{(4)}}{4}\,,\\
\mathcal{D}_6(n,j) &= \frac{(H_n - H_{n-j})^5}{5!} + \frac{(H_n - H_{n-j})^3(H_n^{(2)} - H_{n-j}^{(2)})}{12}\\
&\quad + \frac{(H_n - H_{n-j})^2(H_n^{(3)} - H_{n-j}^{(3)})}{6} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})^2}{8}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(4)} - H_{n-j}^{(4)})}{4} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(3)} - H_{n-j}^{(3)})}{6}\\
&\quad + \frac{H_n^{(5)} - H_{n-j}^{(5)}}{5}\,,\\
\mathcal{D}_7(n,j) &= \frac{(H_n - H_{n-j})^6}{6!} + \frac{(H_n - H_{n-j})^4(H_n^{(2)} - H_{n-j}^{(2)})}{48}\\
&\quad + \frac{(H_n - H_{n-j})^3(H_n^{(3)} - H_{n-j}^{(3)})}{18} + \frac{(H_n - H_{n-j})^2(H_n^{(2)} - H_{n-j}^{(2)})^2}{16}\\
&\quad + \frac{(H_n - H_{n-j})^2(H_n^{(4)} - H_{n-j}^{(4)})}{8} + \frac{(H_n - H_{n-j})(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(3)} - H_{n-j}^{(3)})}{6}\\
&\quad + \frac{(H_n - H_{n-j})(H_n^{(5)} - H_{n-j}^{(5)})}{5} + \frac{(H_n^{(2)} - H_{n-j}^{(2)})^3}{48}\\
&\quad + \frac{(H_n^{(2)} - H_{n-j}^{(2)})(H_n^{(4)} - H_{n-j}^{(4)})}{8} + \frac{(H_n^{(3)} - H_{n-j}^{(3)})^2}{18} + \frac{H_n^{(6)} - H_{n-j}^{(6)}}{6}\,.
\end{aligned}
$$

It is interesting to observe that the number of terms of each of the right-hand sides of $\mathcal{D}_r(n,j)$ is equal to the number of partitions of r ($1 \le r \le 7$), respectively. In addition, the same terms of generalized harmonic numbers appear in [10,11]

$$
\begin{aligned}
&\sum_{n=1}^\infty \frac{H_n}{(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^\infty \frac{(H_n)^2 - H_n^{(2)}}{2(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^\infty \frac{(H_n)^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}}{3!(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^\infty \frac{(H_n)^4 - 6(H_n)^2 H_n^{(2)} + 8H_n H_n^{(3)} + 3(H_n^{(2)})^2 - 6H_n^{(4)}}{4!(n+1)(n+2)} = 1\,,\\
&\sum_{n=1}^\infty \frac{1}{5!(n+1)(n+2)}\Big((H_n)^5 - 10(H_n)^3 H_n^{(2)} + 20(H_n)^2 H_n^{(3)}\\
&\quad + 15H_n(H_n^{(2)})^2 - 30H_n H_n^{(4)} - 20H_n^{(2)}H_n^{(3)} + 24H_n^{(5)}\Big) = 1\,,\\
&\sum_{n=1}^\infty \frac{1}{6!(n+1)(n+2)}\Big((H_n)^6 - 15(H_n)^4 H_n^{(2)} + 40(H_n)^3 H_n^{(3)}\\
&\quad + 45(H_n)^2(H_n^{(2)})^2 - 90(H_n)^2 H_n^{(4)} - 120H_n H_n^{(2)} H_n^{(3)} + 144H_n H_n^{(5)}\\
&\quad - 15(H_n^{(2)})^3 + 90H_n^{(2)}H_n^{(4)} + 40(H_n^{(3)})^2 - 120H_n^{(5)}\Big) = 1\,.
\end{aligned}
$$

Some Initial Trials

For observation, we calculate the expressions of $\mathcal{D}_r(n,j)$ for small r, one by one, using the method in [12]. Here, we mention the cases for $r = 1, 2, 3$.

When $r = 1$, we find the following relation. Thus, by Theorem 1 in the next section, we can obtain $\mathfrak{D}_1(n,j) = 1$.

Lemma 1.

$$\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l} = 1\,. \tag{4}$$

When $r = 2$, we find the following relation. Here, $(n)_j = n(n-1)\cdots(n-j+1)$ ($j \ge 1$) is the falling factorial with $(n)_0 = 1$, and $\left[{n \atop k}\right]$ denotes the (unsigned) Stirling number of the first kind, arising from the relation $(x)_n = \sum_{k=0}^n (-1)^{n-k}\left[{n \atop k}\right]x^k$.

Lemma 2.

$$
\begin{aligned}
\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{n-l} &= H_n - H_{n-j}\\
&= \frac{1}{(n)_j}\sum_{\nu=0}^{j-1}(-1)^{j-\nu-1}(\nu+1)\left[{j \atop \nu+1}\right]n^\nu\,.
\end{aligned} \tag{5}
$$

Note that

$$\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{n-l} \ne \frac{l+1}{(n)_j}\left[{j \atop l+1}\right]n^l\,.$$

Hence, we have the following formula.

Corollary 1.

$$\sum_{k=0}^{n} H_k^{(2)} \binom{n}{k} (1-q)^k q^{n-k} = H_n^{(2)} - \sum_{j=1}^{n} (H_n - H_{n-j}) \frac{q^j}{j}$$
$$= H_n^{(2)} - \sum_{j=1}^{n} \left(\frac{1}{(n)_j} \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} (\nu+1) \begin{bmatrix} j \\ \nu+1 \end{bmatrix} n^\nu \right) \frac{q^j}{j} . \quad (6)$$

Proof of Lemma 1. Put

$$A(n,j) = \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l},$$
$$B(n,j) = \sum_{l=0}^{j} (-1)^{j-l} \binom{n-l}{n-j} \binom{n}{l}.$$

Since

$$B(n+1,j) = \frac{n+1}{n-j+1} B(n,j) = \frac{(n+1)n}{(n-j+1)(n-j)} B(n-1,j)$$
$$= \cdots = \frac{(n+1)n\cdots(j+1)}{(n-j+1)!} B(j,j)$$
$$= \frac{(n+1)!}{(n-j+1)!j!} \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l}$$
$$= \binom{n+1}{j} (1-1)^j = 0,$$

we have

$$A(n,j+1) - A(n,j) = B(n,j) = 0.$$

Hence, we obtain

$$A(n,j) = A(n,j-1) = \cdots = A(n,1) = (-1)^0 \binom{n-1}{n-1} \binom{n}{0} = 1.$$

□

Proof of Lemma 2. The Formula (5) is yielded from the definition of the Stirling numbers of the first kind:

$$(x)_j = \sum_{k=0}^{j} (-1)^{j-k} \begin{bmatrix} j \\ k \end{bmatrix} x^k$$
$$= \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} \begin{bmatrix} j \\ \nu+1 \end{bmatrix} x^{\nu+1} \quad (\text{if } j \geq 1).$$

Differentiatingboth sides with respect to x gives

$$(x)_j \sum_{l=0}^{j-1} \frac{1}{x-l} = \sum_{\nu=0}^{j-1} (-1)^{j-\nu-1} (\nu+1) \begin{bmatrix} j \\ \nu+1 \end{bmatrix} x^\nu .$$

Thus, the right-hand side of (5) is equal to

$$\sum_{l=0}^{j-1} \frac{1}{n-l} = H_n - H_{n-j} .$$

Put the left-hand side of (5) as

$$C(n,j) := \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l} \frac{1}{n-l} .$$

Then,

$$\begin{aligned}
&C(n,j) - C(n,j-1) \\
&= \sum_{l=0}^{j-2} (-1)^{j-l-1} \left(\binom{n-l-1}{n-j} \binom{n}{l} + \binom{n-l-1}{n-j+1} \binom{n}{l} \right) \frac{1}{n-l} \\
&\quad + \binom{n}{j-1} \frac{1}{n-j+1} \\
&= \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l}{n-j+1} \binom{n}{l} \frac{1}{n-l} \\
&= \binom{n}{j-1} \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{j-1}{l} \frac{1}{n-l} .
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{j-1}{l} \frac{1}{n-l} \\
&= \int \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{j-1}{l} x^{n-l-1} dx \bigg|_{x=1} \\
&= \int x^{n-1} \left(1 - \frac{1}{x}\right)^{j-1} dx \bigg|_{x=1} \\
&= \left(1 - \frac{1}{x}\right)^{j} \frac{{}_2F_1(-j+1, n-j+1; n-j+2; x)}{(x-1)^j (n-j+1)} \bigg|_{x=1} \\
&= \frac{\Gamma(j)\Gamma(n-j+2)}{(n-j+1)\Gamma(n+1)} = \frac{(j-1)!(n-j)!}{n!} ,
\end{aligned}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function. Hence,

$$C(n,j) - C(n,j-1) = \frac{1}{n-j+1} .$$

Therefore,

$$\begin{aligned}
C(n,j) &= C(n,j-1) + \frac{1}{n-j+1} \\
&= C(n,j-2) + \frac{1}{n-j+2} + \frac{1}{n-j+1} \\
&= \cdots \\
&= C(n,1) + \frac{1}{n-1} + \cdots + \frac{1}{n-j+2} + \frac{1}{n-j+1} \\
&= \sum_{l=0}^{j-1} \frac{1}{n-l} .
\end{aligned} \tag{7}$$

□

When $r = 3$, we have the following.

Lemma 3.

$$\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^2}$$
$$=\frac{(H_n-H_{n-j})^2}{2}+\frac{H_n^{(2)}-H_{n-j}^{(2)}}{2}.$$

Therefore, we have the following formula.

Corollary 2.

$$\sum_{k=0}^{n}H_k^{(3)}\binom{n}{k}(1-q)^k q^{n-k}$$
$$=H_n^{(3)}-\sum_{j=1}^{n}\left(\frac{(H_n-H_{n-j})^2}{2}+\frac{H_n^{(2)}-H_{n-j}^{(2)}}{2}\right)\frac{q^j}{j}.$$

Proof of Lemma 3. Put

$$D(n,j):=\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^2}.$$

Then,

$$D(n,j)-D(n,j-1)=\binom{n}{j-1}\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{j-1}{l}\frac{1}{(n-l)^2}. \tag{8}$$

Weshall prove that

$$D(n,j)-D(n,j-1)=\frac{1}{n-j+1}(H_n-H_{n-j}). \tag{9}$$

By (9), we obtain

$$\begin{aligned}D(n,j)&=\left(\frac{H_n}{n-j+1}-\frac{H_{n-j}}{n-j+1}\right)+\left(\frac{H_n}{n-j+2}-\frac{H_{n-j+1}}{n-j+2}\right)\\&\quad+\cdots+\left(\frac{H_n}{n-1}-\frac{H_{n-2}}{n-1}\right)+\left(\frac{H_n}{n}-\frac{H_{n-1}}{n}\right)\\&=H_n\left(\frac{1}{n-j+1}+\frac{1}{n-j+2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right)\\&\quad-\left(\frac{H_{n-j}}{n-j+1}+\frac{H_{n-j+1}}{n-j+2}+\cdots+\frac{H_{n-2}}{n-1}+\frac{H_{n-1}}{n}\right)\\&=H_n(H_n-H_{n-j})-\left(\frac{H_n^2-H_n^{(2)}}{2}-\frac{H_{n-j}^2-H_{n-j}^{(2)}}{2}\right)\\&=\frac{(H_n-H_{n-j})^2}{2}+\frac{H_n^{(2)}-H_{n-j}^{(2)}}{2}.\end{aligned}$$

In order to prove (9), we put

$$E(n,j)=(n-j+1)\big(D(n,j)-D(n,j-1)\big).$$

Then, by (8) and Lemma 1 (4), we have

$$\begin{aligned}
&E(n,j)-E(n,j-1)\\
&=\frac{1}{n-j+1}\sum_{l=0}^{j-2}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}+\frac{1}{n-j+1}\binom{n}{j-1}\\
&=\frac{1}{n-j+1}\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}=\frac{1}{n-j+1}\,.
\end{aligned}$$

Hence, by $D(n,1)=1/n^2$, we obtain

$$\begin{aligned}
D(n,j)-D(n,j-1)&=\frac{E(n,j)}{n-j+1}\\
&=\frac{1}{n-j+1}\left(\frac{1}{n-j+1}+\frac{1}{n-j+2}+\cdots+\frac{1}{n-1}+E(n,1)\right)\\
&=\frac{1}{n-j+1}(H_n-H_{n-j})\,.
\end{aligned}$$

which is the right-hand side of (9). □

3. Expressions (Main Results)

Let n, j, r be positive integers.

Theorem 1.

$$\mathcal{D}_r(n,j)=\sum_{l=0}^{j-1}(-1)^{j-l-1}\binom{n-l-1}{n-j}\binom{n}{l}\frac{1}{(n-l)^{r-1}}\,.$$

Theorem 2. *For $r \geq 1$,*

$$\mathcal{D}_{r+1}(n,j)=\sum_{j_1=1}^{j}\sum_{j_2=1}^{j_1}\cdots\sum_{j_r=1}^{j_{r-1}}\frac{1}{(n-j_1+1)(n-j_2+1)\cdots(n-j_r+1)}\,.$$

$\mathcal{D}_r(n,j)$ $(r \geq 2)$ can be expressed in terms of the determinant [13] ([Ch. I S2]). See also [14,15].

Theorem 3.

$$\mathcal{D}_{r+1}(n,j)$$
$$=\frac{1}{r!}\begin{vmatrix}
H_n-H_{n-j} & -1 & 0 & \cdots & 0\\
H_n^{(2)}-H_{n-j}^{(2)} & H_n-H_{n-j} & -2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
H_n^{(r-1)}-H_{n-j}^{(r-1)} & H_n^{(r-2)}-H_{n-j}^{(r-2)} & H_n^{(r-3)}-H_{n-j}^{(r-3)} & \cdots & -r+1\\
H_n^{(r)}-H_{n-j}^{(r)} & H_n^{(r-1)}-H_{n-j}^{(r-1)} & H_n^{(r-2)}-H_{n-j}^{(r-2)} & \cdots & H_n-H_{n-j}
\end{vmatrix}.$$

Remark 1. *By using the inversion formula (see, e.g., ([Lemma 1] [16]), ([13] (p. 28)) regarding (12) below, we also have*

$$
(-1)^{r-1}(H_n^{(r)} - H_{n-j}^{(r)}) = \begin{vmatrix} \mathcal{D}_2(n,j) & 1 & 0 & \cdots & 0 \\ 2\mathcal{D}_3(n,j) & \mathcal{D}_2(n,j) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (r-1)\mathcal{D}_r(n,j) & \mathcal{D}_{r-1}(n,j) & \mathcal{D}_{r-2}(n,j) & \cdots & 1 \\ r\mathcal{D}_{r+1}(n,j) & \mathcal{D}_r(n,j) & \mathcal{D}_{r-1}(n,j) & \cdots & \mathcal{D}_2(n,j) \end{vmatrix}.
$$

$\mathcal{D}_r(n,j)$ ($r \ge 2$) can be expressed by a combinatorial sum ([Proposition 1 (17)] [11]):

Theorem 4.

$$
\mathcal{D}_{r+1}(n,j) = \sum_{i_1+2i_2+3i_3+\cdots=r} \frac{1}{i_1!i_2!i_3!\cdots} \left(\frac{H_n - H_{n-j}}{1}\right)^{i_1} \left(\frac{H_n^{(2)} - H_{n-j}^{(2)}}{2}\right)^{i_2} \left(\frac{H_n^{(3)} - H_{n-j}^{(3)}}{3}\right)^{i_3} \cdots.
$$

Remember that the (complete exponential) Bell polynomial $\mathbf{Y}_n(x_1, x_2, \dots, x_n)$ is defined by

$$
\exp\left(\sum_{m=1}^\infty x_m \frac{t^m}{m!}\right) = 1 + \sum_{n=1}^\infty \mathbf{Y}_n(x_1, x_2, \dots, x_n) \frac{t^n}{n!}
$$

(see, e.g., ([Ch.3.3] [17])). That is,

$$
\mathbf{Y}_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^n \sum \frac{n!}{i_1!i_2!\cdots i_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{i_1} \left(\frac{x_2}{2!}\right)^{i_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{i_{n-k+1}}
$$

with $\mathbf{Y}_0 = 1$. Here, the second sum satisfies the following conditions:

$$
i_1 + 2i_2 + 3i_3 + \cdots + (n-k+1)i_{n-k+1} = n, \quad i_1 + i_2 + i_3 + \cdots = k.
$$

Theorem 5. *For $r \ge 1$, we have*

$$
\mathcal{D}_{r+1}(n,j) = \frac{1}{r!} \mathbf{Y}_r\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}), \dots\big).
$$

4. Proof

Proof of Theorem 1. We shall show that

$$
\sum_{k=0}^n H_k^{(r)} \binom{n}{k} (1-q)^k q^{n-k} = H_n^{(r)} - \sum_{j=1}^n \binom{n}{j} \left(\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{j-1}{l} \frac{1}{(n-l)^r}\right) q^j. \tag{10}
$$

We have

$$
\begin{aligned}
\sum_{k=0}^{n} H_k^{(r)} \binom{n}{k} (1-q)^k q^{n-k} &= \sum_{k=0}^{n} H_k^{(r)} \binom{n}{k} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} q^{n-l} \\
&= \sum_{l=0}^{n} q^{n-l} \binom{n}{l} \sum_{k=l}^{n} (-1)^{k-l} \binom{n-l}{n-k} H_k^{(r)} \\
&= \sum_{j=0}^{n} \binom{n}{j} q^j \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} H_{n-\nu}^{(r)} \\
&= H_n^{(r)} - \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} q^j \sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} \sum_{l=0}^{n-1} \frac{1}{(n-l)^r} \,.
\end{aligned}
$$

Since

$$
\sum_{\nu=0}^{l} (-1)^{\nu} \binom{j}{\nu} = (-1)^l \binom{j-1}{l} \quad \text{(proved by induction on } l (\geq 0))
$$

and

$$
\sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} = (1-1)^j = 0\,,
$$

we have

$$
\begin{aligned}
&\sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} \sum_{l=0}^{n-1} \frac{1}{(n-l)^r} \\
&= \sum_{l=0}^{j-1} \left(\sum_{\nu=0}^{l} (-1)^{\nu} \binom{j}{\nu} \right) \frac{1}{(n-l)^r} + \sum_{l=j}^{n-1} \left(\sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} \right) \frac{1}{(n-l)^r} \\
&= \sum_{l=0}^{j-1} (-1)^l \binom{j-1}{l} \frac{1}{(n-l)^r} \,.
\end{aligned}
$$

By (10),

$$
\begin{aligned}
\mathcal{D}_r(n,j) &= j! \binom{n}{j} \left(\sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{j-1}{l} \frac{1}{(n-l)^r} \right) \\
&= \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l} \frac{1}{(n-l)^{r-1}} \,.
\end{aligned}
$$

□

Proof of Theorem 2. By Theorem 1,

$$
\begin{aligned}
&\mathcal{D}_r(n,j) - \mathcal{D}_r(n,j-1) \\
&= \sum_{l=0}^{j-1} (-1)^{j-l-1} \left(\binom{n-l-1}{n-j} + \binom{n-l-1}{n-j+1} \right) \binom{n}{l} \frac{1}{(n-l)^{r-1}} \\
&= \frac{1}{n-j+1} \sum_{l=0}^{j-1} (-1)^{j-l-1} \binom{n-l-1}{n-j} \binom{n}{l} \frac{1}{(n-l)^{r-2}} \\
&= \frac{\mathcal{D}_{r-1}(n,j)}{n-j+1} \,.
\end{aligned}
$$

Hence, by $\mathcal{D}_r(n,0) = 0$, we have

$$\begin{aligned}
\mathcal{D}_{r+1}(n,j) &= \mathcal{D}_{r+1}(n,j-1) + \frac{\mathcal{D}_r(n,j)}{n-j+1} \\
&= \mathcal{D}_{r+1}(n,j-2) + \frac{\mathcal{D}_r(n,j-1)}{n-j+2} + \frac{\mathcal{D}_r(n,j)}{n-j+1} \\
&= \cdots = \sum_{j_1=1}^{j} \frac{\mathcal{D}_r(n,j_1)}{n-j_1+1} \\
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{\mathcal{D}_{r-1}(n,j_2)}{n-j_2+1} \\
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{1}{n-j_2+1} \sum_{j_3=1}^{j_2} \frac{\mathcal{D}_{r-2}(n,j_3)}{n-j_3+1} \\
&= \cdots \\
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{1}{n-j_2+1} \cdots \sum_{j_r=1}^{j_{r-1}} \frac{\mathcal{D}_1(n,j_r)}{n-j_r+1} \\
&= \sum_{j_1=1}^{j} \frac{1}{n-j_1+1} \sum_{j_2=1}^{j_1} \frac{1}{n-j_2+1} \cdots \sum_{j_r=1}^{j_{r-1}} \frac{1}{n-j_r+1} .
\end{aligned}$$

□

In order to prove Theorem 4 and Theorem 3, we need the following relations.

Lemma 4. *For the sequences* $\{p_n\}_{n\geq 1}$ *and* $\{h_n\}_{n\geq 1}$*, we have*

$$\begin{aligned}
(-1)^{n-1} p_n &= \begin{vmatrix} h_1 & 1 & 0 & \cdots & 0 \\ 2h_2 & h_1 & 1 & & \vdots \\ 3h_3 & h_2 & & \ddots & \\ \vdots & \vdots & & & 1 \\ nh_n & h_{n-1} & \cdots & h_2 & h_1 \end{vmatrix} \\
\iff n! h_n &= \begin{vmatrix} p_1 & -1 & 0 & \cdots & 0 \\ p_2 & p_1 & -2 & & \vdots \\ p_3 & p_2 & & \ddots & 0 \\ \vdots & \vdots & & & -n+1 \\ p_n & p_{n-1} & \cdots & p_2 & p_1 \end{vmatrix} \\
&= \sum_{\substack{i_1+2i_2+\cdots+ni_n \\ i_1,i_2,\ldots,i_n \geq 0}} \frac{n!}{i_1! i_2! \cdots i_n!} \left(\frac{p_1}{1}\right)^{i_1} \left(\frac{p_2}{2}\right)^{i_2} \cdots \left(\frac{p_n}{n}\right)^{i_n} .
\end{aligned}$$

Proof. The last identity is a simple modification of Trudi's formula (([18] (Volume 3, p. 214)), [19]):

$$\begin{vmatrix} a_1 & a_0 & \cdots & & 0 \\ a_2 & a_1 & \cdots & & \\ \vdots & \vdots & \ddots & & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$
$$= \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(i_1+\cdots+i_n)!}{i_1!\cdots i_n!}(-a_0)^{n-i_1-\cdots-i_n} a_1^{i_1} a_2^{i_2}\cdots a_n^{i_n}\,.$$

Notice that the expansion of the second determinant is equivalent to the following relation:

$$nh_n = \sum_{i=1}^{n} p_i h_{n-1} \quad \text{with} \quad h_0 = 1\,. \tag{11}$$

Byapplying the inversion formula (see, e.g., ([Lemma 1] [16])), we can obtain the first identity. □

Proof of Theorem 3. The determinant in Theorem 3 is equivalent to the recurrence relation:

$$\mathcal{D}_{r+1}(n,j) = \frac{1}{r}\sum_{i=1}^{r}(H_n^{(r-i+1)} - H_{n-j}^{(r-i+1)})\mathcal{D}_i(n,j)\,. \tag{12}$$

Byapplying the relation (11) in the first identity of the second part of Lemma 4 to (12), we can obtain the desired determinant identity. The identity of this remark can be given from the first part of Lemma 4. □

Proof of Theorem 4. The result follows from the second part of Lemma 4 by setting $h_r = \mathcal{D}_{r+1}(n,j)$ and $p_i = H_n^{(i)} - H_{n-j}^{(i)}$, satisfying (12). □

Proof of Theorem 5. Since Bell polynomials satisfy the following recurrence relation:

$$\mathbf{Y}_r(x_1,x_2,\ldots,x_r) = \sum_{i=1}^{r}\binom{r-1}{i-1} x_{r-i+1}\mathbf{Y}_{i-1}(x_1,x_2,\ldots,x_{i-1})$$

(see, e.g., [17]), by setting $x_\ell = (\ell-1)!(H_n^{(\ell)} - H_{n-j}^{(\ell)})$, we have

$$\begin{aligned} &\frac{1}{r!}\mathbf{Y}_r\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}),\ldots\big) \\ &= \frac{1}{r}\sum_{i=1}^{r}(H_n^{(r-i+1)} - H_{n-j}^{(r-i+1)}) \\ &\quad \times \frac{\mathbf{Y}_{i-1}\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}),\ldots\big)}{(i-1)!}\,. \end{aligned}$$

Since

$$\begin{aligned} \mathcal{D}_2(n,j) &= H_n - H_{n-j} \\ &= \mathbf{Y}_1\big(H_n - H_{n-j}, 1!(H_n^{(2)} - H_{n-j}^{(2)}), 2!(H_n^{(3)} - H_{n-j}^{(3)}),\ldots\big)\,, \end{aligned}$$

for $r \ge 1$, we can write the form in Theorem 5. □

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