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Bi-Fuzzy S-Approximation Spaces

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Abstract: The S-approximation spaces are significant extension of the rough set model and have been widely applied in intelligent decision-making. However, traditional S-approximation spaces are limited to two crisp universes, whereas bi-fuzzy universes (i.e., two distinct fuzzy domains) are more prevalent in practical applications. To bridge this gap, this study introduces the bi-fuzzy S-approximation spaces (BFS approximation spaces) as an advancement of knowledge space theory's fuzzy extension. Upper and lower approximation operators are formally defined, and the properties of BFS approximation spaces under various operations, such as complement, intersection and union are systematically explored. Special attention is given to a significant form of these operators, under which the monotonicity and complementary compatibility of BFS approximation spaces are rigorously analyzed. These results not only extend the theoretical framework of S-approximation spaces but also pave the way for further exploration of fuzzy extensions within knowledge space theory.

Keywords: S-approximation spaces; rough sets; fuzzy sets; bi-fuzzy S-approximation spaces (BFS approximation spaces); monotonicity; complementary compatibility

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1. Introduction

Given the inherent incompleteness and uncertainty of information, effectively managing uncertain information has become a crucial research focus across diverse applications in data processing and knowledge discovery. Introduced by Pawlak in 1982 [1], rough set theory offers a robust framework for addressing these issues [2]. By leveraging lower and upper approximations, it introduces an innovative method for analyzing information. Grounded in equivalence relations, the rough set model efficiently handles classification and reduction tasks within information systems and has found widespread applications in areas such as data mining, knowledge acquisition, feature selection, and medical diagnosis [3–6].

As the volume of uncertain data grows, Pawlak's rough set model faces notable constraints in handling complex information due to its reliance on single universes, strict equivalence relations, and inclusion-based operators, limiting its flexibility in processing uncertainty. To address these challenges, researchers have expanded the model by generalizing its universes, equivalence frameworks, and inclusion principles [7–10]. Wong et al. introduced compatible relations to propose the two-universe rough set model, effectively breaking the traditional rough set model's reliance on a single universe [7]. Building on this foundation, S-approximation spaces further extend the two-universe rough set model by incorporating multi-valued mappings, offering a more flexible approximation approach that significantly improves the model's adaptability to complex information.

Introduced by Hooshmandasl et al. [11] and grounded in Dempster's multi-valued mapping theory [12], S-approximation spaces provide a novel method for approximate analysis. Unlike traditional rough set models, S-approximation spaces remove restrictions on universes, equivalence relations, and inclusion relations, allowing rough sets and their extended models, such as two-universe rough sets [7], variable precision rough sets [13], and T-rough sets [14], to be represented within the S-approximation framework [12,15,16]. This adaptability and flexibility make S-approximation spaces particularly well-suited for processing uncertain data. In recent years, researchers have explored S-approximation spaces and their extensions from various perspectives. Hooshmandasl et al. examined the topological properties of S-approximation spaces, highlighting their potential to handle uncertainty independently of inclusion relations [17]. Shakiba et al. studied the relationship between evidence theory and S-approximation spaces, showing that belief structures can be derived from S-approximation spaces under monotonicity conditions [18]. The integration of three-way decision theory with S-approximation spaces has further expanded their applicability in complex decision support systems [15,19–22].

Originally proposed by Zadeh, fuzzy set theory is a fundamental mathematical tool for managing fuzzy information [23] and has found wide application in data science and artificial intelligence [24]. The integration of fuzzy set theory with rough set theory enhances their combined ability to address both fuzzy and uncertain information [25–27]. Shakiba et al. incorporated fuzzy set theory into S-approximation spaces by representing one universe W as a fuzzy power set and extending the mapping S from $\{0,1\}$ to $[0,1]$, thereby developing both the intuitionistic fuzzy S-approximation spaces [28] and the fuzzy S-approximation space model [29]. They analyzed the properties of the fuzzy S-approximation spaces model under conditions of partial monotonicity and complementary compatibility, and explored its applications in medical diagnosis [29]. However, the fuzzy S-approximation spaces are primarily constructed based on a single fuzzy universe, whereas two fuzzy universes are more common in practical applications, which limits the model's applicability in scenarios involving two fuzzy universes. Constructing an S-approximation spaces model based on two fuzzy universes and defining the corresponding approximation operators would further advance the integration of fuzzy sets with S-approximation spaces, expanding its range of applications.

In related studies, rough set theory and knowledge space theory (KST) have been shown to be closely connected [30,31], with each theory's structure and properties enriching the other [32,33]. KST, proposed by mathematical psychologists Doignon and Falmagne, is a significant model in mathematical psychology designed to assess a learner's knowledge state and provide personalized learning guidance. KST has found extensive applications in fields like assisted learning and adaptive testing [34–37]. Knowledge structures and skill maps are core concepts within knowledge space theory, and constructing knowledge structures and skill maps is a key issue in knowledge space research [38]. Recently, researchers have increasingly integrated KST with fuzzy set theory [38]. Traditional skill maps have been extended to fuzzy skill maps [39–41], and dichotomous knowledge structures have been expanded to polytomous knowledge structures [42]. Fuzzy knowledge structures, combined with fuzzy sets, form a special type of polytomous knowledge structures [43–46]. Zhou et al.'s research on fuzzy skill maps and fuzzy knowledge structures [43] provides a valuable perspective for constructing bi-fuzzy S-approximation spaces (BFS approximation spaces) models and defining corresponding approximation operators.

Building on the fuzzy extension model of knowledge space theory, this paper further extends S-approximation spaces to bi-fuzzy S-approximation spaces and defines the corresponding upper and lower approximation operators. This study investigates the properties of BFS approximation spaces under various decision mapping conditions, with a particular focus on an important form of the approximation operators, exploring the behavior and characteristics of BFS approximation spaces under monotonicity and complement compatibility conditions. BFS approximation spaces enable the modeling and analysis of scenarios involving two fuzzy universes, such as layered knowledge structures in e-learning, multi-criteria decision-making, etc. This extension provides new theoretical support for handling uncertain and fuzzy data in broader application scenarios.

The structure of this paper is as follows: Section 2 discusses the core concepts of fuzzy sets, S-approximation spaces, and fuzzy extensions in knowledge space theory. Section 3 elaborates on bi-fuzzy S-approximation spaces (BFS approximation spaces), defines the upper and lower approximation operators, and examines their properties under complement, intersection and union of BFS approximation spaces. Section 4 explores the monotonicity properties of BFS approximation spaces and provides rigorous proofs. Section 5 introduces the notion of complementary compatibility within BFS approximation spaces and conducts an in-depth analysis of its properties. Finally, Section 6 summarizes the study's findings and suggests potential future research directions.

2. Preliminaries

This section provides an overview of essential concepts, including fuzzy sets [39], S-approximation spaces [11,29], and the fuzzy extension of knowledge spaces [43].

Definition 1. ([39]). *Let U be a nonempty set. If $A : U \to [0,1]$, then A is referred to as a fuzzy set. The collection of all fuzzy sets on U is denoted as $\mathcal{F}(U)$ and is expressed as $\mathcal{F}(U) = \{A | A : U \to [0,1]\}$. $\forall x \in U$, $A(x)$ represents the membership degree of x in the fuzzy set A. Generally, a fuzzy set A on U can be written as $A = \{(x, A(x)) | x \in U\}$.*

Definition 2. ([11]). *Let $G = (U, W, T, S)$ be a quadruple, where U and W are nonempty finite sets, 2^W denotes the power set of W, $T : U \to 2^W$ is a knowledge mapping, and $S : 2^W \times 2^W \to \{0,1\}$ is a decision mapping. Then, G is called an S-approximation space. $\forall A \in 2^W$, the lower and upper approximations of A are defined as:*

$$\underline{G}(A) = \{x \in U | S(T(x), A) = 1\},$$
$$\overline{G}(A) = \{x \in U | S(T(x), A^c) = 0\}.$$

Here, A^c denotes the complement of A with respect to W.

In essence, S captures the decision criterion for inclusion relations or degree of overlap in rough set approximations by extending classical equivalence relations [11].

Following this, we introduce the notions of complement, intersection, and union for S-approximation spaces.

Definition 3. ([11]). *Let $G = (U, W, T, S)$ be an S-approximation space. Define $H = (U, W, T, S')$ as the complement of G, where $\forall A, B \in 2^W$, $S'(A, B) = 1 - S(A, B)$.*

Definition 4. ([11]). *Let W be a nonempty finite set, and define:*

$$\mathbb{S}_S(W) = \left\{ S \,\middle|\, S : 2^W \times 2^W \to \{0,1\} \right\}.$$

For $S_1, S_2 \in \mathbb{S}_S(W)$ and $A, B \in 2^W$, define:

$$(S_1 \wedge S_2)(A,B) = S_1(A,B) \wedge S_2(A,B) = min\{S_1(A,B), S_2(A,B)\},$$
$$(S_1 \vee S_2)(A,B) = S_1(A,B) \vee S_2(A,B) = max\{S_1(A,B), S_2(A,B)\}.$$

Definition 5. ([11]). *Let $G_1 = (U, W, T, S_1)$ and $G_2 = (U, W, T, S_2)$ be S-approximation spaces. The intersection $G_1 \wedge G_2$ and union $G_1 \vee G_2$ of G_1 and G_2 are defined as follows:*

$$(G_1 \wedge G_2) = (U, W, T, S_1 \wedge S_2),$$
$$(G_1 \vee G_2) = (U, W, T, S_1 \vee S_2).$$

Next, we introduce relevant concepts of fuzzy S-approximation spaces.

Definition 6. ([29]). *Let $FG = (U, W, T, S)$ be a quadruple, where U and W are nonempty finite sets, $T : U \to \mathcal{F}(W)$, and $S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0,1]$, with $\mathcal{F}(W) = \{A | A : W \to [0,1]\}$. Then, FG is called a fuzzy S-approximation space. $\forall A \in \mathcal{F}(W)$, the lower and upper approximations of A are defined as follows:*

$$\underline{FG}(A)(x) = S(T(x), A),$$
$$\overline{FG}(A)(x) = 1 - S(T(x), A^c).$$

Definition 7. ([29]). *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space, and let (a, r) be a pair of thresholds where $0 \le r < a \le 1$. The lower and upper approximations of a set $A \in \mathcal{F}(W)$ with respect to these thresholds are defined as follows:*

$$\underline{FG_a}(A) = \{x \in U | S(T(x), A) \ge a\},$$
$$\overline{FG_r}(A) = \{x \in U | S(T(x), A^c) \le r\}.$$

The properties of the lower and upper approximation operators in fuzzy S-approximation spaces primarily depend on decision mappings. By investigating different decision mappings, two special types of fuzzy S-approximation spaces have been identified: monotonic fuzzy approximation spaces and weakly complement-compatible fuzzy approximation spaces.

Definition 8. ([29]). *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space. If $\forall A, B, X \in \mathcal{F}(W)$, the following condition holds:*

$$A \subseteq B \Rightarrow S(X, A) \le S(X, B),$$

then FG is called a monotonic fuzzy S-approximation space.

Definition 9. ([29]). *Let $FG = (U, W, T, S)$ be a fuzzy S-approximation space. If $\forall x \in U$ and $A \in \mathcal{F}(W)$, the following condition holds:*

$$S(T(x), A) + S(T(x), A^c) \le 1,$$

then FG is called a weakly complement-compatible fuzzy S-approximation space.

Below, we introduce the concepts of skill mapping, fuzzy knowledge state, and fuzzy skill mapping in KST.

Definition 10. ([43]). *A skill mapping is defined as the triple $(Q \times L^+, S, \tau)$, where Q is a nonempty finite set of items, $Q \times L^+ = \{(q,l) | q \in Q,\ l \in L^+\}$, $L \subseteq [0,1]$ with $0, 1 \in L$, and $L^+ = L \setminus \{0\} \subseteq (0,1]$, S is a nonempty finite set of skills, and $\tau : Q \times L^+ \to 2^S \setminus \{\varnothing\}$.*

Definition 11. ([43]). *Let $(Q \times L^+, S, \tau)$ be a skill mapping and $M \in 2^S \setminus \{\varnothing\}$. Define $K = \{(q, K(q)) | q \in Q\}$, where the fuzzy knowledge state of item q under the conjunctive model is $K(q) = \sup\{\, l \in L^+ | \tau(q, l) \subseteq M\}$. For convenience, $\tau(q, l)$ is used to represent $\tau((q, l))$. The conjunctive model assumes that "all the skills associated with an item are necessary for solving it."*

Definition 12. ([43]). *Let $(Q \times L^+, S, \tau)$ be a skill mapping and $M \in 2^S \setminus \{\varnothing\}$. Define $K = \{(q, K(q)) | q \in Q\}$, where the fuzzy knowledge state of item q under the disjunctive model is $K(q) = \sup\{\, l \in L^+ | \tau(q, l) \cap M \neq \varnothing\}$. The disjunctive model assumes that "having any of the skills associated with an item is sufficient for solving it."*

Definition 13. ([43]). *A fuzzy skill mapping is defined as the triple $(Q \times L^+, S, \tau)$, where Q is a nonempty finite set of items, S is a nonempty finite set of skills, $L \subseteq [0, 1]$ with $0, 1 \in L$, $L^+ = L \setminus \{0\} \subseteq (0, 1]$, and $\tau : Q \times L^+ \to \mathcal{F}(S) \setminus \{\varnothing\}$.*

3. Bi-Fuzzy S-Approximation Spaces

By simultaneously handling two fuzzy universes, bi-fuzzy S-approximation spaces surmount the single-fuzzy-universe constraint, making them suitable for layered or multi-criteria domains.

In this section, building upon the fuzzy extension of knowledge space theory, we present the concept of bi-fuzzy S-approximation spaces and establish the corresponding lower and upper approximation operators. Additionally, we explore their properties under various decision mappings.

Definition 14. *The quadruple $BFG = (U \times L^+, W, T, S)$ is defined as follows: U and W are nonempty finite sets, $T : U \times L^+ \to \mathcal{F}(W)$ is a knowledge mapping, and $S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0, 1]$ is a decision mapping, where $\mathcal{F}(W) = \{A | A : W \to [0, 1]\}$, $U \times L^+ = \{(x, l) | x \in U,\ l \in L^+\}$ with $L \subseteq [0, 1]$, $0, 1 \in L$, and $L^+ = L \setminus \{0\} \subseteq (0, 1]$. In this study, L is set to be finite; therefore, L^+ is also finite. The quadruple BFG is called a bi-fuzzy S-approximation space, abbreviated as BFS approximation space.*

Remark 1. *The crisp set U can be regarded as a special fuzzy set, where the membership degree of $x \in U$ is 1. Therefore, if $L = \{0, 1\}$, then $U \times L^+ = \{(x, 1) | x \in U\} = U$. Thus, when $L = \{0, 1\}$, the bi-fuzzy S-approximation space degenerates into a fuzzy S-approximation space.*

Remark 2. *For convenience, $T(x, l)$ is used to represent $T((x, l))$.*

Example 1. *Let $U = \{x_1, x_2\}$, $W = \{y_1, y_2, y_3, y_4\}$, $L^+ = \{0.6, 1.0\}$, $U \times L^+ = \{(x_1, 0.6), (x_1, 1.0), (x_2, 0.6), (x_2, 1.0)\}$. The knowledge mapping $T : U \times L^+ \to \mathcal{F}(W)$ is defined as:*

$$T(x_1, 0.6) = \{(y_1, 0.5), (y_2, 0.3), (y_4, 1.0)\},$$
$$T(x_1, 1.0) = \{(y_1, 0.7), (y_2, 0.3), (y_3, 0.6), (y_4, 1.0)\},$$
$$T(x_2, 0.6) = \{(y_1, 0.2), (y_3, 0.55)\},$$
$$T(x_2, 1.0) = \{(y_1, 0.2), (y_2, 0.1), (y_3, 1.0)\}.$$

The decision mapping $S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0, 1]$ is defined as:

$$S(A, B) = \max_{w \in W} \left\{ \frac{A(w) + B(w)}{2} \right\}, \text{ where } A, B \in \mathcal{F}(W).$$

Then $(U \times L^+, W, T, S)$ is a BFS approximation space.

According to Definitions 11 and 12, let $(Q \times L^+, S, \tau)$ be a skill mapping and $M \in 2^S \setminus \{\varnothing\}$. Define $K = \{(q, K(q)) | q \in Q\}$, where the fuzzy knowledge state of item q under the conjunctive model is $K(q) = \sup\{\, l \in L^+ | \tau(q, l) \subseteq M\}$; under the disjunctive model, the fuzzy knowledge state of item q is $K(q) = \sup\{ l \in L^+ | \tau(q,l) \cap M \neq \varnothing \}$. In relation to the definition of S-approximation spaces and their upper and lower approximation operators in Definition 2, $A, B \in 2^W$, with the decision mapping $S(A,B) = \begin{cases} 1 \text{ if } A \subseteq B \\ 0 \text{ otherwise} \end{cases}$. Let $\tau = T$, $q = x$, $M = A$, where T and A are the terms defined in Definition 2.

For the conjunctive model, we have:

$$K(q) = \sup\{ l \in L^+ | \tau(q,l) \subseteq M \} = \sup\{ l \in L^+ | S(T(x,l), A) = 1 \}.$$

For the disjunctive model, we have:

$$K(q) = \sup\{ l \in L^+ | \tau(q,l) \cap M \neq \varnothing \} = \sup\{ l \in L^+ | S(T(x,l), A^c) = 0 \}.$$

This provides an important approach to defining lower and upper approximation operators. Subsequently, we extend this to a broader framework and define these operators $\forall A \in \mathcal{F}(W)$ within the BFS approximation space.

Definition 15. *Let $BFG = (U \times L^+, W, T, S)$ be a BFS approximation space, with $D_1, D_2 \subseteq [0,1]$, $x \in U$, and $\forall A \in \mathcal{F}(W)$; the lower and upper approximations of A are defined as follows:*
$\forall x \in U$,

$$\underline{BFG}(A)(x) = \sup\{ l \in L^+ | S(T(x,l), A) \in D_1 \},$$
$$\overline{BFG}(A)(x) = \sup\{ l \in L^+ | S(T(x,l), A^c) \in D_2 \}.$$

If $\forall l \in L^+$, $S(T(x,l), A) \notin D_1$, then $\underline{BFG}(A)(x) = 0$.
If $\forall l \in L^+$, $S(T(x,l), A^c) \notin D_2$, then $\overline{BFG}(A)(x) = 0$.

It is evident that specific choices of L, the decision mapping S, and the intervals D_1 and D_2 can yield special forms for the lower and upper approximations defined above.

Within this context, D_1 controls the degree of similarity between elements, while D_2 regulates the degree of dissimilarity. These parameters are independent and can be set based on specific application requirements, allowing for flexibility in defining the lower and upper approximations. Accurately determining the parameters D_1 and D_2 is essential for the effective functioning of the BFS approximation space. Three primary approaches can be employed to establish appropriate values for these parameters:

(1) Empirical Methods. Utilize historical data and empirical observations relevant to the specific application domain to estimate suitable values for D_1 and D_2. Statistical analysis can help identify patterns and optimal thresholds that best capture the degrees of similarity and dissimilarity.

(2) Domain-Specific Knowledge. Leverage expert knowledge and insights from the relevant field to set D_1 and D_2. Experts can provide valuable information on what constitutes significant similarity or dissimilarity within the context of the application, thereby guiding the selection of these parameters.

(3) Optimization Techniques. Apply optimization algorithms to determine the optimal values of D_1 and D_2 that maximize the performance metrics of the approximation space. Techniques such as gradient descent, genetic algorithms, or other heuristic methods can be utilized to fine-tune these parameters for enhanced performance.

Proposition 1. *Let $BFG = (U \times L^+, W, T, S)$ be a BFS approximation space. $\forall A \in \mathcal{F}(W)$, the following results hold:*

(1) If $L = \{0,1\}$*, then*

$$\underline{BFG}(A) = \{x \in U | S(T(x), A) \in D_1\},$$
$$\overline{BFG}(A) = \{x \in U | S(T(x), A^c) \in D_2\}.$$

(2) If $S : \mathcal{F}(W) \times \mathcal{F}(W) \to \{0,1\}$*, then*

$$\underline{BFG}(A) = \{x \in U \big| \exists l \in L^+, S(T(x,l), A) \in D_1\},$$
$$\overline{BFG}(A) = \{x \in U \big| \exists l \in L^+, S(T(x,l), A^c) \in D_2\}.$$

(3) If $L = \{0,1\}$*,* $S : \mathcal{F}(W) \times \mathcal{F}(W) \to \{0,1\}$*,* $D_1 = \{1\}$*, and* $D_2 = \{0\}$*, then*

$$\underline{BFG}(A) = \{x \in U | S(T(x), A) = 1\},$$
$$\overline{BFG}(A) = \{x \in U | S(T(x), A^c) = 0\}.$$

Remark 3. *It can be seen that if* $L = \{0,1\}$*,* $S : \mathcal{F}(W) \times \mathcal{F}(W) \to \{0,1\}$*,* $D_1 = \{1\}$*, and* $D_2 = \{0\}$*, the BFS approximation space is reduced to an S-approximation space. In this case, the lower and upper approximations defined in Definition 15 align with those of the S-approximation space. If* $L = \{0,1\}$*, the lower and upper approximations defined in Definition 15 generalize two specific forms of approximations in fuzzy S-approximation spaces.*

Next, we investigate the complement of a BFS approximation space and its properties.

Definition 16. *Let* $BFG = (U \times L^+, W, T, S)$ *be a BFS approximation space. Define* $BFH = (U \times L^+, W, T, S')$ *as the complement of* BFG*,* $\forall A, B \in \mathcal{F}(W)$*,* $S'(A,B) = 1 - S(A,B)$*.*

It is easy to see that $BFH = (U \times L^+, W, T, S')$ is also a BFS approximation space. Next, we define the complement of an interval D.

Definition 17. $\forall D \subseteq [0,1]$*, the complement interval of* D*, denoted by* $\overline{D}$*, is defined as:*

$$\overline{D} = \{1 - a | a \in D\}.$$

Proposition 2 describes the relationship between the upper and lower approximation operators of a BFS approximation space and its complement.

Proposition 2. *Let* $BFG = (U \times L^+, W, T, S)$ *be a BFS approximation space, and let* $BFH = (U \times L^+, W, T, S')$ *be the complement of* BFG*, where* $D_1, D_2 \subseteq [0,1]$ *and* $D_2 = \overline{D_1}$*.* $\forall A \in \mathcal{F}(W)$*, the following relationships hold:*

(1) $\underline{BFH}(A) = \overline{BFG}(A^c)$,
(2) $\overline{BFH}(A) = \underline{BFG}(A^c)$.

Proof. (1) Suppose $S'(T(x,l), A) \notin D_1$ for all $l \in L^+$ and $x \in U$. Then by Definition 15, $\underline{BFH}(A)(x) = 0$. Since $S'(T(x,l), A) = 1 - S(T(x,l), A)$ and it does not belong to D_1, we get $S(T(x,l), A) \notin \overline{D_1} = D_2$. Consequently, by Definition 15, if $S(T(x,l), A) \notin D_2$ for all $l \in L^+$, we must have $\overline{BFG}(A^c)(x) = 0$. Hence, in this case, $\underline{BFH}(A)(x) = \overline{BFG}(A^c)(x) = 0$.

Otherwise, there exists at least one $l \in L^+$ such that $S'(T(x,l), A) = 1 - S(T(x,l), A) \in D_1$. Equivalently, $S(T(x,l), A) \in \overline{D_1} = D_2$. Therefore, by taking the supremum over all such $l \in L^+$, we get

$$\underline{BFH}(A)(x) = \sup\{\, l \in L^+ \big| 1 - S(T(x,l), A) \in D_1\} = \sup\{\, l \in L^+ \big| S(T(x,l), A) \in D_2\}.$$

By Definition 15, this is exactly the condition for $\overline{BFG}(A^c)(x) = \sup\{l \in L^+ | S(T(x,l), A) \in D_2\}$. Hence, $\underline{BFH}(A)(x) = \overline{BFG}(A^c)(x)$.

Combining both cases, we conclude that $\underline{BFH}(A) = \overline{BFG}(A^c)$.

(2) Suppose $S'(T(x,l), A^c) \notin D_2$ for all $l \in L^+$ and $x \in U$. Then, $\overline{BFH}(A)(x) = 0$. Since $S'(T(x,l), A^c) = 1 - S(T(x,l), A^c)$ and it does not belong to D_2, we have $S(T(x,l), A^c) \notin \overline{D_2} = D_1$. Consequently, if $S(T(x,l), A^c) \notin D_1$ for all $l \in L^+$, it follows from Definition 15 that $\underline{BFG}(A^c)(x) = 0$. Hence, $\overline{BFH}(A)(x) = \underline{BFG}(A^c)(x) = 0$.

Otherwise, there exists at least one $l \in L^+$ such that $S'(T(x,l), A^c) = 1 - S(T(x,l), A^c) \in D_2$. Equivalently, $S(T(x,l), A^c) \in \overline{D_2} = D_1$. Therefore,

$$\overline{BFH}(A)(x) = \sup\{ l \in L^+ | S'(T(x,l), A^c) \in D_2\} = \sup\{ l \in L^+ | 1 - S(T(x,l), A^c) \in D_2\}.$$

Since $1 - S(T(x,l), A^c) \in D_2 \Leftrightarrow S(T(x,l), A^c) \in \overline{D_2} = D_1$, we see by Definition 15 that

$$\underline{BFG}(A^c)(x) = \sup\{ l \in L^+ | S(T(x,l), A^c) \in D_1\} \quad (1)$$

Thus, $\overline{BFH}(A)(x) = \underline{BFG}(A^c)(x)$.

Combining both cases, we conclude that $\overline{BFH}(A) = \underline{BFG}(A^c)$. □

Definition 18. *Let W be a nonempty finite set. Define:*

$$\mathbb{S}_S(W) = \{S | S : \mathcal{F}(W) \times \mathcal{F}(W) \to [0,1]\}.$$

For $S_1, S_2 \in \mathbb{S}_S(W)$, $A, B \in \mathcal{F}(W)$, and $D \subseteq [0,1]$, the operations of intersection and union for decision mappings are defined as:

$$(S_1 \wedge S_2)(A,B) = S_1(A,B) \wedge S_2(A,B),$$
$$(S_1 \vee S_2)(A,B) = S_1(A,B) \vee S_2(A,B).$$

Here, $S_1(A,B), S_2(A,B) \in D$, and therefore $(S_1 \wedge S_2)(A,B)$, $(S_1 \vee S_2)(A,B) \in D$.

Next, we introduce the concepts of intersection and union for BFS approximation spaces.

Definition 19. *Let $BFG_1 = (U \times L^+, W, T, S_1)$ and $BFG_2 = (U \times L^+, W, T, S_2)$ be BFS approximation spaces. Define the intersection $BFG_1 \wedge BFG_2$ and union $BFG_1 \vee BFG_2$ of BFG_1 and BFG_2 as:*

$$BFG_1 \wedge BFG_2 = (U \times L^+, W, T, S_1 \wedge S_2),$$
$$BFG_1 \vee BFG_2 = (U \times L^+, W, T, S_1 \vee S_2).$$

It is evident that both $BFG_1 \wedge BFG_2$ and $BFG_1 \vee BFG_2$ are BFS approximation spaces.

Proposition 3. *Let $BFG_1 = (U \times L^+, W, T, S_1)$ and $BFG_2 = (U \times L^+, W, T, S_2)$ be BFS approximation spaces. Define $BFG = (U \times L^+, W, T, S_1 \wedge S_2)$ and $BFM = (U \times L^+, W, T, S_1 \vee S_2)$, with $D_1, D_2 \subseteq [0,1]$. Then $\forall A \in \mathcal{F}(W)$, the following properties hold:*

(1) $\underline{BFG}(A) = \underline{BFG_1}(A) \cap \underline{BFG_2}(A)$,
(2) $\overline{BFG}(A) = \overline{BFG_1}(A) \cap \overline{BFG_2}(A)$,
(3) $\underline{BFM}(A) = \underline{BFG_1}(A) \cup \underline{BFG_2}(A)$,
(4) $\overline{BFM}(A) = \overline{BFG_1}(A) \cup \overline{BFG_2}(A)$.

Proof. (1) $\forall l \in L^+$, if $S_1(T(x,l), A) \notin D_1$, then $\underline{BFG_1}(A)(x) = 0$. Since $S(A,B) = S_1(A,B) \wedge S_2(A,B)$, it follows that $S(T(x,l), A) \notin D_1$, and thus $\underline{BFG}(A)(x) = 0$. Similarly, $\forall l \in L^+$, if $S_2(T(x,l), A) \notin D_1$, then $\underline{BFG_2}(A)(x) = 0$. Otherwise:

$$\underline{BFG}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l),A) \in D_1 \} =$$
$$\sup\{\, l \in L^+ \big| (S_1(T(x,l),A) \wedge S_2(T(x,l),A)) \in D_1 \} =$$
$$\sup\{\, l \in L^+ \big| S_1(T(x,l),A) \in D_1 \} \wedge \sup\{\, l \in L^+ \big| S_2(T(x,l),A) \in D_1 \} =$$
$$\underline{BFG_1}(A)(x) \wedge \underline{BFG_2}(A)(x).$$

Hence, $\underline{BFG}(A) = \underline{BFG_1}(A) \cap \underline{BFG_2}(A)$.

(2) $\forall l \in L^+$, if $S_1(T(x,l),A^c) \notin D_2$, then $\overline{BFG_1}(A)(x) = 0$. Since $S(A,B) = S_1(A,B) \wedge S_2(A,B)$, it follows that $S(T(x,l),A^c) \notin D_2$, and thus $\overline{DFG}(A)(x) = 0$. Similarly, if $S_2(T(x,l),A^c) \notin D_2$, then $\overline{BFG_2}(A)(x) = 0$. Otherwise:

$$\overline{BFG}(A)(x) = sup\{\, l \in L^+ \big| S(T(x,l),A^c) \in D_2 \} =$$
$$\sup\{\, l \in L^+ \big| (S_1(T(x,l),A^c) \wedge S_2(T(x,l),A^c)) \in D_2 \} = \sup\{\, l \in L^+ \big| S_1(T(x,l),A^c) \in D_2 \} \wedge$$
$$\sup\{\, l \in L^+ \big| S_2(T(x,l),A^c) \in D_2 \} = \overline{BFG_1}(A)(x) \wedge \overline{BFG_2}(A)(x).$$

Hence, $\overline{BFG}(A) = \overline{BFG_1}(A) \cap \overline{BFG_2}(A)$.

(3) $\forall l \in L^+$, if $S_1(T(x,l),A) \notin D_1$ and $S_2(T(x,l),A) \notin D_1$, then $\underline{BFG_1}(A)(x) = 0$ and $\underline{BFG_2}(A)(x) = 0$. Since $S(A,B) = S_1(A,B) \vee S_2(A,B)$, it follows that $S(T(x,l),A) \notin D_1$, and thus $\underline{BFM}(A)(x) = 0$. Otherwise:

$$\underline{BFM}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l),A) \in D_1 \} =$$
$$\sup\{\, l \in L^+ \big| (S_1(T(x,l),A) \vee S_2(T(x,l),A)) \in D_1 \} = \sup\{\, l \in L^+ \big| S_1(T(x,l),A) \in D_1 \} \vee$$
$$\sup\{\, l \in L^+ \big| S_2(T(x,l),A) \in D_1 \} = \underline{BFG_1}(A)(x) \vee \underline{BFG_2}(A)(x).$$

Hence, $\underline{BFM}(A) = \underline{BFG_1}(A) \cup \underline{BFG_2}(A)$.

(4) $\forall l \in L^+$, if $S_1(T(x,l),A^c) \notin D_2$ and $S_2(T(x,l),A^c) \notin D_2$, then $\overline{BFG_1}(A)(x) = 0$ and $\overline{BFG_2}(A)(x) = 0$. Since $S(A,B) = S_1(A,B) \vee S_2(A,B)$, it follows that $S(T(x,l),A^c) \notin D_2$, and thus $\overline{BFM}(A)(x) = 0$. Otherwise:

$$\overline{BFM}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l),A^c) \in D_2 \} =$$
$$\sup\{\, l \in L^+ \big| (S_1(T(x,l),A^c) \vee S_2(T(x,l),A^c)) \in D_2 \} =$$
$$sup\{\, l \in L^+ \big| S_1(T(x,l),A^c) \in D_2 \} \vee \sup\{\, l \in L^+ \big| S_2(T(x,l),A^c) \in D_2 \} =$$
$$\overline{BFG_1}(A)(x) \vee \overline{BFG_2}(A)(x).$$

Hence, $\overline{BFM}(A) = \overline{BFG_1}(A) \cup \overline{BFG_2}(A)$. □

Based on Proposition 3 and employing the method of induction, the following corollary is established.

Corollary 1. *Let $BFG_i = (U \times L^+, W, T, S_i)$ for $i = 1, 2 \cdots n$ be BFS approximation spaces, and define $BFG = \left(U \times L^+, W, T, \wedge_{i=1}^n S_i\right)$ and $BFM = \left(U \times L^+, W, T, \vee_{i=1}^n S_i\right)$, where $D_1, D_2 \subseteq [0,1]$. Then, the following statements hold $\forall A \in \mathcal{F}(W)$:*

(1) $\underline{BFG}(A) = \cap_{i=1}^n \underline{BFG_i}(A)$,
(2) $\overline{BFG}(A) = \cap_{i=1}^n \overline{BFG_i}(A)$,
(3) $\underline{BFM}(A) = \cup_{i=1}^n \underline{BFG_i}(A)$,
(4) $\overline{BFM}(A) = \cup_{i=1}^n \overline{BFG_i}(A)$.

In the definitions of the upper and lower approximation operators provided in Definition 15, the intervals D_1 and D_2 are general subsets of $[0,1]$. From the perspectives of rough sets and fuzzy set theory, it is observed that this model gains wider applicability when the intervals are specified as $D_1 = [a,1]$ and $D_2 = [0,b]$, where $a, b \in [0,1]$. Therefore, in subsequent discussions, we adopt these intervals and analyze the properties of the upper and lower approximation operators under this setting in BFS approximation spaces.

4. Monotonicity

This section introduces specific conditions for BFS approximation spaces to construct monotonic BFS approximation spaces and prove their properties. Throughout this section, the intervals for the upper and lower approximation operators are set as $D_1 = [a, 1]$ and $D_2 = [0, b]$, where $a, b \in [0, 1]$.

Definition 20. *Let $BFG = (U \times L^+, W, T, S)$ be a BFS approximation space. For $A, B, C \in \mathcal{F}(W)$ and $\forall l \in L^+$, the following conditions hold:*

(1) $A \subseteq B \Rightarrow S(T(x,l), A) \leq S(T(x,l), B)$,
(2) *If* $l_1 \leq l_2$, $S(T(x,l_1), C) \leq S(T(x,l_2), C)$.

Then, BFG is called a monotonic BFS approximation space.

Proposition 4. *Let $BFG = (U \times L^+, W, T, S)$ be a monotonic BFS approximation space. $\forall A, B \in \mathcal{F}(W)$, the following properties hold:*

(1) $A \subseteq B \Rightarrow \underline{BFG}(A) \subseteq \underline{BFG}(B)$,
(2) $A \subseteq B \Rightarrow \overline{BFG}(A) \subseteq \overline{BFG}(B)$,
(3) $\underline{BFG}(A \cup B) \supseteq \underline{BFG}(A) \cup \underline{BFG}(B)$,
(4) $\underline{BFG}(A \cap B) \subseteq \underline{BFG}(A) \cap \underline{BFG}(B)$,
(5) $\overline{BFG}(A \cup B) \supseteq \overline{BFG}(A) \cup \overline{BFG}(B)$,
(6) $\overline{BFG}(A \cap B) \subseteq \overline{BFG}(A) \cap \underline{BFG}(B)$.

Proof. (1) If $\forall l \in L^+$, $S_1(T(x,l), A) \notin D_1$, then by Definition 15, $\underline{BFG}(A)(x) = 0$. Consequently, $\underline{BFG}(A)(x) \leq \underline{BFG}(B)(x)$. Otherwise,

$$\underline{BFG}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l), A) \in D_1 \} = \sup\{\, l \in L^+ \big| S(T(x,l), A) \geq a \}.$$

Since BFG is monotonic,

$$\sup\{\, l \in L^+ \big| S(T(x,l), A) \geq a \} \leq \sup\{\, l \in L^+ \big| S(T(x,l), B) \geq a \} = \underline{BFG}(B)(x).$$

Thus, $A \subseteq B \Rightarrow \underline{BFG}(A) \subseteq \underline{BFG}(B)$.

(2) If $\forall l \in L^+$, $S_1(T(x,l), A^c) \notin D_2$, then by Definition 15, $\overline{BFG}(A)(x) = 0$. Consequently, $\overline{BFG}(A)(x) \leq \overline{BFG}(B)(x)$. Otherwise, since BFG is monotonic and $A \subseteq B$, we have $S(T(x,l), A) \leq S(T(x,l), B)$. This implies $B^c \subseteq A^c$, and $S(T(x,l), B^c) \leq S(T(x,l), A^c)$. Therefore,

$$\overline{BFG}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l), A^c) \leq b \} \leq \sup\{\, l \in L^+ \big| S(T(x,l), B^c) \leq b \} = \overline{BFG}(B)(x).$$

Thus, $A \subseteq B \Rightarrow \underline{BFG}(A) \subseteq \underline{BFG}(B)$.

(3) Since $A \subseteq A \cup B$, by Property (1), $\underline{BFG}(A) \subseteq \underline{BFG}(A \cup B)$. Similarly, $B \subseteq A \cup B \Rightarrow \underline{BFG}(B) \subseteq \underline{BFG}(A \cup B)$.

Therefore, $\underline{BFG}(A \cup B) \supseteq \underline{BFG}(A) \cup \underline{BFG}(B)$.

(4) Since $A \cap B \subseteq A$, by Property (1), $\underline{BFG}(A \cap B) \subseteq \underline{BFG}(A)$. Similarly, $A \cap B \subseteq B \Rightarrow \underline{BFG}(A \cap B) \subseteq \underline{BFG}(B)$.

Therefore, $\underline{BFG}(A \cap B) \subseteq \underline{BFG}(A) \cap \underline{BFG}(B)$.

(5) Since $A \subseteq A \cup B$, by Property (2), $\overline{BFG}(A) \subseteq \overline{BFG}(A \cup B)$. Similarly, $B \subseteq A \cup B \Rightarrow \overline{BFG}(B) \subseteq \overline{BFG}(A \cup B)$.

Therefore, $\overline{BFG}(A \cup B) \supseteq \overline{BFG}(A) \cup \overline{BFG}(B)$.

(6) Since $A \cap B \subseteq A$, by Property (2), $\overline{BFG}(A \cap B) \subseteq \overline{BFG}(A)$. Similarly, $A \cap B \subseteq B \Rightarrow \overline{BFG}(A \cap B) \subseteq \overline{BFG}(B)$.

Therefore, $\overline{BFG}(A \cap B) \subseteq \overline{BFG}(A) \cap \overline{BFG}(B)$. □

Based on Proposition 4 and the method of induction, the following corollary is derived.

Corollary 2. *Let* $BFG = (U \times L^{+}, W, T, S)$ *be a monotonic BFS approximation space.* $\forall A \in \mathcal{F}(W)$*, the following statements hold:*

(1) $\underline{BFG}\left(\cup_{i=1}^{n} A_i\right) \supseteq \cup_{i=1}^{n} \underline{BFG}(A_i)$,
(2) $\underline{BFG}\left(\cap_{i=1}^{n} A_i\right) \subseteq \cap_{i=1}^{n} \underline{BFG}(A_i)$,
(3) $\overline{BFG}\left(A \cup_{i=1}^{n} A_i\right) \supseteq \cup_{i=1}^{n} \overline{BFG}(A_i)$,
(4) $\overline{BFG}\left(\cap_{i=1}^{n} A_i\right) \subseteq \cap_{i=1}^{n} \overline{BFG}(A_i)$

Proof. (1) For $n = 2$, Proposition 4(3) implies:

$$\underline{BFG}(A_1 \cup A_2) \supseteq \underline{BFG}(A_1) \cup \underline{BFG}(A_2).$$

Assume the statement holds for $n = k$, i.e.,

$$\underline{BFG}\left(\cup_{i=1}^{k} A_i\right) \supseteq \cup_{i=1}^{k} \underline{BFG}(A_i).$$

For $n = k + 1$, let $A = \cup_{i=1}^{k} A_i$. Using Proposition 4(3),

$$\underline{BFG}\left(\cup_{i=1}^{k+1} A_i\right) = \underline{BFG}(A \cup A_{k+1}) \underline{\supseteq BFG}(A) \cup \underline{BFG}(A_{k+1}).$$

By the induction hypothesis,

$$\underline{BFG}(A) \supseteq \cup_{i=1}^{k} \underline{BFG}(A_i).$$

Thus, $\underline{BFG}\left(\cup_{i=1}^{k+1} A_i\right) \supseteq \cup_{i=1}^{k+1} \underline{BFG}(A_i)$.
This completes the induction step.
(2) For $n = 2$, Proposition 4(4) implies:

$$\underline{BFG}(A_1 \cap A_2) \subseteq \underline{BFG}(A_1) \cap \underline{BFG}(A_2).$$

Assume the statement holds for $n = k$, i.e.,

$$\underline{BFG}\left(\cap_{i=1}^{k} A_i\right) \subseteq \cap_{i=1}^{k} \underline{BFG}(A_i).$$

For $n = k + 1$, let $A = \cup_{i=1}^{k} A_i$. Using Proposition 4(4),

$$\underline{BFG}\left(\cap_{i=1}^{k+1} A_i\right) = \underline{BFG}(A \cap A_{k+1}) \subseteq \underline{BFG}(A) \cap \underline{BFG}(A_{k+1}).$$

By the induction hypothesis,

$$\underline{BFG}(A) \subseteq \cap_{i=1}^{k} \underline{BFG}(A_i).$$

Thus,$\underline{BFG}\left(\cap_{i=1}^{k+1} A_i\right) \subseteq \cap_{i=1}^{k+1} \underline{BFG}(A_i)$.
This completes the induction step.

The proofs of statements (3) and (4) follow similarly from Proposition 4(5) and (6), using induction on n. The base cases $n = 2$ hold by Proposition 4, and the induction steps are analogous to those for statements (1) and (2). □

Definition 21. *Let U be a nonempty finite set. Define:*

$$\ell_T(U) = \{T \,|\, T : U \times L^+ \to \mathcal{F}(W)\}.$$

This represents the set of all knowledge mappings defined on $U \times L^+$ and taking values in $\mathcal{F}(W)$.

Next, we introduce the concepts of intersection and union of BFS approximation spaces concerning different knowledge mappings.

To define the intersection and union of these BFS approximation spaces concerning different knowledge mappings, we introduce two operators, "$\diamond$" for intersection and "$\circ$" for union. For all $(x,l) \in U \times L^+$ and $w \in W$, $T_1, T_2 \in \ell_T(U)$, the operator "$\diamond$" aggregates the knowledge mappings by taking the pointwise minimum of their membership degrees:

$$((T_1 \diamond T_2)(x,l))(w) = \min\{(T_1(x,l))(w), (T_2(x,l))(w)\}.$$

Similarly, the operator "$\circ$" aggregates them by taking the pointwise maximum:

$$((T_1 \circ T_2)(x,l))(w) = \max\{(T_1(x,l))(w), (T_2(x,l))(w)\}.$$

Definition 22. *Let $BFG_1 = (U \times L^+, W, T_1, S)$ and $BFG_2 = (U \times L^+, W, T_2, S)$ be BFS approximation spaces, where $T_1, T_2 \in \ell_T(U)$. The intersection $BFG_1 \diamond BFG_2$ and the union $BFG_1 \circ BFG_2$ of BFG_1 and BFG_2, with respect to knowledge mappings, are defined as follows:*

$$BFG_1 \diamond BFG_2 = (U \times L^+, W, T_1 \diamond T_2, S),$$
$$BFG_1 \circ BFG_2 = (U \times L^+, W, T_1 \circ T_2, S).$$

These resulting spaces are also valid BFS approximation spaces since the aggregated knowledge mappings $T_1 \diamond T_2$ and $T_1 \circ T_2$ remain within the set of allowable knowledge mappings $\ell_T(U)$.

Definition 23. *For $BFG_{1\diamond 2} = BFG_1 \diamond BFG_2 = (U \times L^+, W, T_1 \diamond T_2, S)$, let $D_1 = [a,1]$, $D_2 = [0,b]$, and $x \in U$. $\forall A \in \mathcal{F}(W)$, the lower and upper approximations are defined as follows:*

$$\underline{BFG_{1\diamond 2}}(A)(x) = \sup\{\, l \in L^+ \,|\, (S(T_1(x,l), A) \wedge S(T_2(x,l), A)) \in D_1\},$$
$$\overline{BFG_{1\diamond 2}}(A)(x) = \sup\{\, l \in L^+ \,|\, (S(T_1(x,l), A^c) \wedge S(T_2(x,l), A^c)) \in D_2\}.$$

For $BFG_{1\circ 2} = BFG_1 \circ BFG_2 = (U \times L^+, W, T_1 \circ T_2, S)$, let $D_1 = [a,1]$, $D_2 = [0,b]$, and $x \in U$. $\forall A \in \mathcal{F}(W)$, the lower and upper approximations are defined as follows:

$$\underline{BFG_{1\circ 2}}(A)(x) = \sup\{\, l \in L^+ \,|\, (S(T_1(x,l), A) \vee S(T_2(x,l), A)) \in D_1\},$$
$$\overline{BFG_{1\circ 2}}(A)(x) = \sup\{\, l \in L^+ \,|\, (S(T_1(x,l), A^c) \vee S(T_2(x,l), A^c)) \in D_2\}.$$

Example 2. *Consider two BFS approximation spaces $BFG_1 = (U \times L^+, W, T_1, S)$and $BFG_2 = (U \times L^+, W, T_2, S)$ where $U = \{x_1, x_2\}$, $W = \{y_1, y_2\}$, $L^+ = \{0.5, 1.0\}$. The knowledge mappings T_1 and T_2 are given by:*

$$T_1(x_1, 0.5) = \{(y_1, 0.3), (y_2, 0.3)\},$$

$$T_1(x_1, 1.0) = \{(y_1, 0.5), (y_2, 0.5)\},$$

$$T_1(x_2, 0.5) = \{(y_1, 0.3), (y_2, 0.6)\},$$

$$T_1(x_2,1.0) = \{(y_1, 0.4), (y_2, 0.6)\}$$

$$T_2(x_1,0.5) = \{(y_1, 0.2), (y_2, 0.4)\},$$

$$T_2(x_1,1.0) = \{(y_1, 0.3), (y_2, 0.7)\},$$

$$T_2(x_2,0.5) = \{(y_1, 0.3), (y_2, 0.2)\},$$

$$T_2(x_2,1.0) = \{(y_1, 0.5), (y_2, 0.4)\}.$$

The decision mapping S is defined as:

$$S(A, B) = \max_{w \in W}\left\{\frac{A(w) + B(w)}{2}\right\}, \text{ where } A, B \in \mathcal{F}(W).$$

Knowledge Mappings for $BFG_1 \diamond BFG_2$ *and* $BFG_1 \circ BFG_2$ *are shown in Table 1.*

Table 1. Knowledge mappings for $BFG_1 \diamond BFG_2$ and $BFG_1 \circ BFG_2$.

(x,l)	w	$((T_1 \diamond T_2)(x,l))(w)$	$((T_1 \bigcirc T_2)(x,l))(w)$
$(x_1, 0.5)$	y_1	$\min\{0.3, 0.2\} = 0.2$	$\max\{0.3, 0.2\} = 0.3$
$(x_1, 0.5)$	y_2	$\min\{0.3, 0.4\} = 0.3$	$\max\{0.3, 0.4\} = 0.4$
$(x_1, 1.0)$	y_1	$\min\{0.5, 0.3\} = 0.3$	$\max\{0.5, 0.3\} = 0.5$
$(x_1, 1.0)$	y_2	$\min\{0.5, 0.7\} = 0.5$	$\max\{0.5, 0.7\} = 0.7$
$(x_2, 0.5)$	y_1	$\min\{0.3, 0.3\} = 0.3$	$\max\{0.3, 0.3\} = 0.3$
$(x_2, 0.5)$	y_2	$\min\{0.6, 0.2\} = 0.2$	$\max\{0.6, 0.2\} = 0.6$
$(x_2, 1.0)$	y_1	$\min\{0.4, 0.5\} = 0.4$	$\max\{0.4, 0.5\} = 0.5$
$(x_2, 1.0)$	y_2	$\min\{0.6, 0.4\} = 0.4$	$\max\{0.6, 0.4\} = 0.6$

Let $a = 0.5, b = 0.5$. *Then* $D_1 = [0.5, 1]$, $D_2 = [0, 0.5]$. *let* $A = \{(y_1, 0.6), (y_2, 0.4)\} \in \mathcal{F}(W)$, *thus* $A^c = \{(y_1, 0.4), (y_2, 0.6)\}$. *Decision mapping values for* T_1 *and* T_2 *with* A *and* A^c *are shown in Table 2.*

Table 2. Decision mapping values for T_1 and T_2 with A and A^c.

(x,l)	$S(T_1(x,l),A)$	$S(T_2(x,l),A)$	$S(T_1(x,l),A^c)$	$S(T_2(x,l),A^c)$
$(x_1, 0.5)$	0.45	0.45	0.45	0.50
$(x_1, 1.0)$	0.55	0.55	0.55	0.65
$(x_2, 0.5)$	0.50	0.45	0.60	0.40
$(x_2, 1.0)$	0.50	0.55	0.60	0.50

Here, $W = \{y_1, y_2\}$. *Hence, for each pair* (x, l),*we evaluate* $\frac{A(w)+B(w)}{2}$ *over* $w \in \{y_1, y_2\}$ *and take the maximum. As an illustration,* $T_1(x_1, 0.5) = \{(y_1, 0.3), (y_2, 0.3)\}$ *implies*

$$S(T_1(x_1, 0.5), A) = \max\left\{\frac{0.3 + 0.6}{2}, \frac{0.3 + 0.4}{2}\right\} = max\{0.45, 0.35\} = 0.45.$$

Then we obtain the intersection-based and union-based lower and upper approximations as shown in Table 3 below.

Table 3. Intersection-based and union-based lower and upper approximations.

x	$\underline{BFG_{1\diamond 2}}(A)(x)$	$\overline{BFG_{1\diamond 2}}(A)(x)$	$\underline{BFG_{1\diamond 2}}(A)(x)$	$\overline{BFG_{1\diamond 2}}(A)(x)$
x_1	1.0	0.5	1.0	0.5
x_2	1.0	1.0	1.0	0

In Table 3, we illustrate the case $x = x_1$ *for* $\underline{BFG_{1\diamond 2}}(A)(x)$ *and* $\overline{BFG_{1\diamond 2}}(A)(x)$.

For the lower approximation $\underline{BFG_{1\diamond 2}}(A)(x_1)$*, we require*

$$(S(T_1(x_1,l),A) \wedge S(T_2(x_1,l),A)) = \min(S(T_1(x_1,l),A), S(T_2(x_1,l),A)) \in D_1 = [0.5,1].$$

As $\min(S(T_1(x_1,0.5),A), S(T_2(x_1,0.5),A)) = \min(0.45,0.45) = 0.45 < 0.5$ *and* $\min(S(T_1(x_1,1.0),A), S(T_2(x_1,1.0),A)) = \min(0.55,0.55) = 0.55 \geq 0.5$*, then we have* $\underline{BFG_{1\diamond 2}}(A)(x_1) = 1.0$.

For the upper approximation $\overline{BFG_{1\diamond 2}}(A)(x_1)$*, we require*

$$(S(T_1(x_1,l),A) \wedge S(T_2(x_1,l),A^c)) = \min(S(T_1(x_1,l),A), S(T_2(x_1,l),A^c)) \in D_2 = [0,0.5].$$

As $\min(S(T_1(x_1,0.5),A^c), S(T_2(x_1,0.5),A^c)) = \min(0.45,0.50) = 0.45 \leq 0.5$ *and* $\min(S(T_1(x_1,1.0),A^c), S(T_2(x_1,1.0),A^c)) = \min(0.55,0.65) = 0.55 > 0.5$*, then we have* $\overline{BFG_{1\diamond 2}}(A)(x_1) = 0.5$.

In practice, the intersection-based and union-based BFS approximation operators illustrated here can be used to aggregate multiple criteria or multi-faceted fuzzy knowledge from different data sources. For instance, in decision-making, one could treat different knowledge mappings T_1 and T_2 as separate experts or distinct decision criteria. The intersection-based BFS approximation space models the scenario where decisions must satisfy all criteria, whereas union-based BFS approximation space interprets the union of expert opinions. In pattern recognition, different knowledge mappings can represent fuzzy feature sets extracted from an image or signals. By taking intersections and unions of BFS approximation spaces, one could refine or expand the matching criteria for a pattern, thus improving classification results.

Given the monotonicity property and the definitions of lower and upper approximation operators, the following properties hold.

Proposition 5. *Let* $BFG_1 = (U \times L^+, W, T_1, S)$ *and* $BFG_2 = (U \times L^+, W, T_2, S)$ *be monotonic BFS approximation spaces with* $D_1 = [a,1]$ *and* $D_2 = [0,b]$*. Let* $BFG_1 \diamond BFG_2 = (U \times L^+, W, T_1 \diamond T_2, S)$ *and* $BFG_1 \circ BFG_2 = (U \times L^+, W, T_1 \circ T_2, S)$*. Then,* $\forall A, B \in \mathcal{F}(W)$*, the following properties hold:*

(1) $A \subseteq B \Rightarrow \underline{BFG_{1\diamond 2}}(A) \supseteq \underline{BFG_1}(A) \cap \underline{BFG_2}(B)$,
(2) $A \subseteq B \Rightarrow \overline{BFG_{1\diamond 2}}(A) \supseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$,
(3) $A \subseteq B \Rightarrow \underline{BFG_{1\circ 2}}(A) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$,
(4) $A \subseteq B \Rightarrow \overline{BFG_{1\circ 2}}(A) \supseteq \overline{BFG_1}(A) \cup \overline{BFG_2}(B)$,
(5) $\underline{BFG_{1\diamond 2}}(A \cup B) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$,
(6) $\overline{BFG_{1\diamond 2}}(A \cap B) \subseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$,
(7) $\underline{BFG_{1\circ 2}}(A \cup B) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$,
(8) $\overline{BFG_{1\circ 2}}(A \cap B) \subseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$.

Proof. (1) The lower approximation under $\diamond$ is:

$$\underline{BFG_{1\diamond 2}}(A)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l),A) \wedge S(T_2(x,l),A)) \in D_1 \}.$$

Monotonicity ensures

$$A \subseteq B \Rightarrow \ S(T_1(x,l),A) \leq S(T_1(x,l),B), S(T_2(x,l),A) \leq S(T_2(x,l),B).$$

Thus, $\underline{BFG_{1\diamond 2}}(A)(x) \supseteq \underline{BFG_1}(A)(x) \cap \underline{BFG_2}(B)(x)$.
Hence, $A \subseteq B \Rightarrow \underline{BFG_{1\diamond 2}}(A) \supseteq \underline{BFG_1}(A) \cap \underline{BFG_2}(B)$.

(2) The upper approximation under $\diamond$ is:

$$\overline{BFG_{1\diamond 2}}(A)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A^c) \wedge S(T_2(x,l), A^c)) \in D_2 \}.$$

Monotonicity ensures

$$A \subseteq B \Rightarrow A^c \supseteq B^c \ \Rightarrow S(T_1(x,l), A^c) \geq S(T_1(x,l), B^c), S(T_2(x,l), A^c) \geq S(T_2(x,l), B^c).$$

Thus, $\overline{BFG_{1\diamond 2}}(A)(x) \supseteq \overline{BFG_1}(A)(x) \cap \overline{BFG_2}(B)(x)$.
Hence, $A \subseteq B \Rightarrow \overline{BFG_{1\diamond 2}}(A) \supseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$.
(3) The lower approximation under $\circ$ is:

$$\underline{BFG_{1\circ 2}}(A)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A) \vee S(T_2(x,l), A)) \in D_1 \}.$$

Monotonicity ensures

$$A \subseteq B \Rightarrow \ S(T_1(x,l), A) \leq S(T_1(x,l), B), S(T_2(x,l), A) \leq S(T_2(x,l), B).$$

Thus, $\underline{BFG_{1\circ 2}}(A)(x) \supseteq \underline{BFG_1}(A)(x) \cup \underline{BFG_2}(B)(x)$.
Hence, $A \subseteq B \Rightarrow \underline{BFG_{1\circ 2}}(A) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$.
(4) The upper approximation under $\circ$ is:

$$\overline{BFG_{1\circ 2}}(A)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A^c) \vee S(T_2(x,l), A^c)) \in D_2 \}.$$

Monotonicity ensures

$$A \subseteq B \Rightarrow A^c \supseteq B^c \ \Rightarrow S(T_1(x,l), A^c) \geq S(T_1(x,l), B^c), S(T_2(x,l), A^c) \geq S(T_2(x,l), B^c).$$

Thus, $\overline{BFG_{1\circ 2}}(A)(x) \supseteq \overline{BFG_1}(A)(x) \cup \overline{BFG_2}(B)(x)$.
Hence, $A \subseteq B \Rightarrow \overline{BFG_{1\circ 2}}(A) \supseteq \overline{BFG_1}(A) \cup \overline{BFG_2}(B)$.
(5) The lower approximation under $\diamond$ for $A \cup B$ is:

$$\underline{BFG_{1\diamond 2}}(A \cup B)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A \cup B) \wedge S(T_2(x,l), A \cup B)) \in D_1 \}.$$

Since $A \subseteq A \cup B$ and $A \subseteq A \cup B$, monotonicity ensures:

$$\underline{BFG_1}(A)(x) \subseteq \underline{BFG_{1\diamond 2}}(A \cup B)(x),\ \underline{BFG_2}(B)(x) \subseteq \underline{BFG_{1\diamond 2}}(A \cup B)(x).$$

Thus, $\underline{BFG_{1\diamond 2}}(A \cup B)(x) \supseteq \underline{BFG_1}(A)(x) \cup \underline{BFG_2}(B)(x)$.
Hence, $\underline{BFG_{1\diamond 2}}(A \cup B) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$.
(6) The upper approximation under $\diamond$ for $A \cap B$ is:

$$\overline{BFG_{1\diamond 2}}(A \cap B)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A^c \cup B^c) \wedge S(T_2(x,l), A^c \cup B^c)) \in D_2 \}.$$

Since $A \cap B \subseteq A$ and $A \cap B \subseteq A$, monotonicity ensures:

$$\overline{BFG_{1\diamond 2}}(A \cap B)(x) \subseteq \overline{BFG_1}(A)(x) \cap \overline{BFG_2}(B)(x).$$

Hence, $\overline{BFG_{1\diamond 2}}(A \cap B) \subseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$.
(7) The lower approximation under $\circ$ for $A \cup B$ is:

$$\underline{BFG_{1\circ 2}}(A \cup B)(x) = \sup\{\, l \in L^+ \big| (S(T_1(x,l), A \cup B) \vee S(T_2(x,l), A \cup B)) \in D_1 \}.$$

Since $A \subseteq A \cup B$ and $A \subseteq A \cup B$, monotonicity ensures:

$$\underline{BFG_{1\circ 2}}(A \cup B)(x) \supseteq \underline{BFG_1}(A)(x) \cup \underline{BFG_2}(B)(x)$$

Hence, $\underline{BFG_{1\circ 2}}(A \cup B) \supseteq \underline{BFG_1}(A) \cup \underline{BFG_2}(B)$.

(8) The upper approximation under $\circ$ for $A \cap B$ is:

$$\overline{BFG_{1\circ 2}}(A \cap B)(x) = \sup\left\{ l \in L^+ \middle| (S(T_1(x,l), A^c \cup B^c) \vee S(T_2(x,l), A^c \cup B^c)) \in D_2 \right\}.$$

Since $A \cap B \subseteq A$ and $A \cap B \subseteq A$, monotonicity ensures:

$$\overline{BFG_{1\circ 2}}(A \cap B)(x) \subseteq \overline{BFG_1}(A)(x) \cap \overline{BFG_2}(B)(x).$$

Hence, $\overline{BFG_{1\circ 2}}(A \cap B) \subseteq \overline{BFG_1}(A) \cap \overline{BFG_2}(B)$. □

Monotonicity is a crucial property in BFS approximation spaces, ensuring that the inclusion relationships between fuzzy sets are reasonably reflected in their approximation results. Specifically, the monotonicity condition guarantees that if a fuzzy set A is included in another fuzzy set B (i.e., $A \subseteq B$), then the lower approximation of A does not exceed that of B, and similarly, the upper approximation of A does not exceed that of B. This property plays a significant role in various practical applications, including:

(1) Maintaining consistency in knowledge hierarchies. In applications such as knowledge assessment and skill evaluation, knowledge or skills often possess hierarchical structures. For instance, a partial mastery of a skill is naturally contained within a near-complete mastery level. Monotonicity ensures that the lower approximation result of partial mastery does not exceed that of near-complete mastery, aligning with our intuitive understanding of knowledge hierarchy. This consistency is crucial for building reliable assessment models and ensuring the rationality of evaluation outcomes.

(2) Enhancing consistency in decision-making processes. In multi-criteria decision-making and intelligent decision support systems, decision-makers often need to balance multiple fuzzy criteria. Monotonicity ensures that as a criterion set expands (i.e., more options are included), its approximation results appropriately reflect this change, avoiding contradictions and inconsistencies in the decision-making process. This is vital for maintaining the coherence and reliability of decision-making procedures.

5. Complementary Compatibility

In the fuzzy S-approximation space, the concept of weak complementary compatibility was introduced [29] to ensure that the lower approximation of a set is always contained within its upper approximation. Similarly, this section introduces the concept of complementary compatibility in the BFS approximation space.

Definition 24. *Let $BFG = (U \times L^+, W, T, S)$ be a BFS approximation space. If $\forall x \in U$, $\forall l \in L^+$, and $\forall A \in \mathcal{F}(W)$, the intervals for the upper and lower approximation operators are set as $D_1 = [a, 1]$ and $D_2 = [0, b]$, where $a, b \in [0, 1]$ and $a + b = 1$. The following condition is satisfied:*

$$S(T(x,l), A) + S(T(x,l), A^c) \leq 1,$$

then BFG is said to be complementary compatible.

Proposition 6. *Let $BFG = (U \times L^+, W, T, S)$ be a complementary compatible BFS approximation space. Then, $\forall A \in \mathcal{F}(W)$, $\underline{BFG}(A) \subseteq \overline{BFG}(A)$.*

Proof. If $\forall l \in L^+$, $S(T(x,l),A) \notin D_1$, then, $\underline{BFG}(A)(x) = 0$. Thus, $\underline{BFG}(A)(x) \leq \overline{BFG}(A)(x)$. Otherwise, suppose there exists $l \in L^+$ with $S(T(x,l),A) \in D_1$. Since BFG is complementary compatible, that implies $S(T(x,l),A) + S(T(x,l),A^c) \leq a + b = 1$. Hence

$$S(T(x,l),A) \geq a \Rightarrow S(T(x,l),A^c) \leq b.$$

Taking the supremum over all such $l \in L^+$, it follows

$$\underline{BFG}(A)(x) = \sup\{\, l \in L^+ \big| S(T(x,l),A) \in D_1\} \leq \sup\{\, l \in L^+ \big| S(T(x,l),A) \in D_2\} = \overline{BFG}(A)(x).$$

Hence, $\underline{BFG}(A)(x) \leq \overline{BFG}(A)(x)$.

Combining the above results, we have $\underline{BFG}(A) \subseteq \overline{BFG}(A)$.□

Example 3. *Let us consider the BFS approximation space* $(U \times L^+, W, T, S)$ *in Example 1. Take* $a = 0.6$ *and* $b = 0.4$*. Let*

$$A = \{(y_1, 0.2), (y_2, 0.8), (y_3, 0.1), (y_4, 0.5)\} \in F(W),$$

then its complement is

$$A^c = \{(y_1, 0.8), (y_2, 0.2), (y_3, 0.9), (y_4, 0.5)\}.$$

Taking $(x_1, 0.6)$*as an example, according to the definition in Example 1,* $T(x_1, 0.6) = \{(y_1, 0.5), (y_2, 0.3), (y_3, 0), (y_4, 1)\}$*. Then,*

$$S(T((x_1,0.6)),A) = \max\left\{\tfrac{0.5+0.2}{2}, \tfrac{0.3+0.8}{2}, \tfrac{0+0.1}{2}, \tfrac{1+0.5}{2}\right\} = \max\left\{0.35, 0.55, 0.05, 0.75\right\} = 0.75.$$
$$S(T((x_1,0.6)),A^c) = \max\left\{\tfrac{0.5+0.8}{2}, \tfrac{0.3+0.2}{2}, \tfrac{0+0.9}{2}, \tfrac{1+0.5}{2}\right\} = \max\left\{0.65, 0.25, 0.45, 0.75\right\} = 0.75.$$
$$S(T((x_1,0.6)),A) + S(T((x_1,0.6)),A^c) = 0.75 + 0.75 = 1.5 \geq 1.$$

Therefore, the BFS approximation space in Example 1 does not satisfy the complementary compatibility condition. To achieve complementary compatibility, one may consider adjusting the knowledge mapping T *or redefining the decision mapping* S*.*

In data mining tasks, BFS approximation spaces allow flexible thresholds through D_1 and D_2 to discover fuzzy patterns of co-occurrence or to identify partial associations between concepts in large-scale data. Moreover, complementary compatibility considerations ensure that the lower and upper approximations are internally consistent, which is crucial in intelligent decision support systems. For example, when mining customer preferences (fuzzy sets of features), BFS approximation can unify partial knowledge across multiple product categories, enabling more robust recommendation and segmentation strategies.

6. Conclusions

Building on the fuzzy extension of knowledge space theory, this paper extends S-approximation spaces to BFS approximation spaces and defines their corresponding upper and lower approximation operators. Through rigorous analysis, the study demonstrates that these operators exhibit distinct mathematical properties under various operations and assumptions. Additionally, the paper explores a significant form of the approximation operators, highlighting the monotonicity and complementary compatibility inherent in BFS approximation spaces. These findings not only expand the theoretical framework of S-approximation spaces but also provide fresh insights into the fuzzy extension of knowledge space theory. Looking ahead, future research could delve deeper into the connections between BFS approximation spaces, fuzzy skill mappings, and fuzzy knowledge structures.

Such endeavors would further enrich the theoretical foundation of BFS approximation spaces and enhance their practical applications in the broader context of fuzzy knowledge space theory.

Building on these results, BFS approximation spaces play a key role in facilitating the transition from single-fuzzy to bi-fuzzy universes. Nonetheless, there remain several aspects that warrant additional investigation. First, the choice of intervals D_1 and D_2 may be application-dependent and can require extensive parameter tuning. Second, in high-dimensional or large-scale settings, computing these approximations for multiple fuzzy universes simultaneously might become computationally intensive, thus prompting a need for efficient algorithms and distributed implementations. Third, complementary compatibility imposes specific constraints on the sum of membership evaluations, which may not always hold in real-world fuzzy data. In future work, more advanced parameter optimization or machine-learning-based calibration methods could be investigated to automatically set a and b. Moreover, a tighter integration with fuzzy skill maps in knowledge space theory might yield deeper insights, and exploring BFS approximation spaces in emerging fields such as explainable AI, cognitive modeling, or big data analytics could further broaden their applicability.

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