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On the Oscillation of Fourth-Order Delay Differential Equations via Riccati Transformation

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Abstract: This paper deals with the oscillatory behavior of solutions of a general class of fourth-order non-linear delay differential equations. New oscillation criteria are established using Riccati transformation and a Philos-type technique. The obtained results not only improve and extend some published results in the literature, but also relax some traditional conditions on the function $\psi(\chi(\iota))$. Three examples are provided to illustrate the main results.

Keywords: delay differential equations; oscillation; Riccati transformation

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1. Introduction

In this work, we consider a fourth-order differential equation of the type

$$[F(\iota)\psi(\chi(\iota))\chi'''(\iota)]' + \aleph(\iota)\chi(\delta(\iota)) = 0, \quad \iota \geq \iota_0, \tag{1}$$

under the canonical and non-canonical cases

$$\int_{\iota_0}^{\infty} \frac{1}{F^{\frac{1}{\alpha}}(s)} ds = \infty, \tag{2}$$

and

$$\int_{\iota_0}^{\infty} \frac{1}{F^{\frac{1}{\alpha}}(s)} ds < \infty, \tag{3}$$

respectively. Moreover, we assume that

(A_1) $\aleph, \delta \in C([\iota_0, \infty), \mathbb{R})$ such that $\aleph(\iota) > 0$, $\delta(\iota) \leq \iota$ and $\lim_{\iota\to\infty} \delta(\iota) = \infty$;
(A_2) $F(\iota) \in C([\iota_0, \infty), \mathbb{R})$, $F(\iota) > 0$ for all $\iota \geq \iota_0$;
(A_3) $0 < \xi_1(\iota) \leq \psi(\chi(\iota)) \leq \xi_2(\iota) < \infty$.

We say that a solution $\chi(\iota)$ of (1) is oscillatory if it has an infinite set of zeros, otherwise it is called non-oscillatory. In the last few decades, there has been considerable interest in studying delay differential equations, since they are critical in modeling systems where the future state depends not only on the current state but also on past states. In fact, delay differential equations play essential role in various fields such as biology, engineering, economics and physics, where processes often involve time delays, like population dynamics where the birth rate at any given time depends on the population size at a previous time, or in control systems where the response to an input is delayed due to processing time. In fact, they provide a more accurate and realistic description of these systems than ordinary differential equations, capturing the intrinsic time-lagged interactions within the system. Therefore, understanding and solving delay differential equations is crucial for predicting and controlling the behavior of such time-dependent processes. Recently, the study of the oscillatory behavior of delayed differential equations has received great interest from many authors (see [1–24] and the references therein). There has been considerable interest in the existence and nonexistence of solutions of fourth-order differential equations; (see [25]) and for a fourth-order boundary value problem (see [26]). Moreover, for those who are concerned with fractional calculus, see [27–30]. For instance, we mention here some of the related works which motivated our results.

In [14], the authors investigated the oscillatory behavior of all solutions of the fourth-order functional differential equations

$$(F(\iota)(\chi'(\iota))^{\alpha})''' + \aleph(\iota)M(\chi(g(\iota))) = 0,$$

and

$$(F(\iota)(\chi'(\iota))^{\alpha})''' = \aleph(\iota)M(\chi(g(\iota))) + p(\iota)h(\chi(\delta(\iota))),$$

in the non-canonical case (3). Zhang et al. [24] discussed the oscillation of a certain class of fourth-order delay differential equations of the type

$$(F(\iota)(\chi'''(\iota))^{\alpha})' + \aleph(\iota)\chi^{\alpha}(\delta(\iota)) = 0.$$

They established some new oscillation criteria (including Hille- and Nehari-type criteria). Although their results improve some of those given by Zhang et al. [23], if one is concerned with the case $\alpha = 1$, our results include their equation, since they considered only the special case $\psi(\chi(\iota)) = 1$. In [18], Moaaz et al. studied the oscillatory behavior of solutions of the fourth-order non-linear differential equations

$$(F(\iota)(\chi'''(\iota))^{\alpha})' + \aleph(\iota)\chi^{\gamma}(\delta(\iota)) = 0.$$

They obtained new oscillation criteria by employing a refinement of Riccati transformations to complement and improve some of the results reported in the literature. Comparing with the work of [18], we note that although our results here are restricted to the case $\alpha = \gamma = 1$, however their results are applicable only for the special case $\psi(\chi(\iota)) = 1$. Moreover, we considered here both canonical and non-canonical cases, while the authors there were concerned with canonical case only.

Hou and Cheng [15] discussed the asymptotic behavior of the fourth-order differential equation

$$\chi^{(4)}(\iota) + p(\iota)\chi'(\iota) + \aleph(\iota)M(\chi(\delta(\iota))) = 0.$$

They deduced that all solutions converged to zero or oscillated. They employed novel Riccati-type techniques involving third-order linear differential equations. The importance of their results appears in the particular case when one interprets the solution $\chi(\iota)$ as a deflection from the equilibrium position of a horizontal beam at the spatial coordinate, through which the middle term $p(\iota)\chi'(\iota)$ acts as a control of the slope of the beam under consideration at the time coordinate ι.

Džurina and Jadlovská [11] discussed the oscillation of a fourth-order linear delay differential equation with a negative middle term in the form

$$\chi^{(4)}(\iota) - p(\iota)\chi'(\iota) + \aleph(\iota)\chi(\delta(\iota)) = 0,$$

under the assumption that all solutions of an auxiliary third-order differential equation are non-oscillatory. Their work can be considered as a continuation of the recent works of [15], but in case where the middle term is negative, which from the point of view of [11] there are no other results of this kind. Although there exists a very large body of literature devoted to the corresponding two-term fourth-order equation of the type

$$\chi^{(4)}(\iota) + \aleph(\iota)\chi(\delta(\iota)) = 0,$$

however, to the best of our knowledge, there do not appear to be any oscillation results for equations of the type (1) through which the function $\psi(\chi(\iota))$ is bounded by functions of time (ι) and not necessary by constants. The main objective of this paper is to present such oscillation criteria for (1) which relax this restriction. The effectiveness of the newly obtained criteria is illustrated by several examples.

In [2], Agarwal et al. established the oscillation criteria for all bounded solutions of the fourth-order differential equation of the form

$$\left(\frac{1}{F_3(\iota)}\left(\left(\frac{1}{F_2(\iota)}\left(\left(\frac{1}{F_1(\iota)}(\chi'(\iota))^{\alpha_1}\right)'\right)^{\alpha_2}\right)'\right)^{\alpha_3}\right)' + \aleph(\iota)M(\chi(\delta(\iota))) = 0.$$

Also, they gave some comparison results with first- and second-order equations. In [23], Zhang et al. studied the oscillation of a higher-order half-linear differential equation of the form

$$(F(\iota)(\chi^{(n-1)}(\iota))^{\alpha})' + \aleph(\iota)\chi^{\gamma}(\delta(\iota)) = 0.$$

Although they gave new oscillation criteria, the obtained results cannot be applied for $\delta(\iota) = \iota$. Our aim in the present article is to establish new oscillation criteria for the fourth-order differential Equation (1), which to the best of our knowledge has not been discussed before with our condition on the function $\psi(\chi(\iota))$.

2. Preliminaries

Throughout this section, we outline some notations and results which are needed for our main results. Below, we define the Philos-type integral conditions.

Let $D = \{(\iota, s) : -\infty < s \leq \iota < \infty\}$. We say that a function $\varepsilon(\iota, s)$ belongs to the class W if

(i) $\varepsilon \in C(D, [0, \infty))$; $\varepsilon(\iota, s) \geq 0$ for $-\infty < s < \iota < \infty$; and $\varepsilon(\iota, \iota) = 0$;

(ii) *ε has continuous partial derivative* $\frac{\partial \varepsilon}{\partial s}$ *satisfying*

$$\frac{\partial \varepsilon}{\partial s} = -\epsilon(\iota, s)\sqrt{\varepsilon(\iota, s)},$$

where $\epsilon \in L_{loc}(D, \mathbb{R})$

(iii) $0 < \inf_{s \geq \iota_0}\left[\liminf_{\iota \to \infty} \frac{\varepsilon(\iota, s)}{\varepsilon(\iota, \iota_0)}\right] \leq \infty$, see[31].

Lemma 1 (see [22]). *Let $\alpha \geq 1$ be a ratio of two odd positive integer numbers, where X and Y are constants. Then*

$$X\nu - Y\nu^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{X^{\alpha+1}}{Y^{\alpha}}, \quad Y > 0.$$

Lemma 2 (see [32]). *If the function χ satisfies $\chi^{(j)} > 0$, $j = 0, 1, \ldots, m$, and $\chi^{(m+1)} < 0$, then*

$$\frac{\chi(\iota)}{\iota^m/m!} \geq \frac{\chi'(\iota)}{\iota^{m-1}/(m-1)!}.$$

Lemma 3 (see [1]). *Let $\chi \in C^j([\iota_0,\infty),(0,\infty))$. Suppose that $\chi^{(j)}(\iota)$ has a fixed sign and on $[\iota_0,\infty)$, $\chi^{(j)}(\iota)$) not identically zero, and that there exists a $\iota_1 \geq \iota_0$ such that $\chi^{(j-1)}(\iota)\chi^{(j)}(\iota) \leq 0$ for all $\iota \geq \iota_1$. If $\lim_{\iota\to\infty}\chi(\iota) \neq 0$, then for every $\beta \in (0,1)$ there exists $\iota_\beta \geq \iota_0$ such that*

$$\chi(\iota) \geq \frac{\beta}{(j-1)!}\iota^{j-1}\left|\chi^{(j-1)}(\iota)\right|,$$

and

$$\chi'(\frac{\iota}{2}) \geq \frac{\beta}{(j-2)!}\iota^{j-2}\left|\chi^{(j-1)}(\iota)\right|.$$

Lemma 4. *Let $\chi(\iota)$ be an eventually positive solution of (1). Then, there exist two possible cases:*
(C_1) $\chi^{(j)}(\iota) > 0$ *for* $j = 0,1,2,3,$
(C_2) $\chi^{(j)}(\iota) > 0$ *for* $j = 0,1,3,$ *and* $\chi''(\iota) < 0,$
for all sufficiently large $\iota \geq \iota_1 \geq \iota_0$.

3. Main Results

Theorem 1. *Assume that (A_1)–(A_3) hold. If there exist positive functions $\varrho_1,\varrho_2 \in C^1([\iota_0,\infty);\mathbb{R})$ such that*

$$\lim_{\iota\to\infty}\int_{\iota_0}^{\iota}\left[\frac{\delta^3(s)}{s^3}\varrho_1(s)\aleph(s) - \frac{(\varrho_1'(s))^2 F(s)\xi_2(s)}{2\beta s^2\varrho_1(s)}\right]ds = \infty, \tag{4}$$

for some $\beta \in (0,1)$ and

$$\lim_{\iota\to\infty}\int_{\iota_0}^{\iota}\left[\varrho_2(\vartheta)\int_{\vartheta}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv - \frac{(\varrho_2'(\vartheta))^2}{4\varrho_2(\vartheta)}\right]d\vartheta = \infty, \tag{5}$$

then all solutions of (1) are oscillatory.

Proof. Suppose that there exists a non-oscillatory solution $\chi(\iota)$ of (1). For the sake of contradiction, assume that there exists a $\iota_1 \in [\iota_0,\infty)$, such that $\chi(\iota) > 0$, $\chi(\delta(\iota)) > 0$ for all $\iota \geq \iota_1 \geq \iota_0$. Using Lemma 4, we have one of the two cases (C_1) and (C_2).
We first consider case (C_1): Define the Riccati substitution by

$$w_1(\iota) = \varrho_1(\iota)\frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota)}{\chi(\iota)}. \tag{6}$$

It is clear that $w_1 > 0$ and

$$\begin{aligned} w_1'(\iota) &= \varrho_1'(\iota)\left(\frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota)}{\chi(\iota)}\right) + \varrho_1(\iota)\left[\frac{(F(\iota)\psi(\chi(\iota))\chi'''(\iota))'}{\chi(\iota)} - \frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota)\chi'(\iota)}{\chi^2(\iota)}\right], \\ &= \frac{\varrho_1'(\iota)}{\varrho_1(\iota)}w_1(\iota) - \varrho_1(\iota)\aleph(\iota)\frac{\chi(\delta(\iota))}{\chi(\iota)} - \varrho_1(\iota)F(\iota)\psi(\chi(\iota))\frac{\chi'(\iota)\chi'''(\iota)}{\chi^2(\iota)}. \end{aligned}$$

Since for the case (C_1), $\chi''(\iota) > 0$, then clearly $\chi'(\iota) \geq \chi'(\frac{\iota}{2})$. Using Lemma 3, we obtain

$$\chi'(\iota) \geq \frac{\beta}{2}\iota^2\chi'''(\iota). \tag{7}$$

Putting $m = 3$ in Lemma 2, we obtain

$$\frac{\chi'(\iota)}{\iota^2/2!} \leq \frac{\chi}{\iota^3/3!},$$

then

$$\chi'(\iota) \leq \frac{\chi(\iota)}{\iota/3}.$$

So,

$$\frac{\chi'(\iota)}{\chi(\iota)} \leq \frac{3}{\iota}.$$

Integrating from ι to $\delta(\iota)$, we obtain

$$\frac{\chi(\delta(\iota))}{\chi(\iota)} \geq \frac{\delta^3(\iota)}{\iota^3},$$

i.e.,

$$\chi(\delta(\iota)) \geq \frac{\delta^3(\iota)}{\iota^3}\chi(\iota). \tag{8}$$

Thus with (6)–(8), we obtain

$$\begin{aligned} w_1'(\iota) &\leq \frac{\varrho_1'(\iota)}{\varrho_1(\iota)}w_1(\iota) - \frac{\delta^3(\iota)}{\iota^3}\varrho_1(\iota)\aleph(\iota) - \frac{\beta\iota^2}{2}\varrho_1(\iota)F(\iota)\psi(\chi(\iota))\left(\frac{\chi'''(\iota)}{\chi(\iota)}\right)^2, \\ &= \frac{\varrho_1'(\iota)}{\varrho_1(\iota)}w_1(\iota) - \frac{\delta^3(\iota)}{\iota^3}\varrho_1(\iota)\aleph(\iota) - \frac{\beta\iota^2}{2}\frac{w_1^2(\iota)}{\varrho_1(\iota)F(\iota)\psi(\chi(\iota))}. \end{aligned} \tag{9}$$

Using Lemma 1 with $X = \frac{\varrho_1'(\iota)}{\varrho_1(\iota)}$, $Y = \frac{\beta\iota^2}{2}\frac{1}{\varrho_1(\iota)F(\iota)\psi(\chi(\iota))}$ and $\nu = w_1$, we obtain

$$w_1'(\iota) \leq -\frac{\delta^3(\iota)}{\iota^3}\varrho_1(\iota)\aleph(\iota) + \frac{(\varrho_1'(\iota))^2F(\iota)\psi(\chi(\iota))}{2\beta\iota^2\varrho_1(\iota)}.$$

Therefore using (A_3), we obtain

$$w_1'(\iota) \leq -\frac{\delta^3(\iota)}{\iota^3}\varrho_1(\iota)\aleph(\iota) + \frac{(\varrho_1'(\iota))^2F(\iota)\xi_2(\iota)}{2\beta\iota^2\varrho_1(\iota)}.$$

This implies that

$$\int_{\iota_1}^{\iota}\left[\frac{\delta^3(s)}{s^3}\varrho_1(s)\aleph(s) - \frac{(\varrho_1'(s))^2F(s)\xi_2(s)}{2\beta s^2\varrho_1(s)}\right]ds \leq w_1(\iota_1),$$

for any constant $\beta \in (0,1)$. This is a contradiction.
Consider the case (C_2) and define

$$w_2(\iota) = \varrho_2(\iota)\frac{\chi'(\iota)}{\chi(\iota)}, \ \ \iota \geq \iota_1. \tag{10}$$

Then $w_2(\iota) > 0$ for $\iota \geq \iota_1$ and

$$\begin{aligned} w_2'(\iota) &= \varrho_2'(\iota)\frac{\chi'(\iota)}{\chi(\iota)} + \varrho_2(\iota)\left(\frac{\chi''(\iota)\chi(\iota) - (\chi'(\iota))^2}{(\chi(\iota))^2},\right) \\ &= \frac{\varrho_2'(\iota)}{\varrho_2(\iota)}w_2(\iota) + \varrho_2(\iota)\frac{\chi''(\iota)}{\chi(\iota)} - \frac{w_2^2(\iota)}{\varrho_2(\iota)}. \end{aligned} \tag{11}$$

By integrating (1) from ι to u, we have

$$F(u)\psi(\chi(u))\chi'''(u) - F(\iota)\psi(\chi(\iota))\chi'''(\iota) + \int_{\iota}^{u}\aleph(s)\chi(\delta(s))ds = 0. \tag{12}$$

From 2, we obtain

$$\frac{\chi(\delta(\iota))}{\chi(\iota)} \geq \frac{\delta(\iota)}{\iota}. \tag{13}$$

Using (13) then (12) leads to

$$F(u)\psi(\chi(u))\chi'''(u) - F(\iota)\psi(\chi(\iota))\chi'''(\iota) + \int_{\iota}^{u} \aleph(s)\left(\frac{\delta(s)}{s}\right)\chi(\iota)ds \leq 0.$$

Since $\chi'(\iota) > 0$, then

$$F(u)\psi(\chi(u))\chi'''(u) - F(\iota)\psi(\chi(\iota))\chi'''(\iota) + \chi(\iota)\int_{\iota}^{u} \frac{\delta(s)}{s}\aleph(s)ds \leq 0.$$

Letting $u \to \infty$, we obtain

$$-F(\iota)\psi(\chi(\iota))\chi'''(\iota) + \chi(\iota)\int_{\iota}^{u} \frac{\delta(s)}{s}\aleph(s)ds \leq 0.$$

Thus

$$\chi''(\iota) + \chi(\iota)\int_{\iota}^{\infty}\left[\frac{1}{F(v)\psi(\chi(v))}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv \leq 0,$$

i.e.,

$$\begin{aligned} \frac{\chi''(\iota)}{\chi(\iota)} &\leq -\int_{\iota}^{\infty}\left[\frac{1}{F(v)\psi(\chi(v))}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv, \\ &\leq -\int_{\iota}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv. \end{aligned} \tag{14}$$

Hence, by substituting from (14) into (11), we obtain

$$w_2'(\iota) \leq -\varrho_2(\iota)\int_{\iota}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv + \frac{\varrho_2'(\iota)}{\varrho_2(\iota)}w_2(\iota) - \frac{w_2^2(\iota)}{\varrho_2(\iota)}.$$

From Lemma 1, with $X = \frac{\varrho_2'(\iota)}{\varrho_2(\iota)}$, $Y = \frac{1}{\varrho_2(\iota)}$ and $\nu = w_2$, we have

$$w_2'(\iota) \leq -\varrho_2(\iota)\int_{\iota}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv + \frac{(\varrho_2'(\iota))^2}{4\varrho_2(\iota)}.$$

Integrating from ι_1 to ι, we find

$$\int_{\iota_1}^{\iota}\left[\varrho_2(\vartheta)\int_{\vartheta}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_{v}^{\infty}\frac{\delta(s)}{s}\aleph(s)ds\right]dv - \frac{(\varrho_2'(\vartheta))^2}{4\varrho_2(\vartheta)}\right]d\vartheta \leq w_2(\iota_1),$$

which contradicts (5). The proof is complete. □

Remark 1. *Theorem 1 improves and extends Theorem 2.1 of [24].*

Theorem 2. *Let the assumptions (A_1)–(A_3) be satisfied. Moreover, suppose that there exist $\varepsilon \in W$ that satisfy the conditions (i)–(ii) and $\kappa \in C([\iota_0,\infty);\mathbb{R})$ such that (iii) is satisfied. Furthermore, assume that $\chi^{(j)}(\iota) > 0$ for $j = 0,1,2,3$ and for some $\mu > 1$, all $\iota > \iota_0$ and $L \geq \iota_0$. If*

$$\limsup_{\iota\to\infty}\frac{1}{\varepsilon(\iota,L)}\int_{L}^{\iota}\left[\varepsilon(\iota,s)\Phi_1(s) - \frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}\epsilon^2(\iota,s)\right]ds \geq \kappa(L), \tag{15}$$

and

$$\limsup_{\iota\to\infty}\int_{\iota_0}^{\iota}\frac{s^2\kappa_+^2(s)}{\varrho_3(s)F(s)\xi_2(s)} = \infty, \tag{16}$$

where

$$\Phi_1(\iota) = \frac{\delta^3(\iota)}{\iota^3}\varrho_3(\iota)\aleph(\iota) + \frac{\beta\iota^2\varrho_3(\iota)g^2(\iota)}{2F(\iota)\xi_2(\iota)} - \varrho_3(\iota)g'(\iota) - \frac{[\varrho_3'(\iota)F(\iota)\xi_2(\iota) + \beta\iota^2\varrho_3(\iota)g(\iota)]^2}{\beta\iota^2\varrho_3(\iota)F(\iota)\xi_1(\iota)}, \quad (17)$$

for a continuously differentiable function $g \in C^1([\iota_0,\infty);\mathbb{R})$. *Then, Equation (1) is oscillatory.*

Proof. Let $\chi(\iota)$ be a non-oscillatory solution of (1) where $\chi^{(j)}(\iota) > 0$ for $j = 0,1,2,3$ for $\iota \geq T_0 \geq t_0$. Define the generalized Riccati transformation

$$w_3(\iota) = \varrho_3(\iota)\left[\frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota)}{\chi(\iota)} + g(\iota)\right]. \quad (18)$$

Then, $w_3(\iota) > 0$ and

$$\begin{aligned} w_3'(\iota) &= \varrho_3'(\iota)\left[\frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota)}{\chi(\iota)} + g(\iota)\right] + \varrho_3(\iota)\left[\frac{(F(\iota)\psi(\chi(\iota))\chi'''(\iota))'}{\chi(\iota)} + g'(\iota)\right. \\ &\quad \left. - \frac{F(\iota)\psi(\chi(\iota))\chi'''(\iota))\chi'(\iota)}{\chi^2(\iota)}\right], \\ &= \frac{\varrho_3'(\iota)}{\varrho_3(\iota)}w_3(\iota) - \varrho_3(\iota)\aleph(\iota)\frac{\chi(\delta(\iota))}{\chi(\iota)} + \varrho_4(\iota)g'(\iota) - \varrho_3(\iota)F(\iota)\psi(\chi(\iota))\frac{\chi'(\iota)\chi'''(\iota)}{\chi^2(\iota)}. \end{aligned}$$

Thus, with (7), (8) and (18) for all $\iota \geq L_0$, we obtain

$$w_3'(\iota) \leq \frac{\varrho_3'(\iota)}{\varrho_3(\iota)}w_3(\iota) - \frac{\delta^3(\iota)}{\iota^3}\varrho_3(\iota)\aleph(\iota) + \varrho_3(\iota)g'(\iota) - \frac{\beta\iota^2\varrho_3(\iota)}{2F(\iota)\psi(\chi(\iota))}\left[\frac{w_3(\iota)}{\varrho_3(\iota)} - g(\iota)\right]^2. \quad (19)$$

That is,

$$\begin{aligned} w_3'(\iota) &\leq \left[\frac{\varrho_3'(\iota)}{\varrho_3(\iota)} + \frac{\beta\iota^2 g(\iota)}{F(\iota)\psi(\chi(\iota))}\right]w_3(\iota) - \frac{\beta\iota^2}{2\varrho_3(\iota)F(\iota)\psi(\chi(\iota))}w_3^2(\iota) \\ &\quad - \frac{\delta^3(\iota)}{\iota^3}\varrho_3(\iota)\aleph(\iota) - \frac{\beta\iota^2\varrho_3(\iota)g^2(\iota)}{2F(\iota)\psi(\chi(\iota))} + \varrho_3(\iota)g'(\iota). \end{aligned} \quad (20)$$

Using the following inequality, which is valid for all $a > 0$ and $b, z \in \mathbb{R}$,

$$bz - az^2 \leq \frac{b^2}{2a} - \frac{a}{2}z^2, \quad (21)$$

we deduce that

$$\begin{aligned} w_3'(\iota) &\leq \frac{[\varrho_3'(\iota)F(\iota)\psi(\chi(\iota)) + \beta\iota^2\varrho_3(\iota)g(\iota)]^2}{\beta\iota^2\varrho_3(\iota)F(\iota)\psi(\chi(\iota))} - \frac{\beta\iota^2}{4\varrho_3(\iota)F(\iota)\psi(\chi(\iota))}w_3^2(\iota) \\ &\quad - \frac{\delta^3(\iota)}{\iota^3}\varrho_3(\iota)\aleph(\iota) - \frac{\beta\iota^2\varrho_3(\iota)g^2(\iota)}{2F(\iota)\psi(\chi(\iota))} + \varrho_3(\iota)g'(\iota). \end{aligned} \quad (22)$$

In view of (A_3), for all $\iota \geq L_0$, (22) gives

$$w_3'(t) \leq -\Phi_1(\iota) - \frac{\beta\iota^2}{4\varrho_3(\iota)F(\iota)\xi_2(\iota)}w_3^2(\iota), \quad (23)$$

where $\Phi_1(\iota)$ is defined as in (17). Multiplying (23) by $\varepsilon(\iota,s)$, integrating from L to ι, and using $(i) - (ii)$ for all $\iota \geq L \geq L_0$, we find

$$\begin{aligned}\int_L^\iota \varepsilon(\iota,s)\Phi_1(s)ds &\leq -\int_L^\iota \varepsilon(\iota,s)w_3'(s)ds - \int_\varepsilon^\iota \varepsilon(\iota,s)\frac{\beta s^2}{4\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds, \\ &= -\varepsilon(\iota,s)w_3(s)\mid_L^\iota - \int_L^\iota \left[-\frac{\partial\varepsilon(\iota,s)}{\partial s}w_3(s) + \varepsilon(\iota,s)\frac{\beta s^2}{4\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)\right]ds, \\ &= \varepsilon(\iota,L)w_3(L) - \int_L^\iota \left(\epsilon(\iota,s)\sqrt{\varepsilon(\iota,s)}w_3(s) + \varepsilon(\iota,s)\frac{\beta s^2}{4\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)\right)ds.\end{aligned}$$

For any $\mu > 1$, we obtain

$$\begin{aligned}\int_L^\iota \varepsilon(\iota,s)\Phi_1(s)ds \leq \varepsilon(\iota,L)w_3(L) - \int_L^\iota \left(\sqrt{\frac{\beta s^2\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}}w_3(s) + \sqrt{\frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}}\epsilon(\iota,s)\right)^2 ds \\ +\frac{\mu}{\beta}\int_L^\iota \left(\frac{\varrho_3(s)F(s)\xi_2(s)}{s^2}\right)\epsilon^2(\iota,s)ds - \int_L^\iota \frac{\beta s^2(\mu-1)\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds.\end{aligned} \tag{24}$$

Then,

$$\begin{aligned}\int_L^\iota \left[\varepsilon(\iota,s)\Phi_1(s) - \frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}\epsilon^2(\iota,s)\right]ds \leq \varepsilon(\iota,L)w_3(L) \\ -\int_L^\iota \left(\sqrt{\frac{\beta s^2\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}}w_3(s) + \sqrt{\frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}}\epsilon(\iota,s)\right)^2 ds \\ -\int_L^\iota \frac{\beta s^2(\mu-1)\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds.\end{aligned} \tag{25}$$

Thus, for $\iota > L \geq L_0$, we obtain

$$\begin{aligned}\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \left[\varepsilon(\iota,s)\Phi_1(s) - \frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}\epsilon^2(\iota,s)\right]ds \leq w_3(L) \\ -\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \left(\sqrt{\frac{\beta s^2\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}}w_3(s) + \sqrt{\frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}}\epsilon(\iota,s)\right)^2 ds \\ -\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \frac{\beta s^2(\mu-1)\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds,\end{aligned}$$

and

$$\begin{aligned}\limsup_{\iota\to\infty}\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \left[\varepsilon(\iota,s)\Phi_1(s) - \frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}\epsilon^2(\iota,s)\right]ds \\ \leq w_3(L) - \liminf_{\iota\to\infty}\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \frac{\beta s^2(\mu-1)\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds.\end{aligned} \tag{26}$$

This, with (16), leads to

$$w_3(L) \geq \kappa(L) + \liminf_{\iota\to\infty}\frac{1}{\varepsilon(\iota,L)}\int_L^\iota \frac{\beta s^2(\mu-1)\varepsilon(\iota,s)}{4\mu\varrho_3(s)F(s)\xi_2(s)}w_3^2(s)ds.$$

The proof can now be completed as in Theorem 3 in [33]. □

Remark 2.

(1) *Although our technique in Theorem 2 depends on the work of Rogovchenko et al. [33], the authors there were only concerned with the case of a second-order differential equation.*

(2) *When choosing $\varepsilon(\iota,s) = (\iota - s)^n, n > 1$ or $\varepsilon(\iota,s) = (\ln\frac{\iota}{s})^n$ for a positive integer $n \geq 2$, then one can obtain two other oscillation criteria for Equation (1).*

Theorem 3. *Let the assumptions* (A_1)–(A_3) *be satisfied. Assume that there exist functions* $\varepsilon \in W$, $\eta \in C^1([\iota_0,\infty);\mathbb{R})$, *for all* $\iota > \iota_0$ *and* $L \geq \iota_0$. *Furthermore, assume that* $\chi^{(j)}(\iota) > 0$ *for* $j = 0,1,3$, *and* $\chi''(\iota) < 0$. *If*

$$\limsup_{\iota\to\infty} \frac{1}{\varepsilon(\iota,L)} \int_L^{\iota} \left[\Phi_2(s)\varepsilon(\iota,s) - \frac{\varrho_4(s)}{4}\epsilon^2(\iota,s)\right] ds = \infty, \tag{27}$$

then (1) is oscillatory. Where

$$\varrho_4(\iota) = \exp\left(-2\int^{\iota} \frac{1}{\eta(s)} ds\right), \tag{28}$$

$$\Phi_2(\iota) = \varrho_4(\iota)\left[\daleth(\iota) + \frac{\eta'(\iota)+1}{\eta^2(\iota)}\right], \tag{29}$$

and

$$\daleth(\iota) = \int_{\iota}^{\infty} \left[\frac{1}{F(v)\xi_2(v)} \int_v^{\infty} \aleph(s)\left(\frac{\delta(s)}{s}\right) ds\right] dv. \tag{30}$$

Proof. For the sake of contradiction, suppose that $\chi(\iota)$ be a non-zero solution of (1) where $\chi^{(j)}(\iota) > 0$ for $j = 0,1,3$, and $\chi''(\iota) < 0$. Define

$$w_4(\iota) = \varrho_4(\iota)\left[\frac{\chi'(\iota)}{\chi(\iota)} + \frac{1}{\eta(\iota)}\right]. \tag{31}$$

Then, $w_4(\iota) > 0$ and

$$w_4'(\iota) = \varrho_4'(\iota)\left[\frac{\chi'(\iota)}{\chi(\iota)} + \frac{1}{\eta(\iota)}\right] + \varrho_4(\iota)\left[\frac{\chi''(\iota)}{\chi(\iota)} - \left(\frac{\chi'(\iota)}{\chi(\iota)}\right)^2 - \frac{\eta'(\iota)}{\eta^2(\iota)}\right]. \tag{32}$$

From (14), we obtain

$$\frac{\chi''(\iota)}{\chi(\iota)} \leq -\daleth(\iota), \tag{33}$$

where $\daleth(\iota)$ is defined by (30).
Now, substituting from (33) into (32) and using (31), we obtain

$$w_4'(\iota) \leq -\Phi_2(\iota) - \frac{w_4^2(\iota)}{\varrho_4(\iota)}, \tag{34}$$

where $\Phi_2(\iota)$ is given by (29). Multiplying (34) by $\varepsilon(\iota,s)$ and integrating from L to ι, we obtain

$$\begin{aligned}
\int_L^{\iota} \Phi_2(s)\varepsilon(\iota,s)ds &\leq -\int_L^{\iota} w_4'(s)\varepsilon(\iota,s)ds - \int_L^{\iota} \frac{\varepsilon(\iota,s)}{\varrho_4(s)} w_4^2(s)ds \\
&= -w_4(s)\varepsilon(\iota,s)|_L^{\iota} - \int_L^{\iota}\left[-\frac{\partial\varepsilon(\iota,s)}{\partial s} w_4(s) + \frac{\varepsilon(\iota,s)}{\varrho_4(s)} w_4^2(s)\right] ds \\
&= w_4(T)\varepsilon(\iota,s) - \int_L^{\iota}\left[\epsilon(\iota,s)\sqrt{\varepsilon(\iota,s)} w_4(s) + \frac{\varepsilon(\iota,s)}{\varrho_4(s)} w_4^2(s)\right] ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_L^{\iota}\left[\Phi_2(s)\varepsilon(\iota,s) - \frac{\varrho_4(s)}{4}\epsilon^2(\iota,s)\right] ds &\leq w_4(L)\varepsilon(L,s) - \int_L^{\iota}\left(\sqrt{\frac{\varepsilon(\iota,s)}{\varrho_4(s)}} w_4(s) + \sqrt{\frac{\varrho_4(s)}{4}}\epsilon(\iota,s)\right)^2 ds \\
&\leq w_4(L)\varepsilon(L,s).
\end{aligned}$$

and so
$$\frac{1}{\varepsilon(L,s)}\int_L^\iota\left[\Phi_2(s)\varepsilon(\iota,s)-\frac{\varrho_4(s)}{4}\epsilon^2(\iota,s)\right]ds\leq w_4(L)<\infty,$$
which contradicts (27). This completes the proof. □

Remark 3. *The criterion of Theorem 3 is more general than that was established in Theorem 1 of [7].*

4. Examples

Example 1. *Consider the differential equation*
$$\left[\frac{\iota}{2\iota^2+1}\left(\frac{2+\frac{\sin^2(\chi(\iota))}{\iota^2}}{1+\frac{\sin^2(\chi(\iota))}{\iota^2}}\right)\chi'''(\iota)\right]'+e^\iota\chi\left(\frac{\iota}{2}\right)=0. \quad (35)$$

Here, $\psi(\chi(\iota))=(\frac{2+\frac{\sin^2(\chi(\iota))}{\iota^2}}{1+\frac{\sin^2(\chi(\iota))}{\iota^2}})$. *Taking* $\xi_1(\iota)=\frac{2\iota^2}{1+\iota^2}$ *and* $\xi_2(\iota)=2+\frac{1}{\iota^2}$. *If we set* $\varrho_1(\iota)=\varrho_2(\iota)=1$, *then*
$$\lim_{\iota\to\infty}\int_{\iota_0}^{\iota}\left[\frac{\delta^3(s)}{s^3}\varrho_1(s)\aleph(s)-\frac{(\varrho_1'(s))^2F(s)\xi_2(s)}{2\beta s^2\varrho_1(s)}\right]ds=\lim_{\iota\to\infty}\int_{\iota_0}^{\iota}\left(\frac{1}{8}e^s\right)ds=\infty,$$
and
$$\lim_{\iota\to\infty}\int_{\iota_0}^{\iota}\left[\varrho_2(\vartheta)\int_\vartheta^\infty\left[\frac{1}{F(v)\xi_2(v)}\int_v^\infty\frac{\delta(s)}{s}\aleph(s)ds\right]dv-\frac{(\varrho_2'(\vartheta))^2}{4\varrho_2(\vartheta)}\right]d\vartheta=\infty.$$
Thus, from Theorem, 1 Equation (35) oscillates.

Example 2. *For* $\iota\geq 1$, *consider the differential equation*
$$\left[\iota^6\left(\frac{1+\sin^2(\chi(\iota))}{\iota^2+\sin^2(\chi(\iota))}\right))\chi'''(\iota)\right]'+\aleph(\iota)\chi\left(\frac{\iota}{\sqrt[3]{2}}\right)=0. \quad (36)$$

Here, $\xi_1(\iota)=\frac{1}{\iota^2+1}\leq\psi(\chi(\iota))=\frac{1+\sin^2(\chi(\iota))}{\iota^2+\sin^2(\chi(\iota))}\leq\frac{2}{\iota^2}=\xi_2(\iota)$, $g(\iota)=\iota^2$, $\varrho_3(\iota)=1$ *and* $\beta=\frac{1}{2}$. *Then* $\Phi_1(\iota)=e^{-\iota}+\frac{2}{\iota^2}+16$, *where* $\aleph(\iota)=\frac{1}{2}\left[\frac{3}{4}\iota^2+4\iota+33+\frac{4}{\iota^2}+2e^{-\iota}\right]$.
Taking $\varepsilon(\iota,s)=(\iota-s)^2$ *and letting* $\mu=1$, *we get*
$$\limsup_{\iota\to\infty}\frac{1}{\varepsilon(\iota,L)}\int_L^\iota\left[\varepsilon(\iota,s)\Phi_1(s)-\frac{\mu\varrho_3(s)F(s)\xi_2(s)}{\beta s^2}\epsilon^2(\iota,s)\right]ds$$
$$\begin{aligned}&=\limsup_{\iota\to\infty}\iota^{-2}\int_L^\iota\left[(\iota-s)^2\Phi_1(s)-\frac{4\mu}{\beta}\frac{\varrho_3(s)F(s)\xi_2(s)}{s^2}\right]ds\\&=\limsup_{\iota\to\infty}\frac{1}{\iota^2}\int_L^\iota\left[\left(e^{-s}+\frac{2}{s^2}+16\right)(\iota-s)^2-16s^2\right]ds\\&=\frac{2}{L}+e^{-L}-16L=\kappa(L).\end{aligned}$$
Then,
$$\frac{\iota^2\kappa_+(\iota)}{\varrho_3(\iota)F(\iota)\xi_2(\iota)}=O(\iota^2)\quad\text{as }\iota\to\infty.$$
Consequently, using Theorem 2, Equation (36) oscillates.

Example 3. *Consider the differential equation*

$$\left[\iota^2\left(\frac{\ln(2+(\frac{\cos(\chi(\iota))}{\iota})^2}{1+(\frac{\cos(\chi(\iota))}{\iota})^2}\right)(\chi'''(\iota))^\alpha\right]' + \frac{1}{\iota}\chi\left(\frac{\iota}{4}\right) = 0, for\ \iota \geq 1. \quad (37)$$

Using the inequality $\frac{2}{2+u} \leq \frac{\ln(1+u)}{u} \leq \frac{2+u}{2+2u}$, *where* $u = 1 + \frac{\cos^2\chi}{\iota^2}$. *Then*

$$\frac{2}{3+(\frac{\cos(\chi(\iota))}{\iota})^2} \leq \left(\frac{\ln(2+(\frac{\cos(\chi(\iota))}{\iota})^2}{1+(\frac{\cos(\chi(\iota))}{\iota})^2}\right) \leq \frac{3+(\frac{\cos(\chi(\iota))}{\iota})^2}{4+(2\frac{\cos(\chi(\iota))}{\iota})^2}.$$

So,

$$\frac{2}{3+\frac{1}{\iota^2}} \leq \psi(\chi(\iota)) \leq \frac{3+\frac{1}{\iota^2}}{4+0},$$

i.e.,

$$\xi_1(\iota) = \frac{2\iota^2}{1+3\iota^2} \leq \psi(\chi(\iota)) \leq \frac{3}{4} + \frac{1}{4\iota^2} = \xi_2(\iota).$$

Let $\alpha = 2$, *then we have*

$$\int_{\iota_0}^{\infty} \frac{1}{F^{\frac{1}{\alpha}}(s)} ds = \int_{\iota}^{\infty} \frac{1}{s} ds = \infty.$$

For $\varrho_4(\iota) = 1$, *then*

$$\urcorner(\iota) = \int_{\iota}^{\infty}\left[\frac{1}{F(v)\xi_2(v)}\int_v^{\infty}\aleph(s)\left(\frac{\delta(s)}{s}\right)ds\right]dv = \infty.$$

So, condition (27) holds. Therefore, by Theorem 3, Equation (37) oscillates.

5. Conclusions

Throughout this paper, we established new oscillation criteria for a general class of fourth-order non-linear delay differential equations of the form (1). The obtained sufficient conditions improve and extend some known results in the literature and overcome some traditional conditions that the function $\psi(\chi(\iota))$ is bounded by some constants.

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