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Communication

Existence Results for Fractional Neutral Functional Differential Equations with Random Impulses

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Abstract: In this paper, we investigate the existence of solutions for the fractional neutral differential equations with random impulses. The results are obtained by using Krasnoselskii's fixed point theorem. Examples are added to show applications of the main results.

Keywords: fractional neutral differential equations; random impulses; existence; fixed point theorem

1. Introduction

Fractional Differential Equations, in which an unknown function is contained under the operation of a derivative of fractional order, have been of great interest recently. Many papers and books on fractional differential equations have appeared (see [1–6]). In [7], Lakshmikantham and Vatsala derived the basic theory of fractional differential equations. In [8], Hernandez *et al.* proved the existence of solutions of abstract fractional differential equations by using fixed point techniques.

On the other hand, impulsive differential systems are proved to be adequate mathematical models for numerous processes and phenomena studied in population dynamics, physics, chemistry and engineering. In recent years, some impressive results have been obtained in this area (see [7,9]). For the general theory of impulsive differential systems, the reader can refer to [10].

However, actual impulses do not always happen at fixed points but usually at random points. When the impulses exist at random points, the solutions of the differential systems are stochastic processes. Random impulsive systems are more realistic than deterministic impulsive systems. The study of random impulsive differential equations is a new area of research. So far, few results have been discussed in random impulsive systems. The existence and uniqueness of differential system with random impulses is studied by Anguraj *et al.* in [11,12]. In [13], Wu and Duan discussed the oscillation, stability and boundedness of second-order differential systems with random impulses, and in [14,15], the authors proved the existence and stability results of random impulsive semilinear differential systems.

Recently, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works (see [16,17]). Also, several authors [18–20] have studied the behaviour of neutral dfferential equations. The main reason for this interest is that delay differential equations play an important role in applications. For instance, in biological applications, delay equations give a better description of fluctuations in population than the ordinary ones. Also, neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits. In [21], Agarwal, Zhou and He proved the existence results of fractional neutral functional differential equations and also in [22], Anguraj *et al.* proved the existence results for fractional impulsive neutral differential equations. By the motivation of the recent surge in developing the theory of fractional neutral differential equations, we present a new idea of research to prove the existence of fractional neutral differential equations with random impulses.

This paper is divided into four sections. In Secion 2, we recall some basic definitions and preliminary facts. In Section 3, we shall establish the existence theorem for the Equation (1) by using the Krasnoselskii's fixed point theorem and in the final section, an illustrative example is presented.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and Ω a non-empty set. Assume that τ_k is a random variable defined from Ω to $D_k \stackrel{def.}{=} (0, d_k)$ for all $k = 1, 2, ...$ where $0 < d_k < \infty$. Furthermore, assume that τ_i and τ_j are independent of each other as $i \neq j$ for $i, j = 1, 2, ...$ Let $\tau, T \in R$ be two constants satisfying $\tau < T$. We denote $R_\tau = [\tau, T], R^+ = [t_0, \infty)$.

We consider the fractional neutral functional differential equations with random impulses of the form:

$$\begin{cases} {}^cD^\alpha(x(t) - g(t, x_t)) = A(t, x)x(t) + f(t, x_t), & t \in [\tau, T],\ t \neq \zeta_k \\ x(\zeta_k) = b_k(\tau_k)x(\zeta_k^-), & t = t_k,\ k = 1, 2, ... \\ x_{t_0} = \phi \end{cases} \tag{1}$$

where ${}^cD^\alpha$ is the standard Caputo's fractional derivative of order $0 < \alpha < 1$. $f, g : R_\tau \times C \to R^n$, $C = C([-r, 0], R^n)$ are given functions mapping $[-r, 0]$ into R^n with some given $r > 0$. ϕ is a function defined from $[-r, 0]$ to R^n ; x_t is a function when t is fixed, defined by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]; \zeta_0 = t_0$ and $\zeta_k = \zeta_{k-1} + \tau_k$ for $k = 1, 2, ...$ Here $t_0 \in R_\tau$ is an arbitrary given real number. Obviously, $t_0 = \zeta_0 < \zeta_1 < \zeta_2 < ...... < \zeta_k < ...$; $b_k : D_k \to R^{n\times n}$ is a matrix valued function for each $k = 1, 2, ..., x(\zeta_k^-) = \lim_{t\to\zeta_k} x(t)$ with the norm $||x||_t = \sup_{t-r<s<t} ||x(s)||$ for each t satisfying $\tau \le t \le T$ and $T \in R^+$ is a given number, $||.||$ is any given norm in R^n. Let $\mathcal{B}(R^n)$ denote the Banach space of bounded linear operators from R^n to R^n with the norm $||A||_{\mathcal{B}(R^n)} = sup\Big\{||A(y)|| : ||y|| = 1\Big\}$.

Denote $\{B_t, t \ge 0\}$ the simple counting process generated by ζ_n, that is, $\{B_t \ge n\} = \{\zeta_n \le t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{B_t, t \ge 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. For the simplicity, denote the Banach space $\Gamma = \Big\{$ all functions defined from $[t_0 - r, \infty)$ to R^n with the norm defined by $||\chi||_\Gamma = \sup_{t\ge t_0} E||\chi||_t\Big\}$.

Definition 1. *([4]). The fractional integral of order q with the lower limit t_0 for a function f is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{(q-1)} f(s)ds, \quad t > t_0, \quad q > 0$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the gamma function.

Definition 2. *([4]). Riemann-Liouville (R-L) derivative of order q with the lower limit t_0 for a function $f : [t_0, \infty) \longrightarrow R$ can be written as*

$$D^q f(t) = \frac{1}{\Gamma(n - q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t - s)^{(n-q-1)} f(s)ds, \quad t > t_0, \quad n - 1 < q < n.$$

The most important property of R-L fractional derivative is that for $t > t_0$ and $q > 0$, we have $D^q(I^q f(t)) = f(t)$, which means that R-L fractional differentiation operator is a left inverse to the R-L fractional integration operator of the same order q.

Definition 3. *([4]). The Caputo fractional derivative of order q with the lower limit t_0 for a function $f : [t_0, \infty) \longrightarrow R$ can be written as*

$${}^cD^q f(t) = \frac{1}{\Gamma(n - q)} \int_{t_0}^{t} (t - s)^{(n-q-1)} f^{(n)}(s)ds = I^{(n-q)} f^{(n)}(t), t > t_0, n - 1 < q < n.$$

Obviously, Caputo's derivative of a constant is equal to zero.

We shall state some properties of the operators I^α and ${}^cD^\alpha$.

Proposition 4. *([4,15]) For $\alpha, \beta > o$ and f as a suitable function, we have*

(i) $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$

(ii) $I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t)$

(iii) $I^\alpha(f(t) + g(t)) = I^\alpha f(t) + I^\alpha g(t)$

(iv) $I^{\alpha}\, {}^{c}D^{\alpha} f(t) = f(t) - f(0), 0 < \alpha < 1$

(v) ${}^{c}D^{\alpha} I^{\alpha} f(t) = f(t)$

(vi) ${}^{c}D^{\alpha} f(t) = I^{(1-\alpha)} Df(t) = I^{(1-\alpha)} f'(t), 0 < \alpha < 1, D = \frac{d}{dt}$

(vii) ${}^{c}D^{\alpha}\, {}^{c}D^{\beta} f(t) \neq {}^{c}D^{(\alpha+\beta)} f(t)$

(ix) ${}^{c}D^{\alpha}\, {}^{c}D^{\beta} f(t) \neq {}^{c}D^{\beta}\, {}^{c}D^{\alpha} f(t)$

In [7], Balachandran and Trujillo observed that both the R-L and the Caputo fractional differential operators do not possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. For basic facts about fractional integrals and fractional derivatives one can refer to the books [4,6,9].

Definition 5. *For a given $T \in (t_0, \infty)$, a stochastic process $\{x(t), t_0 - r \leq t \leq T\}$ is called a solution to the Equation (1) in $(\Omega, P, \{\mathcal{F}_t\})$, if*
(i) x(t) is $\mathcal{F}_t$-adapted.
(ii)$x(t_0 + s) = \phi(s)$ when $s \in [-r, 0]$, and

$$
\begin{aligned}
x(t) \;\; = \;\; & g(t, x_t) + \sum_{k=0}^{\infty} \Big[\sum_{i=1}^{k} g(t_i, x_{t_i})(b_i(\tau_i) - 1) \prod_{j=i+1}^{k} b_j(\tau_j) + \prod_{i=1}^{k} b_i(\tau_i) \Big[\phi(0) - g(t_0, \phi) \Big] \\
& + \sum_{i=1}^{k} \prod_{j=i}^{k} \frac{b_j(\tau_j)}{\Gamma(\alpha)} \Big\{ \int_{\zeta_{i-1}}^{\zeta_i} (\zeta_i - s)^{\alpha-1} A(s, x) x(s) ds \Big\} \\
& + \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^{t} (t - s)^{\alpha-1} A(s, x) x(s) ds \\
& + \sum_{i=1}^{k} \prod_{j=i}^{k} \frac{b_j(\tau_j)}{\Gamma(\alpha)} \Big\{ \int_{\zeta_{i-1}}^{\zeta_i} (\zeta_i - s)^{\alpha-1} f(s, x_s) ds \Big\} \\
& + \frac{1}{\Gamma(\alpha)} \int_{\zeta_k}^{t} (t - s)^{\alpha-1} f(s, x_s) ds \Big] I_{[\zeta_k, \zeta_{k+1})}(t), \qquad t \in [t_0, T]
\end{aligned} \tag{2}
$$

where $\prod_{j=i}^{k} b_j(\tau_j) = b_k(\tau_k) b_{k-1}(\tau_{k-1}) ... b_i(\tau_i)$, $\prod_{j=m}^{n}(.) = 1$ as $m > n$ and $I_A(.)$ is the index function, i.e.,

$$
I_A(t) = \begin{cases} 1, & if \;\; t \in A \\ 0, & if \;\; t \notin A \end{cases}
$$

Lemma 6. *(Krasnoselskii's Fixed point theorem). Let X be a Banach space, let E be a bounded closed convex subset of X and let S, U be maps of E into X such that $Sx + Uy \in E$ for every pair $x, y \in E$. If S is a contraction and U is Completely continuous, then the equation $Sx + Ux = x$ has a solution on E.*

3. Existence Results

In this section, we discuss the existence of the solutions of the system (1). Before stating and proving the main results, we introduce the following hypothesis.

(H_1) The function f satisfies the Lipschitz condition and there exists a positive constant $L_1 > 0$ such that for $x, y \in C$ and $t \in [\tau, T]$,

$$||f(t, x_t) - f(t, y_t)|| \leq L_1||x - y||.$$

(H_2) The function g is continuous and there exists a constant $L_2 > 0$ such that

$$||g(t, x_t)|| \leq L_2.$$

(H_3) $A : J \times R^n \to \mathcal{B}(R^n)$ is a continuous bounded linear operator and there exists a constant $L_3 > 0$ such that

$$||A(t, x) - A(t, y)|| \leq L_3||x - y||,$$

for all $x, y \in R^n$.

(H_4) The functions f and A are continuous and there exist a non-negative constant k such that

$$||f(t, 0)|| \leq k, ||f(t, x_t)|| \leq L_1||x|| + k$$

$$||A(t, 0)|| \leq k, ||A(t, x)|| \leq L_3||x|| + k.$$

(H_5) $\max_{i,k}\Big\{\prod_{j=i}^{k}||b_j(\tau_j)||\Big\}$ is uniformly bounded. (*i.e.*) there is a $B > 0$ such that

$$\max_{i,k}\Big\{\prod_{j=i}^{k}||b_j(\tau_j)||\Big\} \leq B, \forall \tau_j \in D_j, j = 1, 2, ...$$

(H_6) There exists a constant $N > 0$ such that

$$\max_{k}\Big\{||g(t_i, x_{t_i})(b_i(\tau_i) - 1)||\Big\} \leq N.$$

Theorem 7. *Under the hypotheses* $(H_1) - (H_6)$*, there exists a solution for the equation (1) if*

$$\frac{(T - t_0)^\alpha}{\Gamma(\alpha + 1)} \max\Big\{1, B\Big\}[(L_3 r + k)r + (L_1 r + k)] + L_2 + B\Big(N + E||\phi(0) - g(t_0, \phi)||\Big) \leq r \tag{3}$$

and

$$\frac{(T - t_0)^\alpha}{\Gamma(\alpha + 1)} \max\Big\{1, B\Big\}[2L_3 r + k + L_1] < 1 \tag{4}$$

Proof: Let T be an arbitrary positive number $t_0 < T < \infty$. Let us define an operator $P : \Gamma \to \Gamma$ as follows:

$$Px(t) \;=\; \phi(t-t_0), \;\; t \in [t_0 - r, t_0)$$

and

$$\begin{aligned} Px(t) \;=\;& g(t,x_t) + \sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k} g(t_i,x_{t_i})(b_i(\tau_i)-1)\prod_{j=i+1}^{k} b_j(\tau_j) + \prod_{i=1}^{k} b_i(\tau_i)\Big[\phi(0)-g(t_0,\phi)\Big] \\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\Big\{\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}A(s,x)x(s)ds\Big\} \\ &+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}A(s,x)x(s)ds \\ &+\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\Big\{\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}f(s,x_s)ds\Big\} \\ &+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}f(s,x_s)ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t), \;\; t\in[t_0,T] \end{aligned}$$

Let $B_r = \Big\{x \in \Gamma : \|x\| \le r\Big\}$
We define the operators S and U on B_r as

$$Sx(t) \;=\; \begin{cases} \phi(t-t_0) & t\in[t_0-r,t_0] \\ \sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\Big\{\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}A(s,x)x(s)ds\Big\} & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}A(s,x)x(s)ds & \\ \quad+\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\Big\{\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}f(s,x_s)ds\Big\} & \\ \quad+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}f(s,x_s)ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t), & t\in[t_0,T] \end{cases}$$

and

$$Ux(t) \;=\; \begin{cases} \phi(t-t_0) & t\in[t_0-r,t_0] \\ g(t,x_t)+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k} g(t_i,x_{t_i})(b_i(\tau_i)-1)\prod_{j=i+1}^{k} b_j(\tau_j) & \\ +\prod_{i=1}^{k} b_i(\tau_i)\,[\phi(0)-g(t_0,\phi)\,]\Big]I_{[\zeta_k,\zeta_{k+1})}(t), & t\in[t_0,T] \end{cases}$$

Next, we have to prove that $S+U$ has a fixed point in B_r.
The proof is divided into three steps.

Step I: To prove $Sx+Uy \in B_r$, for all $x, y \in B_r$.
For $x, y \in B_r$, consider,

$$\begin{aligned} \|Sx+Uy\| \;=\;& \|\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}A(s,x)x(s)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}A(s,x)x(s)ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}f(s,x_s)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}f(s,x_s)ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t) \end{aligned}$$

$$
\begin{aligned}
& +g(t,y_t)+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k} g(t_i,y_{t_i})(b_i(\tau_i)-1)\prod_{j=i+1}^{k} b_j(\tau_j) \\
& +\prod_{i=1}^{k} b_i(\tau_i)[\phi(0)-g(t_0,\phi)]\Big] I_{[\zeta_k,\zeta_{k+1})}(t)\| \\
\leq\ & \|g(t,y_t)\|+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\|g(t_i,y_{t_i})(b_i(\tau_i)-1)\|\prod_{j=i+1}^{k}\|b_j(\tau_j)\| \\
& +\prod_{i=1}^{k}\|b_i(\tau_i)\|\|\phi(0)-g(t_0,\phi)\|\Big] I_{[\zeta_k,\zeta_{k+1})}(t) \\
& +\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{\|b_j(\tau_j)\|}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\|A(s,x)\|\,\|x(s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\|A(s,x)\|\,\|x(s)\|ds\Big] I_{[\zeta_k,\zeta_{k+1})}(t) \\
& +\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{\|b_j(\tau_j)\|}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\|f(s,x_s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\|f(s,x_s)\|ds\Big] I_{[\zeta_k,\zeta_{k+1})}(t) \\
\leq\ & L_2+\max_{k}\Big\{\sum_{i=1}^{k}\|g(t_i,y_{t_i})[b_i(\tau_i)-1]\|\Big\}\max_{i,k}\Big\{\prod_{j=i+1}^{k}\|b_j(\tau_j)\|\Big\} \\
& +\max_{k}\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|\Big\}\|\phi(0)-g(t_0,\phi)\| \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\|A(s,x)\|\,\|x(s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x_s)\|ds \\
\leq\ & L_2+NB+B\|\phi(0)-g(t_0,\phi)\| \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[\|A(s,x)-A(s,0)\|+\|A(s,0)\|\Big]\,\|x(s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[\|f(s,x_s)-f(s,0)\|+\|f(s,0)\|\Big]ds \\
\leq\ & L_2+B\big(N+\|\phi(0)-g(t_0,\phi)\|\big) \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_3\|x\|+k\Big]\,\|x(s)\|ds \\
& +\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_1\|x\|+k\Big]ds
\end{aligned}
$$

Now,

$$\begin{aligned}
E\|(Sx+Uy)(t)\| &\leq L_2 + B\big(N + E\|\phi(0)-g(t_0,\phi)\|\big)\\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_3E\|x\|+k\Big]\ E\|x(s)\|ds\\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_1E\|x\|+k\Big]ds\\
\sup_{t_0\leq t\leq T} E\|(Sx+Uy)(t)\| &\leq L_2 + B\big(N + \sup_{t_0\leq t\leq T} E\|\phi(0)-g(t_0,\phi)\|\big)\\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_3\sup_{t_0\leq t\leq T}E\|x\|+k\Big]\sup_{t_0\leq t\leq T}E\|x(s)\|ds\\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_1\sup_{t_0\leq t\leq T}E\|x\|+k\Big]ds\\
&\leq L_2 + B\big(N + \|\phi(0)-g(t_0,\phi)\|\big)\\
&+\frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}\max\Big\{1,B\Big\}\Big[(L_3r+k)r+(L_1r+k)\Big]
\end{aligned}$$

Therefore, by Equation (3) $\|Sx+Uy\| = \sup_{t\geq t_0} E\|Sx+Uy\| \leq r$, which means that $Sx+Uy \in B_r$.

Step II: To prove S is a contraction on B_r.
Let $x,y \in B_r$.
Consider,

$$\begin{aligned}
Sx(t)-Sy(t) &= \sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\big[A(s,x)x(s)-A(s,y)y(s)\big]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\big[A(s,x)x(s)-A(s,y)y(s)\big]ds\\
&+\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{b_j(\tau_j)}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\big[f(s,x_s)-f(s,y_s)\big]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\big[f(s,x_s)-f(s,y_s)\big]ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t).
\end{aligned}$$

Then,
$\|Sx(t)-Sy(t)\|$

$$\begin{aligned}
&\leq \sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{\|b_j(\tau_j)\|}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\big[\|A(s,x)(x(s)-y(s))\|+\|(A(s,x)-A(s,y))y(s)\|\big]ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\big[\|A(s,x)(x(s)-y(s))\|+\|(A(s,x)-A(s,y))y(s)\|\big]ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\prod_{j=i}^{k}\frac{\|b_j(\tau_j)\|}{\Gamma(\alpha)}\int_{\zeta_{i-1}}^{\zeta_i}(\zeta_i-s)^{\alpha-1}\big[\|f(s,x_s)-f(s,y_s)\|\big]ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\zeta_k}^{t}(t-s)^{\alpha-1}\big[\|f(s,x_s)-f(s,y_s)\|\big]ds\Big]I_{[\zeta_k,\zeta_{k+1})}(t) \\
\leq\ & \frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\big[\|A(s,x)(x(s)-y(s))\|+\|(A(s,x)-A(s,y))y(s)\|\big]ds \\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,\prod_{j=i}^{k}\|b_j(\tau_j)\|\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x_s)-f(s,y_s)\|ds \\
\leq\ & \frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[(L_3\|x\|+k)\|x(s)-y(s)\|+L_3\|x-y\|\|y(s)\|\Big]ds \\
&+\frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\Big[L_1\|x-y\|\Big]ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sup_{t_0\leq t\leq T}E\|(Sx(t)-Sy(t)\| \leq\ & \frac{1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}\Big\{(t-s)^{\alpha-1}\Big[(L_3\sup_{t_0\leq t\leq T}E\|x\|_s+k)\times \\
& \sup_{t_0\leq t\leq T}E\|x(s)-y(s)\|_s+L_3\sup_{t_0\leq t\leq T}E\|x-y\|_s\sup_{t_0\leq t\leq T}E\|y(s)\|_s\Big]\Big\}ds \\
& +\frac{L_1}{\Gamma(\alpha)}\max_{i,k}\Big\{1,B\Big\}\int_{t_0}^{t}(t-s)^{\alpha-1}\sup_{t_0\leq t\leq T}E\|x-y\|_s ds \\
\leq\ & \frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}\max\Big\{1,B\Big\}\big[2L_3r+k+L_1\big]\sup_{t_0\leq t\leq T}E\|x-y\|_t.
\end{aligned}
$$

Thus,

$$
\|Sx(t)-Sy(t)\| \leq \Big\{\frac{(T-t_0)^{\alpha}}{\Gamma(\alpha+1)}\max\Big\{1,B\Big\}\big[2L_3r+k+L_1\big]\Big\}\|x-y\|
$$

Therefore, by Equation (4), S is a contraction.

Step III: To prove that U is a completely continuous operator.
For that, first we prove that U is uniformly bounded.
For any $t\in[t_0,T]$, consider

$$
\begin{aligned}
\|Ux(t)\| \leq\ & \|g(t,x_t)\|+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\|g(t_i,x_{t_i})(b_i(\tau_i)-1)\|\prod_{j=i+1}^{k}\|b_j(\tau_j)\| \\
& +\prod_{i=1}^{k}\|b_i(\tau_i)\|\|\phi(0)-g(t_0,\phi)\|\Big]I_{[\zeta_k,\zeta_{k+1})}(t)
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\sup_{t_0\le t\le T} E\|Ux(t)\| &\le L_2+\max_k\Big\{\sum_{i=1}^{k}\|g(t_i,x_{t_i})[b_i(\tau_i)-1]\|\Big\}\max_{i,k}\Big\{\prod_{j=i+1}^{k}\|b_j(\tau_j)\|\Big\}\\
&\quad+\max_k\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|\Big\}\sup_{t_0\le t\le T}E\|\phi(0)-g(t_0,\phi)\|\\
&\le L_2+B\Big(N+\sup_{t_0\le t\le T}E\|\phi(0)-g(t_0,\phi)\|\Big)
\end{aligned}$$

That implies that, $\|Ux(t)\|\le L_2+B\Big(N+\|\phi(0)-g(t_0,\phi)\|\Big)$.
This yields that U is uniformly bounded.
Next, we have to show that $\Big\{Ux : x\in B_r\Big\}$ is equicontinuous.
Let $x\in B_r$ and let $t_0\le t_1<t_2\le T$, then we have
$Ux(t_2)-Ux(t_1)$

$$\begin{aligned}
&= \Big[g(t_2,x_{t_2})+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}g(t_{2i},x_{t_{2i}})(b_i(\tau_i)-1)\prod_{j=i+1}^{k}b_j(\tau_j)+\prod_{i=1}^{k}b_i(\tau_i)[\phi(0)-g(t_0,\phi)]\Big]I_{[\zeta_k,\zeta_{k+1})}(t_2)\Big]\\
&\quad-\Big[g(t_1,x_{t_1})+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}g(t_{1i},x_{t_{1i}})(b_i(\tau_i)-1)\prod_{j=i+1}^{k}b_j(\tau_j)+\prod_{i=1}^{k}b_i(\tau_i)[\phi(0)-g(t_0,\phi)]\Big]I_{[\zeta_k,\zeta_{k+1})}(t_1)\Big]\\
&= g(t_2,x_{t_2})-g(t_1,x_{t_1})+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}g(t_{2i},x_{t_{2i}})(b_i(\tau_i)-1)\prod_{j=i+1}^{k}b_j(\tau_j)\\
&\quad+\prod_{i=1}^{k}b_i(\tau_i)[\phi(0)-g(t_0,\phi)]\Big]\,[I_{[\zeta_k,\zeta_{k+1})}(t_2)-I_{[\zeta_k,\zeta_{k+1})}(t_1)\,]\\
&\quad+\sum_{k=0}^{\infty}\Big[\sum_{i=1}^{k}\big[g(t_{2i},x_{t_{2i}})-g(t_{1i},x_{t_{1i}})\big](b_i(\tau_i)-1)\prod_{j=i+1}^{k}b_j(\tau_j)\Big]I_{[\zeta_k,\zeta_{k+1})}(t_1)
\end{aligned}$$

Then

$$\|Ux(t_2)-Ux(t_1)\| \le \|g(t_2,x_{t_2})-g(t_1,x_{t_1})\|+\|I_1\|+\|I_2\| \tag{5}$$

where,

$$\begin{aligned}
E\|I_1\| &\le E\Big(\max_k\Big\{\sum_{i=1}^{k}\|g(t_{2i},x_{t_{2i}})(b_i(\tau_i)-1)\|\Big\}\max_{i,k}\Big\{\prod_{j=i+1}^{k}\|b_j(\tau_j)\|\Big\}\\
&\quad+\max_k\Big\{\prod_{i=1}^{k}\|b_i(\tau_i)\|\Big\}\|\phi(0)-g(t_0,\phi)\|\,[I_{[\zeta_k,\zeta_{k+1})}(t_2)-I_{[\zeta_k,\zeta_{k+1})}(t_1)\,]\Big)\\
&\le B\Big(N+E(\|\phi(0)-g(t_0,\phi)\|)\Big)E(I_{[\zeta_k,\zeta_{k+1})}(t_2)-I_{[\zeta_k,\zeta_{k+1})}(t_1))\\
&\to 0\ as\ t_2\to t_1
\end{aligned} \tag{6}$$

$$\begin{aligned}
E\|I_2\| &\leq E\Big(\max_k\Big\{\sum_{i=1}^{k}\|\,[g(t_{2i},x_{t_{2i}})-g(t_{1i},x_{t_{1i}})\,](b_i(\tau_i)-1)\|\Big\}\\
&\quad \max_{i,k}\Big\{\prod_{j=i+1}^{k}\|b_j(\tau_j)\|\Big\}I_{[\zeta_k,\zeta_{k+1})}(t_1)\Big)\\
&\leq B\max_k\Big\{\sum_{i=1}^{k}E\Big(\|\,[g(t_{2i},x_{t_{2i}})-g(t_{1i},x_{t_{1i}})\,](b_i(\tau_i)-1)\|\Big)\Big\}\\
\to 0\; as\; t_2 \to t_1 &
\end{aligned} \tag{7}$$

From the Equations (6) and (7), the right hand side of the Equation (5) $\to 0$ as $t_2 \to t_1$.

$$\|Ux(t_2)-Ux(t_1)\| = \sup_{t_0\leq t\leq T} E\|Ux(t_2)-Ux(t_1)\| \to 0$$

as

$$t_2 \to t_1.$$

Thus, U is equicontinuous.

4. Example

Let τ_k be a random variable defined in $D_k \equiv (0,d_k)$ for all $k = 1,2,...$ where $0 < d_k < \infty$. Furthermore, assume that τ_i and τ_j be independent with each other as $i \neq j$ for $i,j = 1,2,...$

Consider, the following fractional differential equation with random impulses of the form:

$$\begin{cases}
{}^cD^\alpha\Big(x(t) - \frac{cost}{(t+3)^2}\frac{x}{1+x}\Big) = \frac{1}{9}sinx(t)x(t) + \frac{1}{t+1}\frac{x}{x+9}, & t\in[t_0,T],\ t\neq\zeta_k\\
x(\zeta_k) = p(k)(\tau_k)x(\zeta_k^-), & k = 1,2,...\\
x_{t_0} = \phi
\end{cases} \tag{8}$$

It is easily seen that the functions f,g and A satisfies the assumptions and clearly, we have $L_1 = L_2 = L_3 = k = \frac{1}{9}$.

Moreover the assumptions (H_5) and (H_6) are satisfied.

Further, if $r = 1$, from the above facts, in view of Theorem (3), we conclude that the Equation (8) has a solution on $[t_0,T]$, provided that the inequalities:

$$\frac{4(T-t_0)^\alpha}{9\Gamma(\alpha+1)}\max\Big\{1,B\Big\} + \frac{1}{9} + B(N + ||\phi(0) - g(t_0,\phi)||) \leq 1 \tag{9}$$

and

$$\frac{4(T-t_0)^\alpha}{9\Gamma(\alpha+1)}\max\Big\{1,B\Big\} < 1 \tag{10}$$

are satisfied.

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Author Contributions

All authors have contributed equally.

Conflicts of Interest

The authors declare no conflict of interest.

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