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Article

Basic Results for Sequential Caputo Fractional Differential Equations

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Abstract: We have developed a representation form for the linear fractional differential equation of order q when $0 < q < 1$, with variable coefficients. We have also obtained a closed form of the solution for sequential Caputo fractional differential equation of order $2q$, with initial and boundary conditions, for $0 < 2q < 1$. The solutions are in terms of Mittag–Leffler functions of order q only. Our results yield the known results of integer order when $q = 1$. We have also presented some numerical results to bring the salient features of sequential fractional differential equations.

Keywords: sequential Caputo fractional derivative; Mittag–Leffler function

1. Introduction

Qualitative properties of non-linear dynamic systems with integer derivatives are well known. See [1–3] for some of the results. However, from the modeling point of view, dynamic systems with fractional derivatives are known to be more useful, suitable and economical. See [4–10] and the references therein for more details. The advantage of using fractional derivative *versus* the integer derivative is that the integer derivative is local in nature, where as the fractional derivative is global in nature. This behavior is very useful in modeling physical problems, which involves past memory, and also equations, which involve delay. In the past three decades, dynamic systems with fractional derivatives have gained importance due to their advantage in applications. See [11–16] for some applications. For applications of fractional calculus for univalent functions, see [15,17]. In the literature, there are several other types of fractional derivatives, such as the Erdélyi–Kober type, Hadamard type and Grunwald–Letnikov type. In this work, we have used the Caputo fractional derivative [8]. In the past 30 years, a vast literature on Caputo fractional differential equations and applications has been developed [9]. The reason is, the results of Caputo derivatives are closer to integer derivative results. Although there are plenty of results available in the literature for the existence and uniqueness of solutions of non-linear fractional differential equations, a vast majority of the results are via some kind of fixed point theorem methods. Unfortunately, the fixed point theorem methods do not guarantee the interval of existence. In order to develop, an iterative method that guarantees the interval of existence using the solution of the corresponding linear equation is very useful. In this work, we consider the linear sequential Caputo fractional differential equation of order q for $0 < q < 1$, with variable coefficients, with initial conditions. We obtain a closed form of the solution for the Caputo fractional differential equation of order q for $0 < q < 1$, in such a way that the results for $q = 1$ will be a special case. Next, we consider the linear sequential Caputo fractional differential equation of order $2q$, of the form $A\ {}^cD^{2q}u + B\ {}^cD^qu + Cu = 0$ when $1 < 2q < 2$. For other known results relative to sequential derivative and sequential fractional differential equations, see [7,8]. We obtain two linearly-independent solutions in terms of the Mittag–Leffler functions [18] of the form $E_{q,1}(\mu t^q)$ of order q, when $0 < q < 1$. All of our results yield the integer results as a special case. The advantage of considering the sequential Caputo fractional derivative is that we can have the solution of the linear fractional differential equation of order $2q$ in terms of the Mittag–Leffler functions of order q. It is to be noted that we cannot use the variation of the parameter method as in the integer case. We have developed numerical results for all of the analytical solutions that we have obtained. We have obtained solutions when the quadratic $A\mu^2 + B\mu + C = 0$ has real and distinct roots, coincident roots and complex roots. Several numerical examples are presented, which bring the salient features of the oscillatory solutions of the sequential fractional differential equation. In addition, heuristically, we have established that $q = 1/2$ is the bifurcation value from oscillation to non-oscillation. We have also obtained a representation form for the linear sequential Caputo boundary value problems in terms of Green's function. This will be useful to develop a monotone method to obtain the solution of the non-linear sequential Caputo boundary value problem.

2. Preliminary Results

In this section, we recall basic definitions of Caputo fractional derivatives, fractional integrals and known results, which play an important role in our main results.

Definition 1. *The Caputo (left-sided) fractional derivative of u(t) of order q, $n-1 < q < n$, is given by equation:*

$$ {}^cD^qu(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-s)^{n-q-1}u^{(n)}(s)ds,\ \ t \in [t_0, t_0+T] \tag{1}$$

and (right-sided):

$$ {}^cD^qu(t) = \frac{(-1)^n}{\Gamma(n-q)}\int_{t}^{t_0+T}(s-t)^{n-q-1}u^{(n)}(s)ds,\ \ t \in [t_0, t_0+T] \tag{2}$$

where $u^{(n)}(t) = \frac{d^n(u)}{dt^n}$.

Further, if $q = n$, an integer, then ${}^cD^q u = u^{(n)}(t)$ and ${}^cD^q u = u'(t)$ if $q = 1$.

In particular, if $0 < q < 1$, we use the following definition.

Definition 2. *The Caputo (left-sided) fractional derivative of order* q *is given by equation:*

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds \tag{3}$$

where $0 < q < 1$.

Definition 3. *The Riemann–Liouville (left-sided) fractional integral is defined by:*

$$ D^{-q} u(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} u(s) ds, \;\; t < T, \;\; 0 < q < 1 \tag{4}$$

One can also define the right-sided Riemann–Liouville fractional integral.

Throughout this paper, we have used the Caputo (left-sided) fractional derivative, except in the section on boundary value problems. Note that $q = 1$ in Definitions 1 and 2 is the special case of the integer derivative. In order to compute the solutions, we introduce the two-parameter Mittag–Leffler functions.

Definition 4. *The Mittag–Leffler function is given by:*

$$ E_{\alpha,\beta}(\lambda(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^{\alpha})^k}{\Gamma(\alpha k + \beta)} \tag{5}$$

where $\alpha, \beta > 0$ *and* λ *is a constant. Furthermore, for* $t_0 = 0$, $\alpha = q$ *and* $\beta = q$, *it reduces to:*

$$ E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + q)} \tag{6}$$

where $q > 0$. *If* $\alpha = q$ *and* $\beta = 1$, *then:*

$$ E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + 1)} \tag{7}$$

where $q > 0$.

If $q = 1$, then $E_{1,1}(\lambda t) = e^{\lambda t}$. See [7,8,10,18] for more details. The work in [18] is exclusively for the study and application of the Mittag–Leffler function. Note that when $q = 1$ in Equation (7) is the special case of the integer derivative, it is the usual exponential function. Since we seek solutions of the sequential Caputo fractional differential equations to yield the integer solutions as a special case, we need the following definition of the sequential Caputo fractional derivative of order nq.

Definition 5. *The Caputo fractional derivative of order* nq, $n - 1 < nq < n$, *is said to be the sequential Caputo fractional derivative, if the relation:*

$$ ({}^cD^{nq})u(t) = {}^cD^q({}^cD^{(n-1)q})u(t) \tag{8}$$

holds for $n = 2, 3, .., etc.$

Consider the linear fractional differential equations of the form:

$$ {}^cD^q u(t) = \lambda u(t) + f(t), \quad u(t_0) = u_0, \ \ on \ \ [t_0, t_0 + T], \ T > 0 \tag{9} $$

where $0 < q < 1$ and λ is a constant and $f(t) \in C([t_0, t_0 + T], \mathbb{R})$. The solution of Equation (9) exists and is unique where λ is constant. The explicit solution of Equation (9) is given by:

$$ u(t) = u_0 E_{q,1}(\lambda(t - t_0)^q) + \int_{t_0}^{t} (t - s)^{q-1} E_{q,q}(\lambda(t - s)^q) f(s) ds \tag{10} $$

See [7,8] for details.

This explicit solution Equation (10) is useful to develop our main result of fractional differential equations of order $2q$, when $1 < 2q < 2$, with constant coefficients.

Definition 6. *We say that $u(t)$ is a C_p continuous function on $([t_0, t_0 + T], \mathbb{R})$, if $(t - t_0)^p u(t)$ is continuous on $[t_0, t_0 + T]$.*

In particular, if u is a continuous function on $[t_0, t_0 + T]$, then it is automatically C_p continuous; see [8–10]. We use this information in our first main result.

3. Main Results

3.1. Solution of the Linear Caputo Fractional Differential Equation in the Space of Continuously Differential Functions

Consider the linear Caputo fractional differential equations:

$$ {}^cD^q u(t) = p(t)u + f(t), \ \ t_0 < t < t_0 + T, \, T > 0, \ \ u(t_0) = u_0 \tag{11} $$

where $p(t)$ and $f(t)$ are continuous on $[t_0, t_0 + T]$. We seek solution $u(t)$ of Equation (11), which is C^1 on $[t_0, t_0 + T]$.

Note that if $u \in C^1([t_0, t_0 + T], \mathbb{R})$, then $u \in C^1_\gamma$ on $([t_0, t_0 + T], \mathbb{R})$, which follows from Definition 6. Furthermore, see page 4 of [8] for details.

In this section, in our first result, we obtain a symbolic representation for the solution of Equation (11). For this purpose, we note that the solution of Equation (11) is also the solution of the corresponding Volterra fractional integral equations:

$$ u(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} p(s) u(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s) ds \tag{12} $$

for $t_0 < t < t_0 + T, \ T > 0$, and *vice versa* [7,8,10]. We use this information to obtain a symbolic representation for the solution of Equation (11). This is precisely the next result.

Theorem 1. *Let $p(t)$ and $f(t) \in C([t_0, t_0 + T], \mathbb{R})$, then the solution of the linear Caputo fractional differential Equation (11) can be symbolically represented as:*

$$ u(t) = u_0 e^{{}^cD^{-q} p(t)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} e^{{}^cD^{-q} p(s)} f(s) ds, \ \ t_0 < t < t_0 + T, \, T > 0 \tag{13} $$

Proof. We achieve this by obtaining a representation form for the solution of Equation (12), which is also the solution of Equation (11). For this purpose, consider the sequence $\{u_n(t)\}$ defined by:

$$u_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) u_{n-1}(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds, \;\; t_0 < t < t_0 + T, \;\; T > 0 \tag{14}$$

Starting with the initial approximation $u_0(t) = u_0$, we get:

$$u_1(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) u_0 ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds, \;\; t_0 < t < t_0 + T, \;\; T > 0 \tag{15}$$

which simplifies to:

$$u_1(t) = u_0[1 + D^{-q}p(t)] + D^{-q}f(t) \tag{16}$$

Since $p(t)$ and $f(t)$ are continuous on $[t_0, t_0 + T]$ and $D^{-q}p(t)$ and $D^{-q}f(t)$ are continuous on a closed and bounded set and, hence, they are uniformly continuous.

If $p(t) \equiv \lambda$, a constant, then:

$$u_1(t) = u_0 \left(1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)}\right) + \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds \tag{17}$$

If $|p(t)| \leq \lambda$, then:

$$|u_1(t)| \leq |u_0| \left(1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)}\right) + \int_{t_0}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} |f(s)| ds \tag{18}$$

This proves that $u_1(t)$ is uniformly continuous on $[t_0, t_0 + T]$.

Continuing this process, we get:

$$u_2(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) u_1(s) ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) ds, \;\; t_0 < t < t_0 + T, \;\; T > 0 \tag{19}$$

This simplifies to,

$$u_2(t) = u_0 \left\{1 + D^{-q}p(t) + D^{-q}(p(t)D^{-q}p(t))\right\} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) D^{-q} f(s) ds + D^{-q}f(t) \tag{20}$$

This is achieved by interchanging the order of integration as:

$$\frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) \left\{\int_{t_0}^{s} (s-\sigma)^{q-1} f(\sigma) d\sigma\right\} ds = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s) \left\{\int_{t_0}^{s} (s-\sigma)^{q-1} p(\sigma) d\sigma\right\} ds \tag{21}$$

Now, we get:

$$u_2(t) = u_0 \left\{1 + D^{-q}p(t) + D^{-q}(p(t)D^{-q}p(t))\right\} + D^{-q}\left\{(1 + D^{-q}p(t))f(t)\right\} \tag{22}$$

If $p(t) \equiv \lambda$, a constant, then:

$$u_2(t) = u_0 \left\{1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)} + \frac{\lambda(t-t_0)^{2q}}{\Gamma(2q+1)}\right\} + \int_{t_0}^{t} (t-s)^{q-1} \left\{\frac{1}{\Gamma(q)} + \frac{\lambda(t-s)^q}{\Gamma(2q)}\right\} f(s) ds \tag{23}$$

If $|p(t)| \leq \lambda$, then:

$$|u_2(t)| \leq |u_0| \left\{1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)} + \frac{\lambda(t-t_0)^{2q}}{\Gamma(2q+1)}\right\} + \int_{t_0}^{t} (t-s)^{q-1} \left\{\frac{1}{\Gamma(q)} + \frac{\lambda(t-s)^q}{\Gamma(2q)}\right\} |f(s)|ds \tag{24}$$

This proves that $u_2(t)$ is uniformly continuous on $[t_0, t_0+T]$.

Thus, in general, we get by induction,

$$u_n(t) = u_0 \sum_{k=0}^{n} \frac{(D^{-q}(p))^k}{k!} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \sum_{k=0}^{n-1} \frac{(D^{-q}(p))^k}{k!} f(s)ds \tag{25}$$

where:

$$\sum_{k=0}^{n} \frac{(D^{-q}(p))^k}{k!} \tag{26}$$

is such that

$$^cD^q \left(\sum_{k=0}^{n} \frac{(D^{-q}(p))^k}{k!}\right) = p \left(\sum_{k=0}^{n} \frac{(D^{-q}(p))^{k-1}}{(k-1)!}\right) \tag{27}$$

We have used relation Equation (49) of [19] to obtain Equation (27).

If $p(t) \equiv \lambda$, a constant, then:

$$u_n(t) = u_0 \left\{1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)} + \frac{\lambda(t-t_0)^{2q}}{\Gamma(2q+1)} + ... + \frac{\lambda(t-t_0)^{nq}}{\Gamma(nq+1)} ...\right\} \tag{28}$$

$$+ \int_{t_0}^{t} (t-s)^{q-1} \left\{\frac{1}{\Gamma(q)} + \frac{\lambda(t-s)^q}{\Gamma(2q)} + ... + \frac{\lambda(t-s)^{nq}}{\Gamma(nq)} + ...\right\} f(s)ds \tag{29}$$

If $|p(t)| \leq \lambda$ on $[t_0, t_0+T],\ T > 0$, then:

$$|u_n(t)| \leq |u_0| \left\{1 + \frac{\lambda(t-t_0)^q}{\Gamma(q+1)} + \frac{\lambda(t-t_0)^{2q}}{\Gamma(2q+1)} + ... + \frac{\lambda(t-t_0)^{nq}}{\Gamma(nq+1)} ...\right\} \tag{30}$$

$$+ \int_{t_0}^{t} (t-s)^{q-1} \left\{\frac{1}{\Gamma(q)} + \frac{\lambda(t-s)^q}{\Gamma(2q)} + ... + \frac{\lambda(t-s)^{nq}}{\Gamma(nq)} + ...\right\} |f(s)|ds \tag{31}$$

$$\leq |u_0| E_{q,1}(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) f(s)ds = F(t) \tag{32}$$

for $n = 0, 1, 2, ...$, which proves that $u_n(t)$ is uniformly continuous on $[t_0, t_0+T]$ and $|u_n(t)| \leq F(t)$ on $[t_0, t_0+T]$, for all n. Thus, $u_n(t)$ converges to, say, $u(t)$ on $[t_0, t_0+T]$. Now, taking the limit as $n \to \infty$ in Equation (25), we obtain the following symbolic representation for $u(t)$ as:

$$u(t) = u_0 e^{^cD^{-q}p(t)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} e^{^cD^{-q}p(s)} f(s)ds,\ \ t_0 < t < t_0+T,\ T > 0 \tag{33}$$

Here:

$$e^{^cD^{-q}p(t)} = 1 + {}^cD^{-q}p + \frac{(^cD^{-q}p)^2}{2!} + ... \frac{(^cD^{-q}p)^n}{n!} + ... \tag{34}$$

where:

$$\frac{(^cD^{-q}p)^n}{n!} = {}^cD^{-q}(p\,{}^cD^{-q}(p\,{}^cD^{-q}(.....(ntimes)) \tag{35}$$

When taking the limit as $n \to \infty$ in Equation (30), we obtain Equation (10). Now, taking the limit as $n \to \infty$ in Equation (14) using the Lebesgue dominated convergence, we get that $u(t)$ is the solution of fractional integral Equation (12). Thus, $u(t)$ is the solution of Equation (11), as well. This concludes the proof. □

Although we have obtained a symbolic representation for the solution of Equation (11), it is very useful in the numerical computation of the solution of Equation (11).

Remark 1. *If $p(t) \equiv \lambda$, a constant, then the symbolic solution of Equation* (11) *results in the solution Equation* (10)*. If $q = 1$ and $f = 0$, we get the solution of the ordinary differential equation: $\frac{du}{dt} = p(t)u(t), u(t_0) = u_0$.*

Next, we prove that the solution of the initial value problem Equation (11) is unique. Let $u_1(t)$ and $u_2(t)$ be any two solutions of the initial value problem Equation (11). Let $U = u_1 - u_2$ be the solution of Equation (11) with $f \equiv 0$, and $U(t_0) = 0$. In this case, we get $U(t) = 0$. This proves that $U = u_1 - u_2 \equiv 0$. This proves our claim.

3.2. Linear Fractional Differential Equations with Constant Coefficients of Order $2q$, Where $1 < 2q < 2$.

Next, we consider the sequential Caputo linear fractional differential equations of order $2q$ with initial conditions of the form:

$$ {}^cD^{2q}u(t) + B\ {}^cD^q u(t) + Cu(t) = 0,\ u(t_0) = u_0,\ {}^cD^q u(t_0) = u_0^q \tag{36} $$

when $1 < 2q < 2$. In this section, throughout, we have used the initial condition as ${}^cD^q u(t_0) = u_0^q$ instead of $u'(t_0) = u_0$. The advantage of this is that we can use this even when $0 < 2q < 1$. We obtain two linearly-independent solutions in terms of Mittag–Leffler functions of order q for Equation (36). Since we assume ${}^cD^{2q}u(t)$ to be sequential, the solutions we seek are of the form that satisfies the composite rule:

$$ {}^cD^{2q}u = {}^cD^q({}^cD^q u) \tag{37} $$

Let $u = E_{q,1}(rt^q)$ be the solutions of Equation (36), then the characteristic equation for Equation (36) is given by:

$$ r^2 + Br + C = 0 \tag{38} $$

If $B^2 - 4C > 0$, we will have two real and distinct roots, and the general solution of Equation (36) is given by:

$$ u(t) = c_1 E_{q,1}(r_1(t-t_0)^q) + c_2 E_{q,1}(r_2(t-t_0)^q) \tag{39} $$

If $B^2 - 4C = 0$, then let $r = r_1$ be the coincident roots. Now, Equation (36) reduces to:

$$ ({}^cD^q - r_1)({}^cD^q - r_1)u = 0 \tag{40} $$

Letting $({}^cD^q - r_1)u = \bar{u}$, we can compute $\bar{u} = E_{q,1}(r_1(t-t_0)^q)$, using Equation (9) with $f(t) \equiv 0$. Now, using Equation (9) again, we obtain the solution of:

$$ ({}^cD^q - r_1)u = E_{q,1}(r_1(t-t_0)^q) \tag{41} $$

The solution of Equation (41) is given by:

$$ u(t) = u_0 E_{q,1}(r_1(t-t_0)^q) + \bar{u_0}\int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(r_1(t-s)^q)E_{q,1}(r_1(s-t_0)^q)ds \tag{42} $$

In this case, the two linearly-independent solutions are:

$$E_{q,1}(r_1(t-t_0)^q) \quad \text{and} \quad \frac{(t-t_0)^q}{q}E_{q,q}(r_1(t-t_0)^q) \tag{43}$$

The second solution is obtained by the closed form of the integral term of Equation (42)

If $B^2 - 4C < 0$, then we have two complex roots for r, say, $r_1 = \lambda_1 + i\lambda_2$ and $r_2 = \lambda_1 - i\lambda_2$.

In this case, the two linearly-independent solutions are:

$$E_{q,1}((\lambda + i\mu)(t-t_0)^q) \quad \text{and} \quad E_{q,1}((\lambda - i\mu)(t-t_0)^q) \tag{44}$$

Note that the usual exponential rule does not hold good for the Mittag–Leffler function. Hence, the form of solution in Equation (44) cannot be simplified further as in the integer case. However, when $\lambda = 0$, the two linearly-independent solutions can be simplified and written as:

$$\sin_q(\mu t) \quad \text{and} \quad \cos_q(\mu t) \tag{45}$$

Here:

$$\sin_q(\mu t) = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k+1}}{\Gamma((2k+1)q+1)}$$

and:

$$\cos_q(\mu t) = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k}}{\Gamma(2kq+1)}$$

For $q = 1$, they are the usual $\sin \mu t$ and $\cos \mu t$ functions.

In the next section, we present some numerical examples and their graphs for our theoretical results developed in this section. All of our numerical results and their graphs are computed using MATLAB.

4. Numerical Results

In this section, we present numerical examples for our explicit computation of solutions of Equations (9) and (11) when $0 < q < 1$ and $1 < 2q < 2$. We have also presented examples when $0 < 2q < 1$. Specially, we have demonstrated that $q = 0.5$ is the bifurcation value where the nature of the solution and its graph changes.

Now, we present two examples when:

$$p(t) = (t-t_0)^q \quad \text{and} \quad p(t) = -(t-t_0)^q \tag{46}$$

when $f(t) \equiv 0$ in Equation (11).

Example 1. *For $p(t) = (t-t_0)^q$, then Equation (11) simplifies to:*

$${}^cD^q u(t) = (t-t_0)^q u(t), \quad u(t_0) = u_0 \ \text{on} \ [t_0, t_0+T], \, T > 0 \tag{47}$$

where $0 < q < 1$.

The solution for Equation (47) is obtained in the form:

$$u(t) = u_0 \left\{ 1 + \sum_{k=1}^{\infty} ((t-t_0)^{2q})^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)} \right\} \tag{48}$$

Example 2. *For* $p(t) = -(t-t_0)^q$, *then Equation* (11) *reduces to:*

$$^cD^q u(t) = -(t-t_0)^q u(t), \quad u(t_0) = u_0 \ \ on \ \ [t_0, t_0+T], \ T > 0 \tag{49}$$

where $0 < q < 1$.

The solution for Equation (49) is obtained in the form:

$$u(t) = u_0 \left\{ 1 + \sum_{k=1}^{\infty} ((t-t_0)^{2q})^k (-1)^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)} \right\} \tag{50}$$

Next, we present graphs of the numerical simulation of the solution of Equations (47) and (49) for different q values when $0 < q < 1$ and for 10 iterations.

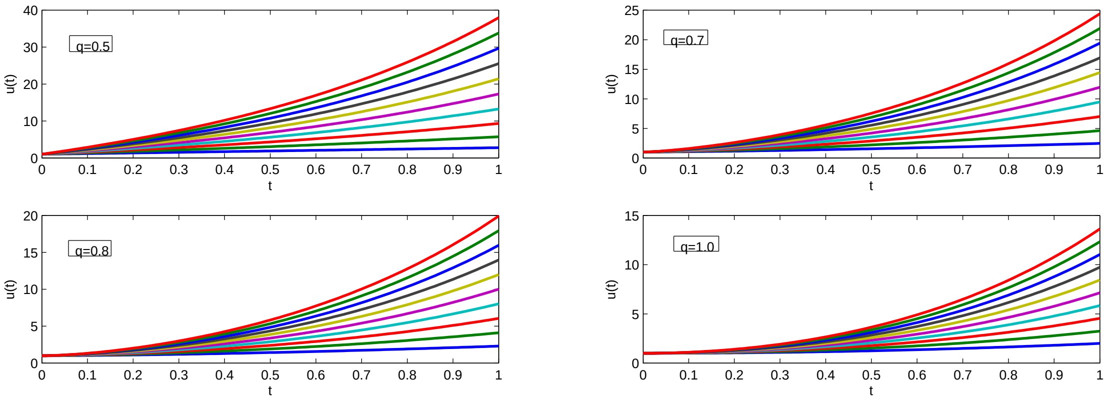

Figure 1. For the special case $\lambda = (t-t_0)^q$ when $q = 0.5, 0.7, 0.8, 1.0$.

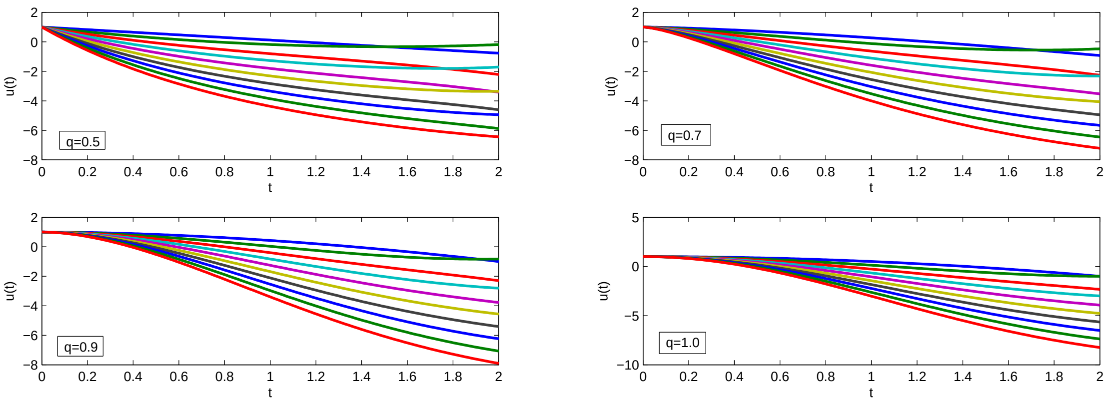

Figure 2. For the special case $\lambda = -(t-t_0)^q$ when $q = 0.5, 0.7, 0.9, 1.0$.

Example 3. *Consider the linear Caputo fractional differential equation of order* $2q$, $1 < 2q < 2$*:*

$$^cD^{2q}u(t) - 3\ {}^cD^q u(t) + 2u(t) = 0 \tag{51}$$

where $u(0) = 0$, $^cD^q(u(0)) = 1$ *for* $0.5 < q < 1$.

Let $u = E_{q,1}(rt^q)$, then the solution for Equation (51) is given by the equation:

$$u(t) = -1E_{q,1}((t - t_0)^q) + E_{q,1}(2(t - t_0)^q) \tag{52}$$

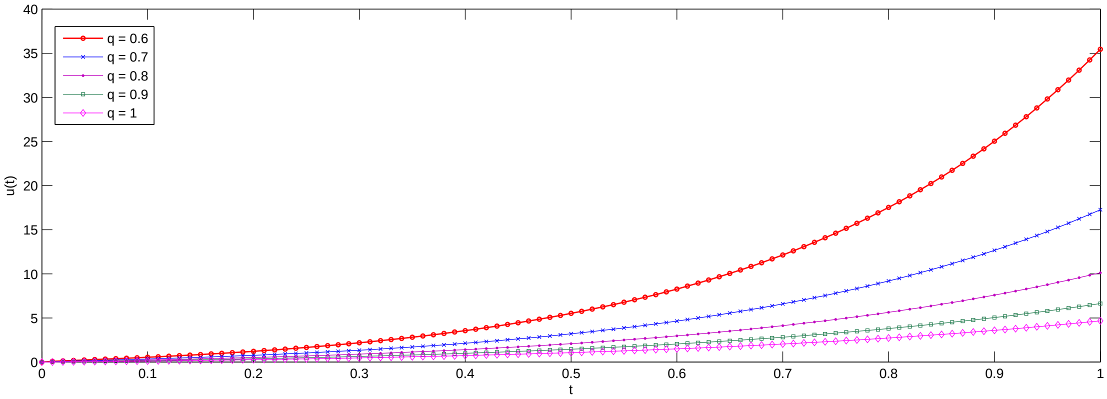

Figure 3. Graph for real and distinct roots when $q = 0.6, 0.7, 0.8, 0.9, 1.0$.

Example 4. *Consider the linear Caputo fractional differential equation of order* $2q$*,* $1 < 2q < 2$*:*

$$^cD^{2q}u(t) - 2\ ^cD^{q}u(t) + u(t) = 0 \tag{53}$$

where $u(0) = 1$*,* $^cD^q(u(0)) = 2$ *for* $0.5 < q < 1$*.*

Let $u = E_{q,1}(rt^q)$, then the solution for Equation (53) is given by the equation:

$$u(t) = c_2E_{q,1}(2(t - t_0)^q) + c_1 \left\{ \int_{t_0}^{t} (t - s)^{q-1} E_{q,q}(2(t - s)^q)E_{q,1}(2(s - t_0)^q)ds \right\} \tag{54}$$

The above expression reduces to:

$$u(t) = E_{q,1}((t - t_0)^q) + \frac{(t - t_0)^q}{q} E_{q,q}((t - t_0)^q) \tag{55}$$

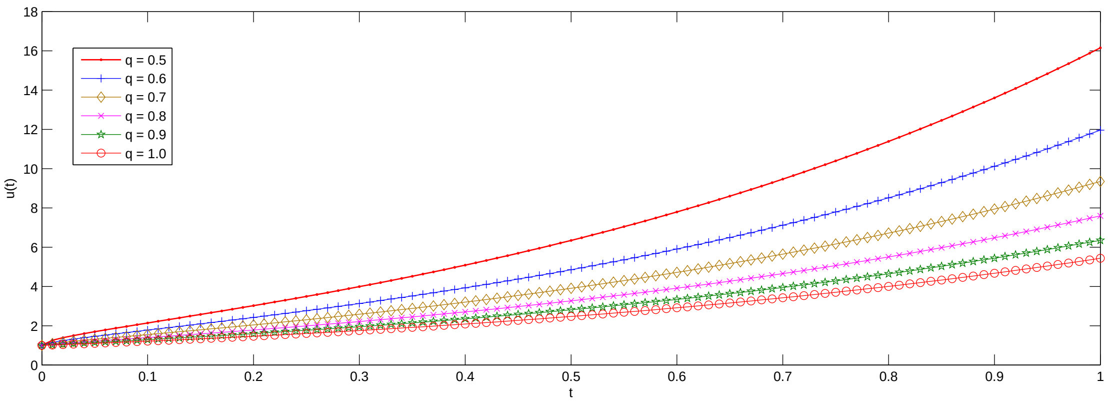

Figure 4. Graph for real and coincident roots when $q = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$.

Example 5. *Consider the linear Caputo fractional differential equation of order* $2q$*,* $1 < 2q < 2$*:*

$$^{c}D^{2q}u(t) + u(t) = 0 \tag{56}$$

where $u(0) = 0$*,* $^{c}D^{q}(u(0)) = 1$ *or* $u(0) = 1$*,* $^{c}D^{q}(u(0)) = 0$ *for* $0.5 < q < 1$*.*

Let $u = E_{q,1}(rt^q)$, then the solution for Equation (56) is given by the equation:

$$\sin_q(t^q) = u(t) = \sum_{k=0}^{\infty} \frac{(t)^{2kq}(-1)^k}{\Gamma(2kq+1)} \tag{57}$$

and:

$$\cos_q(t^q) = v(t) = \sum_{k=0}^{\infty} \frac{(t)^{(2k+1)q}(-1)^k}{\Gamma((2k+1)q+1)} \tag{58}$$

where $t \geq 0$, $0.5 < q < 1$.

In the graph below, when $0.5 < q < 1$, the zeros of $sin_q(t^q)$ and $\cos_q(t^q)$ are approximately close to the zeros of $\sin t$ and $\cos t$ graphs.

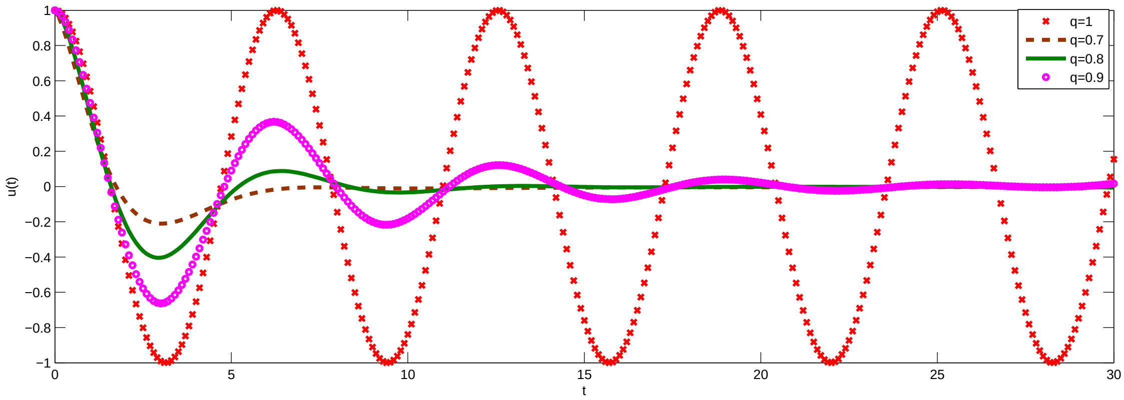

Figure 5. $\cos_q(t^q)$ graph.

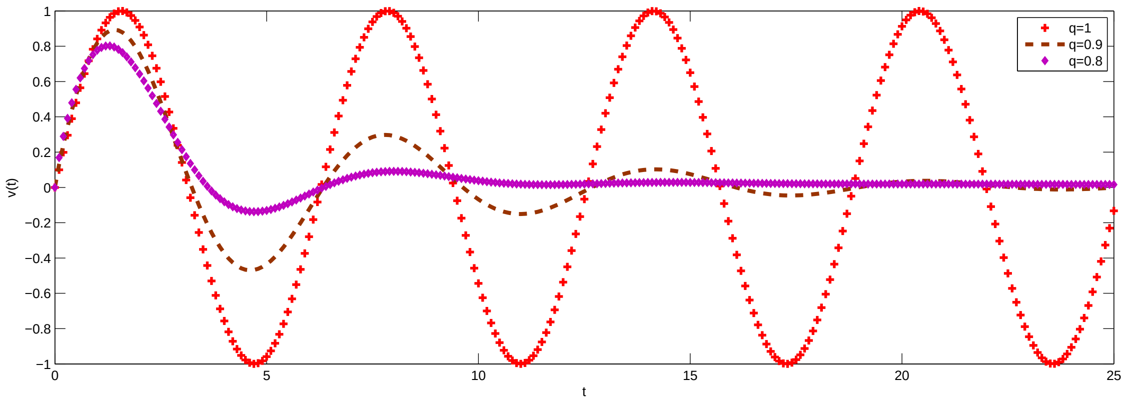

Figure 6. $\sin_q(t^q)$ graph.

When $q = 0.5$, there is a bifurcation in the $\sin_q(t^q)$ and $\cos_q(t^q)$ graph. That is, they no longer are oscillatory solutions.

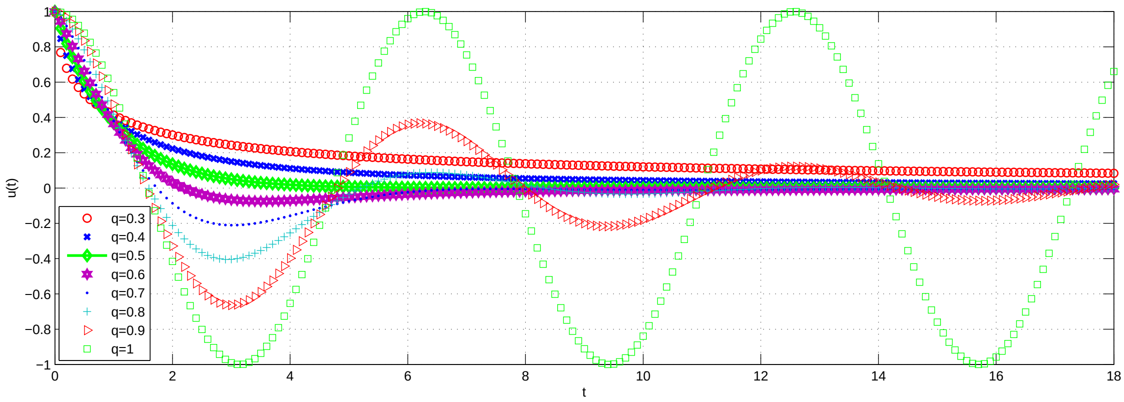

Figure 7. $\cos_q(t^q),\ 0 < q \leq 1$ graph.

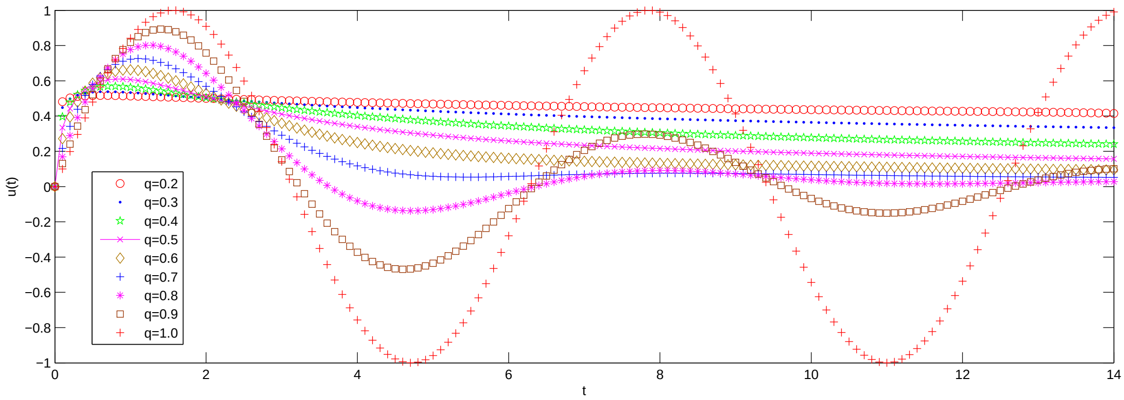

Figure 8. $\sin_q(t^q),\ 0 < q \leq 1$ graph.

When $0 < q < 0.5$, we have the exponentially decaying graph given below.

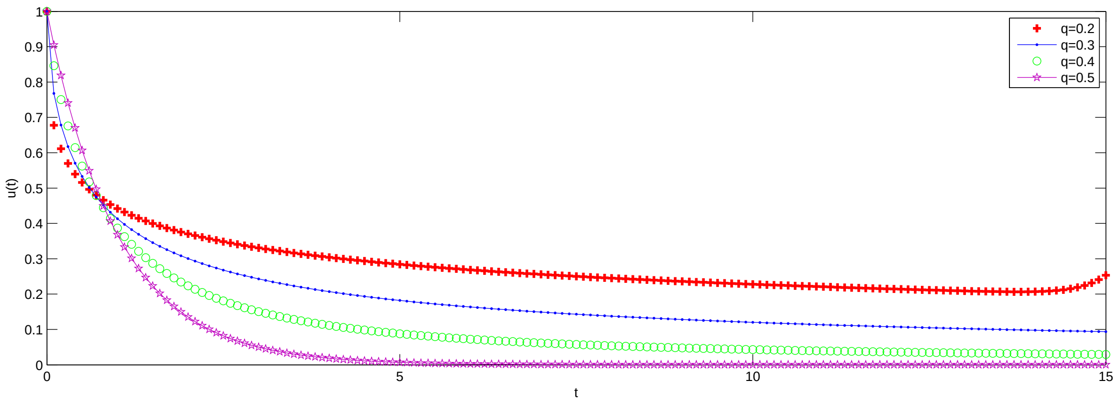

Figure 9. $\cos_q(t^q)$ graph.

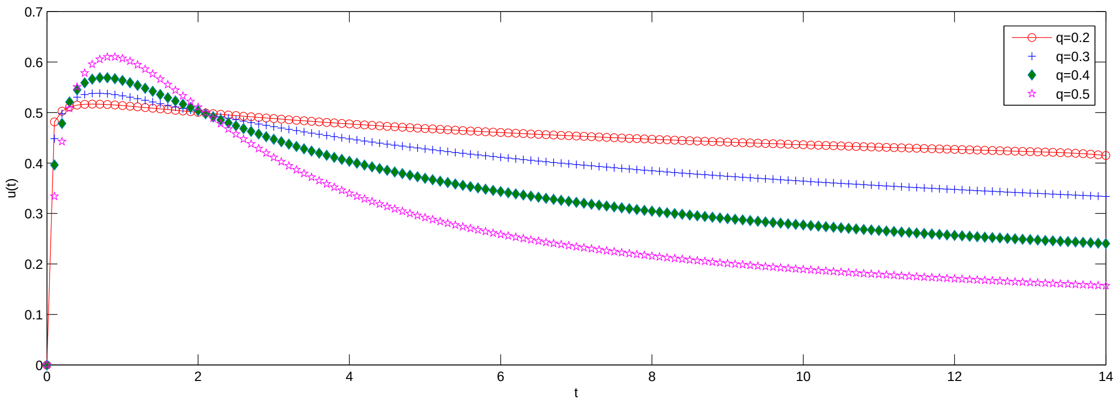

Figure 10. $\sin_q(t^q)$ graph.

Example 6. *Consider the linear Caputo fractional differential equation of order* $2q$*,* $1 < 2q < 2$*:*

$$^cD^{2q}u(t) - 2\ {}^cD^qu(t) + 2u(t) = 0 \tag{59}$$

where $u(0) = 1$*,* $^cD^q(u(0)) = 1$ *for* $0.5 < q < 1$.

Let $u = E_{q,1}(rt^q)$, then the solution for Equation (59) is given by,

$$u(t) = 0.5E_{q,1}((1 + i)t^q) + 0.5E_{q,1}((1 - i)t^q) \tag{60}$$

where $0.5 < q < 1$.

The graph for Example 6 is given below.

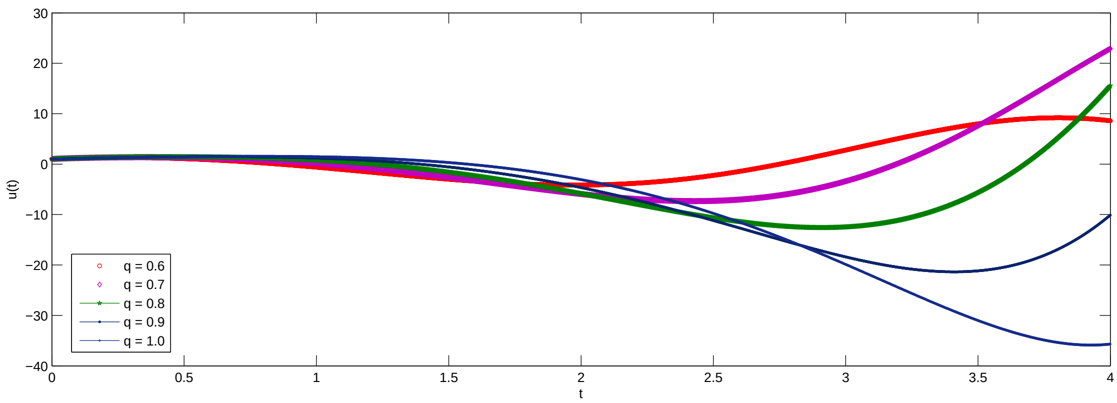

Figure 11. Complex roots graph.

5. Boundary Value Problem for Fractional Differential Equations

In this section, we consider the linear fractional differential equation of order $2q$ with the Dirichlet boundary condition. For that purpose, consider the boundary value problem,

$$^cD^{2q}u + u = f(t) \quad u(0) = A, \quad u(1) = B \tag{61}$$

If $A = 0$ and $B = 0$, then the general solution of Equation (61) is given by:

$$u(t) = c_1 \cos_q t^q + c_2 \sin_q t^q + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s)ds \tag{62}$$

By computing Green's function relative to Equation (61), we obtain the unique solution of the boundary value problem Equation (61) for $A = B = 0$ as:

$$u(t) = \int_0^1 G(t,s)f(s)ds \tag{63}$$

where:

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}\sin_q(1) - (1-s)^{q-1}\sin_q(t^q)}{\Gamma(q)sin_q(1)}, & 0 \le s \le t \le 1 \\ \frac{-(1-s)^{q-1}\sin_q(t^q)}{\Gamma(q)\sin_q(1)}, & 0 \le t \le s \le 1 \end{cases} \tag{64}$$

If $A \neq 0$ and $B \neq 0$, then using Green's function of Equation (61) given above, we obtain the unique solution of the boundary value problem Equation (61) as:

$$u(t) = \int_0^1 G(t,s)f(s)ds + y(t) \tag{65}$$

where $y(t) = A\cos_q(t^q) + (B - A\cos_q(1))\frac{\sin_q(t^q)}{\sin_q(1)}$.

This expression is useful in computing the solution of the linear fractional boundary value problem of order $2q$ for $0 < 2q < 1$, with Dirichlet boundary conditions.

Next, we prove that the solution of the boundary value problem is unique. For that purpose, assume that $u_1(t)$ and $u_2(t)$ are any two solutions of the boundary value problem. That is, $u_1(t)$ and $u_2(t)$ satisfy the following boundary value problem:

$$^cD^{2q}u_1 + u_1 = f(t), \quad u_1(0) = A, \quad u_1(1) = B \tag{66}$$

and:

$$^cD^{2q}u_2 + u_2 = f(t), \quad u_2(0) = A, \quad u_2(1) = B \tag{67}$$

respectively. Then, by setting $U = u_1 - u_2$, it is easy to observe that U satisfies the homogeneous boundary value problem with the homogeneous boundary conditions of the form:

$$^cD^{2q}U + U = 0, \quad U(0) = 0, \quad U(1) = 0 \tag{68}$$

Then, the general solution of Equation (68) is:

$$U(t) = c_1\cos_q t^q + c_2\sin_q t^q \tag{69}$$

Using the homogeneous boundary conditions, we get $c_1 = 0, c_2 = 0$; this proves that $U = u_1 - u_2 \equiv 0$. This proves our claim.

6. Conclusions

We have developed some basic results for sequential Caputo fractional differential equations of order q and $2q$, respectively. We have developed the symbolic representation form for the Caputo linear fractional differential equations of order q, where $0 < q < 1$. This symbolic form can be used to develop an effective numerical scheme to solve the linear fractional differential equation with a variable coefficient and a non-homogeneous term. In addition, our results yield most of the integer results as a special case. Our initial conditions also are assumed in such a way that the initial conditions of the integer results are special cases. We have presented many numerical results and their graphs to justify the analytical solutions for the sequential Caputo fractional differential equations of order q and $2q$, when $0 < q < 1$ and $1 < 2q < 2$, respectively. The interesting observations are that $\sin_q(t^q)$ and $\cos_q(t^q)$ functions are periodic functions similar to the usual $\sin t$ and $\cos t$ functions, but the solutions are decaying without a decay term. This means that there is damping without a damping term. Further, this result can be used as a tool to develop the corresponding eigenvalue results. In addition, the solutions ceases to oscillate when $q = 0.5$; thus, $q = 0.5$ is the bifurcation value where the functions $\sin_q(t^q)$ and $\cos_q(t^q)$ cease to oscillate. That is, when $q = 0.5$, the nature of the solution and its graph changes. We have developed an integral representation form for the solution of the non-homogeneous linear Caputo Dirichlet boundary value problem by using Green's function. Finally, we have proved the uniqueness result for the linear Caputo fractional boundary value problem. This will be a useful tool to develop iterative methods to compute the solution of the corresponding non-linear Caputo fractional boundary value problem.

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Author Contributions

There is equal contribution by the authors.

Conflicts of Interest

The authors declare no conflict of interest.

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