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*Article*

# Subordination Principle for a Class of Fractional Order Differential Equations

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**Abstract:** The fractional order differential equation

$$u'(t) = Au(t) + \gamma D_t^\alpha Au(t) + f(t),\ t > 0, \quad u(0) = a \in X$$

is studied, where $A$ is an operator generating a strongly continuous one-parameter semigroup on a Banach space $X$, $D_t^\alpha$ is the Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$, $\gamma > 0$ and $f$ is an $X$-valued function. Equations of this type appear in the modeling of unidirectional viscoelastic flows. Well-posedness is proven, and a subordination identity is obtained relating the solution operator of the considered problem and the $C_0$-semigroup, generated by the operator $A$. As an example, the Rayleigh–Stokes problem for a generalized second-grade fluid is considered.

**Keywords:** Riemann–Liouville fractional derivative; $C_0$-semigroup of operators; Mittag–Leffler function; completely monotone function; Bernstein function

**MSC classifications:** 26A33; 33E12; 34A08; 35R11; 47D06

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## 1. Introduction

Consider the fractional order differential equation:

$$u'(t) = Au(t) + \gamma D_t^\alpha Au(t) + f(t),\ t > 0, \quad u(0) = a \in X \tag{1}$$

where $D_t^\alpha$ is the Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$, $\gamma > 0$, $A$ is an unbounded closed linear operator defined on a Banach space $X$ and $f$ is an $X$-valued function.

Our motivation for studying this equation comes from recent works where related problems appear in the modeling of unidirectional viscoelastic flows. For example, if the operator $A$ is some realization of the Laplace operator, then Problem (1) is the Rayleigh–Stokes problem for a generalized second-grade fluid; see e.g., [1–3]. Exact solutions of this problem in the form of eigenfunction expansion on a bounded space domain are obtained in [4,5]. Numerical analysis of Problem (1) with $A$ being the one- or two-dimensional Laplacian with Dirichlet boundary conditions is carried out in [6–11]. In [12], a compact Duhamel-type representation of the solution is obtained and used for its numerical computation.

In addition, let us note that the governing equation in (1) can be also considered as a distributed order equation in the Riemann–Liouville sense:

$$u'(t) = \int_0^1 \mu(\beta) D_t^{\beta} Au(t)\, d\beta, \quad t > 0$$

with weight function $\mu(\beta) = \delta(\beta) + \gamma\delta(\beta - \alpha)$; see [13] and the references cited there.

If $\gamma = 0$ and $f \equiv 0$, Problem (1) reduces to the classical abstract Cauchy problem:

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = a \in X \tag{2}$$

There is a vast amount of literature devoted to this problem and its equivalent formulation: the theory of strongly continuous ($C_0$) one-parameter semigroups of operators (see, e.g., [14]). In the present paper, it is assumed that the operator $A$ is a generator of a $C_0$-semigroup, *i.e.*, that the classical Cauchy Problem (2) is well-posed. Under this assumption, it will be proven that Problem (1) is well-posed, and a relationship between the corresponding solution operators of these two problems will be established in the form:

$$S(t) = \int_0^{\infty} \varphi(t,\tau) T(\tau)\, d\tau, \quad t > 0 \tag{3}$$

where $S(t)$ and $T(t)$ are the solution operators of the considered Problem (1) and the classical Cauchy Problem (2), respectively. It appears that the function $\varphi(t,\tau)$ is a probability density with respect to both $t$ and $\tau$. Such a relationship between two problems is called the subordination principle; see [15], Ch. 4.

Representations of the form of (3) are useful in the study of differential equations of fractional order, in particular for the understanding of the regularity and the asymptotic behavior of the solution of the subordinate problem. For example, the subordination principle for fractional evolution equations with the Caputo derivative (see [16], Ch. 3) has been successfully applied to inverse problems [17], for asymptotic analysis of diffusion wave equations [18], for the study of stochastic solutions [19], semilinear equations of fractional order [20], systems of fractional order equations [21], nonlocal fragmentation models with the Michaud time derivative [22], *etc.* This gives the author the motivation to present an analogous principle for Problem (1).

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, Problem (1) is reformulated as a Volterra integral equation, and the properties of its kernel are studied, as well as the solution of the scalar version of the problem, where $A$ is a negative constant. Section 4 contains the main result: the subordination principle for Problem (1) and some corollaries. As an example, the Rayleigh–Stokes problem for a generalized second-grade fluid is considered in Section 5.

## 2. Preliminaries

The sets of positive integers, real and complex numbers are denoted as usual by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$. Denote by $\Sigma_\theta$ the sector:

$$\Sigma_\theta = \{s \in \mathbb{C};\ s \neq 0, |\arg s| < \theta\}$$

For the sake of brevity, the following notation is used:

$$\omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0,\ t > 0 \tag{4}$$

where $\Gamma(.)$ is the Gamma function.

Let $*$ denote the Laplace convolution:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau,\ \ t > 0$$

Denote by $J_t^\alpha$ the Riemann–Liouville fractional integral of order $\alpha$:

$$J_t^\alpha f(t) = \int_0^t \omega_\alpha(t-\tau)f(\tau)\, d\tau = (\omega_\alpha * f)(t),\ \ \alpha > 0$$

and by $D_t^\alpha$ the Riemann–Liouville fractional derivative of order $\alpha$:

$$D_t^\alpha f(t) = \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-\tau)f(\tau)\, d\tau = \frac{d}{dt}J_t^{1-\alpha}f(t),\ \ \alpha \in (0,1) \tag{5}$$

We use the following standard notations for the Laplace transform:

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st}f(t)\, dt$$

Note that:

$$\mathcal{L}\left\{\omega_\alpha\right\}(s) = s^{-\alpha}, \quad \alpha > 0,\ t > 0 \tag{6}$$

Application of the Laplace transform to the Riemann–Liouville fractional differential operator $D_t^\alpha$ gives (see, e.g., [23]):

$$\mathcal{L}\{D_t^\alpha f\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s) - (\omega_{1-\alpha} * f)(0),\ \ \alpha \in (0,1) \tag{7}$$

For functions continuous for $t > 0$ and such that $f(0)$ is finite, the second term in (7) vanishes, and the identity reduces to:

$$\mathcal{L}\{D_t^\alpha f\}(s) = s^\alpha \mathcal{L}\{f(t)\}(s),\ \ \alpha \in (0,1) \tag{8}$$

A $C^\infty$ function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotone if:

$$(-1)^n f^{(n)}(t) \geq 0,\ \text{for all } t > 0,\ n \in \mathbb{N}_0 \tag{9}$$

The characterization of completely monotone functions is given by Bernstein's theorem (see, e.g., [24]), which states that a function $f : (0, \infty) \to \mathbb{R}$ is completely monotone if and only if it can be represented as the Laplace transform of a nonnegative measure.

A $C^\infty$ function $f : (0, \infty) \to \mathbb{R}$ is called a Bernstein function if it is nonnegative and its first derivative $f'(t)$ is a completely monotone function. The classes of completely monotone functions and Bernstein functions will be denoted by $\mathcal{CMF}$ and $\mathcal{BF}$.

The three-parameter Mittag–Leffler function, known as the Prabhakar function, is defined by (e.g., [23,25]):

$$E^{\delta}_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \frac{(\delta)_j}{j!}, \qquad \alpha, \beta, \delta, z \in \mathbb{C},\ \Re\alpha > 0 \tag{10}$$

where $(\delta)_j = \delta(\delta + 1) \cdots (\delta + j - 1),\ j \in \mathbb{N}, \delta \in \mathbb{C},\ (\delta)_0 = 1,\ \delta \in \mathbb{C}\backslash\{0\}$. It was introduced by T.R. Prabhakar in 1971 as a generalization of the classical Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, where:

$$E_\alpha(z) = E^1_{\alpha,1}(z),\ \ E_{\alpha,\beta}(z) = E^1_{\alpha,\beta}(z)$$

It is known that if $t > 0$, then $E_\alpha(-t) \in \mathcal{CMF}$ for $0 < \alpha < 1$ and $E_{\alpha,\beta}(-t) \in \mathcal{CMF}$ for $0 \leq \alpha \leq 1$, $\alpha \leq \beta$; see, e.g., [26]. For complete monotonicity property of three-parameter Mittag–Leffler functions, we refer to the recent paper [27].

The asymptotic behavior of the three-parameter Mittag–Leffler function (10) can be obtained from the identity [28]:

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{(-z)^{-\delta-j}}{\Gamma(\beta - \alpha(\delta + j))} \frac{(\delta)_j}{j!}, \qquad |z| > 1 \tag{11}$$

Other kinds of asymptotic estimates were provided in [29], and on their basis, the convergence of series in Functions (10) in the complex domain, similar to these appearing further in (27) and (34), is studied in the recent papers [25,30].

Recall also the Laplace transform pair [23]:

$$\mathcal{L}\{t^{\beta-1} E^{\delta}_{\alpha,\beta}(-\mu t^\alpha)\}(s) = \frac{s^{\alpha\delta-\beta}}{(s^\alpha + \mu)^\delta} \tag{12}$$

Denote by $\Phi_\beta(z),\ \beta \in (0, 1)$ the following function of the Wright type, also called the Wright M-function or Mainardi function (see, e.g., [31]):

$$\Phi_\beta(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\beta k + 1 - \beta)},\ \ \beta \in (0, 1) \tag{13}$$

The following relationship with the Mittag–Leffler function holds:

$$E_\beta(-z) = \int_0^\infty \Phi_\beta(t) e^{-zt}\, dt,\ \ z \in \mathbb{C}$$

In particular, this identity implies that $\Phi_\beta(t)$ is a probability density function:

$$\Phi_\beta(t) \geq 0,\ t > 0;\ \ \int_0^\infty \Phi_\beta(t)\, dt = 1 \tag{14}$$

Let $X$ be a complex Banach space with norm $\|.\|$. Let $A$ be a closed linear unbounded operator in $X$ with dense domain $D(A)$, equipped with the graph norm $\|.\|_A$, $\|x\|_A := \|x\| + \|Ax\|$. Denote by $\varrho(A)$ the resolvent set of $A$ and by $R(s, A) = (sI - A)^{-1}$ the resolvent operator of $A$.

By integrating both sides of the governing equation in (1), we recast Problem (1) with $f \equiv 0$ into a Volterra integral equation:

$$u(t) = a + \int_0^t k(t-\tau)Au(\tau)\,d\tau \tag{15}$$

where the kernel $k(t)$ is specified later; see (20). Here, we recall some definitions and basic theorems, given in [15], concerning abstract Volterra integral equations.

**Definition 1.** *A function $u \in C(\mathbb{R}_+; X)$ is called a strong solution of the integral Equation (15) if $u \in C(\mathbb{R}_+; D(A))$ and (15) holds on $\mathbb{R}_+$. Problem (15) is said to be well-posed if for each $a \in D(A)$, there is a unique strong solution $u(t; a)$ of (15), and $a_n \in D(A)$, $a_n \to 0$ imply $u(t; a_n) \to 0$ in $X$, uniformly on compact intervals.*

For a well-posed problem, the solution operator $S(t)$ is defined by:

$$S(t)a = u(t; a), \ \ a \in D(A), \ t \geq 0$$

Since $S(t)$ is a bounded operator, it admits extension to all of $X$.

Suppose $\int_0^\infty e^{-st}|k(t)|\,dt < \infty$ for $s > 0$ and $\widehat{k}(s) \neq 0$, $1/\widehat{k}(s) \in \varrho(A)$ for $s > 0$. Then, the Laplace transform of the solution operator $S(t)$ of Problem (15):

$$H(s) = \int_0^\infty e^{-st} S(t)\,dt, \ \ \Re s > 0 \tag{16}$$

is given by

$$H(s) = \frac{g(s)}{s} R(g(s), A), \ \ g(s) = 1/\widehat{k}(s) \tag{17}$$

The generation theorem ([15], Theorem 1.3) states that Problem (15) is well-posed with solution operator $S(t)$ satisfying $\|S(t)\| \leq M, \ t \geq 0$, if and only if:

$$\|H^{(n)}(s)\| \leq M \frac{n!}{s^{n+1}}, \quad \text{for all } s > 0, \ n \in \mathbb{N}_0 \tag{18}$$

Note that the classical Cauchy Problem (2) can be also rewritten as Volterra Equation (15) with kernel $k(t) = 1$, $g(s) = s$. In this case, the generation theorem is known as the classical Hille–Yosida theorem.

## 3. Integral Reformulation of the Problem

Let first $f \equiv 0$. Assume $u, Au \in C(\mathbb{R}_+, X)$. Then, integrating both sides of the governing equation in (1), by the use of (5) and the identity $(J_t^{1-\alpha} Au)(0) = 0$, we obtain:

$$u(t) = a + \int_0^t \left(1 + \gamma\omega_{1-\alpha}(t-\tau)\right) Au(\tau)\,d\tau \tag{19}$$

that is the Volterra integral Equation (15) with kernel $k(t)$ given by:

$$k(t) = 1 + \gamma\omega_{1-\alpha}(t) \tag{20}$$

where the function $\omega_\alpha$ is defined in (4). Conversely, differentiating both sides of (19) and using that:

$$\frac{d}{dt}\left(\omega_{1-\alpha} * Au\right) = \frac{d}{dt}\left(J_t^{1-\alpha} Au\right) = D_t^\alpha Au$$

we get back the governing equation in (1). Since $(k * Au)(0) = 0$, the initial condition is also satisfied. The above observations give rise to following definition.

**Definition 2.** *Problem (1) is said to be well-posed if the Volterra integral Equation (19) is well-posed. For well-posed problems, the solution operator of the integral Equation (19) is called a solution operator of Problem (1).*

Let Problem (1) be well-posed, and let $S(t)$ be the corresponding solution operator. Consider the inhomogeneous Problem (1) with $f \in L^1(\mathbb{R}_+; X)$. In the same way as above, it can be rewritten as the integral equation:

$$u(t) = a + \int_0^t (1 + \gamma\omega_{1-\alpha}(t-\tau))\, Au(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau$$

Then, by the variation of the parameters formula (see e.g., [15]), the solution of the inhomogeneous Problem (1) is given by:

$$u(t) = S(t)a + \int_0^t S(t-\tau) f(\tau)\, d\tau \tag{21}$$

Therefore, it is essential for the homogeneous, as well as for the inhomogeneous problem, to study the solution operator $S(t)$.

We begin with summarizing some properties of the kernel $k(t)$ defined in (20), relevant for further study. Along with the kernel $k(t)$, the related function:

$$g(s) = (\widehat{k}(s))^{-1} = \frac{s}{1 + \gamma s^\alpha} \tag{22}$$

is also of interest, since it appears in (17). Here, $\widehat{k}$ is the Laplace transform of $k(t)$, and the Laplace transform pair (6) is used.

**Theorem 3.** *The functions $k(t)$ and $g(s)$ have the following properties:*
*(a) $k \in L^1_{loc}(\mathbb{R}_+)$;*
*(b) $k(t) \in \mathcal{CMF}$ for $t > 0$;*
*(c) $g(s) \in \mathcal{BF}$ for $s > 0$;*
*(d) $g(s)/s \in \mathcal{CMF}$ for $s > 0$;*
*(e) $g(s)$ admits analytic extension to $\Sigma_\pi$ and:*

$$|\arg g(s)| \le |\arg s|, \quad s \in \Sigma_\pi$$

*(f) the estimate holds true:*

$$|g(s)| \le C \min(|s|, |s|^{1-\alpha}),\ s \in \Sigma_\pi$$

**Proof.** The function $k(t)$ is infinitely continuously differentiable for $t > 0$ with integrable singularity at $t = 0$, and its derivatives satisfy (9); thus, (a) and (b) are fulfilled. To prove (c) and (d), note that by (12):

$$g(s) = s\frac{1}{1 + \gamma s^\alpha} = s\mathcal{L}\{\gamma^{-1} t^{\alpha-1} E_{\alpha,\alpha}(-\gamma^{-1} t^\alpha)\}(s)$$

Since $\alpha \in (0,1)$, the Mittag–Leffler function $E_{\alpha,\alpha}(-x) \in \mathcal{CMF}$ for $x > 0$ (see, e.g., [26]). Then, the function $E(t) = \gamma^{-1}t^{\alpha-1}E_{\alpha,\alpha}(-\gamma^{-1}t^{\alpha}) \in \mathcal{CMF}$ for $t > 0$. Therefore, $E(t)$ is a nonnegative and nonincreasing function. Moreover, $E(t)$ has an integrable singularity at $t = 0$ and $\lim_{t\to+\infty} E(t) = 0$. Then, according to Proposition 4.3 in [15], $g(s) = s\widehat{E}(s) \in \mathcal{BF}$. Moreover, $g(s)/s = \widehat{E}(s) \in \mathcal{CMF}$ for $s > 0$ by Bernstein's theorem. Further, Property (e) holds for $g(s)$, since it holds for $\widehat{k}(s)$ as a Laplace transform of a completely monotone function; see [15], Example 2.2. An alternative direct proof of (e), as well as a proof of Property (f) can be found in [11], Lemma 2.1. □

It is instructive to study first the scalar version of Equation (1), where $A = -\lambda$ is a given negative constant. Consider the problem:

$$u'(t) + \lambda u(t) + \lambda\gamma D_t^{\alpha}u(t) = 0, \quad u(0) = 1 \tag{23}$$

where $\lambda > 0$. Denote its solution by $u(t,\lambda)$. To solve (23), we apply the Laplace transform and use the identities (7) and $\mathcal{L}\{u'\}(s) = s\mathcal{L}\{u\}(s) - u(0)$. In this way, for the Laplace transform of $u(t,\lambda)$, one gets:

$$\int_0^{\infty} e^{-st}u(t,\lambda)\,dt = \frac{1}{s + \gamma\lambda s^{\alpha} + \lambda} \tag{24}$$

**Theorem 4.** *For any $\lambda > 0$, the solution $u(t,\lambda)$ of Problem (23) has the following properties:*
*(a) $u(0,\lambda) = 1$, $u(t,\lambda)$ is a positive nonincreasing function for $t > 0$ and $u(t,\lambda) \to 0$ as $t \to +\infty$ with:*

$$u(t,\lambda) \sim -\frac{\gamma t^{-\alpha-1}}{\lambda\Gamma(-\alpha)}, \quad t \to +\infty \tag{25}$$

*(b) $u(t,\lambda) \in \mathcal{CMF},\ t > 0$,*
*(c) The identity is satisfied:*

$$\int_0^{\infty} u(t,\lambda)\,dt = \frac{1}{\lambda} \tag{26}$$

*(d) The solution admits the following explicit representation:*

$$u(t,\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\gamma^{k+1}\lambda^{k+1}} t^{(\alpha-1)(k+1)} E^{k+1}_{\alpha,\alpha(k+1)-k}(-\gamma^{-1}t^{\alpha}) \tag{27}$$

**Proof.** Properties (a) and (b), except the asymptotic estimate (25), are proven in [11], Theorem 2.2. To prove (25), we apply the Karamata–Feller–Tauberian theorem (e.g., [24]). Since for small $|s|$, the Laplace transform (24) of $u(t,\lambda)$ is dominated by the function:

$$\frac{1}{\lambda\gamma s^{\alpha} + \lambda}$$

applying the asymptotic estimate (11) (note that $\Gamma(0)^{-1} = 0$), we obtain for large $t$:

$$u(t,\lambda) \sim \mathcal{L}^{-1}\left\{\frac{1}{\lambda\gamma s^{\alpha} + \lambda}\right\} = \frac{1}{\lambda\gamma}t^{\alpha-1}E_{\alpha,\alpha}(-\frac{1}{\gamma}t^{\alpha}) \sim -\frac{\gamma t^{-\alpha-1}}{\lambda\Gamma(-\alpha)}, \quad t \to +\infty$$

Identity (26) is obtained taking $s \to 0$ in (24).

Representation (27) in terms of three-parameter Mittag–Leffler functions is obtained by taking the inverse Laplace transform of Function (24). If $|s\lambda^{-1}(\gamma s^{\alpha} + 1)^{-1}| < 1$, then:

$$\frac{1}{s + \gamma\lambda s^{\alpha} + \lambda} = \frac{1}{\lambda(\gamma s^{\alpha} + 1)}\left(\frac{s}{\lambda(\gamma s^{\alpha} + 1)} + 1\right)^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(\gamma\lambda)^{k+1}} \frac{s^k}{(s^{\alpha} + \gamma^{-1})^{k+1}}$$

and applying term-wise the inverse Laplace transform, we get (27) by the use of (12). □

## 4. Subordination Principle

Assume the operator $A$ generates a bounded $C_0$ semigroup $T(t)$. The main goal of this paper is to prove that in this case, Problem (1) is well-posed, and its solution operator $S(t)$ satisfies the relationship:

$$S(t) = \int_0^\infty \varphi(t,\tau)T(\tau)\,d\tau, \quad t > 0 \tag{28}$$

with an appropriate function $\varphi(t,\tau)$. This is the so-called subordination principle: Problem (1) is subordinate to the classical Cauchy Problem (2).

Let the function $\varphi(t,\tau)$ be such that its Laplace transform with respect to $t$ satisfies:

$$\int_0^\infty e^{-st}\varphi(t,\tau)\,dt = \frac{g(s)}{s}e^{-\tau g(s)}, \quad s,\tau > 0 \tag{29}$$

where $g(s)$ is defined in (22). The reason for this is that then the operator $S(t)$, defined by (28), will satisfy (16) and (17). Indeed, by (28) and (29) and the identity for the Laplace transform of a $C_0$-semigroup $\int_0^\infty e^{-\mu\tau}T(\tau)\,d\tau = R(\mu, A)$, it follows:

$$\begin{aligned}\int_0^\infty e^{-st}S(t)\,dt &= \int_0^\infty \left(\int_0^\infty e^{-st}\varphi(t,\tau)\,dt\right)T(\tau)\,d\tau \\ &= \frac{g(s)}{s}\int_0^\infty e^{-\tau g(s)}T(\tau)\,d\tau = \frac{g(s)}{s}R(g(s), A)\end{aligned} \tag{30}$$

Then, by the uniqueness of the Laplace transform, $S(t)$ will be the solution operator of Problem (1).

Identity (29) implies that the function $\varphi(t,\tau)$ can be found by the inverse Laplace integral:

$$\varphi(t,\tau) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st-\tau g(s)}\,\frac{g(s)}{s}\,ds, \quad c > 0,\ t,\tau > 0 \tag{31}$$

Let us check that the function $\varphi(t,\tau)$ is well defined in this way. According to Theorem 3 (e), $\Re\{s\} > 0$ implies $\Re\{g(s)\} > 0$. More precisely, if $s = re^{\mathrm{i}\theta}$, then:

$$\Re\{g(s)\} = \frac{r\cos\theta + \gamma r^{\alpha+1}\cos(1-\alpha)\theta}{1 + 2\gamma r^\alpha\cos\alpha\theta + \gamma^2 r^{2\alpha}}$$

Hence, when $r \to \infty$, $|\theta| \to \pi/2$, the dominant term of $\Re\{g(s)\}$ is $r^{1-\alpha}\sin\alpha\pi/2 > 0$. This together with the estimate (f) of Theorem 3 shows that the integral in (31) is absolutely convergent.

We are ready to formulate the main result of this paper.

**Theorem 5.** *Let $A$ be a generator of a bounded $C_0$ semigroup $T(t)$, such that:*

$$\|T(t)\| \le M, \quad t \ge 0 \tag{32}$$

*Then, Problem (1) is well-posed, and its solution operator $S(t)$ satisfies:*

$$\|S(t)\| \le M, \quad t \ge 0 \tag{33}$$

*Moreover, the subordination identity (28) holds, where the function $\varphi(t,\tau)$ has the representations for $t,\tau > 0$:*

$$\varphi(t,\tau) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st-\tau g(s)}\,\frac{g(s)}{s}\,ds = \sum_{k=0}^\infty (-1)^k\frac{t^{(\alpha-1)(k+1)}}{\gamma^{k+1}}\frac{\tau^k}{k!}E^{k+1}_{\alpha,\alpha(k+1)-k}(-\gamma^{-1}t^\alpha) \tag{34}$$

*where $c > 0$ and $g(s)$ is defined in (22). The function $\varphi(t, \tau)$ is a probability density function with respect to both variables $t$ and $\tau$, i.e., it satisfies the following properties for $t, \tau > 0$:*

$$\varphi(t, \tau) \geq 0 \tag{35}$$

$$\int_0^\infty \varphi(t, \tau)\, d\tau = 1 \tag{36}$$

$$\int_0^\infty \varphi(t, \tau)\, dt = 1 \tag{37}$$

**Proof.** Let us find the Laplace transform of $\varphi(t, \tau)$ with respect to $\tau$. Applying (31) and interchanging the order of integration, it follows:

$$\begin{aligned}\int_0^\infty e^{-\lambda\tau}\varphi(t, \tau)\, d\tau &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st}\frac{g(s)}{s}\left(\int_0^\infty e^{-(\lambda+g(s))\tau}\, d\tau\right) ds \\ &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st}\frac{g(s)}{s(g(s)+\lambda)}\, ds\end{aligned}$$

From the definition of $g(s)$ in (22):

$$\frac{g(s)}{s(g(s)+\lambda)} = \frac{1}{s + \gamma\lambda s^\alpha + \lambda}$$

Therefore, (24) implies that the last integral gives exactly the solution $u(t, \lambda)$ of the scalar Equation (23), *i.e.*,

$$\int_0^\infty e^{-\lambda\tau}\varphi(t, \tau)\, d\tau = u(t, \lambda), \quad \lambda, t > 0 \tag{38}$$

Inserting representation (27) of $u(t, \lambda)$ in (38) and using (6), we deduce the series expansion of the function $\varphi(t, \tau)$ in (34). Alternatively, this expansion can be deduced, inserting the series expansion of the function $\frac{g(s)}{s}e^{-\tau g(s)}$ in (31) and using the Laplace transform pair (12).

The complete monotonicity of $u(t, \lambda)$ for $t > 0$ and (38) imply the positivity of $\varphi(t, \tau)$ by applying Bernstein's theorem. Alternatively, the positivity of $\varphi(t, \tau)$ can be also deduced from (29), since

$$\frac{g(s)}{s}e^{-\tau g(s)} \in \mathcal{CMF}$$

(this follows from Properties (c) and (d) of $g(s)$ in Theorem 3). Further, taking $s \to 0$ in (29) and $\lambda \to 0$ in (38) and noting that $u(t, 0) = 1$, we deduce the integral identities, (36) and (37).

The definition (28), the estimate (32) and the properties, (35) and (36), imply:

$$\|S(t)\| = \int_0^\infty \varphi(t, \tau)\|T(\tau)\|\, d\tau \leq M\int_0^\infty \varphi(t, \tau)\, d\tau = M, \quad t > 0$$

*i.e.*, (33) is established.

Next, we deduce the strong continuity of $S(t)$ at the origin from the strong continuity of $T(t)$ at the origin:

$$\lim_{t\downarrow 0} T(t)a = a \tag{39}$$

On the basis of the dominated convergence theorem and by the change of variables $\sigma = t^{\alpha-1}\tau$ in (28), we obtain:

$$\lim_{t\downarrow 0} S(t)a = \lim_{t\downarrow 0}\int_0^\infty t^{1-\alpha}\varphi(t, \sigma t^{1-\alpha})T(\sigma t^{1-\alpha})a\, d\sigma \tag{40}$$

For the function under the integral sign, we get from (34):

$$t^{1-\alpha}\varphi(t,\sigma t^{1-\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\sigma^k}{\gamma^{k+1}} E^{k+1}_{\alpha,\alpha(k+1)-k}(-\gamma^{-1}t^{\alpha})$$

and thus:

$$\lim_{t\downarrow 0} \left(t^{1-\alpha}\varphi(t,\sigma t^{1-\alpha})\right) = \sum_{k=0}^{\infty} \frac{(-1)^k\sigma^k}{\gamma^{k+1}k!\Gamma(\alpha(k+1)-k)} = \frac{1}{\gamma}\Phi_{1-\alpha}\left(\frac{\sigma}{\gamma}\right)$$

where $\Phi_\beta(z)$, $\beta \in (0,1)$, is the Mainardi function (13). Therefore, (40) together with (39) and the integral identity in (14) for the Mainardi function imply:

$$\lim_{t\downarrow 0} S(t)a = \int_0^\infty \frac{1}{\gamma}\Phi_{1-\alpha}\left(\frac{\sigma}{\gamma}\right)\, d\sigma a = a$$

In this way, we proved that $S(t)$, defined by (28), is a strongly continuous operator-valued function, satisfying (33). Moreover, in (30), we proved that the Laplace transform of $S(t)$ satisfies:

$$\int_0^\infty e^{-st} S(t)\, dt = H(s) \tag{41}$$

where

$$H(s) = \frac{g(s)}{s} R(g(s), A)$$

After easily justified differentiation under the integral sign in (41), we obtain from (33) the estimates (18) and, thus, the well-posedness of Problem (1). Then, Identity (16) implies by the uniqueness of the Laplace transform that $S(t)$ is exactly the solution operator of (1). The proof of the theorem is completed. □

The positivity of the function $\varphi(t,\tau)$ in (28) has an important implication related to the positivity of the solution operator $S(t)$.

Let $X$ be an ordered Banach space (for a simple introduction, see, e.g., [14], p. 353). For example, such are the spaces of type $L^p(\Omega)$ or $C_0(\Omega)$ for some $\Omega \in \mathbb{R}^d$, $d \in \mathbb{N}$, with the canonical ordering: a function $a \in X$ is positive (in symbols: $a \geq 0$) if $a(x) \geq 0$ for (almost) all $x \in \Omega$.

A solution operator $S(t)$ in an ordered Banach space $X$ is called positive if $a \geq 0$ implies $S(t)a \geq 0$ for any $t \geq 0$. In other words, the positivity of a solution operator means that the positivity of the initial condition is preserved in time.

As a direct consequence of the subordination identity (28), we obtain:

**Corollary 6.** *If the operator $A$ is a generator of positive $C_0$-semigroup, then the solution operator $S(t)$ of Problem (1) is positive.*

Another implication of the subordination principle follows from Identities (28) and (37):

**Corollary 7.** *If $\int_0^\infty T(t)\, dt < \infty$, then $\int_0^\infty S(t)\, dt = \int_0^\infty T(t)\, dt$.*

In the case of a contraction solution operator of Problem (1), there holds some inversion of the subordination principle in the following sense. Let us suppose that for some $\alpha \in (0,1)$, Problem (1) is well-posed and admits a contraction solution operator $S(t)$, $\|S(t)\| \leq 1$, $t \geq 0$. Then, (18) with $n = 0$ gives:

$$\|R(g(s), A)\| \leq \frac{1}{g(s)}, \quad s > 0$$

Since $g(s)$ is a Bernstein function for $s > 0$, it is positive and strongly increasing. This, together with $g(0) = 0$ and $g(+\infty) = +\infty$, implies that $g(s) : \mathbb{R}_+ \to \mathbb{R}_+$ is a one-to-one mapping. Therefore,

$$\|R(s, A)\| \leq \frac{1}{s}, \quad s > 0$$

and the Hille–Yosida theorem implies that $A$ generates a contraction semigroup. Then, by the subordination principle in Theorem 5, it follows that for any $\alpha \in (0, 1)$, Problem (1) is well-posed and admits a contraction solution operator.

**Corollary 8.** *If (1) is well-posed for some $\alpha \in (0, 1)$ and admits a contraction solution operator $S(t)$, $\|S(t)\| \leq 1$, $t \geq 0$, then for all $\alpha \in (0, 1)$, Problem (1) is well-posed with a contraction solution operator.*

Note that, if the $C_0$-semigroup generated by the operator $A$ is moreover analytic on some sector $\Sigma_\theta$, then the solution operator $S(t)$ of Problem (1) admits analytic extension to the same sector $\Sigma_\theta$. This result follows from Corollary 2.4 in [15] and is based on Properties (a) and (b) of the kernel $k(t)$, given in Theorem 3.

Next, we compare the subordination principle formulated in Theorem 5 with the analogous subordination principle for fractional evolution equations with the Caputo fractional derivative (see [16]), which can be written in the form:

$$u'(t) = D_t^{1-\alpha} A u(t), \quad u(0) = a \in X, \quad \alpha \in (0, 1) \tag{42}$$

Note that for such equations, the function $\varphi(t, \tau)$ in (28) is given by $\varphi(t, \tau) = t^{-\alpha}\Phi_\alpha(\tau t^{-\alpha})$, where $\Phi_\alpha$ is the Mainardi function (13). In the case of Equation (42), the subordinate solution operator $S(t)$ is always analytic in some sector without assuming the analyticity of the $C_0$-semigroup $T(t)$. It seems that this is not true for the here considered Equation (1). This difference is due to the following: the kernel corresponding to Equation (42) is given by $k(t) = \omega_\alpha(t)$, and it is sectorial kernel of angle $\alpha\pi/2 < \pi/2$, while the kernel (20) associated with Problem (1) is sectorial of angle $\pi/2$. A kernel $k(t)$ is called sectorial of angle $\theta > 0$ if ([15], Ch. 3):

$$|\arg \widehat{k}(s)| \leq \theta, \quad \text{for all } \Re s > 0$$

Another difference is that identity (37) does not hold for Equation (42).

## 5. Example

The following problem, which is a particular case of Problem (1), is considered in [11]. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and $T > 0$. Consider the initial-boundary value problem:

$$\begin{aligned} &\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) + \gamma D_t^\alpha \Delta u(x,t) + f(x,t), \;\; x \in \Omega, \;\; t \in (0,T); \\ &u(x,t) = 0, \;\; x \in \partial\Omega, \; t \in (0,T); \\ &u(x,0) = a(x), \;\; x \in \Omega \end{aligned} \tag{43}$$

where $\Delta$ is the Laplace operator acting on space variables.

Let $X = L^2(\Omega)$. Define the operator $A$ by $A = \Delta$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

If $\{-\lambda_n, \phi_n\}_{n=1}^\infty$ is the eigensystem of the operator $A$, then $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lambda_n \to \infty$ as $n \to \infty$, and $\{\phi_n\}_{n=1}^\infty$ form an orthonormal basis of $X = L^2(\Omega)$. Applying eigenfunction decomposition and Laplace transform, the solution of Problem (43) is obtained in the form:

$$u(x,t) = \sum_{n=1}^\infty a_n u_n(t)\phi_n(x) + \sum_{n=1}^\infty \left( \int_0^t u_n(t-\tau) f_n(\tau)\, d\tau \right) \phi_n(x) \tag{44}$$

where $a_n = (a, \phi_n)$, $f_n(t) = (f(.,t), \phi_n)$, and $u_n(t) = u(t, \lambda_n)$: the solution of the scalar Equation (23) with $\lambda = \lambda_n$, $n \in \mathbb{N}$. Therefore, the solution operator for this problem admits the representation:

$$S(t)a = \sum_{n=1}^\infty a_n u_n(t)\phi_n(x). \tag{45}$$

The $C_0$-semigroup $T(t)$ generated by the operator $A$ (corresponding to the solution of (43) with $\gamma = 0$) is given by:

$$T(t)a = \sum_{n=1}^\infty a_n e^{-\lambda_n t}\phi_n(x) \tag{46}$$

Theorem 5 implies the well-posedness of Problem (43) from the well-posedness of the corresponding initial boundary value problem for the diffusion Equation ((43) with $\gamma = 0$) and gives the relationship between the solution operators $S(t)$ and $T(t)$. Note that the subordination Identity (28) in this case can be obtained from Identity (38). Since the $C_0$-semigroup $T(t)$ gives the solution of a diffusion equation, it is positive, *i.e.*, it preserves the positivity of the initial function $a$. Applying Corollary 6, we deduce that the solution operator of Problem (43) is also positive: *i.e.*, if $a(x) \geq 0$ for a.a.$x \in \Omega$, then the solution of Problem (43) with $f \equiv 0$ given by $u(x,t) = S(t)a$ is positive for a.a. $x \in \Omega$, and all $t \geq 0$.

In [11], various estimates in Sobolev spaces are obtained for the homogeneous Equation (43). Let us consider here the inhomogeneous equation, *i.e.*, we set in (43) $a = 0$ and $f \neq 0$, $f \in L^2(0,T; L^2(\Omega))$. Based on some properties of the solution of the scalar equation summarized in Theorem 4, we find a maximal regularity estimate for the inhomogeneous equation. From (44), we have for its solution:

$$u(x,t) = \sum_{n=1}^\infty \left( \int_0^t u_n(t-\tau) f_n(\tau)\, d\tau \right) \phi_n(x) \tag{47}$$

We prove first that the solution $u(x,t)$ satisfies the estimate:

$$\|\Delta u\|_{L^2(0,T;L^2(\Omega))} \leq \|f\|_{L^2(0,T;L^2(\Omega))} \tag{48}$$

Recall that the norm in the space $L^2(0,T;L^2(\Omega))$ of a function $f(x,t)$ is given by:

$$\|f\|^2_{L^2(0,T;L^2(\Omega))} = \int_0^T \|f\|^2_{L^2(\Omega)}\,dt = \int_0^T \sum_{n=1}^{\infty} |f_n(t)|^2\,dt = \sum_{n=1}^{\infty} \|f_n(t)\|^2_{L^2(0,T)}$$

where $f_n(t) = (f(.,t),\phi_n)$. From Theorem 4, we know that $u_n(t) > 0$ and $\int_0^T u_n(t)\,dt < 1/\lambda_n$.

Applying the Young inequality for the convolution $\|k * f\|_{L^2} \leq \|k\|_{L^1}\|f\|_{L^2}$, it follows:

$$\left\| \int_0^t u_n(t-\tau) f_n(\tau)\,d\tau \right\|^2_{L^2(0,T)} \leq \left( \int_0^T u_n(t)\,dt \right)^2 \int_0^T |f_n(t)|^2\,dt \leq \frac{1}{\lambda_n^2} \int_0^T |f_n(t)|^2\,dt \tag{49}$$

Therefore,

$$\begin{aligned} \|\Delta u\|^2_{L_2(0,T;L^2(\Omega))} &\leq \sum_{n=1}^{\infty} \lambda_n^2 \left\| \int_0^t u_n(t-\tau) f_n(\tau)\,d\tau \right\|^2_{L^2(0,T)} \\ &\leq \sum_{n=1}^{\infty} \int_0^T |f_n(t)|^2\,dt = \|f\|^2_{L^2(0,T;L^2(\Omega))} \end{aligned}$$

In this way, (48) is proven. In fact, the following maximal regularity estimate also holds:

$$\|\partial u/\partial t\|_{L^2(0,T;L^2(\Omega))} + \|\Delta u\|_{L^2(0,T;L^2(\Omega))} + \|D_t^\alpha \Delta u\|_{L^2(0,T;L^2(\Omega))} \leq C\|f\|_{L^2(0,T;L^2(\Omega))} \tag{50}$$

*i.e.*, all terms in the governing equation of Problem (43), $\frac{\partial u}{\partial t}$, $\Delta u$ and $D_t^\alpha \Delta u$, have the same smoothness as the function $f$. Indeed, since $u_n(0) = 1$, then:

$$\frac{d}{dt} \int_0^t u_n(t-\tau) f_n(\tau)\,d\tau = \int_0^t u_n'(t-\tau) f_n(\tau)\,d\tau + f_n(t)$$

and by the complete monotonicity of $u_n(t)$ (see Theorem 4), we obtain:

$$\int_0^T |u_n'(t)|\,dt = -\int_0^T u_n'(t)\,dt = 1 - u_n(T) < 1$$

Then, in an analogous way as in (49), we deduce the estimate:

$$\left\| \frac{d}{dt} \int_0^t u_n(t-\tau) f_n(\tau)\,d\tau \right\|^2_{L^2(0,T)} \leq C_1 \int_0^T |f_n(t)|^2\,dt$$

which gives:

$$\|\partial u/\partial t\|_{L^2(0,T;L^2(\Omega))} \leq C_2 \|f\|_{L^2(0,T;L^2(\Omega))}$$

This estimate together with (48) and the identity $D_t^\alpha \Delta u = \gamma^{-1}(\partial u/\partial t - \Delta u)$ implies (50).

## 6. Conclusions

An abstract fractional order differential equation is studied, which contains as particular cases the Rayleigh–Stokes problem for a generalized second-grade fluid with a fractional derivative model and a two-term fractional diffusion equation in the Riemann–Liouville sense. Well-posedness is proven, and a subordination identity is presented relating the solution operator of the considered equation and the solution operator of the classical Cauchy problem. The subordination identity contains a special function represented as a series of three-parameter Mittag–Leffler functions. As a limiting case, the Mainardi function is recovered.

A feasible generalization of the presented results is an extension of the subordination principle to the general case when $T(t)$ is an exponentially-bounded $C_0$-semigroup: $\|T(t)\| \leq Ce^{\omega t}$, $\omega > 0$. Then, estimates for the solution of the scalar equation with $A = \omega > 0$ will be necessary.

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## Conflicts of Interest

The author declare no conflict of interest.

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