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*Article*

# A Class of Extended Mittag–Leffler Functions and Their Properties Related to Integral Transforms and Fractional Calculus

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**Abstract:** In a joint paper with Srivastava and Chopra, we introduced far-reaching generalizations of the extended Gammafunction, extended Beta function and the extended Gauss hypergeometric function. In this present paper, we extend the generalized Mittag–Leffler function by means of the extended Beta function. We then systematically investigate several properties of the extended Mittag–Leffler function, including, for example, certain basic properties, Laplace transform, Mellin transform and Euler-Beta transform. Further, certain properties of the Riemann–Liouville fractional integrals and derivatives associated with the extended Mittag–Leffler function are investigated. Some interesting special cases of our main results are also pointed out.

**Keywords:** extended Beta function; extended hypergeometric functions; extended confluent hypergeometric function; Mittag–Leffler function; generalized Mittag–Leffler function; Laplace transform; Mellin transform; Euler-Beta transforms; Wright hypergeometric function; Fox H-function; fractional calculus operators

**MSC classifications:** Primary 26A33, 33E12, 33C05, 33C15, 33C20; Secondary 33C65, 33C90.

## 1. Introduction, Definitions and Preliminaries

Several interesting generalizations of the familiar Euler-Gamma function $\Gamma(z)$, Euler-Beta function $B(\alpha, \beta)$, the Gauss hypergeometric functions ${}_2F_1$ and the generalized hypergeometric functions ${}_rF_s$ with

$r$ numerator and $s$ denominator were studied and investigated by various authors (see, for example, [1–7] and the references cited in each of these papers). For example, for an appropriately-bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ of arbitrary (real or complex) numbers, Srivastava *et al.* [8] (p. 243, Equation (2.1)) recently considered the function:

$$\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z\right) := \begin{cases} \sum\limits_{\ell=0}^{\infty} \kappa_\ell \dfrac{z^\ell}{\ell!} & (|z| < R;\ 0 < R < \infty;\ \kappa_0 := 1) \\ \mathfrak{M}_0\, z^\omega \exp(z) \left[1 + O\left(\dfrac{1}{z}\right)\right] & \left(\Re(z) \to \infty;\ \mathfrak{M}_0 > 0;\ \omega \in \mathbb{C}\right) \end{cases} \tag{1}$$

for some suitable constants $\mathfrak{M}_0$ and $\omega$ depending essentially on the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. In terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z\right)$ defined by Equation (1), in a joint paper with Srivastava and Chopra [8], we introduced far-reaching generalizations of the extended Gamma function, extended Beta function and the extended Gauss hypergeometric function by:

$$\Gamma_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z) = \int_0^\infty t^{z-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t - \frac{p}{t}\right) dt \tag{2}$$

$$\left(\Re(z) > 0;\ \Re(p) \geqq 0\right)$$

$$\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha, \beta; p) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{p}{t(1-t)}\right) dt \tag{3}$$

$$\left(\min\{\Re(\alpha), \Re(\beta)\} > 0;\ \Re(p) \geqq 0\right)$$

and:

$$\mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(a, b; c; z) := \sum_{n=0}^{\infty} (a)_n \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n, c-b; p)}{B(b, c-b)} \frac{z^n}{n!} \tag{4}$$

$$\left(|z| < 1;\ \Re(c) > \Re(b) > 0;\ \Re(p) \geqq 0\right)$$

respectively, provided that the defining integrals in the definitions (Equations (2)–(4)) exist.

For various special choices of the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$, the definition in Equations (2)–(4) would reduce to (known or new) extensions of the Gamma, Beta and hypergeometric functions. In particular, if we set:

$$\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell} \qquad (\ell \in \mathbb{N}_0) \tag{5}$$

the definition (Equations (2)–(4)) immediately reduces to the extended Gamma function $\Gamma_p^{(\rho,\sigma)}(z)$, the extended Beta function $B^{(\rho,\sigma)}(\alpha, \beta; p)$ and the extended hypergeometric function $F_p^{(\rho,\sigma)}(a, b; c; z)$ introduced by Özergin *et al.* [7]:

$$\Gamma_p^{(\rho,\sigma)}(z) := \int_0^\infty t^{z-1}\, {}_1F_1\left(\rho; \sigma; -t - \frac{p}{t}\right) dt \tag{6}$$

$$\left(\min\{\Re(z), \Re(\rho), \Re(\sigma)\} > 0;\ \Re(p) \geqq 0\right)$$

$$B^{(\rho,\sigma)}(\alpha, \beta; p) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, {}_1F_1\left(\rho; \sigma; -\frac{p}{t(1-t)}\right) dt \tag{7}$$

$$\left(\min\{\Re(\alpha), \Re(\beta), \Re(\rho), \Re(\sigma)\} > 0;\ \Re(p) \geqq 0\right)$$

and:

$$F_p^{(\rho,\sigma)}(a,b;c;z) := \frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \, B^{(\rho,\sigma)}(b+n, c-b; p) \, \frac{z^n}{n!} \tag{8}$$

$$\big(|z| < 1; \; \min\{\Re(\rho), \Re(\sigma)\} > 0; \; \Re(c) > \Re(b) > 0; \; \Re(p) \geqq 0\big)$$

respectively. Furthermore, for the sequence:

$$\kappa_\ell = 1 \tag{9}$$

the definition (Equations (2)–(4)) reduces immediately to the generalized gammafunction, extended betafunction and extended Gauss hypergeometric function studied earlier by Chaudhry and Zubair [3] (p. 9, Equation (1.66)), Chaudhry *et al.* [1] and Chaudhry *et al.* [2]:

$$\Gamma_p(z) := \int_0^\infty t^{z-1} \exp\left(-t - \frac{p}{t}\right) dt \qquad (\Re(p) > 0; \; z \in \mathbb{C}) \tag{10}$$

$$B(x, \, y\,; \, p) = \int_0^1 t^{x-1} \, (1-t)^{y-1} \, \exp\left(-\frac{p}{t(1-t)}\right) dt \quad (\Re(p) > 0) \tag{11}$$

and:

$$F_p\,(a, \, b; \, c; \, z) := \sum_{n=0}^{\infty} (a)_n \, \frac{B(b+n, \, c-b\,; \, p)}{B(b, \, c-b)} \, \frac{z^n}{n!} \tag{12}$$

$$(p \geq 0, \;\; |z| < 1\,; \, \Re(c) > \Re(b) > 0)$$

respectively. For $p = 0$ or (alternatively) for:

$$\kappa_\ell = 0 \qquad (\ell \in \mathbb{N})$$

the definitions (Equations (2)–(4)) would reduce immediately to classical Gamma, Beta and Gauss hypergeometric functions (see, for details, [9,10]), respectively.

The one-parameter Mittag–Leffler function:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha \in \mathbb{C}, \, \Re(\alpha) > 0, \;\; z \in \mathbb{C}) \tag{13}$$

and its two-parameter extension, nowadays called the Mittag–Leffler function:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (\alpha, \beta \in \mathbb{C}; \;\; \Re(\alpha) > 0, \;\; \Re(\beta) > 0) \tag{14}$$

were introduced and studied by Mittag–Leffler [11,12], Wiman [13,14], Agarwal [15], Humbert [16] and Humbert and Agarwal [17].

In 1971, Prabhakar [18] introduced the three-parameter generalization of Equation (14) as:

$$E^\gamma_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \quad (\alpha, \; \beta, \; \gamma \in \mathbb{C}; \;\; \Re(\alpha) > 0, \, \Re(\beta) > 0) \tag{15}$$

called usually the Prabhakar function. Further, various authors studied and investigated generalized Mittag–Leffler functions (see, for details, [19–25]). Motivated essentially by the demonstrated potential for applications of these extended hypergeometric functions, we extend the generalized Mittag–Leffler function (Equation (15)) by means of the extended Beta function $\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(x,\, y\,;\, p)$ defined by Equation (3) and investigate certain basic properties, including differentiation formulas and the integral property, Laplace transform, Euler-Beta transform and Mellin transform with their several special cases and relationships with generalized hypergeometric function ${}_pF_q$ and $H$-function. Further, certain relations between the extended generalized Mittag–Leffler function and the Riemann–Liouville fractional integrals and derivatives are investigated. Some interesting special cases of our main results are also considered.

## 2. A Class of Extended Mittag–Leffler Functions

In terms of the extended Beta function $\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(x,\, y\,;\, p)$ defined by Equation (3), we propose a different extension of the generalized Mittag–Leffler function by replacing:

$$\frac{(\gamma)_n}{(1)_n} = \frac{B(\gamma+n,\, 1-\gamma)}{B(\gamma,\, 1-\gamma)} \to \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\, 1-\gamma\,;\, p)}{B(\gamma,\, 1-\gamma)}$$

in Equation (15) as follows:

$$E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p) = \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\, 1-\gamma\,;\, p)}{B(\gamma,\, 1-\gamma)} \frac{z^n}{\Gamma(\alpha n+\beta)} \tag{16}$$

$$(z,\, \beta,\, \gamma \in \mathbb{C};\; \Re(\alpha)>0,\; \Re(\beta)>0,\; \Re(\gamma)>1;\; p\geq 0)$$

Remark 1. The special case of Equation (16) when we set the sequence $\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell}$ $(\ell \in \mathbb{N}_0)$, yields another form of the extended generalized Mittag–Leffler function:

$$E_{\alpha,\beta}^{(\rho,\sigma);\,\gamma}(z;p) = \sum_{n=0}^{\infty} \frac{B^{(\rho,\sigma)}(\gamma+n,\, 1-\gamma\,;\, p)}{B(\gamma,\, 1-\gamma)} \frac{z^n}{\Gamma(\alpha n+\beta)} \tag{17}$$

$$(z,\, \beta,\, \gamma \in \mathbb{C};\, \Re(\rho)>0,\, \Re(\sigma)>0,\, \Re(\alpha)>0,\, \Re(\beta)>0,\, \Re(\gamma)>1;\, p\geq 0)$$

Again, the sequence $\kappa_\ell = 1$ $(\ell \in \mathbb{N})$ yields the known definition of Özarslan and Yilmaz [26] (with $c = 1$):

$$E_{\alpha,\beta}^{\gamma}(z;p) = \sum_{n=0}^{\infty} \frac{B(\gamma+n,\, 1-\gamma\,;\, p)}{B(\gamma,\, 1-\gamma)} \frac{z^n}{\Gamma(\alpha n+\beta)} \tag{18}$$

$$(z,\, \beta,\, \gamma \in \mathbb{C};\; \Re(\alpha)>0,\; \Re(\beta)>0,\; \Re(\gamma)>1;\; p\geq 0)$$

For $p = 0$ or (alternatively) for $\kappa_\ell = 0$ $(\ell \in \mathbb{N})$, this immediately reduces to Prabhakar's definition (Equation (15)).

Remark 2. The special case for $\alpha = \beta = 1$ in Equations (16)–(18) can be expressed in terms of the extended confluent hypergeometric functions as:

$$E_{1,1}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p) = \; \Phi_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma;1;z)$$

$$E_{1,1}^{(\rho,\sigma);\,\gamma}(z;p) = \; \Phi_p^{(\rho,\sigma)}(\gamma;1;z)$$

and:

$$E_{1,1}^{\gamma}(z;p) = \Phi_p(\gamma;1;z)$$

## 3. Basic Properties of $E_{\alpha,\beta}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p)$

In this section, we obtain certain basic properties, including the differentiation formula and the integral property of the extended generalized Mittag–Leffler function in Equation (16).

Theorem 1. The following differentiation formula for the extended generalized Mittag–Leffler function in Equation (16) holds true:

$$E_{\alpha,\beta}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) = \beta E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) \tag{19}$$

$$(\alpha,\,\beta,\,\gamma \in \mathbb{C};\; \Re(\alpha) > 0,\, \Re(\beta) > 0,\, \Re(p) > 0)$$

In particular, we have:

$$E_{\alpha,\beta}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma}(z) \tag{20}$$

**Proof.** Using the definition (Equation (16)) in right-hand side of Equation (19), we have:

$$\begin{aligned}
&\beta E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) \\
&\quad = \beta E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) + \alpha z \frac{d}{dz} \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\gamma+n,\,1-\gamma\,;p)}{B(\gamma,\,1-\gamma)} \frac{z^n}{\Gamma(\alpha n+\beta+1)} \\
&\quad = \beta E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) + \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\gamma+n,\,1-\gamma\,;p)}{B(\gamma,\,1-\gamma)} \frac{\alpha n\; z^n}{\Gamma(\alpha n+\beta+1)} \\
&\quad = \beta E_{\alpha,\beta+1}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p) + \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\gamma+n,\,1-\gamma\,;p)}{B(\gamma,\,1-\gamma)} \frac{(\alpha n+\beta-\beta) z^n}{\Gamma(\alpha n+\beta+1)} \\
&\quad = E_{\alpha,\beta}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(z;p).
\end{aligned}$$

The relation Equation (20) follows from Equation (19) when $p = 0$ or for $\kappa_\ell = 0$ $(\ell \in \mathbb{N})$. □

Theorem 2. The following derivative formulas for the extended generalized Mittag–Leffler function in Equation (16) are satisfied:

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(\omega z^{\alpha};p)\right] = z^{\beta-m-1} E_{\alpha,\beta-m}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma)}(\omega z^{\alpha};p) \quad (\Re(\beta-m) > 0,\; m \in \mathbb{N}) \tag{21}$$

*where* $\alpha,\,\beta,\,\gamma,\,\omega \in \mathbb{C};\, \Re(\alpha) > 0,\, \Re(\beta) > 0;\; \Re(p) > 0.$

In particular, we have:

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega z^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m}^{\gamma}(\omega z^{\alpha}) \tag{22}$$

**Proof.** Using Equation (16) and employing term-wise differentiation $m$ times on the left-hand side of Equation (21) under the summation sign, which is possible in accordance with the uniform convergence of the series in Equation (16), we get:

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega z^\alpha;p)\right]$$
$$= \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)} \frac{\omega^n}{\Gamma(\alpha n+\beta)} \left(\frac{d}{dz}\right)^m [\,z^{\alpha\, n+\beta-1}\,]$$
$$= \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta-m)} \omega^n \frac{z^{\alpha n+\beta-1-m}}{\Gamma(\alpha n+\beta)}$$
$$= z^{\beta-m-1} E_{\alpha,\beta-m}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega z^\alpha;p)$$

The special cases of Equation (21) when $p = 0$ or for $\kappa_\ell = 0$ $(\ell \in \mathbb{N})$ are easily seen to yield Equation (22). □

Corollary 1. The following integral property for the extended generalized Mittag–Leffler function in Equation (16) holds true:

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega t^\alpha;p)\,dt = z^\beta\, E_{\alpha,\beta+1}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega z^\alpha;p) \tag{23}$$

*where* $\alpha,\ \beta,\ \gamma,\ \omega \in \mathbb{C};\ \Re(\alpha) > 0,\ \Re(\beta) > 0;\ \Re(p) > 0.$

In particular, we have:

$$\int_0^z t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega t^\alpha)\,dt = z^\beta\, E_{\alpha,\beta+1}^{\gamma}(\omega z^\alpha) \tag{24}$$

## 4. Integral Transforms of $E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)$

In this section, we obtain the Laplace transform, Mellin transform representations and the Euler-Beta transform, alternatively called the Erdélyi–Kober fractional integral for the extended generalized Mittag–Leffler function $E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)$, in Equation (16) as follows.

### *4.1. Laplace Transform*

The Laplace transform (see, e.g., [27]) of the function $f(z)$ is defined, as usual, by:

$$L\{f(z)\} = \int_0^\infty e^{-sz} f(z)\,dz \tag{25}$$

Theorem 3. The following Laplace transform representation for the extended generalized Mittag–Leffler function in Equation (16) holds true:

$$L\{z^{\beta-1} E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p)\} := \frac{1}{s^\beta}\, \mathfrak{F}_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)} \left(1,\gamma;1;\frac{x}{s^\alpha}\right) \tag{26}$$

$$(\Re(p) > 0;\ \Re(s) > 0,\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(\gamma) > 0)$$

**Proof.** Using the definition (Equation (25)) of the Laplace transform, we find from Equation (16):

$$L\{z^{\beta-1}E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p)\} := \int_0^\infty z^{\beta-1}e^{-sz}E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p)dz$$
$$= \int_0^\infty z^{\beta-1}e^{-sz}\left(\sum_{n=0}^\infty \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n z^{\alpha n}}{\Gamma(\alpha n+\beta)}\right)dz$$
$$= \sum_{n=0}^\infty \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n}{\Gamma(\alpha n+\beta)}\int_0^\infty z^{\alpha n+\beta+1}e^{-sz}dz$$
$$= \sum_{n=0}^\infty \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\alpha n+\beta)}{s^{\alpha n+\beta}}$$
$$= \frac{1}{s^\beta}\sum_{n=0}^\infty \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\left(\frac{x}{s^\alpha}\right)^n$$
$$= \frac{1}{s^\beta}\sum_{n=0}^\infty (1)_n\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{\left(\frac{x}{s^\alpha}\right)^n}{n!}$$

Now, using the definition (Equation (12)) to express the involved sum as an extended hypergeometric function, we are led to the desired result. □

Remark 3. The special case of Equation (26) when $p = 0$ or for $\kappa_\ell = 0$ $(\ell \in \mathbb{N})$ is seen to yield the known Laplace transform of the generalized Mittag–Leffler function (see [18] (p. 8, Equation (2.5)); see also [23] (p. 37, Equation (2.19))):

$$\int_0^\infty z^{\beta-1}e^{-sz}E_{\alpha,\beta}^{\gamma}(xz^\alpha)dz = \frac{1}{s^\beta}\left(1-\frac{x}{s^\alpha}\right)^{-\gamma}$$

### *4.2. Mellin Transform*

The Mellin transform [27] of a suitably-integrable function $f(t)$ with index $s$ is defined, as usual, by:

$$\mathcal{M}\{f(\tau):\tau\to s\} := \int_0^\infty \tau^{s-1}\,f(\tau)\,d\tau \tag{27}$$

whenever the improper integral in Equation (27) exists.

Theorem 4. The following Mellin transform representation for the extended generalized Mittag–Leffler function in Equation (16) holds true:

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p):p\to s\right\} := \frac{\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)\Gamma(1-\gamma+s)}{\Gamma(\gamma)\Gamma(1-\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(1,1),(\gamma+s,1);\\(1+2s,1),(\beta,\alpha);\end{array}\ z\right] \tag{28}$$

$$(\Re(s)>0\quad and\quad \Re(1-\gamma+s)>0)$$

where $\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)$ is the specialized case in Equation (2) for $p = 0$.

**Proof.** Using the definition (Equation (27)) of the Mellin transform, we find from Equation (16):

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p):p\to s\right\}:=\int_0^\infty p^{s-1}E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)dp$$
$$=\int_0^\infty p^{s-1}\left(\sum_{n=0}^\infty \frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{z^n}{\Gamma(\alpha n+\beta)}\right)dp \tag{29}$$

Upon interchanging the order of integration and summation in Equation (29), which can easily be justified by uniform convergence under the constraints stated with Equation (28), we get:

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p):p\to s\right\}:=\frac{1}{B(\gamma,\,1-\gamma)}\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+\beta)}$$
$$\int_0^\infty p^{s-1}\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)dp \tag{30}$$

Using the easily-derivable result as in Section 4 of Srivastava *et al.* [8] (p. 251, Theorem 5):

$$\int_0^\infty p^{s-1}\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(x,\,y\,;\,p)\,dp=\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)\;B(x+s,\,y+s)\qquad(\Re(s)>0) \tag{31}$$

we obtain:

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p):p\to s\right\}=\frac{\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)}{B(\gamma,\,1-\gamma)}\sum_{n=0}^\infty B(\gamma+n+s,\,1-\gamma+s)\frac{z^n}{\Gamma(\alpha n+\beta)}$$
$$=\frac{\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)\Gamma(1-\gamma+s)}{\Gamma(\gamma)\Gamma(1-\gamma)}\sum_{n=0}^\infty\frac{\Gamma(\gamma+s+n)}{\Gamma(1+2s+n)}\frac{z^n}{\Gamma(\beta+\alpha n)} \tag{32}$$

Using the definition of the Wright generalized hypergeometric function ${}_p\Psi_q(z)$ (see, e.g., [28,29]) in Equation (32), we get the desired representation Equation (28). □

Corollary 2. The following Mellin transform representation is expressed in terms of generalized hypergeometric functions in Equation (28) as follows:

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)\right\}:=\frac{\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)B(\gamma+s,\,1-\gamma+s)}{B(\gamma,\,1-\gamma)}$$
$${}_2F_{1+\alpha}\left[\begin{array}{c}1,\,\gamma+s;\;\dfrac{z}{\alpha^\alpha}\\ 1+2s,\,\Delta(\alpha;\beta);\end{array}\right] \tag{33}$$

where $\alpha\in\mathbb{N}$ and $\Delta(\alpha;\beta)$ is an array of $\alpha$ parameters $\frac{\beta}{\alpha},\frac{\beta+1}{\alpha},\cdots,\frac{\beta+\alpha-1}{\alpha}$.

Remark 4. The Wright generalized hypergeometric function ${}_p\Psi_q(z)$ (see, e.g., [28,29]) is expressible in terms of Fox $H$-function $H_{p,q}^{m,n}(z)$ (see, e.g., [30–32]) as follows (see, e.g., [32] (p. 25, Equation (1.140)) and [31] (p. 11, Equation (1.7.8)):

$${}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\cdots,(a_p,A_p);\\(b_1,B_1),\cdots,(b_q,B_q);\end{array}z\right]=H_{p,q+1}^{1,p}\left[-z\;\middle|\;\begin{array}{c}(1-a_1,A_1),\cdots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\cdots,(1-b_q,B_q)\end{array}\right] \tag{34}$$

Now, applying the relationship Equation (34) to Equation (28), we can deduce an interesting representation for the extended Mittag–Leffler function in Equation (16) asserted by Corollary 3 below. We state here the resulting representation without proof.

Corollary 3. The following Mellin transform representations is expressed in terms of Fox $H$-functions in Equation (28) as follows:

$$\mathcal{M}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)\right\} := \frac{\Gamma_0^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(s)\Gamma(1-\gamma+s)}{\Gamma(\gamma)\Gamma(1-\gamma)} H_{2,3}^{1,2}\left[-z \,\middle|\, \begin{array}{c} (0,1),(1-\gamma-s,1) \\ (0,1),(0,1),(-2s,1),(1-\beta,\alpha) \end{array}\right] \tag{35}$$

### *4.3. Euler-Beta Transform*

The Euler-Beta transform [27], alternatively called the Erdélyi–Kober fractional integral of the function $f(z)$, is defined, as usual, by:

$$\mathcal{B}\{f(z);a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\,dz \tag{36}$$

Theorem 5. The following Euler-Beta transform or Erdélyi–Kober fractional integral representation for the extended generalized Mittag–Leffler function in Equation (16) holds true:

$$\mathcal{B}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p) : \beta, b\right\} = \Gamma(b)\; E_{\alpha,\beta+b}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(x;p) \tag{37}$$

$$(\Re(p)>0;\ \Re(b)>0,\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\gamma)>0)$$

**Proof.** Using the definition (Equation (36)) of the Euler-Beta transform, we find from Equation (16):

$$\begin{aligned}\mathcal{B}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p) : \beta, b\right\} &:= \int_0^1 z^{\beta-1}(1-z)^{b-1} E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p)dz \\ &= \int_0^1 z^{\beta-1}(1-z)^{b-1}\left(\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n z^{\alpha n}}{\Gamma(\alpha n+\beta)}\right)\end{aligned} \tag{38}$$

Upon interchanging the order of integration and summation in Equation (38), which can easily be justified by uniform convergence under the constraint state with Equation (37), we get:

$$\begin{aligned}&\mathcal{B}\left\{E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(xz^\alpha;p) : \beta, b\right\} \\ &= \sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n}{\Gamma(\alpha n+\beta)}\left(\int_0^1 z^{\beta+\alpha n-1}(1-z)^{b-1}dz\right) \\ &= \sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\alpha n+\beta)\Gamma(b)}{\Gamma(\alpha n+\beta+b)} \\ &= \Gamma(b)\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma;\,p)}{B(\gamma,\,1-\gamma)}\frac{x^n}{\Gamma(\alpha n+\beta+b)}\end{aligned}$$

Using the definition (Equation (16)), we get the desired representation Equation (37). □

Corollary 4. In a similar manner, we obtain:

$$\int_0^1 z^{a-1}(1-z)^{\beta-1}E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(x(1-z)^\alpha;p)dz=\Gamma(a)E_{\alpha,\beta+a}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(x;p) \tag{39}$$

In general, we have:

$$\int_t^x (z-u)^{a-1}(z-t)^{\beta-1}E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(w(z-t)^\alpha;p)dz=\Gamma(\alpha)(x-t)^{\beta+a-1}E_{\alpha,\beta+a}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(w(x-t)^\alpha;p) \tag{40}$$

## 5. Fractional Calculus Operators of $E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)$

In this section, we derive certain interesting properties of the extended generalized Mittag–Leffler function $E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(z;p)$ in Equation (16) associated with right-sided Riemann–Liouville fractional integral operator $I_{a+}^\mu$ and the right-sided Riemann-Liouville fractional derivative operator $D_{a+}^\mu$, which are defined as (see, e.g., [33,34]):

$$\left(I_{a+}^\mu\varphi\right)(x)=\frac{1}{\Gamma(\mu)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\mu}}dt\quad\left(\mu\in\mathbb{C},\,\Re(\mu)>0\right) \tag{41}$$

and:

$$\left(D_{a+}^\mu\varphi\right)(x)=\left(\frac{d}{dx}\right)^n\left(I_{a+}^{n-\mu}\varphi\right)(x)\quad\left(\mu\in\mathbb{C},\,\Re(\mu)>0;\,n=[\Re(\mu)]+1\right) \tag{42}$$

where $[x]$ means the greatest integer not exceeding real $x$.

A generalization of Riemann–Liouville fractional derivative operator $D_{\alpha+}^\mu$ in Equation (42) by introducing a right-sided Riemann–Liouville fractional derivative operator $D_{a+}^{\mu,\nu}$ of order $0<\mu<1$ and type $0\leqq\nu\leqq 1$ with respect to $x$ by Hilfer (see, e.g., [35]) is as follows:

$$\left(D_{a+}^{\mu,\nu}\varphi\right)(x)=\left(I_{a+}^{\nu(1-\mu)}\frac{d}{dx}\right)\left(I_{a+}^{(1-\nu)(1-\mu)}\varphi\right)(x)\quad\left(\mu\in\mathbb{C},\,\Re(\mu)>0;\,n=[\Re(\mu)]+1\right) \tag{43}$$

The generalization Equation (43) yields the classical Riemann–Liouville fractional derivative operator $D_{a+}^\mu$ when $\nu=0$.

Theorem 6. Let $a\in\mathbb{R}_+=[0,\infty)$, $\alpha,\,\beta,\,\gamma,\,\mu,\,\omega\in\mathbb{C}$ *and* $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\mu)>0$. Then, for $x>a$, the relation holds:

$$\left(I_{a+}^\mu\left[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)\right]\right)(x)=(x-a)^{\beta+\mu-1}\,E_{\alpha,\beta+\mu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(x-a)^\alpha;p) \tag{44}$$

$$\left(D_{a+}^\mu\left[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)\right]\right)(x)=(x-a)^{\beta-\mu-1}\,E_{\alpha,\beta-\mu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(x-a)^\alpha;p) \tag{45}$$

and

$$\left(D_{a+}^{\mu,\nu}\left[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)\right]\right)(x)$$
$$=(x-a)^{\beta-\mu-1}\,E_{\alpha,\beta-\mu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(x-a)^\alpha;p) \tag{46}$$

**Proof.** By virtue of the formulas (Equations (41) and (16)), the term-by-term fractional integration and the application of the relation [34]:

$$\left(I_{a+}^{\alpha}[(t-a)^{\beta-1}]\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\qquad\left(\alpha,\beta\in\mathbb{C},\,\Re(\alpha)>0,\,\Re(\beta)>0\right) \tag{47}$$

yield for $x>a$:

$$\left(I_{a+}^{\mu}[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)]\right)(x)$$
$$=\left(I_{a+}^{\mu}\left[\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{\omega^n\,(t-a)^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)n!}\right]\right)(x)$$
$$=(x-a)^{\beta+\mu-1}\,E_{\alpha,\beta+\mu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(x-a)^\alpha;p) \tag{48}$$

Next, by Equations (42) and (16), we find that:

$$\left(D_{a+}^{\mu}[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)]\right)(x))$$
$$=\left(\frac{d}{dx}\right)^n\left(I_{a+}^{n-\mu}[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)]\right)(x)$$
$$=\left(\frac{d}{dx}\right)^n\left[(x-a)^{\beta+n-\mu-1}\,E_{\alpha,\beta+n-\mu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(x-a)^\alpha;p)\right] \tag{49}$$

Applying Equation (21), we are led to the desired result Equation (45).

Finally, by Equations (43) and (16), we have:

$$\left(D_{a+}^{\mu,\nu}\left[(t-a)^{\beta-1}\,E_{\alpha,\beta}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\,\gamma\right)}(\omega(t-a)^\alpha;p)\right]\right)(x))$$
$$=\left(D_{a+}^{\mu,\nu}\left[\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{\omega^n\,(t-a)^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}\right]\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\gamma+n,\,1-\gamma\,;\,p)}{B(\gamma,\,1-\gamma)}\frac{\omega^n}{\Gamma(\alpha n+\beta)}\left(D_{a+}^{\mu,\nu}\left[(t-a)^{\alpha n+\beta-1}\right]\right)(x) \tag{50}$$

Using the known relation of Srivastava and Tomovski [36] (p. 203, Equation (2.18)):

$$\left(D_{\alpha+}^{\mu,\nu}\left[(t-\alpha)^{\lambda-1}\right]\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}(x-\alpha)^{\lambda-\mu-1}\;\left(x>\alpha;\,0<\mu<1;\,0\leqq\nu\leqq1;\,\Re(\lambda)>0\right) \tag{51}$$

in Equation (50), we are led to the desired result Equation (46). □

Remark 4. The special cases of the results presented here when $p = 0$ or for $\kappa_\ell = 0$ $(\ell \in \mathbb{N})$ would reduce to the corresponding well-known results for the generalized Mittag–Leffler function (see, for details, [18] and [23]).

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**Conflicts of Interest**

The author declares no conflict of interest.

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