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Gauge Invariance and Symmetry Breaking by Topology and Energy Gap

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Abstract: For the description of observables *and* states of a quantum system, it may be convenient to use a canonical Weyl algebra of which only a subalgebra $\mathcal{A}$, with a non-trivial center $\mathcal{Z}$, describes observables, the other Weyl operators playing the role of intertwiners between inequivalent representations of $\mathcal{A}$. In particular, this gives rise to a gauge symmetry described by the action of $\mathcal{Z}$. A distinguished case is when the center of the observables arises from the fundamental group of the manifold of the positions of the quantum system. Symmetries that do not commute with the topological invariants represented by elements of $\mathcal{Z}$ are then spontaneously broken in each irreducible representation of the observable algebra, compatibly with an energy gap; such a breaking exhibits a mechanism radically different from Goldstone and Higgs mechanisms. This is clearly displayed by the quantum particle on a circle, the Bloch electron and the two body problem.

Keywords: Weyl-polymer quantization; symmetry breaking by topology; energy gap

1. Introduction

The mathematical foundations of quantum mechanics rely on the Dirac–von Neumann axioms (for a critical review, see [1,2]), and the equivalence between the Heisenberg formulation in terms of canonical operators and the Schrödinger formulation in terms of wave functions is provided by the Stone–von Neumann theorem, which states the uniqueness of the Schrödinger representation of the canonical algebra under general regularity conditions.

Technically, this result is achieved by introducing the Weyl unitary operators (formally, the exponentials of the canonical variables q, p) and the corresponding Weyl algebra, defined by algebraic relations, which encode the canonical commutations relations of Heisenberg canonical variables.

The use of the Weyl algebra is usually motivated by the better behavior and mathematical control of the unitary Weyl operators with respect to the Heisenberg canonical variables, which are necessarily represented by unbounded operators.

It is usually taken for granted that the Dirac–Heisenberg quantization, in terms of the canonical commutation relations of the q's and p's, and the Weyl quantization, in terms of the commutation relations of the Weyl operators, are equivalent under the implicit assumption that one is interested only in regular representations of the Weyl algebra, to which the Stone–von Neumann uniqueness theorem applies.

The point is that the Dirac–Heisenberg quantization implicitly assumes that all of the canonical variables describe observables so that the regularity of their exponentials (Weyl operators) is required by their existence as (unbounded) operators in the Hilbert representation space.

The regularity condition at the basis of the Stone–von Neumann theorem is standard in the mathematical analysis and classification of Lie group representations, and, in, the quantum mechanical case, it amounts to considering the strongly (equivalently weakly) -continuous (unitary) representations of the Heisenberg group.

However, for a class of physically-interesting systems, especially in connection with quantum gauge theories, it has become apparent [3–6] that the Dirac–Heisenberg quantization is not compatible with a gauge-invariant ground state, only the Weyl quantization being allowed.

In these cases, the inequivalence of the two quantization methods arises by the lack of regularity of one-parameter groups generated by Weyl operators, so that the corresponding generators, *i.e.*, the corresponding Heisenberg canonical variables, cannot be defined as (self-adjoint) operators in the Hilbert space of states.

The physical reason at the basis of such a lack of regularity is that the QM description of a class of physical systems involves canonical variables, not all of which correspond to observable quantities; some of them are introduced for the description of the states, namely of the inequivalent representations of the algebra of observables, for which purpose only their exponentials are needed to exist as well-defined operators in the Hilbert space of states, with the role of intertwiners between inequivalent representations.

Thus, the larger algebra $\mathcal{F}$, in which the algebra of observables is embedded, may be characterized as the minimal algebra, such that all of the automorphisms of $\mathcal{A}$ are inner. This means that $\mathcal{F}$ contains the intertwiners between the inequivalent representations of $\mathcal{A}$, *i.e.*, all of the "variables" needed for a complete description of the observables and the states of the given quantum system. Such an embedding provides a natural C^* norm for $\mathcal{A}$. Clearly, the basis of such a structure, beyond the Stone–von Neumann unique characterization of quantum mechanics, is the existence of inequivalent representations of $\mathcal{A}$. This typically occurs in the case of quantum systems on manifolds with a non-trivial fundamental group.

Thus, what might at first sight look to be an uninteresting singular, if not pathological, case turns out to be crucial for the quantum description of physically-interesting systems.

In general, this lack of regularity may be related to the existence of a gauge group. Typically, one has the Weyl algebra $\mathcal{A}_W$ generated by the exponentials of the full set of canonical variables needed for the description of the observables and states of the system, but only a subalgebra $\mathcal{A}$ describes observables.

Generically, $\mathcal{A}$ has a non-trivial center $\mathcal{Z}$, which generates transformations having the meaning of gauge symmetries (gauge transformations). Thus, the algebra $\mathcal{A}_W$ of canonical variables contains both gauge-dependent and gauge-invariant (*i.e.*, observable) variables.

Clearly, the regularity condition must be satisfied by the exponentials of the observable variables, otherwise the representation is not physically acceptable, but there is no physical reason for the regularity condition of the gauge-dependent Weyl operators, playing the role of intertwiners.

As we shall see, the representation of the Weyl algebra by a gauge-invariant ground state in general requires the non-regularity of the gauge-dependent Weyl operators and implies the impossibility of defining the corresponding generators as well-defined operators in the corresponding Hilbert space (non-regular representation of the Heisenberg group or of the Weyl algebra).

Relevant quantum mechanical examples of such a structure are the electron in a periodic potential (Bloch electron), the quantum Hall electron, the particle on a circle, where the gauge transformations are, respectively, the lattice translations, the magnetic translations and the rotations of 2π. In gauge quantum field theories (GQFT), the need of a non-regular Weyl quantization arises in (positive) representations of the field algebra defined by a gauge-invariant vacuum state.

The non-regular Weyl representations for the quantization of systems with a gauge symmetry exhibit the following characteristic structures, which do not have a counterpart in the Dirac–Heisenberg canonical quantization:

 (i) a gauge invariance constraint in operator form compatible with canonical Weyl quantization (avoiding the mathematically-unacceptable recourse to non-normalizable states)
 (ii) super-selected charges defined by the center of the observable algebra
(iii) gauge-invariant ground states, defining inequivalent representations of the observable algebra, labeled by the spectrum of the super-selected charges (θ sectors)
(iv) the absence of "Goldstone states" associated with the spontaneous breaking of symmetries (canonically) conjugated to the gauge transformations.

Particularly relevant is the case in which the above structure of the algebra of observables for a (finite dimensional) quantum system arises as a consequence of the non-trivial topology of the manifold $\mathcal{M}$ of the particle positions. The mechanism is that the fundamental group of the manifold, which has been shown in [7] to be the only source of topological effects, gives rise to elements of the center of the observable algebra and a corresponding gauge group. This leads to a mechanism of symmetry breaking, which is substantially different from the standard Nambu–Goldstone case (with associated Goldstone bosons) and from the Higgs mechanism (characterized by a dynamics that induces a long-range Coulomb-like delocalization): the symmetry breaking is forced by the topology in any irreducible representation of the observables, with a compatible energy gap related to the spectrum of the first homology group of the manifold (symmetry breaking by topology and energy gap; see Section 5 below).

In our opinion, the realization of such general structures and their clear and simple realization in (finite dimensional) QM mechanical models, fully under control, shed light on the more difficult infinite dimensional GQFT models, discussed within very specific approximations.

The physical relevance of non-regular representations of the Heisenberg group raises the problem of the their classification, *i.e.*, a generalization of the classical Stone–von-Neumann theorem [9–12], which characterizes the regular ones.

Such a generalization may be obtained by exploiting the simple form of the Gelfand spectrum of the maximal abelian subalgebra $\mathcal{A}_Z$ of the Weyl algebra generated by the pairs $U_i(-2\pi/\lambda)$, $V_i(\lambda)$, $i = 1, ...d$ (d the space dimensions), formally corresponding to the exponentials $\exp -i(2\pi/\lambda)q_i$, $\exp i\lambda\, p_i$. Such pairs of operators were introduced by Zak [13] in order to discuss the dynamics of electrons in solids in external fields; the crucial distinctive property (see [14]) of the corresponding C^*-algebra, called the Zak algebra, is that its Gelfand spectrum Σ is given by d copies of the two-dimensional torus $\Sigma = (\mathbf{T}^2)^d$.

Then, one may prove [14] that all of the representations of the Weyl algebra that are spectrally multiplicity free as representations of its Zak algebra (a condition that generalizes irreducibility) and are strongly measurable (a condition that replaces regularity in non-separable spaces) are unitarily equivalent to a representation of the Weyl algebra of the same form of the standard Schrödinger representation on $L^2(\Sigma, d\mu)$, with $d\mu$ a (positive) translationally-invariant Borel measure, which reduces to the Lebesgue measure iff the regularity condition is satisfied.

The conditions that yield such a classification are satisfied by all of the non-regular representations of physical interest mentioned above; thus, they all have the above form, each with a corresponding Borel measure on $(\mathbf{T}^2)^d$.

This represents a radical departure from the standard structure of quantum mechanics, since it requires non-separable Hilbert spaces, very discontinuous expectations of one-parameter groups of unitary operators (vanishing for all non-zero values of the parameters, so that the corresponding generators do not exist), *etc*. Such features have been regarded as pathological and to be avoided for an acceptable physical interpretation. Actually, they have a very sound mathematical status; there is no problem for the physical interpretation, and they explain the important role of the gauge group and of topology for the evasion of the Goldstone theorem in the case of gauge symmetry breaking.

The role of topology in (flat space) GQFT, especially in quantum chromodynamics (QCD), had been realized a long time ago, but in terms of topological classification of the Euclidean configurations (the instanton solution in QCD) assumed to dominate the functional integral within a semiclassical approximation. Such a classification requires the regularity and continuity of such Euclidean configurations, and it is well known that regular configurations have zero functional measure. Apart from this consistency problem, the above exploitation of topology appears to be strictly bound to very special mechanisms, and the question arises whether there is an underlying general framework, a point not clearly addressed in the literature.

One of the aims of this paper is to emphasize, on the basis of quantum mechanical models recognized to mimic the basic structures of GQFT, that the crucial and basic ingredient is the existence of a nontrivial center of the observables and its topological origin. This clarification, in terms of the topology of the gauge group, provides a general abstraction of the mechanism, similar to that provided by studying the properties of a Lie group rather than those of the vectors of one of its representations (Section 2).

Such a topological origin of the center of the observable algebra arises as a consequence of the topology of the manifold of the QM configurations (Section 3 and examples that are reviewed under this more general perspective).

The main focus of the paper is to study the implications on spontaneous breaking of symmetries not commuting with the topological invariants, which define elements of the center $\mathcal{Z}$ of the observables. The conclusion (Theorem 3) is that such symmetries are broken in any irreducible (or factorial) representation of the observables and that (contrary to the classical Goldstone theorem) the energy spectrum may have a gap given by the spectrum of the topological invariants belonging to $\mathcal{Z}$.

2. Gauge Invariance, Super-Selection Rules and Non-Regular Canonical Quantization

A general case leading to non-regular representations is when:

(i) a quantum system is described by canonical variables, generating a Heisenberg group G_H, but only a subset of them, and consequently, only a subgroup $G_{obs} \subset G_H$, describes observable quantities, the other canonical variables providing the intertwiners for the description of the inequivalent representations of the observable algebra,

(ii) G_{obs} is generated by a Heisenberg subgroup and by an Abelian subgroup $\mathcal{G}$, which commutes with G_{obs}.

Then, $\mathcal{G}$ generates a group of transformations α_g, $g \in \mathcal{G}$, which leave the observables pointwise invariant:

$$\alpha_g(A) = A, \quad \forall A \in G_{obs}, \quad \forall g \in \mathcal{G} \tag{1}$$

i.e., $\mathcal{G}$ has the meaning of a gauge group.

The elements of G_H generate a C^*-algebra $\mathcal{F}_W$, called field algebra, and the elements of G_{obs} generate a C^*-algebra $\mathcal{A}$ of observables, characterized by gauge invariance, Equation (1). $\mathcal{A}$ has a non-trivial center generated by the elements of $\mathcal{G}$. A representation of $\mathcal{F}_W$ is physical if G_{obs} is regularly represented.

In the irreducible representations of $\mathcal{A}$, $\mathcal{Z}$ is represented by multiples of the identity. The generators of $\mathcal{G}$ have the meaning of super-selected charges and the points θ of the spectrum $\sigma(\mathcal{Z})$ of $\mathcal{Z}$ label inequivalent representations $(\mathcal{H}_\theta, \pi_\theta)$ of $\mathcal{A}$, called $\boldsymbol{\theta}$ sectors. The Stone–von Neumann uniqueness theorem does not apply, and this can be traced back to the fact that, contrary to the Weyl C^*-algebra, $\mathcal{A}$ is not simple.

By definition, a gauge-invariant state ω on $\mathcal{F}_W$ satisfies

$$\omega(\alpha_g(F)) = \omega(F), \quad \forall F \in \mathcal{F}_W,$$

and therefore, in the Gelfand-Naimark-Segal (GNS) representation π_ω of $\mathcal{F}_W$ defined by ω, the gauge transformations are implemented by unitary operators $U(g)$ defined by (Ψ_ω denotes the vector, which represents ω):

$$U(g)\Psi_\omega = \Psi_\omega, \quad U(g)\pi_\omega(F)\Psi_\omega = \pi_\omega(\alpha_g(F))\,\Psi_\omega, \quad \forall F \in \mathcal{F}_W$$

Let $V(g)$ denote the element of $\mathcal{G}$, which defines α_g:

$$\alpha_g(F) = V(g)\,F\,V(g)^{-1}, \quad \forall F \in \mathcal{F}_W$$

then $\pi_\omega(V(g))U(g)^*$ commutes with $\mathcal{F}_W$ and, in each irreducible representation of $\mathcal{F}_W$, $\pi_\omega(V(g))\,U(g)^* = e^{i\theta(g)}\mathbf{1}$. Hence, Ψ_ω is an eigenvector of $\pi_\omega(V(g))$, with eigenvalue $e^{i\theta(g)}$,

$$\pi_\omega(V(g))\Psi_\omega = \pi_\omega(V(g))U(g)^*\Psi_\omega = e^{i\theta(g)}\Psi_\omega \tag{2}$$

Thus, the GNS representation π_ω of $\mathcal{F}_W$, equivalently of G_H, defined by a gauge invariant state ω is non-regular,

$$\mathcal{H}_{\pi_\omega} = \sum_{\theta\in\sigma(\mathcal{Z})} \oplus\,\mathcal{H}_\theta$$

and the subspaces $\mathcal{H}_\theta$ carrying disjoint irreducible representations of $\mathcal{A}$ are proper subspaces of the non-separable space $\mathcal{H}_{\pi_\omega}$.

Summarizing, one has:

Proposition 1. *Let G_H be the Heisenberg group defined by the set of canonical variables $\{q_i,\,p_i\}$, $\mathcal{F}_W$ the corresponding canonical C^*-algebra, $\mathcal{A} \subset \mathcal{F}_W$ the C^*-subalgebra of observables and $\mathcal{G}$ the commutative group of gauge transformations, defined by a subgroup $\mathcal{G} \subset G_H$.*

Then, the GNS representation of $\mathcal{F}_W$ defined by a gauge-invariant state is a non-regular representation of $\mathcal{F}_W$, as well as of the Heisenberg group G_H, and the elements of $\mathcal{G}$ define super-selection rules.

Relevant examples of such a structure are provided by quantum mechanical models, in particular those exhibiting strong analogies with gauge quantum field theories (as discussed below).

The prototype is provided by the non-regular representation [15] of the Weyl algebra $\mathcal{A}_W(\mathbb{R}^2)$ generated by the one-parameter unitary groups $U(\alpha), V(\beta)$, $\alpha, \beta \in \mathbb{R}$, with $U(\alpha)V(\beta) = V(\beta)U(\alpha)e^{-i\alpha\beta}$, $V(\beta)$ describing gauge transformations.

Proposition 2. *The GNS representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{A}_W(\mathbb{R}^2)$ defined by a pure gauge-invariant state ω_0 is unitarily equivalent to the following representation:*

$$\omega_0(U(\alpha)\,V(\beta)) = 0, \quad if \ \ \alpha \neq 0, \quad \omega_0(V(\beta)) = e^{i\beta\bar{p}}, \ \ \bar{p}\in\mathbb{R} \tag{3}$$

Thus, the one-parameter group $U(\alpha)$ is non-regularly represented. The GNS representation space $\mathcal{H}_0$ contains as representative of ω_0 a cyclic vector Ψ_0, such that (denoting by the same symbols the elements of the Weyl algebra and their representatives):

$$V(\beta)\Psi_0 = e^{i\beta\bar{p}}\,\Psi_0, \quad (U(\alpha)\,\Psi_0,\, U(\alpha')\,\Psi_0) = 0, \ \ if \ \ \alpha \neq \alpha' \tag{4}$$

The linear span D of the vectors $U(\alpha)\Psi_0$, $\alpha \in \mathbb{R}$ is dense in $\mathcal{H}_0$, which is therefore non-separable.

The generator of the one-parameter group $U(\alpha)$ does not exist, but nevertheless, a generic vector of D:

$$\Psi_A = A\Psi_0, \quad A = \sum_{n\in\mathbb{Z}} a_n U(\alpha_n), \ \ \{a_n\} \in l^2$$

may be represented by a wave function $\psi_A(x) = \sum_{n\in\mathbb{Z}} a_n\,e^{i\alpha_n x}$, with the scalar product given by the ergodic mean:

$$(\psi_A, \psi_A) = \sum_{n\in\mathbb{Z}} |a_n|^2 = \lim_{L\to\infty} (2L)^{-1}\int_{-L}^{L} dx\,\bar{\psi}_A(x)\,\psi_A(x) \tag{5}$$

The spectrum of $V(\beta)$ is a pure point spectrum.

Proof. By the gauge invariance of ω_0 and the Weyl relations, one has, $\forall \gamma \in \mathbb{R}$,

$$\omega_0(U(\alpha)V(\beta)) = \omega_0(V(\gamma)U(\alpha)V(\beta)V(\gamma)^*) =$$
$$= \omega_0(U(\alpha)V(\gamma)V(\beta)V(\gamma)^*)e^{i\alpha\gamma} = \omega_0(U(\alpha)V(\beta))e^{i\alpha\gamma}$$

which proves the first Equation (3); furthermore, by Equation (2) and the group law:

$$V(\beta)\Psi_0 = e^{i\bar{p}\beta}\Psi_0, \ \bar{p} \in \mathbb{R}$$

The rest of the proposition easily follows. □

Example 1. *Gauge invariance in the two-body problem:*

The quantum description of two interacting particles is given by the Weyl algebra $\mathcal{A}_W$, corresponding to the two particle canonical variables q_1, q_2, p_1, p_2; however, for the discussion of the bound state spectrum of the two-body problem and, in particular, the lowest energy level, the position of the center of mass is irrelevant.

It is therefore natural to consider as observable C^*-algebra $\mathcal{A}$ the algebra generated by the relative canonical variable q, p and by the center of mass momentum P.

Hence, the translations $v(\beta)$ of the center of mass have the meaning of gauge transformations. The lowest energy state ω_0 must satisfy $\omega_0(P^2) = 0$, so that the corresponding vector Ψ_0 satisfies $P^2\Psi_0 = 0$, *i.e.*, it is gauge invariant $v(\beta)\Psi_0 = \Psi_0$. This condition is incompatible with the canonical commutation relations in the Heisenberg form. It has been suggested to bypass such incompatibility by allowing Ψ_0 to be non-normalizable [16].

In our opinion, such a choice would have catastrophic consequence on the GNS representation defined by such a ground state; by the cyclicity of Ψ_0, all vectors of such a representation would be non-normalizable; all matrix elements (including the ground state expectations of gauge-invariant operators) would be divergent; and one could not extract finite results in a consistent mathematical way.

A canonical quantization is not forbidden, provided it is done in terms of the Weyl algebra, rather than of the Heisenberg algebra, and it is given by Proposition 2.

In our opinion, from a mathematical point of view, the non-regularity of the representation is a much better price to pay, rather than living with non-normalizable state vectors.

The advantages of such a quantization is that the states are described by normalizable vectors of a Hilbert space; the basic quantum mechanical rules are not violated; the observable subalgebra $\mathcal{A}$ is regularly represented in the standard way; the canonical variables that are not gauge invariant are non-regularly represented, only their exponentials being well defined.

Thus, the model suggests a general strategy for quantizing systems with a gauge-invariance constraint.

Example 2. *Bloch electron:*

Another relevant quantum mechanical example is provided by an electron in a periodic bounded measurable potential $W(q) = W(q + a)$. The periodic structure of the system leads to considering as observable C^*-algebra $\mathcal{A}$ the subalgebra of the Weyl algebra $\mathcal{F}_W$, generated by the translations $V(\beta)$ and by the periodic functions of the position $U(2\pi n/a), n \in \mathbb{Z}$.

The center $\mathcal{Z}$ of $\mathcal{A}$ is generated by the translations $V(a)$, and the irreducible representations of $\mathcal{A}$ are given by the subspaces $\mathcal{H}_\theta$ (θ sectors), corresponding to the GNS representations of $\mathcal{A}$ defined by the states invariant under the gauge group of translations $V(a)$.

The operators $U(\alpha/a), \alpha \neq 2\pi n$ do not commute with the center of $\mathcal{A}$ and, therefore, intertwine between the inequivalent representations π_θ and $\pi_{\theta+\alpha}$; the corresponding one-parameter group is non-regularly represented in the representation of $\mathcal{F}_W$ defined by the gauge-invariant ground state [17].

3. Topology, Gauge Groups and Weyl Non-Regular Quantization

The structure of an observable algebra $\mathcal{A}$ naturally embedded in a larger Weyl algebra $\mathcal{F}_W$, with the non-trivial center of $\mathcal{A}$ generating gauge transformations on $\mathcal{F}_W$, arises for example as a consequence of the non-trivial topology of the manifold $\mathcal{M}$, which describes the configurations of the quantum system.

In fact, the quotient of the fundamental group $\pi_1(\tilde{\mathcal{M}})$, of the universal covering space $\tilde{\mathcal{M}}$ of the manifold, with its commutator group, *i.e.*, the first homology group $H_1(\tilde{\mathcal{M}})$ of $\tilde{\mathcal{M}}$, defines topological invariants, which are represented by elements of the center of the observable algebra [7,8].

For simplicity, we consider the case of a quantum particle on a circle, where the observable algebra $\mathcal{A}$ can be taken as the C^*-subalgebra of the standard Weyl (field) algebra $\mathcal{F}_W$, generated by $U(n) = e^{in\varphi}$, $n \in \mathbb{Z}$ and $V(\beta) = e^{i\beta p}$, $\beta \in \mathbb{R}$; with the canonical commutation relations:

$$U(n)\,V(\beta) = e^{-in\beta}\,V(\beta)\,U(n) \tag{6}$$

The rotations of 2π define elements of the observable algebra $\mathcal{A}$, which may actually be characterized as the subalgebra of $\mathcal{F}_W$ invariant under the translations γ^n of $2\pi n$, $n \in \mathbb{Z}$, which, therefore, get the meaning of gauge transformations.

The structure fits into the general discussion of Heisenberg group G_H, observable subgroup G_{obs} and gauge group $\mathcal{G}$, with corresponding C^*-algebras $\mathcal{F}_W$, $\mathcal{A}$ and a non-trivial center $\mathcal{Z}$ of $\mathcal{A}$, as discussed above. The non-trivial center of $\mathcal{A}$ may be traced back to the non-trivial fundamental group of the circle: $\pi_1(S^1) = \mathbb{Z}$. The representation π of $\mathcal{A}$ is regular if $\pi(V(\beta))$ is a weakly-continuous group of unitary operators.

In each irreducible representation of $\mathcal{A}$, the element $V(2\pi)$ is a multiple of the identity, say $e^{i\theta}$, $\theta \in [0, 2\pi)$.

Theorem 1. *[1] For any given θ, all of the irreducible regular representations π_θ of $\mathcal{A}$ with $\pi_\theta(e^{i2\pi p}) = e^{i\theta}$ are unitary equivalent.*

The Hilbert space $\mathcal{H}$ of the unique regular representation π_S of the Weyl algebra $\mathcal{F}_W$ decomposes as a direct integral:

$$\mathcal{H} = \int_0^{2\pi} d\theta\,\mathcal{H}_\theta$$

over the spectrum of $\pi_S(V(2\pi))$. There is a unique irreducible representation π of $\mathcal{F}_W$, whose Hilbert space decomposes as a direct sum of the irreducible representations of the observable algebra $\mathcal{A}$; in such a representation, $V(\beta)$ is regularly represented, but the algebra generated by $U(\alpha)$, $\alpha \in \mathbb{R}$ is not. The operators $U(\alpha)$ intertwine between inequivalent representations of $\mathcal{A}$.

A similar structure is displayed by an electron in a periodic crystal, *i.e.*, subject to a periodic potential. For simplicity, we consider the one-dimensional case. In this case, the Hamiltonian is $H = -d^2/dx^2 + W(x)$, with the potential satisfying the periodicity condition $W(x + a) = W(x)$, for a suitable a. The field algebra $\mathcal{F}_W$ is generated by the Weyl operators $U(\alpha), V(\beta), \alpha, \beta \in \mathbb{R}$.

As in the case of a particle on a circle, the center of the observable algebra may be viewed as arising from the non-trivial topology of $\mathbb{R}/[0, a]$. In this case, the non-regular representations of $\mathcal{F}_W$ are defined by the gauge-invariant ground states (θ *vacua*).

Proposition 3. *[17] Let $W(x)$ be a bounded measurable periodic potential, $W(x) = W(x + a)$, then there exists one and only one irreducible representation $(\pi, \mathcal{K})$ of the Weyl algebra $\mathcal{A}_W$ in which the Hamiltonian*

$$H = p^2/2 + W(x)$$

is well defined, as a strong limit of elements of $\mathcal{A}_W$ (on a dense domain), and has a ground state $\Psi_0 \in \mathcal{K}$.

Moreover, such a representation is independent of W, in the class mentioned above, and it is the unique non-regular representation π_0 in which the subgroup $V(\beta)$, $\beta \in \mathbb{R}$ is regularly represented; its generator p has a discrete spectrum.

The Hilbert space $\mathcal{K}$ of π_0 consists of the formal sums:

$$\psi(x) = \sum_{n \in \mathbb{Z}} c_n \, e^{i\alpha_n x}, \quad \{c_n\} \in l^2(\mathbf{C}), \quad x \in \mathbb{R}, \quad \alpha_n \in \mathbb{R} \tag{7}$$

with the scalar product given by the ergodic mean:

$$(\psi, \psi) = \sum_{n \in \mathbb{Z}} |c_n|^2 = \lim_{L \to \infty} (2L)^{-1} \int_L^L dx \, \bar{\psi}(x) \, \psi(x) \tag{8}$$

The Weyl operators are represented by:

$$(\pi_o(U(\alpha))\psi)(x) = e^{i\alpha x}\psi(x), \quad (\pi_0(V(\beta))\psi)(x) = \psi(x + \beta) \tag{9}$$

The (orthogonal) decomposition of $\mathcal{K}$ over the spectrum of $V(a)$ is:

$$\mathcal{K} = \bigoplus_{\theta \in [0, 2\pi)} \mathcal{H}_\theta, \quad V(a) \, \mathcal{H}_\theta = e^{i\theta} \, \mathcal{H}_\theta, \quad \theta \in [0, 2\pi) \tag{10}$$

The spectrum of p in $\mathcal{H}_\theta$ is $\sigma(p)|_{\mathcal{H}_\theta} = \{2\pi n/a + \theta/a, \ n \in \mathbb{Z}\}$, and the wave functions $\psi_\theta \in \mathcal{H}_\theta$ are quasi-periodic of the form:

$$\psi_\theta(x) = e^{i\theta x/a} \sum c_n e^{i2\pi nx/a} \tag{11}$$

(Bloch waves). The unique ground state is a vector of $\mathcal{H}_{\theta=0}$.

4. Non-Regular Representations and Symmetry Breaking

We briefly recall that, given a C^*-algebra $\mathcal{A}$, an algebraic symmetry is an automorphism β of $\mathcal{A}$; given a state ω, the symmetry is unbroken in the corresponding representation space if β is implemented by a unitary operator $T(\beta)$ there, *i.e.*,

$$\pi_\omega(\beta(A)) = T(\beta)\,\pi_\omega(A)\,T(\beta)^*, \quad \forall A \in \mathcal{A} \tag{12}$$

This means that the representation defined by the state ω_β, $\omega_\beta(A) \equiv \omega(\beta(A))$ is unitary equivalent to π_ω: $\pi_{\omega_\beta}(A) = T(\beta)\,\pi_\omega(A)\,T^*(\beta)$ and that ω and ω_β are described by vectors of the same Hilbert space.

In this case, β gives rise to a Wigner symmetry in $\mathcal{H}_\omega$, *i.e.*, all transition amplitudes are invariant. Otherwise, if there is no unitary operator that implements β in $\mathcal{H}_\omega$, by the Wigner theorem on symmetries, at least one transition amplitude is not invariant, and the symmetry β is said to be broken in $\mathcal{H}_\omega$.

A one-parameter group β^λ, $\lambda \in \mathbb{R}$, of symmetries shall be called a continuous symmetry. A symmetry is called internal if it commutes with the one-parameter group α_t, $t \in \mathbb{R}$, of the time translations. In the following, the breaking of an internal symmetry shall be called spontaneous symmetry breaking. An algebraic symmetry is said to be regular if it maps regular representations into regular ones. For a discussion of the meaning and the mechanism of spontaneous symmetry breaking, see [18].

In the case of quantum systems described by the canonical Weyl algebra $\mathcal{A}_W$, any regular algebraic symmetry of $\mathcal{A}_W$ is unbroken in any regular irreducible representation, since, by the Stone–von Neumann theorem, all such representations are unitarily equivalent. Thus, the important phenomenon of symmetry breaking, in the strong sense of a loss of symmetry as defined above (which goes much beyond the mere non-invariance of the ground state), cannot appear in the case of Heisenberg quantization, more generally in the case of regular Weyl quantization.

The situation drastically changes in the case of quantum systems whose algebra of observables $\mathcal{A}$ has a non-trivial center. A distinguished case is when one has the structure discussed in Section 2, namely a canonical algebra $\mathcal{F}_W$ and an observable (gauge-invariant) subalgebra $\mathcal{A}$, with a non-trivial center $\mathcal{Z} \subset \mathcal{A}$.

Clearly, any symmetry β of $\mathcal{A}$, defined by an element of $\mathcal{F}_W$, is implemented by a unitary operator $T(\beta)$ in the non-regular representation π of $\mathcal{F}_W$, defined by a gauge-invariant state ω_θ, $\theta \in \sigma(\mathcal{Z})$.

However, if β does not commute with the gauge group $\mathcal{G}$, β is broken in each irreducible representation $\mathcal{H}_\theta$ of the observable subalgebra $\mathcal{A}$, *i.e.*, β fails to define a Wigner symmetry of the gauge invariant states of $\mathcal{H}_\theta = \overline{\mathcal{A}\,\Psi_{\omega_\theta}}$, because $T(\beta)$ does not leave $\mathcal{H}_\theta$ invariant.

In the regular irreducible representation, π_r of $\mathcal{F}_W$, the symmetry β is unbroken; however the elements of $\mathcal{Z}$ have a continuous spectrum in $\mathcal{H}_{\pi_r}$, and there is not a gauge-invariant (proper) state vector in $\mathcal{H}_{\pi_r}$.

Proposition 4. *Let $\mathcal{F}_W$ denote the canonical field C^*-algebra defined by a Heisenberg group G_H, $\mathcal{A}$ the observable C^*-subalgebra, $\mathcal{Z}$ the non-trivial center of $\mathcal{A}$ generated by the commutative subgroup $\mathcal{G} \subset \mathcal{G}_H$ (gauge group), then:*

(i) any algebraic symmetry β of $\mathcal{A}$, defined by an element of G_H that does not commute with each element of $\mathcal{G}$, is spontaneously broken in each irreducible representation of $\mathcal{A}$ (θ sector);

(ii) *in any representation of $\mathcal{F}_W$ defined by a gauge-invariant state ω, the one-parameter subgroups that do not commute with $\mathcal{G}$ are non-regularly represented, so that the corresponding generators cannot be defined as operators in $\mathcal{H}_\omega$; only their exponentials exist.*

Proof

(i) In fact, $\beta^\lambda(\mathcal{Z}) \subseteq \mathcal{Z}$, and since $\mathcal{Z}$ is not left pointwise-invariant under β^λ, there is at least one $V \in \mathcal{Z}$, which may be taken as unitary, such that $V_\lambda \equiv \beta^\lambda(V) \neq V$, and in a given irreducible representation of $\mathcal{A}$, V_λ and V are different multiples of the identity. Then, the symmetry breaking condition is realized.

(ii) Furthermore, if $U(\lambda)$ denotes the one-parameter unitary group that implements β^λ, $R(\lambda, V) \equiv V^{-1} U(\lambda)^{-1} V U(\lambda) \in \mathcal{Z}$ and is a multiple of the identity $e^{i\theta(\lambda,V)}$ in each irreducible representation of $\mathcal{A}$. Hence, $\forall A \in \mathcal{A}$, with $< A > \neq 0$, using $V U(\lambda) = U(\lambda) V R(\lambda, V)$,

$$< U(\lambda) A >=< V U(\lambda) A V^{-1} >=< U(\lambda) A R(\lambda, V) >=$$

$$e^{i\theta(\lambda,V)} < U(\lambda) A >$$

so that:

$$< U(\lambda) A >= \begin{cases} 0, & \text{if } \lambda \neq 0, \\ < A >, & \text{if } \lambda = 0, \end{cases}$$

i.e., $U(\lambda)$ is not weakly continuous in λ.

□

For representations of $\mathcal{A}$ defined by a ground state ω_0, (more generally, by a state ω invariant under time translations), the non-invariance of ω_0,

$$< A >\equiv \omega_0(A) \neq \omega_0(\beta(A)), \quad \text{for some} \quad A \in \mathcal{A} \tag{13}$$

is still compatible with β giving rise to a Wigner symmetry in the GNS representation space $\mathcal{H}_{\omega_0}$. In this case, if β commutes with the dynamics, Equation (13) implies degeneracy of the ground state. This is what happens if Equation (13) holds for β defined by an element of the field algebra $\mathcal{F}_W$, which commutes also with the gauge group.

5. Symmetry Breaking by Topology and Energy Spectrum

The spontaneous breaking of a continuous symmetry in the quantum theory of infinitely-extended systems is usually accompanied by a strong constraint on the energy spectrum.

In fact, if the symmetry commutes with the dynamics (*i.e.*, if the Hamiltonian is symmetric) and it is generated by a conserved current at all times, the Goldstone theorem predicts the absence of an energy gap with respect to the ground state, in the channels related to the ground state by the broken generators and by the operator that provides the order parameter. For a review and critical discussion of the Goldstone theorem, see [18].

The existence of more than one representation for finite-dimensional quantum systems, corresponding to a non-trivial center of the algebra of observables leading to non-regular representations of canonical

Weyl field algebra, opens the possibility of spontaneous symmetry breaking also in the case of quantum systems described by a finite number of canonical variables. In this case, the question arises of the implications on the energy spectrum.

For this purpose, given a C^*-algebra $\mathcal{A}$, a one-parameter group β^λ, $\lambda \in \mathbb{R}$ of automorphisms of $\mathcal{A}$ commuting with the time translations α_t and a representation π of $\mathcal{A}$ defined by a ground state ω_0, we consider:

(i) the infinitesimal variation of a generic element $F = \pi(A)$, $A \in \mathcal{A}$,

$$\delta F = \delta(\pi(A)) = \frac{d\,\pi(\beta^\lambda(A))}{d\lambda}|_{\lambda=0}$$

(ii) the generation of the continuous symmetry β^λ by elements of the strong closure $\pi(\mathcal{A})''$ of $\pi(\mathcal{A})$, in the sense that there is a sequence $Q_n = Q_n^* \in \pi(\mathcal{A})''$, $n = 1, ..$, such that:

$$\delta F = i \lim_{n\to\infty} [\, Q_n,\, F\,] \tag{14}$$

If, in the GNS representation defined by ω_0, there is a sequence Q_n that generates β^λ and converges weakly to a self-adjoint operator Q, then β^λ is implementable by the unitary operator $e^{i\lambda Q}$, the symmetry is not broken and $< \delta F > \equiv \omega_0(F) \neq 0$ implies that ω_0 is not invariant, *i.e.*, $Q\Psi_0 \neq 0$. Furthermore, if β^λ commutes with the time translations α_t, $\Psi_\lambda \equiv e^{i\lambda Q}\Psi_0$ is a family of degenerate ground states.

However, if there is no sequence $Q_n \in \pi(\mathcal{A})''$ that generates β^λ and converges weakly to a self-adjoint operator Q, then, if $< \delta F > \neq 0$, the symmetry is broken, and $< \delta F >$ has the meaning of a symmetry breaking order parameter.

Furthermore, by the invariance of the ground state under α_t, one has:

$$\lim_{n\to\infty} < [Q_n(t),\, F\,] >= \lim_{n\to\infty} < [\, \alpha_t(Q_n),\, F\,] >=$$

$$= \lim_{n\to\infty} < [\, Q_n,\, \alpha_{-t}(F)\,] >=< \delta(\alpha_{-t}(F)) >=< \delta F >=$$

$$= \lim_{n\to\infty} < [\, Q_n,\, F\,] >$$

where the commutation $\beta^\lambda \alpha_t = \alpha_t \beta^\lambda$ has been used in the last, but one equality.

It is worthwhile to stress that such a time independence of the Ward identity holds also in the more general case in which the symmetry does not commute with the Hamiltonian, but commutes with the time translations in the ground state expectations of the order parameter F. For example, $\lim_{n\to\infty}[Q_n(t),\, H] = \delta H \neq 0$, but $< [\, \delta H, F\,] >= 0$.

This is, e.g., the case in which the Hamiltonian is invariant up to a time derivative, which commutes with F on the ground state (see the example of the quantum particle on a circle discussed below).

In conclusion, by the above arguments, for finite dimensional quantum systems one has the following analog of the Goldstone theorem.

Theorem 2. *(Goldstone) Let $\mathcal{A}$ be a C^*-algebra, α_t the one-parameter group of automorphisms of $\mathcal{A}$ describing the time translations and π the representation of $\mathcal{A}$ defined by a state ω_0, invariant under α_t; if:*

(i) β^λ, $\lambda \in \mathbb{R}$, *is a one-parameter group of automorphisms of the algebra* $\mathcal{A}$,

(ii) *there is one* $F \in \pi(\mathcal{A})$, *such that,*

$$< \delta\, F > \equiv d < \beta^\lambda(F) > /d\lambda|_{\lambda=0} \neq 0$$

(iii) *and:*

$$< \delta\, F > = i \lim_{n\to\infty} < [\, Q_n,\, F\,] > = i \lim_{n\to\infty} < [\, Q_n(t),\, F\,] > \tag{15}$$

for a suitable sequence of $Q_n = Q_n^* \in \pi(\mathcal{A})''$, $Q_n(t) \equiv \alpha_t(Q_n)$, *the limit being understood in the sense of the convergence of tempered distributions in the variable* t, *then there is no energy gap above the ground state. Actually, there is a state (Goldstone-like state) orthogonal to the ground state, with the ground state energy.*

Proof. In fact, Equation (15) implies that, putting $J_n(t) \equiv 2\,\mathrm{Im} < Q_n U(-t)F >$,

$$\lim_{n\to\infty} \tilde{J}_n(\omega) = < \delta F > \delta(\omega)$$

therefore the spectral measure of $U(t)$ contains a $\delta(\omega)$. $\square$

It is worthwhile to stress that the non-invariance of the ground state expectation of a field F does not guarantee that one can write a corresponding Ward identity, Equation (15), a crucial ingredient for the Goldstone theorem.

The interplay between gauge invariance and the breaking of a continuous symmetry provides a mechanism for evading the conclusions of the Goldstone theorem, *i.e.*, for allowing an energy gap in the presence of symmetry breaking.

This is typically the case in which the gauge group arises as a consequence of a non-trivial topology (symmetry breaking by topology).

The prototypic realization of such a structure is when the manifold of the configurations of the quantum system has a nontrivial fundamental group leading through its topological invariants (corresponding to its first homology group) to elements of the center of the observable algebra.

The role of the topology in triggering symmetry breaking and affecting its consequences has been realized in specific models of GQFT, through a topological classification of Euclidean field configurations. The following theorem provides the general mechanism with no reference to specific (mathematically questionable) ingredients, involving the semiclassical approximation.

Theorem 3. *(Spontaneous symmetry breaking and energy gap)*

Let $\mathcal{F}_W$ *be a canonical (field) algebra,* $\mathcal{A}$ *the observable subalgebra and* $\mathcal{Z}$ *the center of* $\mathcal{A}$ *generating gauge transformations on* $\mathcal{F}_W$; *if an automorphism* β *of* $\mathcal{A}$ *does not leave* $\mathcal{Z}$ *pointwise invariant, in particular if the topological invariants that define elements of* $\mathcal{Z}$ *are not invariant under* β, *then* β *is spontaneously broken in each irreducible representation of* $\mathcal{A}$. *Furthermore, if:*

(i) β^λ *is a one-parameter group of automorphisms of* $\mathcal{A}$ *defined by elements of* $\mathcal{F}_W$,

(ii) *in the irreducible representation* π_θ *of* $\mathcal{A}$ *defined by a gauge invariant ground state* ω_θ *(θ vacuum),* $\theta \in \sigma(\mathcal{Z})$, *there is a non-symmetric order parameter:*

$$\omega_\theta(\beta^\lambda(A)) \neq \omega_\theta(A), \quad A \in \mathcal{A} \tag{16}$$

(iii) β^λ commutes with the time translations in the ground state expectations of the order parameter
   A, i.e.,

$$\omega_\theta(\beta^\lambda(\alpha_t(A))) = \omega_\theta(\beta^\lambda(A)) \tag{17}$$

(iv) and β^λ does not leave $\mathcal{Z}$ pointwise invariant,

then β^λ *cannot be generated by elements of $\pi(\mathcal{A})''$, in π_θ, so that the crucial condition of the Goldstone*
theorem fails and an energy gap is allowed.

Proof. The inevitable breaking of β in any irreducible representation of $\mathcal{A}$ follows as in Proposition 4.
Let us assume that β^λ is generated in π_θ by an element of the strong closure of $\pi(\mathcal{F}_W)''$, *i.e.*, there is an
operator $Q \in \pi(\mathcal{F}_W)''$, such that:

$$\delta A = i\,[\,Q,\,A\,], \quad \pi_\theta(\delta A) = i\,[\pi_\theta(Q\,A) - \pi_\theta(A\,Q)] \quad \forall A \in \mathcal{A} \tag{18}$$

If β^λ does not leave $\mathcal{Z}$ pointwise invariant, there is at least one unitary $V \in \mathcal{Z}$, such that:

$$i[\,Q,\,V\,] = \delta V \neq 0$$

Now, since $[V,\,[\,Q,\,A\,]\,] = 0$, one has:

$$[[\,Q,\,V\,],\,A\,] = -[[\,A,\,Q\,],\,V\,] - [[\,V,\,A\,],\,Q\,] = 0$$

i.e., $[\,Q,\,V\,] \in \mathcal{Z}$.

Therefore, by the gauge invariance of ω_θ, one has:

$$\omega_\theta(Q\,A) = \omega_\theta(V\,Q\,A\,V^{-1}) = \omega_\theta(Q\,V\,A\,V^{-1} + [\,Q,\,V\,]A V^{-1}) =$$

$$= \omega_\theta(Q\,A) + \omega_\theta([\,Q,\,V\,]\,V^{-1})\,\omega_\theta(A)$$

Thus, the expectations $\omega_\theta(Q\,A)$ cannot be defined, and one cannot write the symmetry breaking
Ward identities, which are crucial for the conclusion of the Goldstone theorem. An energy gap is
therefore compatible with the spontaneous breaking of β^λ in irreducible representations of the observable
algebra $\mathcal{A}$. $\square$

In the irreducible regular representation π_r of the field algebra $\mathcal{F}_W$, β^λ is implemented by a (weakly
continuous) group of unitary operators $T(\lambda)$; all of the matrix elements are invariant, but there is no
gauge-invariant state (proper) vector invariant under time translations. The symmetry gets broken by the
direct integral decomposition of $\mathcal{H}_{\pi_r}$ over the spectrum of $\mathcal{Z}$, but one cannot write a symmetry breaking
Ward identity for the expectation on the gauge-invariant ground state.

On the other side, in the representation of $\mathcal{F}_W$ defined by a gauge-invariant ground state ω_θ, the
one-parameter group $T(\lambda)$ is not regularly represented. Therefore, its generator cannot be defined as
an operator in $\mathcal{H}_{\omega_\theta}$ and $\omega_\theta(\delta F) \neq 0$ cannot be written in terms of a commutator. In conclusion, the
symmetry breaking Ward identity cannot be written in terms of expectations on θ states.

This is well displayed by the following examples.

Example 3. *Quantum particle on a circle:*

As discussed above, the observable algebra $\mathcal{A}$ is a subalgebra of canonical Weyl (field) algebra $\mathcal{F}_W$ and has a non-trivial center generated by the translations γ^n corresponding to the non-trivial fundamental group of the circle. The one-parameter group of automorphisms of $\mathcal{A}$:

$$\beta^\lambda(U(n)) = U(n), \quad \beta^\lambda(V(\beta)) = e^{-i\,\lambda\,\beta}\,V(\beta + \lambda) \tag{19}$$

is realized by the adjoint action of $U(\lambda) \in \mathcal{F}_W$:

$$\beta^\lambda(U(n)\,V(\beta)) = U(\lambda)\,U(n)\,V(\beta)\,U(-\lambda)$$

Actually, $\mathcal{F}_W$ is the minimal extension of $\mathcal{A}$, such that the automorphisms β^λ are inner and $\mathcal{F}_W$ includes the operators, which intertwine between inequivalent representations of $\mathcal{A}$. This justifies the use of the (canonical) field algebra $\mathcal{F}_W$ for a canonical description of both the observables and all the states of the quantum system.

In the representation π_θ of $\mathcal{A}$, there is a non-symmetric order parameter:

$$\omega_\theta(\beta^\lambda(V(\beta))) = e^{-i\,\lambda\,\beta}\,\omega_\theta(V(\beta)), \quad \omega_\theta(V(\beta)) = e^{i\,\theta\,\beta/2\pi}$$

and β^λ commutes with the time translations $e^{i\,Ht}$ $H = p^2/2m$ in expectations of the order parameter: $\omega_\theta(\beta^\lambda\,\alpha_t(V(\beta))) = \omega_\theta(\alpha_t\beta^\lambda(V(\beta)))$.

Furthermore, β^λ does not leave the center of $\mathcal{A}$ pointwise invariant. Theorem 3 applies, and an energy gap is allowed; in fact, in the representation π_θ of $\mathcal{A}$, the spectrum of the Hamiltonian is discrete, given by:

$$\sigma^\theta(H) = \frac{(n + \theta/2\pi)^2}{2m}, \quad n \in \mathbb{Z}$$

and displays an energy gap above the ground state energy $(\theta/2\pi)^2/2m$, compatibly with the spontaneous symmetry breaking of β^λ. It is worthwhile to note that the energy gap is provided by the spectrum of the center, *i.e.*, by the (discrete) spectrum of the fundamental group.

Example 4. *Bloch electron:*

For simplicity, we consider the one-dimensional case. The field algebra $\mathcal{F}_W$ is the canonical Weyl algebra, and the observable subalgebra is characterized by its pointwise invariance under the gauge transformations defined by the adjoint action of the center of $\mathcal{A}$, which arises from the non-trivial topology of $\mathbb{R}/[0, a]$. The automorphisms β^λ defined by the Weyl operators $U(\lambda)$, do not leave the center pointwise invariant. In the representation π_θ, defined by the gauge-invariant states ω_θ, there is a symmetry breaking order parameter $\omega_\theta(V(\beta))$.

Furthermore, β^λ commutes with $T(\tau) \equiv e^{-i\tau\,p^2/2m}$ on expectations of the order parameter:

$$\omega_\theta(\beta^\lambda(T(\tau)\,V(\beta)\,T(\tau)^{-1})) = \omega_\theta(\beta^\lambda(V(\beta))) =$$

$$= \omega_\theta((T(\tau)\,\beta^\lambda(V(\beta))\,T(\tau)^{-1})$$

since the states ω_θ are invariant under the one-parameter group $T(\tau)$. Thus, if β^λ would be generated by q in the representation π_θ, one could apply the Goldstone theorem and conclude that the spectrum of p^2

does not have a gap above the ground state value. However, Theorem 3 applies, and in fact, the spectrum of p^2 has a gap given by the spectrum of the fundamental group of $\mathbb{R}/[0, a]$.

Author Contributions

Both authors equally gave significant contribution to this paper and the final form of this paper is approved by both.

Conflicts of Interest

The authors declare no conflict of interest.

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