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Nevanlinna's Five Values Theorem on Annuli

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Abstract: By using the second main theorem of the meromorphic function on annuli, we investigate the problem on two meromorphic functions partially sharing five or more values and obtain some theorems that improve and generalize the previous results given by Cao and Yi.

Keywords: meromorphic function; Nevanlinna theory; the annuli

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1. Introduction and Main Results

The purpose of this paper is to study the uniqueness of two meromorphic functions sharing five or more values. Thus, we always assumed that the reader is familiar with the notations of the Nevanlinna theory, such as $T(r,f), m(r,f), N(r,f)$, and so on (see [1–4]). We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane and $\mathbb{X}$ to denote the subset of $\mathbb{C}$.

In 1929, R. Nevanlinna (see [5]) first investigated the uniqueness of meromorphic functions in the whole complex plane and obtained the well-known theorem: the five IMtheorem:

Theorem 1.1. *(see [5]). If f and g are two non-constant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, a_5 IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.*

After his theorem, there are vast references on the uniqueness of meromorphic functions sharing values and sets in the whole complex plane (see [3]). It is an interesting topic how to extend some important uniqueness results in the complex plane to an angular domain or the unit disc. In the past several decades, the uniqueness of meromorphic functions in the value distribution attracted many investigations. For example, I. Lahiri, H.X. Yi, X.M. Li and A. Banerjee (including [3,6–8]) studied the uniqueness for meromorphic functions on the whole complex plane sharing one, two, three or some sets; M.L. Fang, H.F. Liu, Z.Q. Mao and H.Y. Xu (including [9–11]) investigated the shared value of meromorphic functions in the unit disc; J.H. Zheng, Q.C. Zhang, T.B. Cao and W.C. Lin (including [12–16]) considered many uniqueness problem on meromorphic functions on the angular domain.

In 2009, Z.Q. Mao and H.F. Liu [10] gave a different method to investigate the uniqueness problem of meromorphic functions in the unit disc and obtained the following results.

Theorem 1.2. *(see [10]). Let f, g be two meromorphic functions in $\mathbb{D}$, $a_j \in \overline{\mathbb{C}}(j = 1, 2, \ldots, 5)$ be five distinct values and $\Delta(\theta_0, \delta) = \{z : |z| < 1\} \bigcap \{z : |\arg z - \theta_0| < \delta\}$, $0 \le \theta_0 \le 2\pi, 0 < \delta < \pi$ be an angular domain, such that for some $a \in \overline{\mathbb{C}}$,*

$$\limsup_{r\to 1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z) = a)}{\log \frac{1}{1-r}} = \tau > 1. \tag{1}$$

If f and g share $a_j (j = 1, 2, \ldots, 5)$ IM in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.

In the same year, T.B. Cao and H.X. Yi [12] investigated the uniqueness problem of two transcendental meromorphic functions sharing five distinct values in an angular domain and obtained the following theorem:

Theorem 1.3. *(see [12], Theorem 1.3). Let f and g be two transcendental meromorphic functions. Given one angular domain $X = \{z : \alpha < \arg z < \beta\}$ with $0 < \beta - \alpha \leq 2\pi$, we assume that f and g share five distinct values $a_j \in \overline{\mathbb{C}}(j = 1, 2, 3, 4, 5)$ IM in X. Then, $f(z) \equiv g(z)$, provided that:*

$$\lim_{r \to \infty} \frac{S_{\alpha,\beta}(r, f)}{\log(rT(r, f))} = \infty, \quad (r \notin E),$$

where $S_{\alpha,\beta}(r, f)$ is used to denote the angular characteristic function of meromorphic function f.

Remark 1.1. *We may denote Theorem 1.3 by the five IM theorem in an angular domain.*

In 2003, J.H. Zheng [15,16] firstly took into account the value distribution of meromorphic functions in an angular domain. In 2010, J.H. Zheng [17] investigated the uniqueness of the meromorphic function sharing five values in an angular domain, by using Tsjui's characteristic function.

Theorem 1.4. *(see [17]). Let f and g be two nonconstant meromorphic functions in an angular domain $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$, and:*

$$\limsup_{r \to \infty} \frac{\mathfrak{T}_{\alpha,\beta}(r, f)}{\log r} = \infty.$$

If f and g share five distinct values $a_j \in \overline{\mathbb{C}}(j = 1, 2, 3, 4, 5)$ IM in $\Omega(\alpha, \beta)$, then $f(z) \equiv g(z)$.

Remark 1.2. *$\mathfrak{T}_{\alpha,\beta}(r, f)$ is Tsjui's characteristic function of f in the angular domain $\Omega(\alpha, \beta)$, which is introduced in [17].*

However, the whole complex plane, the unit disc and the angular domain can all be regarded as a simply-connected region; in other words, the theorems stated in the above references are only regarded as the uniqueness results in a simply-connected region. In fact, there exists many other sub-regions in the whole complex plane, such as: the annuli, the m-punctured complex plane, *etc.*

Recently, there have been some results focusing on the Nevanlinna theory of meromorphic functions on the annulus (see [18–23]). The annulus can be regarded as the doubly-connected region. From the doubly-connected mapping theorem [24], we can get that each doubly-connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \leq r < R \leq +\infty$. For two cases: $r = 0$, $R = +\infty$, simultaneously, and $0 < r < R < +\infty$; the latter case, the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$ in two cases. In 2005, Khrystiyanyn and Kondratyuk [18,19] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [25]). The basic notions of the Nevanlinna theory on annuli will be shown in the next section. Lund and Ye [21] in 2009 studied meromorphic functions on annuli with the form $\{z : R_1 < |z| < R_2\}$, where $R_1 \geq 0$ and $R_2 \leq \infty$. In 2009 and 2011, Cao [26–28] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna's famous five-value theorem.

Theorem 1.5. *(see [26], Corollary 3.4). Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\overline{\mathbb{C}}$ and $k_j(j = 1, 2, \ldots, q)$ be positive integers or ∞, such that:*

$$k_1 \geq k_2 \geq \cdots \geq k_q. \tag{2}$$

and

$$\overline{E}_{k_j)}(a_j, f_1) = \overline{E}_{k_j)}(a_j, f_2), \quad (j = 1, 2, \ldots, q). \tag{3}$$

Then:

(i) if $q = 7$, *then* $f_1(z) \equiv f_2(z)$.
(ii) if $q = 6$ *and* $k_3 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.
(iii) if $q = 5$, $k_3 \geq 3$ *and* $k_5 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.
(iv) if $q = 5$ *and* $k_4 \geq 4$, *then* $f_1(z) \equiv f_2(z)$.
(v) if $q = 5$, $k_3 \geq 5$ *and* $k_4 \geq 3$, *then* $f_1(z) \equiv f_2(z)$.
(vi) if $q = 5$, $k_3 \geq 6$ *and* $k_4 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

From Theorem 1.5, we can get the following theorem immediately.

Theorem 1.6. *(see [26], Theorem 3.2). Let* f_1 *and* f_2 *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* a_j $(j = 1, 2, 3, 4, 5)$ *be five distinct complex numbers in* $\overline{\mathbb{C}}$. *If* $\overline{E}(a_j, f_1) = \overline{E}(a_j, f_2)$ *for* $j = 1, 2, 3, 4, 5$, *then* $f_1(z) \equiv f_2(z)$.

Remark 1.3. *Write* $E(a, f) = \{z \in \mathbb{A} : f(z) - a = 0\}$, *where each zero with multiplicity* m *is counted* m *times. If we ignore the multiplicity, then the set is denoted by* $\overline{E}(a, f)$. *We use* $\overline{E}_{k)}(a, f)$ *to denote the set of zeros of* $f - a$ *with multiplicities no greater than* k, *in which each zero is counted only once.*

In this paper, we will further investigate the problem on the five values for meromorphic functions on annuli. To state our main theorem, we first introduce the following definition.

Definition 1.1. *For* $B \subset \mathbb{A}$ *and* $a \in \overline{\mathbb{C}}$, *we denote by* $\overline{N}_0^B(r, \frac{1}{f-a})$ *the reduced counting function of those zeros of* $f - a$ *on* $\mathbb{A}$, *which belong to the set* B.

Theorem 1.7. *Let* f *and* g *be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, \ldots, a_q (q \geq 5)$ *be* q *distinct complex numbers or* ∞. *Suppose that* $k_1 \geq k_2 \geq \cdots \geq k_q, m$ *are positive integers or infinity;* $1 \leq m \leq q$ *and* $\delta_j (\geq 0)(j = 1, 2, \ldots, q)$ *are such that:*

$$(1 + \frac{1}{k_m}) \sum_{j=m}^{q} \frac{1}{1 + k_j} + 3 + \sum_{j=1}^{q} \delta_j < (q - m - 1)(1 + \frac{1}{k_m}) + m. \tag{4}$$

Let $B_j = \overline{E}_{k_j)}(a_j, f) \setminus \overline{E}_{k_j)}(a_j, g)$ *for* $j = 1, 2, \ldots, q$. *If:*

$$\overline{N}_0^{B_j}(r, \frac{1}{f - a_j}) \leq \delta_j T_0(r, f) \tag{5}$$

and:

$$\begin{aligned} &\liminf_{r \to \infty} \frac{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{f-a_j})}{\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r, \frac{1}{g-a_j})} \\ > &\frac{k_m}{(1 + k_m) \sum_{j=m}^{q} \frac{k_j}{1+k_j} - 2(1 + k_m) + (m - 2 - \sum_{j=1}^{q} \delta_j) k_m}, \end{aligned} \tag{6}$$

then $f(z) \equiv g(z)$.

From Theorem 1.7, we can get the following consequences.

Corollary 1.1. *Let* $m = 1$, $k_j = \infty$ *for* $j = 1, 2, \ldots, q$ *and:*

$$\gamma = \liminf_{r \to R_0} \frac{\sum_{j=1}^{q} \overline{N}_0(r, \frac{1}{f-a_j})}{\sum_{j=1}^{q} \overline{N}_0(r, \frac{1}{g-a_j})} > \frac{1}{q - 3}.$$

If $\overline{N}_0^{B_j}(r, \frac{1}{f-a_j}) \leq \delta_j T_0(r, f)$ *where* $\delta_j (\geq 0)$ *satisfy* $0 \leq \sum_{j=1}^{q} \delta_j < k - 3 - \frac{1}{\gamma}$, *then* $f(z) \equiv g(z)$.

If we take $q = 5$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$, then $B_j = \emptyset$ for $j = 1, 2, \ldots, 5$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, \ldots, 5$ and take any constant γ, such that $0 \leq 2 - \frac{1}{\gamma}$ in Corollary 1.1; we can get that $f \equiv g$. Especially, if $q = 5$ and $\overline{E}(a_j, f) = \overline{E}(a_j, g)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, \ldots, 5$. We can obtain $f \equiv g$. Therefore, Corollary 1.1 is an improvement of Theorem 1.6.

Corollary 1.2. *Let f and g be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, \ldots, a_q (q \geq 5)$ *be q distinct complex numbers or* ∞. *Suppose that* $k_1, k_2, \cdots, k_q$ *are positive integers or infinity with* $k_1 \geq k_2 \geq \cdots \geq k_q$, *if* $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ *and:*

$$\sum_{j=2}^{q} \frac{k_j}{1 + k_j} - \frac{k_1}{\gamma(1 + k_1)} - 2 > 0,$$

where γ *is stated as in Corollary 1.1; then,* $f(z) \equiv g(z)$.

Corollary 1.3. *Under the assumptions of Corollary 1.2, if* $\overline{E}_{k_j)}(a_j, f) = \overline{E}_{k_j)}(a_j, g)$ *and:*

$$\sum_{j=2}^{q} \frac{k_j}{1 + k_j} - \frac{k_1}{1 + k_1} - 2 > 0,$$

then we have $f(z) \equiv g(z)$.

Corollary 1.4. *Let f and g be two transcendental or admissible meromorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *Let* $a_1, \ldots, a_q (q \geq 5)$ *be q distinct complex numbers or* ∞. *Suppose that* $k_1, k_2, \cdots, k_q$ *are positive integers or infinity with* $k_1 \geq k_2 \geq \cdots \geq k_q$, *if* $\overline{E}_{k_j)}(a_j, f) \subseteq \overline{E}_{k_j)}(a_j, g)$ *and:*

$$\sum_{j=m}^{q} \frac{k_j}{1 + k_j} - 2 + \frac{(m - 2 - \frac{1}{\gamma}) k_m}{1 + k_m} > 0, \tag{7}$$

where γ *is stated as in Corollary 1.1; then,* $f(z) \equiv g(z)$.

Remark 1.4. *If* $\overline{E}_{k_j)}(a_j, f) = \overline{E}_{k_j)}(a_j, g)$ *and taking* $m = 3$ *in Corollary 1.4, thus Equation (5) becomes:*

$$\sum_{j=3}^{q} \frac{k_j}{1 + k_j} > 2.$$

Then, we can get Theorem 1.5 easily. Hence, Theorem 1.7 is an improvement of Theorem 1.5.

Remark 1.5. *Throughout our article, we can see that our theorem and corollaries also hold for transcendental meromorphic function in the whole complex plane, which are also extensions of some results given by Nevanlinna, Yi and Cao [3,5,26].*

2. Preliminaries and Some Lemmas

Next, we will introduce the basic notations and conclusion about meromorphic functions on annuli.

For a meromorphic function f on whole plane $\mathbb{C}$, the classical notations of the Nevanlinna theory are denoted as follows:

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$
$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad T(r, f) = N(r, f) + m(r, f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \leq t\}$.

Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$; the notations of the Nevanlinna theory on annuli will be introduced as follows. Let:

$$N_1(r, f) = \int_{\frac{1}{r}}^{1} \frac{n_1(t, f)}{t} dt, \quad N_2(r, f) = \int_{1}^{r} \frac{n_2(t, f)}{t} dt,$$
$$m_0(r, f) = m(r, f) + m(\frac{1}{r}, f) - 2m(1, f), \quad N_0(r, f) = N_1(r, f) + N_2(r, f),$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function f in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. Similarly, for $a \in \overline{\mathbb{C}}$, we have:

$$\begin{aligned} \overline{N}_0(r, \frac{1}{f-a}) &= \overline{N}_1(r, \frac{1}{f-a}) + \overline{N}_2(r, \frac{1}{f-a}) \\ &= \int_{\frac{1}{r}}^{1} \frac{\overline{n}_1(t, \frac{1}{f-a})}{t} dt + \int_{1}^{r} \frac{\overline{n}_2(t, \frac{1}{f-a})}{t} dt \end{aligned}$$

in which each zero of the function $f - a$ is counted only once. In addition, we use $\overline{n}_1^{k)}(t, \frac{1}{f-a})$ (or $\overline{n}_1^{(k}(t, \frac{1}{f-a})$) to denote the counting function of poles of the function $\frac{1}{f-a}$ with multiplicities $\leq k$ (or $> k$) in $\{z : t < |z| \leq 1\}$, each point counted only once. Similarly, we have the notations $\overline{N}_1^{k)}(t, f)$, $\overline{N}_1^{(k}(t, f)$, $\overline{N}_2^{k)}(t, f)$, $\overline{N}_2^{(k}(t, f)$, $\overline{N}_0^{k)}(t, f)$, $\overline{N}_0^{(k}(t, f)$.

The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by:

$$T_0(r, f) = m_0(r, f) + N_0(r, f). \tag{8}$$

For a nonconstant meromorphic function f on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < r < R_0 \leq +\infty$, the following properties will be used in this paper (see [18]):

$$\begin{aligned} &(i) \;\; T_0(r, f) = T_0\left(r, \frac{1}{f}\right), \\ &(ii) \;\; \max\{T_0(r, f_1 \cdot f_2), T_0(r, \frac{f_1}{f_2}), T_0(r, f_1 + f_2)\} \leq T_0(r, f_1) + T_0(r, f_2) + O(1), \\ &(iii) \;\; T_0(r, \frac{1}{f-a}) = T_0(r, f) + O(1), \; for \;\; every \;\; fixed \;\; a \in \mathbb{C}, \end{aligned}$$

where (iii) can be called the first fundamental theorem on annuli.

In 2005, the lemma on the logarithmic derivative on the the annulus $\mathbb{A}$ was obtained by Khrystiyanyn and Kondratyuk [19].

Lemma 2.1. *(see [19], the lemma on the logarithmic derivative). Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then:*

$$m_0\left(r, \frac{f'}{f}\right) = S_1(r, f),$$

where (i) in the case $R_0 = +\infty$,

$$S_1(r, *) = O(\log(rT_0(r, *))) \tag{9}$$

for $r \in (1, +\infty)$, except for the set $\triangle_r$, such that $\int_{\triangle_r} r^{\lambda-1} dr < +\infty$;
(ii) if $R_0 < +\infty$, then:

$$S_1(r, *) = O(\log(\frac{T_0(r, *)}{R_0 - r})) \tag{10}$$

for $r \in (1, R_0)$, *except for the set* $\triangle'_r$, *such that* $\int_{\triangle'_r} \frac{dr}{(R_0-r)^{\lambda-1}} < +\infty$.

Definition 2.1. *Let* $f(z)$ *be a non-constant meromorphic function on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R_0 \leq +\infty$. *The function* f *is called a transcendental or admissible meromorphic function on the annulus* $\mathbb{A}$ *provided that:*

$$\limsup_{r\to\infty} \frac{T_0(r,f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty \tag{11}$$

or:

$$\limsup_{r\to R_0} \frac{T_0(r,f)}{-\log(R_0-r)} = \infty, \quad 1 < r < R_0 < +\infty, \tag{12}$$

respectively.

Then, for a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, $S_1(r,f) = o(T_0(r,f))$ holds for all $1 < r < R_0$, except for the set $\triangle_r$ or the set $\triangle'_r$ mentioned in Lemma 2.1, respectively.

The following lemma plays an important role in the proof process of Theorem 1.6, which was given by Cao, Yi and Xu [26].

Lemma 2.2. ([26], Theorem 2.3) (The second fundamental theorem). Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let a_1, $a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Then:

$$(q-2)T_0(r,f) < \sum_{j=1}^{q} \overline{N}_0(r, \frac{1}{f-a_j}) + S_1(r,f), \tag{13}$$

where $S_1(r,f)$ is stated as in Lemma 2.1.

Lemma 2.3. *(see [26]). Let* f *be a nonconstant meromorphic function on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, *where* $1 < R < R_0 \leq +\infty$. *Let* a *be an arbitrary complex number and* k *be a positive integer. Then:*

$$(i) \quad \overline{N}_0(R, \frac{1}{f-a}) \leq \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f-a}) + \frac{1}{k+1}N_0(R, \frac{1}{f-a}),$$

$$(ii) \quad \overline{N}_0(R, \frac{1}{f-a}) \leq \frac{k}{k+1}\overline{N}_0^{k)}(R, \frac{1}{f-a}) + \frac{1}{k+1}T_0(R,f) + O(1).$$

3. The Proof of Theorem 1.7

Proof of Theorem 1.7. Suppose that $f \not\equiv g$. Then, by Lemma 2.2 and Lemma 2.3, for any integer $m(1 \leq m \leq q)$, we have:

$$
\begin{aligned}
(q-2)T_0(r,f) &\leq \sum_{j=1}^{q} \overline{N}_0(r,\frac{1}{f-a_j}) + S_1(r,f) \\
&= \sum_{j=1}^{q} \left\{ \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + \overline{N}_0^{(k_j+1}(r,\frac{1}{f-a_j}) \right\} + S_1(r,f) \\
&\leq \sum_{j=1}^{q} \left\{ \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + \frac{1}{1+k_j} N_0^{(k_j+1}(r,\frac{1}{f-a_j}) \right\} + S_1(r,f) \\
&\leq \sum_{j=1}^{q} \left\{ \frac{k_j}{1+k_j} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + \frac{1}{1+k_j} N_0(r,\frac{1}{f-a_j}) \right\} + S_1(r,f) \\
&\leq \sum_{j=1}^{q} \frac{k_j}{1+k_j} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + \left(\sum_{j=1}^{q} \frac{1}{1+k_j} \right) T_0(r,f) + S_1(r,f) \\
&\leq \sum_{j=1}^{m-1} \left(\frac{k_j}{1+k_j} - \frac{k_m}{1+k_m} \right) \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + \left(\sum_{j=1}^{q} \frac{1}{1+k_j} \right) T_0(r,f) \\
&\quad + \sum_{j=1}^{q} \frac{k_m}{1+k_m} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + S_1(r,f) \\
&\leq \sum_{j=1}^{q} \frac{k_m}{1+k_m} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) \\
&\quad + \left(m-1-\frac{(m-1)k_m}{1+k_m} + \sum_{j=m}^{q} \frac{1}{1+k_j} \right) T_0(r,f) + S_1(r,f).
\end{aligned}
$$

that is,

$$
\left(\sum_{j=m}^{q} \frac{k_j}{1+k_j} - 2 + \frac{(m-1)k_m}{1+k_m} \right) T_0(r,f) \leq \sum_{j=1}^{q} \frac{k_m}{1+k_m} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) + S_1(r,f). \tag{14}
$$

$$
\left(\sum_{j=m}^{q} \frac{k_j}{1+k_j} - 2 + \frac{(m-1)k_m}{1+k_m} \right) T_0(r,g) \leq \sum_{j=1}^{q} \frac{k_m}{1+k_m} \overline{N}_0^{k_j)}(r,\frac{1}{g-a_j}) + S_1(r,g). \tag{15}
$$

Since $B_j = \overline{E}_{k_j)}(a_j,f) \setminus \overline{E}_{k_j)}(a_j,g)$, let $D_j = \overline{E}_{k_j)}(a_j,f) \setminus B_j$ for $j = 1,2,\ldots,q$. Thus, it follows from Equation (3) that:

$$
\begin{aligned}
\sum_{j=1}^{q} \overline{N}_0^{k_j)}(r,\frac{1}{f-a_j}) &= \sum_{j=1}^{q} \overline{N}_0^{B_j}(r,\frac{1}{f-a_j}) + \sum_{j=1}^{q} \overline{N}_0^{D_j}(r,\frac{1}{f-a_j}) \\
&\leq \sum_{j=1}^{q} \delta_j T_0(r,f) + N_0(r,\frac{1}{f-g}) \\
&\leq \left(1 + \sum_{j=1}^{q} \delta_j \right) T_0(r,f) + T_0(r,g) + O(1),
\end{aligned}
$$

and since f, g are transcendental or admissible, it follows from Equations (5) and (6) that:

$$\begin{aligned}&\left(\sum_{j=m}^{q}\frac{k_j}{1+k_j}-2+\frac{(m-1)k_m}{1+k_m}+o(1)\right)\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,\frac{1}{f-a_j})\\ \leq&\left(1+\sum_{j=1}^{q}\delta_j\right)\sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}(r,\frac{1}{f-a_j})+(1+o(1))\sum_{j=1}^{q}\frac{k_m}{1+k_m}\overline{N}_0^{k_j)}(r,\frac{1}{g-a_j}),\end{aligned}\tag{16}$$

as $r \to R_0$.

Since:

$$1 \geq \frac{k_1}{k_1+1} \geq \frac{k_2}{k_2+1} \geq \cdots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2},\tag{17}$$

it follows from Equation (7) that:

$$\begin{aligned}&\left\{\sum_{j=m}^{q}\frac{k_j}{1+k_j}-2+\frac{(m-1)k_m}{1+k_m}-\frac{k_m}{1+k_m}\left(1+\sum_{j=1}^{q}\delta_j\right)+o(1)\right\}\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,\frac{1}{f-a_j})\\ \leq&(1+o(1))\frac{k_m}{1+k_m}\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,\frac{1}{g-a_j}),\end{aligned}$$

which implies:

$$\liminf_{r\to R_0}\frac{\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,\frac{1}{f-a_j})}{\sum_{j=1}^{q}\overline{N}_0^{k_j)}(r,\frac{1}{g-a_j})} \leq \frac{\frac{k_m}{1+k_m}}{\sum_{j=m}^{q}\frac{k_j}{1+k_j}-2+(m-2-\sum_{j=1}^{q}\delta_j)\frac{k_m}{1+k_m}}.$$

This is a contradiction to Equation (4). Thus, we have $f(z) \equiv g(z)$.

Therefore, we complete the proof of Theorem 1.7. □

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