mathematics

Article

Qualitative Properties of Difference Equation of Order Six

Abdul Khaliq [1,*] and E.M. Elsayed [1,2]

1 Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; emmelsayed@yahoo.com
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
* Correspondence: khaliqsyed@gmail.com; Tel.: +96-590-77-4089

Academic Editor: Johnny Henderson
Received: 10 February 2016; Accepted: 1 April 2016; Published: 12 April 2016

Abstract: In this paper we study the qualitative properties and the periodic nature of the solutions of the difference equation

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-2}^2}{\gamma x_{n-2} + \delta x_{n-5}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and $\alpha,\ \beta,\ \gamma,\ \delta$ are positive constants. In addition, we derive the form of the solutions of some special cases of this equation.

Keywords: periodicity; global attractor; boundedness; rational difference equations

Mathematics Subject Classification: 39A10

---

1. Introduction

This paper deals with behavior of the solutions of the difference equation

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-2}^2}{\gamma x_{n-2} + \delta x_{n-5}}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and $\alpha,\ \beta,\ \gamma,\ \delta$ are constants. In addition, we obtain the form of solution of some special cases.
Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, and so forth.

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution (see [1–9] and the references cited therein). Recently, a great effort has been made in studying the qualitative analysis of rational difference equations and rational difference system (see [10–23]).

Elabbasy *et al.* [8] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}$$

Elabbasy and Elsayed [9] investigated the local and global stability, boundedness, and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} + cx_{n-q}}.$$

In [13], Elsayed investigated the solution of the following non-linear difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}^2}.$$

Keratas *et al.* [24] obtained the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Saleh *et al.* [25] investigated the dynamics of the solution of difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}.$$

Yalçınkaya [26] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

For other related work on rational difference equations, see [24–49].

Below, we outline some basic definitions and some theorems that we will need to establish our results.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \;\; n = 0, 1, ..., \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\overline{x} \in I$ is called an equilibrium point of Equation (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Equation (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

DEFINITION 1.1. *(Equilibrium Point)*

A point $\overline{x} \in I$ is called an equilibrium point of Equation (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Equation (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

DEFINITION 1.2. *(Periodicity)*

A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

DEFINITION 1.3. *(Fibonacci Sequence)*

The sequence

$$\begin{aligned}\{F_m\}_{m=1}^{\infty} &= \{1,2,3,5,8,13,....\} \quad \textit{i.e.,} \;\; F_m = F_{m-1} + F_{m-2} \geq 0, \\ F_{-2} &= 0, \;\; F_{-1} = 1,\end{aligned}$$

is called Fibonacci Sequence.

DEFINITION 1.4. *(Stability)*

(i) The equilibrium point $\overline{x}$ of Equation (1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, \; x_{-k+1}, ..., \; x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{ for all } \;\; n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Equation (1.2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Equation (1.2) and there exists $\gamma > 0$, such that for all $x_{-k}, \; x_{-k+1}, ..., \; x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Equation (1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Equation (1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Equation (1.2).

(v) The equilibrium point $\overline{x}$ of Equation (1.2) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of Equation (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Theorem A. [38] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + p x_n + q x_{n-k} = 0, \quad n = 0, 1, ... \, .$$

The following theorem will be useful for establishing the results in this paper.

Theorem B. [39] Let $[\alpha, \beta]$ be an interval of real numbers assume that $g : [\alpha, \beta]^2 \to [\alpha, \beta]$, is a continuous function and consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}), \qquad n = 0, 1, ..., \tag{1.4}$$

satisfying the following conditions :

(a) $g(x, y)$ is non-decreasing in $x \in [\alpha, \beta]$ for each fixed $y \in [\alpha, \beta]$ and $g(x, y)$ is non-increasing in $y \in [\alpha, \beta]$ for each fixed $x \in [\alpha, \beta]$.

(b) For any $(m, M) \in [\alpha, \beta] \times [\alpha, \beta]$ that is a solution of the system

$$M = g(M, m) \qquad \textit{and} \qquad m = g(m, M),$$

we have that

$$m = M.$$

Then Equation (1.4) has a unique equilibrium $\overline{x} \in [\alpha, \beta]$ and every solution of Equation (1.4) converges to $\overline{x}$.

2. Local Stability of the Equilibrium Point of Equation (1.1)

In this section we study the local stability properties of the equilibrium point of Equation (1.1). The equilibrium points of Equation (1.1) are given by the relation

$$\overline{x} = \alpha\overline{x} + \frac{\beta\overline{x}^2}{\gamma\overline{x} + \delta\overline{x}},$$

or

$$\overline{x}^2(1-\alpha)(\gamma+\delta) = \beta\overline{x}^2.$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$, then the unique equilibrium point is $\overline{x} = 0$.
Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a continuously differentiable function defined by

$$f(u, v) = \alpha u + \frac{\beta u^2}{\gamma u + \delta v}. \tag{2.1}$$

Therefore

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = \alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}, \quad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = \frac{-\beta\delta}{(\gamma+\delta)^2}.$$

Then the linearized equation of Equation (1.1) about $\overline{x}$ is

$$y_{n+1} - \left(\alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}\right) y_n + \left(\frac{\beta\delta}{(\gamma+\delta)^2}\right) y_{n-1} = 0. \tag{2.2}$$

Theorem 1. *Assume that* $\beta(\gamma + 3\delta) < (\gamma+\delta)^2(1-\alpha)$, $\alpha < 1$. *Then the equilibrium point* $\overline{x} = 0$ *of Equation (1.1) is locally asymptotically stable.*

Proof: From Theorem A, it follows that Equation (2.2) is asymptotically stable if

$$\left|\alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

or

$$\alpha + \frac{\beta\gamma + 3\beta\delta}{(\gamma+\delta)^2},$$

and so

$$\beta(\gamma + 3\delta) < (\gamma+\delta)^2(1-\alpha),$$

which completes the proof.

3. Global Attractivity of the Equilibrium Point of Equation (1.1)

In this section we investigate the global attractivity character of solutions of Equation (1.1).

Theorem 2. *The equilibrium point $\overline{x}$ of Equation (1.1) is a global attractor if* $\gamma(1-\alpha) \neq \beta$.

Proof: Let $\alpha,\ \beta$ are real numbers and assume that $g : [\alpha, \beta]^2 \to [\alpha, \beta]$, be a function defined by Equation (2.1).

Suppose that (m, M) is a solution of the system

$$M = g(M, m) \quad \textit{and} \quad m = g(m, M).$$

Then from Equation (1.1), we see that

$$M = \alpha M + \frac{\beta M^2}{\gamma M + \delta m}, \quad m = \alpha m + \frac{\beta m^2}{\gamma m + \delta M}.$$

Therefore

$$M(1-\alpha) = \frac{\beta M^2}{\gamma M + \delta m}, \quad m(1-\alpha) = \frac{\beta m^2}{\gamma m + \delta M},$$

or,

$$\gamma(1-\alpha)(M^2 - m^2) = b(M^2 - m^2), \qquad c(1-\alpha) \neq \beta,$$

thus

$$M = m.$$

From Theorem B, it follows that $\overline{x}$ is a global attractor of Equation (1.1) and then the proof is complete.

4. Boundedness of Solutions of Equation (1.1)

In this section we study the boundedness of solution of Equation (1.1)

Theorem 3. *Every solution of Equation(1.1) is bounded if* $\left(\alpha + \dfrac{\beta}{\gamma}\right) < 1$.

Proof: Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Equation (1.1). It follows from Equation (1.1) that

$$x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-2}^2}{\gamma x_{n-2} + \delta x_{n-5}} \leq \alpha x_{n-2} + \frac{\beta x_{n-2}^2}{\gamma x_{n-2}} = \left(\alpha + \frac{\beta}{\gamma}\right) x_{n-2}.$$

Then

$$x_{n+1} \leq x_{n-2}, \ \text{ for all } n \geq 0.$$

Then the sub-sequences $\{x_{3n-1}\}_{n=-5}^{\infty}$, $\{x_{3n-2}\}_{n=-5}^{\infty}$ and $\{x_{3n}\}_{n=-5}^{\infty}$ are decreasing and so are bounded from above by $M = \max\{x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$.

In order to confirm the result in this section we consider some numerical examples for $x_{-5} = 10,\ x_{-4} = 5,\ x_{-3} = 8,\ x_{-2} = 2,\ x_{-1} = 9,\ x_0 = 7,\ \alpha = 0.5,\ \beta = 6,\ \gamma = 9,\ \delta = 10$. (See Figure 1) and $x_{-5} = 10,\ x_{-4} = 5,\ x_{-3} = 8,\ x_{-2} = 2,\ x_{-1} = 9,\ x_0 = 7,\ \alpha = 0.6,\ \beta = 6,\ \gamma = 7,\ \delta = 12$. (See Figure 2).

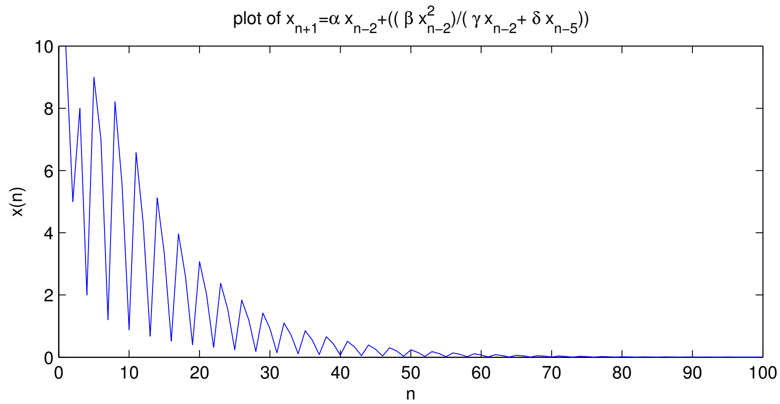

Figure 1. Expresses the solution of $x_{n+1} = \alpha x_{n-2} + \dfrac{\beta x_{n-2}^2}{\gamma x_{n-2} + \delta x_{n-5}}$, when we put initials and constants $x_{-5} = 10,\ x_{-4} = 5,\ x_{-3} = 8,\ x_{-2} = 2,\ x_{-1} = 9,\ x_0 = 7,\ \alpha = 0.5,\ \beta = 6,\ \gamma = 9,\ \delta = 10$.

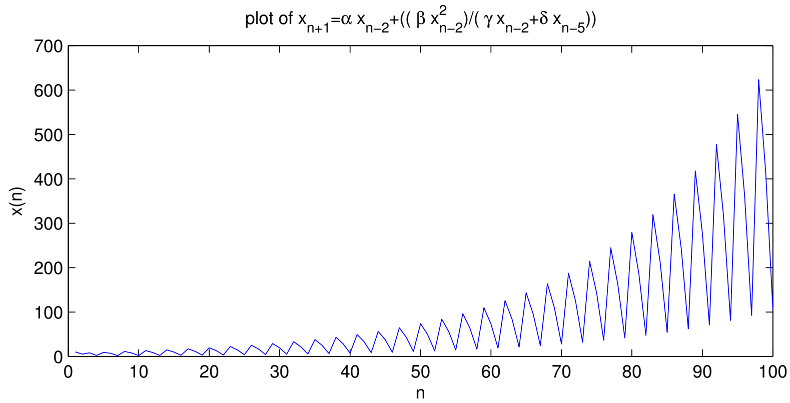

Figure 2. Represents behavior of Equation (1.1) when $x_{-5} = 10,\ x_{-4} = 5,\ x_{-3} = 8,\ x_{-2} = 2,\ x_{-1} = 9,\ x_0 = 7,\ \alpha = 0.6,\ \beta = 6,\ \gamma = 7,\ \delta = 12$.

5. Special Cases of Equation (1.1)

5.1. First Equation

In this section we study the following special case of Equation (1.1)

$$x_{n+1} = x_{n-2} + \frac{x_{n-2}^2}{x_{n-2} + x_{n-5}} \tag{5.1}$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary real numbers.

Theorem 4. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Equation (5.1) then for $n = 0, 1, 2, \ldots$*

$$\begin{aligned} x_{3n-1} &= k\prod_{i=1}^{n}\left(\frac{f_{2i+1}k + f_{2i}q}{f_{2i}k + f_{2i-1}q}\right), \quad x_{3n-2} = h\prod_{i=1}^{n}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right), \\ x_{3n} &= r\prod_{i=1}^{n}\left(\frac{f_{2i+1}r + f_{2i}t}{f_{2i}r + f_{2i-1}t}\right), \end{aligned}$$

where $x_{-5} = t$, $x_{-4} = q$, $x_{-3} = p$, $x_{-2} = r$, $x_{-1} = k$, $x_0 = h$, $\{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$.

Proof: We prove that the forms given are solutions of Equation (5.1) by using mathematical induction. First, we let $n = 0$, then the result holds. Second, we assume that the expressions are satisfied for $n-1,\ n-2$. Our objective is to show that the expressions are satisfied for n. That is;

$$\begin{aligned} x_{3n-8} &= r\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right), & x_{3n-7} &= k\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right), \\ x_{3n-6} &= h\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}h+f_{2i}p}{f_{2i}h+f_{2i-1}p}\right), & x_{3n-5} &= r\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right), \\ x_{3n-4} &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right), & x_{3n-3} &= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h+f_{2i}p}{f_{2i}h+f_{2i-1}p}\right). \end{aligned}$$

Now, it follows from Equation (5.1) that,

$$\begin{aligned} x_{3n-1} &= x_{3n-4}+\frac{x_{3n-4}^2}{x_{3n-4}+x_{3n-7}} \\ &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)+\frac{k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)}{k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)+k\prod_{i=1}^{n-2}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)} \\ &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)+\frac{\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)\left(\frac{f_{2n-1}k+f_{2n-2}q}{f_{2n-2}k+f_{2n-3}q}\right)}{\left(\frac{f_{2n-1}k+f_{2n-2}q}{f_{2n-2}k+f_{2n-3}q}\right)+1} \\ &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)\left(1+\frac{f_{2n-1}k+f_{2n-2}q}{f_{2n-1}k+f_{2n-2}q+f_{2n-2}k+f_{2n-3}q}\right) \\ &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)\left(1+\frac{f_{2n-1}k+f_{2n-2}q}{f_{2n}k+f_{2n-1}q}\right) \\ &= k\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right)\left(\frac{f_{2n+1}k+f_{2n}q}{f_{2n}k+f_{2n-1}q}\right). \end{aligned}$$

Therefore

$$x_{3n-1} = k\prod_{i=1}^{n}\left(\frac{f_{2i+1}k+f_{2i}q}{f_{2i}k+f_{2i-1}q}\right).$$

Also, we see from Equation (5.1) that,

$$\begin{aligned} x_{3n-2} &= x_{3n-5}+\frac{x_{3n-5}^2}{x_{3n-5}+x_{3n-8}} \\ &= r\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right)+\frac{r\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right)\left(\frac{f_{2n-1}r+f_{2n-2}t}{f_{2n-2}r+f_{2n-3}t}\right)}{\left(\frac{f_{2n-1}r+f_{2n-2}t}{f_{2n-2}r+f_{2n-3}t}\right)+1} \\ &= r\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right)\left(1+\frac{f_{2n-1}r+f_{2n-2}t}{f_{2n-1}r+f_{2n-2}t+f_{2n-2}r+f_{2n-3}t}\right) \\ &= r\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}r+f_{2i}t}{f_{2i}r+f_{2i-1}t}\right)\left(\frac{f_{2n+1}r+f_{2n}t}{f_{2n}r+f_{2n-1}t}\right). \end{aligned}$$

Then

$$x_{3n-2} = h\prod_{i=1}^{n}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right).$$

Also, wee see from Equation (5.1) that,

$$\begin{aligned}
x_{3n} &= x_{3n-3} + \frac{x_{3n-3}^2}{x_{3n-3} + x_{3n-6}} \\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}r + f_{2i-1}t}\right) + \frac{h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right)\left(\frac{f_{2n-1}h + f_{2n-2}p}{f_{2n-2}h + f_{2n-3}p}\right)}{\left(\frac{f_{2n-1}h + f_{2n-2}p}{f_{2n-2}h + f_{2n-3}p}\right) + 1} \\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right)\left(1 + \frac{f_{2n-1}h + f_{2n-2}p}{f_{2n-1}h + f_{2n-2}p + f_{2n-2}h + f_{2n-3}p}\right) \\
&= h\prod_{i=1}^{n-1}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right)\left(\frac{f_{2n+1}h + f_{2n}p}{f_{2n}h + f_{2n-1}p}\right)
\end{aligned}$$

Thus

$$x_{3n} = h\prod_{i=1}^{n}\left(\frac{f_{2i+1}h + f_{2i}p}{f_{2i}h + f_{2i-1}p}\right).$$

Hence, the proof is complete.

We will confirm our result by considering some numerical examples assume $x_{-5} = 3$, $x_{-4} = 5$, $x_{-3} = 9$, $x_{-2} = 2$, $x_{-1} = 9$, $x_0 = 4$. (See Figure 3).

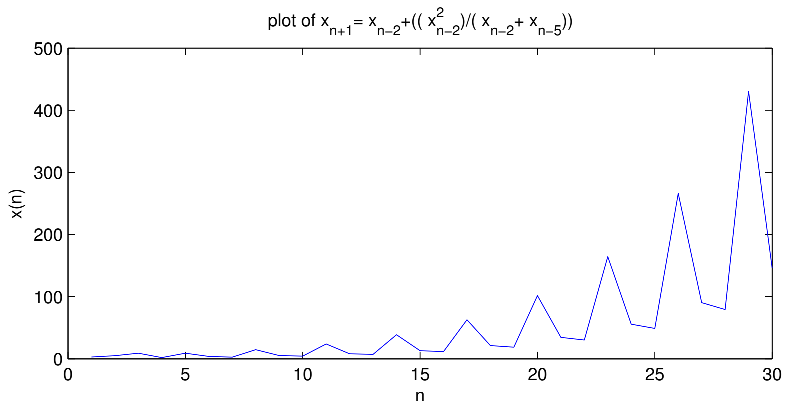

Figure 3. Shows the behavior for Equation (5.1) with $x_{-5} = 3$, $x_{-4} = 5$, $x_{-3} = 9$, $x_{-2} = 2$, $x_{-1} = 9$, $x_0 = 4$.

5.2. Second Equation

In this section we solve a more specific form of Equation (1.1)

$$x_{n+1} = x_{n-2} + \frac{x_{n-2}^2}{x_{n-2} - x_{n-5}}, \tag{5.2}$$

where the initial conditions x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary real numbers.

Theorem 5. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of Equation (5.3). Then for* $n = 0, 1, 2, ...$

$$
\begin{aligned}
x_{3n-1} &= k\prod_{i=1}^{n}\left(\frac{f_{i+2}k - f_i q}{f_i k - f_{i-2}q}\right), \quad x_{3n-2} = h\prod_{i=1}^{n}\left(\frac{f_{i+2}h - f_i p}{f_i h - f_{i-2}p}\right), \\
x_{3n} &= r\prod_{i=1}^{n}\left(\frac{f_{i+2}r - f_i t}{f_i r - f_{i-2}t}\right)
\end{aligned}
$$

where $x_{-5} = t,\ x_{-4} = q,\ x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: The proof is the same as for Theorem 4 and is therefore omitted.

To confirm our result assume $x_{-5} = 3,\ x_{-4} = 5,\ x_{-3} = 9,\ x_{-2} = 2,\ x_{-1} = 9,\ x_0 = 4$. (See Figure 4).

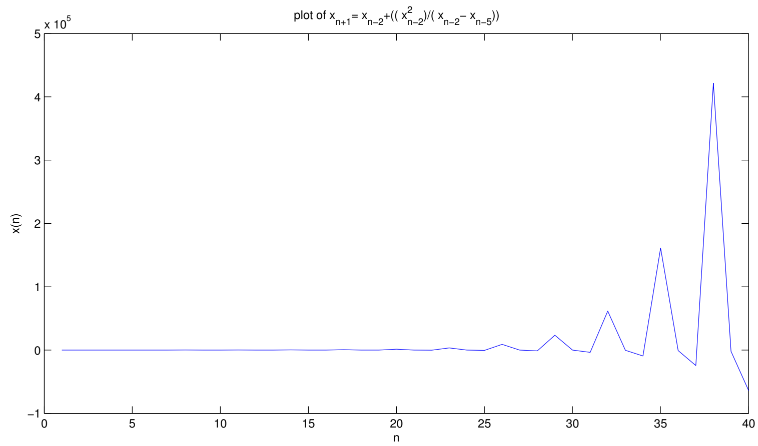

Figure 4. Shows solution of Equation (5.2) with $x_{-5} = 3,\ x_{-4} = 5,\ x_{-3} = 9,\ x_{-2} = 2,\ x_{-1} = 9, x_0 = 4$.

5.3. Third Equation

In this section we deal with the following special case of Equation (1.1)

$$
x_{n+1} = x_{n-2} - \frac{x_{n-2}^2}{x_{n-2} + x_{n-5}}, \tag{5.3}
$$

where the initial conditions $x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary real numbers.

Theorem 6. *Let*$\{x_n\}_{n=-5}^{\infty}$*be a solution of Equation(5.5) then for* $n = 0, 1, 2, ...$

$$
x_{3n-1} = \frac{kq}{f_n k + f_{n+1}q}, \quad x_{3n-2} = \frac{rt}{f_n r + f_{n+1}t}, \quad x_{3n} = \frac{hp}{f_n h + f_{n+1}p},
$$

where $x_{-5} = t,\ x_{-4} = q,\ x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ \{f_m\}_{m=-1}^{\infty} = \{1, 0, 1, 1, 2, 3, 5, 8, 13, ...\}$.

Proof: For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-2,\ n-3$. That is,

$$
\begin{aligned}
x_{3n-8} &= \frac{rt}{f_{n-2}r + f_{n-1}t}, \qquad x_{3n-7} = \frac{kq}{f_{n-2}k + f_{n-1}q}, \\
x_{3n-6} &= \frac{hp}{f_{n-2}h + f_{n-1}p}, \qquad x_{3n-5} = \frac{rt}{f_{n-1}r + f_{n}t}, \\
x_{3n-4} &= \frac{kq}{f_{n-1}k + f_{n}q}, \qquad x_{3n-3} = \frac{hp}{f_{n-1}k + f_{n}q}.
\end{aligned}
$$

Now, it follows from Equation (5.3) that,

$$
\begin{aligned}
x_{3n-1} &= x_{3n-4} - \frac{x_{3n-4}^2}{x_{3n-4} + x_{3n-7}} \\
&= \frac{kq}{f_{n-1}k + f_{n}q} - \left(\frac{\frac{kq}{f_{n-2}k + f_{n-1}q}\frac{kq}{f_{n-2}k + f_{n-1}q}}{\frac{kq}{f_{n-2}k + f_{n-1}q} + \frac{kq}{f_{n-2}k + f_{n-1}q}} \right) \\
&= \frac{kq}{f_{n-1}k + f_{n}q} - \left(\frac{\frac{kq}{f_{n-2}k + f_{n-1}q}\left(f_{n-2}k + f_{n-1}q\right)}{f_{n-2}k + f_{n-1}q + f_{n-1}k + f_{n}q} \right) \\
&= \frac{kq}{f_{n-1}k + f_{n}q} \left(1 - \frac{f_{n-2}k + f_{n-1}q}{f_{n-2}k + f_{n-1}q + f_{n-1}k + f_{n}q} \right) \\
&= \frac{kq}{f_{n-1}k + f_{n}q} \left(\frac{f_{n-2}k + f_{n-1}q + f_{n-1}k + f_{n}q + f_{n-2}k + f_{n-1}q}{f_{n-2}k + f_{n-1}q + f_{n-1}k + f_{n}q} \right) \\
&= \frac{kq}{f_{n-1}k + f_{n}q} \left(\frac{f_{n-1}k + f_{n}q}{f_{n}k + f_{n+1}q} \right) = \frac{kq}{f_{n}k + f_{n+1}q}.
\end{aligned}
$$

Also, from Equation (5.3), we see that,

$$
\begin{aligned}
x_{3n} &= x_{3n-3} - \frac{x_{3n-3}^2}{x_{3n-3} + x_{3n-6}} \\
&= \frac{hp}{f_{n-1}k + f_{n}q} - \left(\frac{\frac{hp}{f_{n-1}k + f_{n}q}\frac{hp}{f_{n-1}k + f_{n}q}}{\frac{hp}{f_{n-1}k + f_{n}q} + \frac{hp}{f_{n-2}h + f_{n-1}p}} \right) \\
&= \frac{hp}{f_{n-1}k + f_{n}q} - \left(\frac{\frac{hp}{f_{n-2}h + f_{n-1}p}\left(f_{n-2}h + f_{n-1}p\right)}{f_{n-2}h + f_{n-1}p + f_{n-1}h + f_{n}p} \right) \\
&= \frac{hp}{f_{n-1}h + f_{n}p} \left(1 - \frac{f_{n-2}h + f_{n-1}p}{f_{n-2}h + f_{n-1}p + f_{n-1}h + f_{n}p} \right) \\
&= \frac{hp}{f_{n-1}h + f_{n}p} \left(\frac{f_{n-1}h + f_{n}p}{f_{n}h + f_{n+1}p} \right) = \frac{hp}{f_{n}h + f_{n+1}p}.
\end{aligned}
$$

Also, from Equation (5.3), we get,

$$\begin{aligned}
x_{3n-2} &= x_{3n-5} - \frac{x_{3n-5}^2}{x_{3n-5} + x_{3n-8}} \\
&= \frac{rt}{f_{n-1}r + f_n t} - \left(\frac{\frac{rt}{f_{n-1}r + f_n t} \frac{rt}{f_{n-1}r + f_n t}}{\frac{rt}{f_{n-1}r + f_n t} + \frac{rt}{f_{n-2}r + f_{n-1}t}} \right) \\
&= \frac{rt}{f_{n-1}r + f_n t} - \left(\frac{\frac{rt}{f_{n-2}r + f_{n-1}t} (f_{n-2}r + f_{n-1}t)}{f_{n-2}r + f_{n-1}t + f_{n-1}r + f_n t} \right) \\
&= \frac{rt}{f_{n-1}r + f_n t} \left(\frac{f_{n-1}r + f_n t}{f_n r + f_{n+1}t} \right) = \frac{rt}{f_n r + f_{n+1}t}.
\end{aligned}$$

Hence, the proof is complete.

We consider a numerical example of this special case, assume $x_{-5} = 2, x_{-4} = 5, x_{-3} = 6$, $x_{-2} = 12$, $x_{-1} = 9, x_0 = 18$. (See Figure 5).

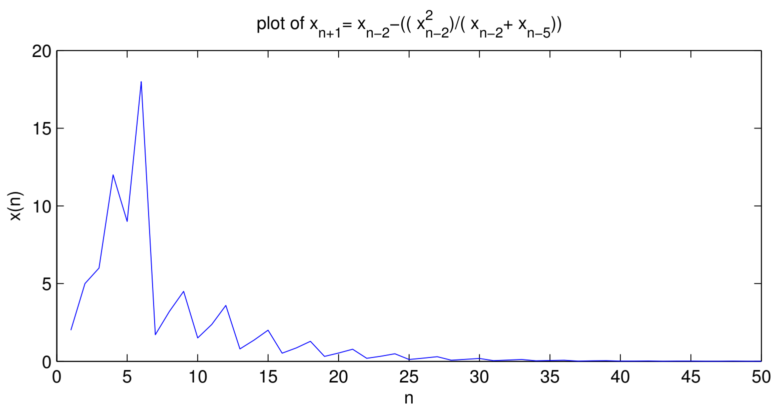

Figure 5. Shows the dynamics of Equaton (5.3) when $x_{-5} = 2$, $x_{-4} = 5$, $x_{-3} = 6$, $x_{-2} = 12$, $x_{-1} = 9$, $x_0 = 18$.

5.4. Fourth Equation

In this section we deal with the form of solution of the following equation

$$x_{n+1} = x_{n-2} - \frac{x_{n-2}^2}{x_{n-2} - x_{n-5}}, \tag{5.4}$$

where the initial conditions x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 are arbitrary real numbers.

Theorem 7. *Let $\{x_n\}_{n=-5}^{\infty}$be a solution of Equation (5.4) Then every solution of Equation (5.4) is periodic with period 18. Moreover, $\{x_n\}_{n=-5}^{\infty}$ takes the form*

$$\left\{ \begin{array}{c} t, q, p, r, k, h, \frac{-rt}{r-t}, \frac{-kq}{k-q}, \frac{-hp}{h-p}, -t, -q, -p, \\ -r, -k, -h, \frac{rt}{r-t}, \frac{kq}{k-q}, \frac{hp}{h-p}, t, q, p, r, k, h, \ldots \end{array} \right\},$$

or,

$$\begin{array}{rclllll} x_{18n-5} & = & t, & x_{18n-4} = q, & x_{18n-3} = p, & x_{18n-2} = r, & x_{18n-1} = k, \\ x_{18n} & = & h, & x_{18n+1} = \dfrac{-rt}{r-t}, & x_{18n+2} = \dfrac{-kq}{k-q}, & x_{18n+3} = \dfrac{-hp}{h-p}, & \\ x_{18n+4} & = & -t, & x_{18n+5} = -q, & x_{18n+6} = -p, & x_{18n+7} = -r, & x_{18n+8} = -k, \\ x_{18n+9} & = & -h, & x_{18n+10} = \dfrac{rt}{r-t}, & x_{18n+11} = \dfrac{kq}{k-q}, & x_{18n+12} = \dfrac{hp}{h-p}, & \end{array}$$

where $x_{-5} = t,\ x_{-4} = q,\ x_{-3} = p,\ x_{-2} = r,\ x_{-1} = k,\ x_0 = h,\ x_{-5} \neq x_{-2},\ x_{-4} \neq x_0,\ x_{-3} \neq x_{-1}$.
Proof: The proof is the same as the proof of Theorem 6 and thus will be omitted.

Figure 6 shows the solution when $x_{-5} = 4,\ x_{-4} = 7,\ x_{-3} = 5,\ x_{-2} = 14,\ x_{-1} = 19,\ x_0 = 11$.

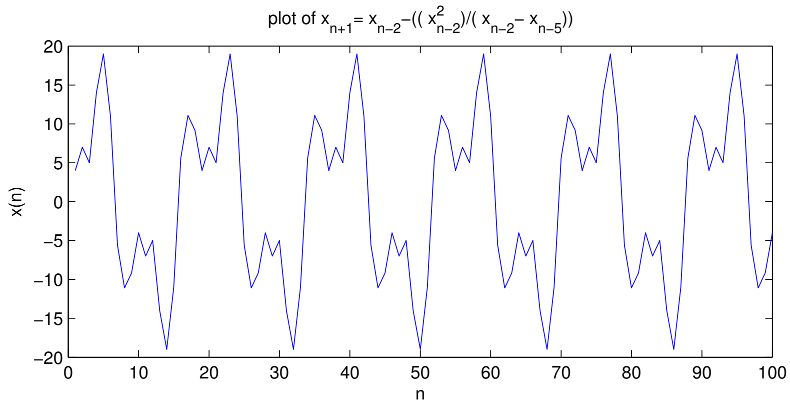

Figure 6. Shows the periodic behavior of solution of $x_{n+1} = \alpha x_{n-2} - \dfrac{\beta x_{n-2}^2}{\gamma x_{n-2} - \delta x_{n-5}}$, with $x_{-5} = 4,\ x_{-4} = 7,\ x_{-3} = 5,\ x_{-2} = 14,\ x_{-1} = 19,\ x_0 = 11$.

6. Conclusions

In this paper we investigated the global attractivity, boundedness and the solutions of some special cases of Equation (1.1). In Section 2 we proved when $\beta(\gamma + 3\delta) < (\gamma + \delta)^2(1 - \alpha)$, Equation (1.1) has local stability. In Section 3 we showed that the unique equilibrium of Equation (1.1) is globally asymptotically stable if $\gamma(1 - \alpha) \neq \beta$. In Section 4 we investigated that the solution of Equation (1.1) is bounded if $\left(\alpha + \dfrac{\beta}{\gamma}\right) < 1$. In Section 5 we obtained the form of the solution of four special cases of Equation (1.1) and gave numerical examples of each of the case with different initial values.

Acknowledgments: The authors thank the main editor and anonymous referees for their valuable suggestions and comments leading to improvement of this paper.

Author Contributions: All authors carried out the proofs of the main results and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ahmed, A.M.; Youssef, A.M. A solution form of a class of higher-order rational difference equations. *J. Egyp. Math. Soc.* **2013**, *21*, 248–253.
2. Alghamdi, M.; Elsayed, E.M.; El-Dessoky, M.M. On the Solutions of Some Systems of Second Order Rational Difference Equations. *Life Sci. J.* **2013**, *10*, 344–351.
3. Asiri, A.; Elsayed, E.M.; El-Dessoky, M.M. On the Solutions and Periodic Nature of Some Systems of Difference Equations. *J. Comput. Theor. Nanosci.* **2015**, *12*, 3697–3704.
4. Das, S.E.; Bayram, M. On a System of Rational Difference Equations. *World Appl. Sci. J.* **2010**, *10*, 1306–1312.

5. Din, Q. Qualitative nature of a discrete predator-prey system. *Contemp. Methods Math. Phys. Gravit.* **2015**, *1*, 27–42.
6. Din, Q. Stability analysis of a biological network. *Netw. Biol.* **2014**, *4*, 123–129.
7. Din, Q.; Elsayed, E.M. Stability analysis of a discrete ecological model. *Comput. Ecol. Softw.* **2014**, *4*, 89–103.
8. Elabbasy, E.M.; El-Metwally, H.; Elsayed, E.M. On the Difference Equation $x_{n+1} = \frac{\alpha x_{n-l} + \beta x_{n-k}}{A x_{n-l} + B x_{n-k}}$. *Acta Math. Vietnam.* **2008**, *33*, 85–94.
9. Elabbasy, E.M.; Elsayed, E.M. Dynamics of a rational difference equation. *Chin. Ann. Math. Ser. B* **2009**, *30*, 187–198.
10. El-Dessoky, M.M.; Elsayed, E.M. On the solutions and periodic nature of some systems of rational difference equations. *J. Comput. Anal. Appl.* **2015**, *18*, 206–218.
11. El-Metwally, H.; Elsayed, E.M. Qualitative Behavior of some Rational Difference Equations. *J. Comput. Anal. Appl.* **2016**, *20*, 226–236.
12. El-Moneam, M.A. On the dynamics of the solutions of the rational recursive sequences. *Br. J. Math. Comput. Sci.* **2015**, *5*, 654–665.
13. Elsayed, E.M. Qualitative behaviour of difference equation of order two. *Math. Comput. Model.* **2009**, *50*, 1130–1141.
14. Elsayed, E.M. Solution and attractivity for a rational recursive sequence. *Discret. Dyn. Nat. Soc.* **2011**, *2011*, 982309.
15. Elsayed, E.M. Solutions of rational difference system of order two. *Math. Comput. Mod.* **2012**, *55*, 378–384.
16. Elsayed, E.M. Behavior and expression of the solutions of some rational difference equations. *J. Comput. Anal. Appl.* **2013**, *15*, 73–81.
17. Elsayed, E.M. Solution for systems of difference equations of rational form of order two. *Comput. Appl. Math.* **2014**, *33*, 751–765.
18. Elsayed, E.M. On the solutions and periodic nature of some systems of difference equations. *Int. J. Biomath.* **2014**, *7*, 1450067.
19. Elsayed, E.M. New method to obtain periodic solutions of period two and three of a rational difference equation. *Nonlinear Dyn.* **2015**, *79*, 241–250.
20. Elsayed, E.M. Dynamics and Behavior of a Higher Order Rational Difference Equation. *J. Nonlinear Sci. Appl.* **2016**, *9*, 1463–1474.
21. Elsayed, E.M.; Ahmed, A.M. Dynamics of a three-dimensional systems of rational difference equations. *Math. Methods Appl. Sci.* **2016**, *39*, 1026–1038.
22. Elsayed, E.M.; Alghamdi, A. Dynamics and Global Stability of Higher Order Nonlinear Difference Equation. *J. Comput. Anal. Appl.* **2016**, *21*, 493–503.
23. Elsayed, E.M.; El-Dessoky, M.M. Dynamics and global behavior for a fourth-order rational difference equation. *Hacet. J. Math. Stat.* **2013**, *42*, 479–494.
24. Karatas, R.; Cinar, C.; Simsek, D. On positive solutions of the difference equation $x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}$. *Int. J. Contemp. Math. Sci.* **2006**, *1*, 495–500.
25. Saleh, M.; Aloqeili, M. On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. *Appl. Math. Comput.* **2006**, *176*, 359–363.
26. Yalçınkaya, I. On the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$. *Discret. Dyn. Nat. Soc.* **2008**, *2008*, 805460, doi:10.1155/2008/805460.
27. Elsayed, E.M.; El-Dessoky, M.M.; Alzahrani, E.O. The Form of The Solution and Dynamics of a Rational Recursive Sequence. *J. Comput. Anal. Appl.* **2014**, *17*, 172–186.
28. Elsayed, E.M.; El-Dessoky, M.M.; Asiri, A. Dynamics and Behavior of a Second Order Rational Difference equation. *J. Comput. Anal. Appl.* **2014**, *16*, 794–807.
29. Elsayed, E.M.; El-Metwally, H. Stability and solutions for rational recursive sequence of order three. *J. Comput. Anal. Appl.* **2014**, *17*, 305–315.
30. Elsayed, E.M.; El-Metwally, H. Global behavior and periodicity of some difference equations. *J. Comput. Anal. Appl.* **2015**, *19*, 298–309.
31. Elsayed, E.M.; Ibrahim, T. F. Solutions and periodicity of a rational recursive sequences of order five. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 95–112.

32. Elsayed, E.M.; Ibrahim, T.F. Periodicity and solutions for some systems of nonlinear rational difference equations. *Hacet. J. Math. Stat.* **2015**, *44*, 1361–1390.
33. Halim, Y.; Bayram, M. On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences. *Math. Methods Appl. Sci.* **2015**, doi:10.1002/mma.3745.
34. Ibrahim, T.F.; Touafek, N. Max-type system of difference equations with positive two-periodic sequences. *Math. Methods Appl. Sci.* **2014**, *37*, 2562–2569.
35. Ibrahim, T.F.; Touafek, N. On a third order rational difference equation with variable coefficients. *Dyn. Contol Disc Imput Syst. Appl. Algo.* **2013**, *20*, 251–264.
36. Jana, D.; Elsayed, E.M. Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete-time system. *Int. J. Biomath.* **2016**, *9*, 1650004.
37. Khaliq, A.; Elsayed, E.M. The Dynamics and Solution of some Difference Equations. *J. Nonlinear Sci. Appl.* **2016**, *9*, 1052–1063.
38. Khan, A.Q.; Qureshi, M.N. Global dynamics of a competitive system of rational difference equations. *Math. Methods Appl. Sci.* **2015**, *38*, 4786–4796.
39. Khan, A.Q.; Qureshi, M.N.; Din, Q. Asymptotic behavior of an anti-competitive system of rational difference equations. *Life Sci. J.* **2014**, *11*, 16–20.
40. Kocic, V.L.; Ladas, G. *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
41. Kulenovic, M.R.S.; Ladas, G. *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*; Chapman & Hall / CRC Press: London, UK, 2001.
42. Kurbanli, A.S. On the Behavior of Solutions of the System of Rational Difference Equations. *World Appl. Sci. J.* **2010**, *10*, 1344–1350.
43. Simsek, D.; Cinar, C.; Yalcinkaya, I. On the recursive sequence $x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$. *Int. J. Contemp. Math. Sci.* **2006**, *1*, 475–480.
44. Touafek, N. On a second order rational difference equation. *Hacet. J. Math. Stat.* **2012**, *41*, 867–874.
45. Yalcinkaya, I.; Cinar, C. On the dynamics of difference equation $\frac{ax_{n-k}}{b + cx_n^p}$. *Fasc. Math.* **2009**, *42*, 141–148.
46. Yazlik, Y.; Elsayed, E.M.; Taskara, N. On the Behaviour of the Solutions of Difference Equation Systems. *J. Comput. Anal. Appl.* **2014**, *16*, 932–941.
47. Yazlik, Y.; Tollu, D.T.; Taskara, N. On the solutions of a max-type difference equation system. *Math. Methods Appl. Sci.* **2015**, *38*, 4388–4410.
48. Zayed, E.M.E.; El-Moneam, M.A. On the rational recursive sequence $x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$. *Commun. Appl. Nonlinear Anal.* **2005**, *12*, 15–28.
49. Zayed, E.M.E.; El-Moneam, M.A. On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$. *Commun. Appl. Nonlinear Anal.* **2008**, *15*, 47–57.