mathematics

Article

Solution for Rational Systems of Difference Equations of Order Three

Mohamed M. El-Dessoky [1,2]

1 Mathematics Department, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia; dessokym@mans.edu.eg; Tel.: +966-6952297; Fax: +966-6952669
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Academic Editor: Johnny Henderson
Received: 24 April 2016; Accepted: 24 August 2016; Published: 3 September 2016

Abstract: In this paper, we consider the solution and periodicity of the following systems of difference equations: $x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{\pm 1 \pm x_{n-2}y_{n-1}x_n}$, with initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, and y_0 are nonzero real numbers.

Keywords: difference equations; recursive sequences; stability; periodic solution; system of difference equations

MSC: 39A10

1. Introduction

This paper is devoted to study the form of the solution and periodicity of the following third order systems of rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{\pm 1 \pm x_{n-2}y_{n-1}x_n}$$

with initial conditions x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, and y_0 are nonzero real numbers.

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques that can be used in the investigation of equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, etc. There are many papers related to difference equations systems; for example, The global asymptotic behavior of the positive solutions of the rational difference system

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}$$

has been studied by Camouzis et al. in [1].

The periodicity of the positive solutions of the rational difference equations systems

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$$

has been obtained by Cinar in [2].

Elabbasy et al. [3] studied the solutions of particular cases of the following general system of difference equations:

$$x_{n+1} = \frac{a_1 + a_2 y_n}{a_3 z_n + a_4 x_{n-1} z_n}, \; y_{n+1} = \frac{b_1 z_{n-1} + b_2 z_n}{b_3 x_n y_n + b_4 x_n y_{n-1}}, \; z_{n+1} = \frac{c_1 z_{n-1} + c_2 z_n}{c_3 x_{n-1} y_{n-1} + c_4 x_{n-1} y_n + c_5 x_n y_n}$$

Elsayed [4] obtained the solutions of the following system of the difference equations:

$$x_{n+1} = \frac{1}{y_{n-k}}, \;\; y_{n+1} = \frac{y_{n-k}}{x_n y_n}$$

Grove et al. [5] studied existence and behavior of solutions of the rational system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n}, \;\; y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}$$

The behavior of positive solutions of the system,

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \; y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}$$

has been studied by Kurbanli et al. [6].

In addition, Kurbanli [7] investigated the behavior of the solutions of the difference equation system,

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n - 1}, \; y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n - 1}, \;\; z_{n+1} = \frac{1}{z_n y_n}$$

In [8], Ozban studied the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}}, \; y_{n+1} = \frac{by_{n-3}}{x_{n-q}y_{n-q}}$$

In [9], Papaschinopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}$$

Schinas [10] studied some invariants for difference equations and systems of difference equations of rational form.

El-Dessoky et al. [11] obtained the solution of the following system of difference equations

$$x_{n+1} = \frac{x_n y_{n-1}}{\pm x_n \pm y_{n-2}}, \; y_{n+1} = \frac{x_{n-1} y_n}{\pm y_n \pm x_{n-2}}$$

Touafek et al. [12] investigated the periodic nature and gave the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3}y_{n-1}}, \; y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3}x_{n-1}}$$

In [13,14], Zhang et al. studied the boundedness, the persistence, and the global asymptotic stability of the positive solutions of the systems of difference equations:

$$x_n = A + \frac{1}{y_{n-p}}, \quad y_n = A + \frac{y_{n-1}}{x_{n-r}y_{n-s}}$$

and

$$x_{n+} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}$$

In [15], El-Dessoky obtained the form of the solutions and the periodicity character of some systems of rational difference equations:

$$x_{n+1} = \frac{z_{n-3}}{a_1 + b_1 z_n y_{n-1} x_{n-2} z_{n-3}}, \; y_{n+1} = \frac{x_{n-3}}{a_2 + b_2 x_n z_{n-1} y_{n-2} x_{n-3}}, \; z_{n+1} = \frac{y_{n-3}}{a_3 + b_3 y_n x_{n-1} z_{n-2} y_{n-3}}$$

Alzahrani et al. [16] obtained the form of the solution and the qualitative properties of the a rational difference equations of order two:

$$x_{n+1} = \frac{y_n y_{n-1}}{x_n\left(\pm 1 \pm y_n y_{n-1}\right)}, \; y_{n+1} = \frac{x_n x_{n-1}}{y_n\left(\pm 1 \pm x_n x_{n-1}\right)}$$

For similar work to the difference equations and nonlinear systems of rational difference equations investigated herein, see references [1–43].

2. The System: $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}x_n}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}x_n} \tag{1}$$

where $n \in \mathbb{N}_0$, and the initial conditions are arbitrary nonzero real numbers with $y_{-2}x_{-1}y_0 \neq 1, \neq \frac{1}{2}$ and $x_{-2}y_{-1}x_0 \neq \pm 1$.

The following theorem is devoted to the form of the solutions of system (1).

Theorem 1. *Assume that $\{x_n, y_n\}$ are solutions of system (1). Then, for $n = 0, 1, 2, ...$, we see that all solutions of system (1) are periodic with period twelve and*

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n+2} &= -y_{-1}(1+x_{-2}y_{-1}x_0), \quad x_{12n+3} = \frac{-y_0(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n+4} &= \frac{-x_{-2}(1-x_{-2}y_{-1}x_0)}{(1+x_{-2}y_{-1}x_0)}, \quad x_{12n+5} = \frac{x_{-1}}{(-1+2y_{-2}x_{-1}y_0)}, \\
x_{12n+6} &= \frac{-x_0(1+x_{-2}y_{-1}x_0)}{(1-x_{-2}y_{-1}x_0)}, \quad x_{12n+7} = \frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n+8} &= y_{-1}(1-x_{-2}y_{-1}x_0), \quad x_{12n+9} = \frac{-y_0}{(-1+y_{-2}x_{-1}y_0)},
\end{aligned}$$

and

$$\begin{aligned}
y_{12n-2} &= y_{-2}, \; y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}, \\
y_{12n+2} &= \frac{x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)}, \; y_{12n+3} = \frac{x_0}{(1-x_{-2}y_{-1}x_0)}, \quad y_{12n+4} = -y_{-2}, \\
y_{12n+5} &= -y_{-1}, \; y_{12n+6} = -y_0, \quad y_{12n+7} = \frac{-x_{-2}}{(1+x_{-2}y_{-1}x_0)}, \\
y_{12n+8} &= \frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)}, \; y_{12n+9} = \frac{-x_0}{(1-x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{12n-14} &= x_{-2}, \quad x_{12n-13} = x_{-1}, \quad x_{12n-12} = x_0, \quad x_{12n-11} = \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n-10} &= -y_{-1}(1+x_{-2}y_{-1}x_0), \quad x_{12n-9} = \frac{-y_0(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \\
{}_{12n-8} &= \frac{-x_{-2}(1-x_{-2}y_{-1}x_0)}{(1+x_{-2}y_{-1}x_0)}, \quad x_{12n-7} = \frac{x_{-1}}{(-1+2y_{-2}x_{-1}y_0)}, \\
x_{12n-6} &= \frac{-x_0(1+x_{-2}y_{-1}x_0)}{(1-x_{-2}y_{-1}x_0)}, \quad x_{12n-5} = \frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n-4} &= y_{-1}(1-x_{-2}y_{-1}x_0), \quad x_{12n-3} = \frac{-y_0}{(-1+y_{-2}x_{-1}y_0)},
\end{aligned}$$

and

$$\begin{aligned}
y_{12n-14} &= y_{-2},\ y_{12n-13}=y_{-1},\ \ y_{12n-12}=y_0,\ \ \ y_{12n-11}=\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{12n-10} &= \frac{x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)},\ y_{12n-9}=\frac{x_0}{(1-x_{-2}y_{-1}x_0)},\ \ y_{12n-8}=-y_{-2},
\end{aligned}$$

$$\begin{aligned}
y_{12n-7} &= -y_{-1},\ y_{12n-6}=-y_0,\ \ y_{12n-5}=\frac{-x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{12n-4} &= \frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)},\ y_{12n-3}=\frac{-x_0}{(1-x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Now it follows from Equation (1) that

$$\begin{aligned}
x_{12n-2} &= \frac{y_{12n-5}}{-1+y_{12n-5}x_{12n-4}y_{12n-3}}\\
&= \frac{\frac{-x_{-2}}{(1+x_{-2}y_{-1}x_0)}}{\left(-1+\frac{-x_{-2}}{(1+x_{-2}y_{-1}x_0)}y_{-1}(1-x_{-2}y_{-1}x_0)\frac{-x_0}{(1-x_{-2}y_{-1}x_0)}\right)}\\
&= \frac{\frac{-x_{-2}}{(1+x_{-2}y_{-1}x_0)}}{-1+\frac{x_{-2}y_{-1}x_0}{(1+x_{-2}y_{-1}x_0)}}=x_{-2},\\
y_{12n-2} &= \frac{x_{12n-5}}{1+x_{12n-5}y_{12n-4}x_{12n-3}}\\
&= \frac{\frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}}{\left(1+\frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}\frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)}\frac{-y_0}{(-1+y_{-2}x_{-1}y_0)}\right)}\\
&= \frac{\frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}}{\left(1+\frac{y_{-2}x_{-1}y_0}{(-1+y_{-2}x_{-1}y_0)}\right)}=\frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)\left(1+\frac{y_{-2}x_{-1}y_0}{(-1+y_{-2}x_{-1}y_0)}\right)}\\
&= \frac{y_{-2}(-1+2y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0+y_{-2}x_{-1}y_0)}=y_{-2},
\end{aligned}$$

We also see that

$$\begin{aligned}
x_{12n-1} &= \frac{y_{12n-4}}{-1+y_{12n-4}x_{12n-3}y_{12n-2}}\\
&= \frac{\frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)}}{\left(-1+\frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)}\frac{-y_0}{(-1+y_{-2}x_{-1}y_0)}y_{-2}\right)}\\
&= \frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(-1+2y_{-2}x_{-1}y_0)\left(-1+\frac{y_{-2}x_{-1}y_0}{(-1+2y_{-2}x_{-1}y_0)}\right)}\\
&= \frac{-x_{-1}(-1+y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0+y_{-2}x_{-1}y_0)}=x_{-1},\\
y_{12n-1} &= \frac{x_{12n-4}}{1+x_{12n-4}y_{12n-3}x_{12n-2}}\\
&= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{\left(1+y_{-1}(1-x_{-2}y_{-1}x_0)\frac{-x_0}{(1-x_{-2}y_{-1}x_0)}x_{-2}\right)}\\
&= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-x_{-2}y_{-1}x_0)}=y_{-1}.
\end{aligned}$$

We can also prove the other relation. The proof is complete. □

Example 1. *See Figure 1 when we put the initial conditions* $x_{-2} = 5$, $x_{-1} = -0.4$, $x_0 = 0.13$, $y_{-2} = 0.3$, $y_{-1} = -0.9$, *and* $y_0 = -2$ *for the difference system (1).*

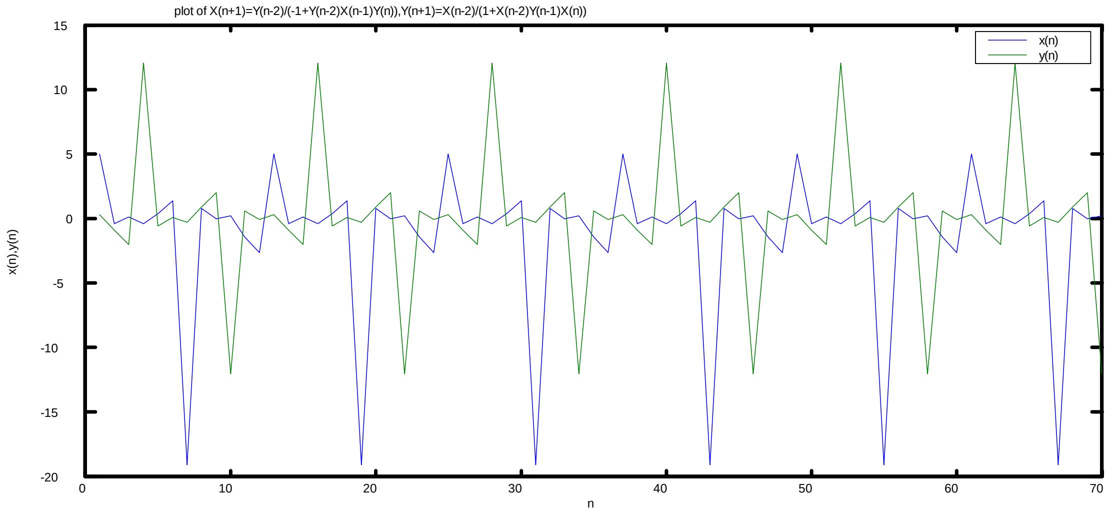

Figure 1. Plot of system $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}x_n}$.

3. The System: $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n}$

In this section, we obtain the form of the solutions of the system of two difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n} \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary non zero real numbers with $y_{-2}x_{-1}y_0 \neq \pm 1$ and $x_{-2}y_{-1}x_0 \neq 1, \neq \frac{1}{2}$.

The following theorem is devoted to the expression of the form of the solutions of System (2).

Theorem 2. *Suppose that* $\{x_n, y_n\}_{n=-2}^{+\infty}$ *are solutions of System (2). Then,* $\{x_n\}_{n=-2}^{+\infty}$ *and* $\{y_n\}_{n=-2}^{+\infty}$ *and are periodic with period twelve and for* $n = 0, 1, 2, \ldots,$

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n+2} &= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(-1+2x_{-2}y_{-1}x_0)}, \quad x_{12n+3} = \frac{-y_0}{(1+y_{-2}x_{-1}y_0)}, \quad x_{12n+4} = -x_{-2}, \\
x_{12n+5} &= -x_{-1}, \quad x_{12n+6} = -x_0, \quad x_{12n+7} = \frac{-y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \\
x_{12n+8} &= \frac{-y_{-1}(1-x_{-2}y_{-1}x_0)}{(-1+2x_{-2}y_{-1}x_0)}, \quad x_{12n+9} = \frac{y_0}{(1+y_{-2}x_{-1}y_0)},
\end{aligned}$$

and

$$\begin{aligned}
y_{12n-2} &= y_{-2}, \quad y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}, \\
y_{12n+2} &= -x_{-1}(-1+y_{-2}x_{-1}y_0), \quad y_{12n+3} = \frac{x_0(-1+2x_{-2}y_{-1}x_0)}{(-1+x_{-2}y_{-1}x_0)}, \\
y_{12n+4} &= \frac{y_{-2}(1+y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \quad y_{12n+5} = \frac{y_{-1}}{(-1+2x_{-2}y_{-1}x_0)}, \\
y_{12n+6} &= \frac{y_0(-1+y_{-2}x_{-1}y_0)}{(1+y_{-2}x_{-1}y_0)}, \quad y_{12n+7} = \frac{-x_{-2}(-1+2x_{-2}y_{-1}x_0)}{(-1+x_{-2}y_{-1}x_0)}, \\
y_{12n+8} &= -x_{-1}(1+y_{-2}x_{-1}y_0), \quad y_{12n+9} = \frac{-x_0}{(1-x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Or, equivalently,

$$\{x_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} x_{-2}, x_{-1}, x_0, \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(-1+2x_{-2}y_{-1}x_0)}, \frac{-y_0}{(1+y_{-2}x_{-1}y_0)}, -x_{-2}, -x_{-1}, \\ -x_0, \frac{-y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \frac{-y_{-1}(1-x_{-2}y_{-1}x_0)}{(-1+2x_{-2}y_{-1}x_0)}, \frac{y_0}{(1+y_{-2}x_{-1}y_0)}, x_{-2}, x_{-1}, x_0, ... \end{array} \right\},$$

and

$$\{y_n\}_{n=-2}^{+\infty} = \left\{ \begin{array}{c} y_{-2}, y_{-1}, y_0, \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}, -x_{-1}(-1+y_{-2}x_{-1}y_0), \frac{x_0(-1+2x_{-2}y_{-1}x_0)}{(-1+x_{-2}y_{-1}x_0)}, \\ \frac{y_{-2}(1+y_{-2}x_{-1}y_0)}{(-1+y_{-2}x_{-1}y_0)}, \frac{y_{-1}}{(-1+2x_{-2}y_{-1}x_0)}, \frac{y_0(-1+y_{-2}x_{-1}y_0)}{(1+y_{-2}x_{-1}y_0)}, \frac{-x_{-2}(-1+2x_{-2}y_{-1}x_0)}{(-1+x_{-2}y_{-1}x_0)}, \\ -x_{-1}(1+y_{-2}x_{-1}y_0), \frac{-x_0}{(1-x_{-2}y_{-1}x_0)}, y_{-2}, y_{-1}, y_0, ... \end{array} \right\}.$$

Proof. The proof follows the form of the proof of Theorem 1, and so will be omitted. □

Example 2. *We assume the initial conditions* $x_{-2} = 0.5$, $x_{-1} = 4$, $x_0 = -2.13$, $y_{-2} = 0.3$, $y_{-1} = 9$, *and* $y_0 = 2$ *for difference system (2); see Figure 2.*

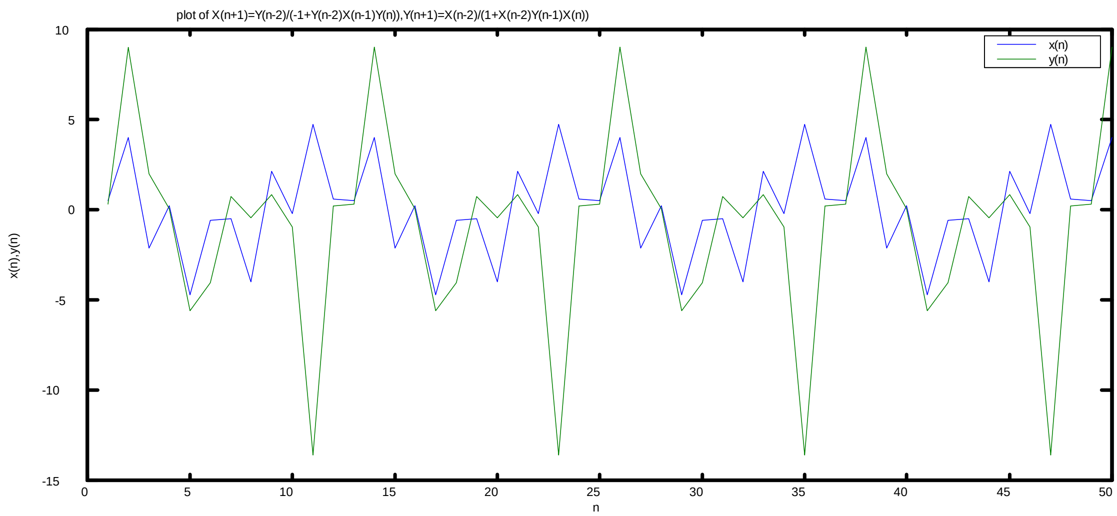

Figure 2. Plot of system $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n}$.

4. The System: $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}x_n}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}x_n} \tag{3}$$

where $n \in \mathbb{N}_0$, and the initial conditions are arbitrary nonzero real numbers such that $x_{-2}y_{-1}x_0 \neq 1$ and $y_{-2}x_{-1}y_0 \neq 1$.

Theorem 3. *If* $\{x_n, y_n\}$ *are solutions of difference equation system (3), then every solution of system (3) is periodic with period six, and takes the form for* $n = 0, 1, 2, ...,$

$$\begin{aligned} x_{6n-2} &= x_{-2}, \quad x_{6n-1} = x_{-1}, \quad x_{6n} = x_0, \quad x_{6n+1} = \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}, \\ x_{6n+2} &= y_{-1}(-1+x_{-2}y_{-1}x_0), \quad x_{6n+3} = \frac{y_0}{(-1+y_{-2}x_{-1}y_0)}, \end{aligned}$$

and

$$\begin{aligned} y_{6n-2} &= y_{-2}, \ y_{6n-1} = y_{-1}, \ y_{6n} = y_0, \ y_{6n+1} = \frac{x_{-2}}{(-1 + x_{-2}y_{-1}x_0)}, \\ y_{6n+2} &= x_{-1}(-1 + y_{-2}x_{-1}y_0), \ y_{6n+3} = \frac{x_0}{(-1 + x_{-2}y_{-1}x_0)}. \end{aligned}$$

Proof. The proof follows the form of the proof of Theorem 1, and so will be omitted. □

Example 3. *We consider an interesting numerical example for difference system (3) with the initial conditions* $x_{-2} = 0.15$, $x_{-1} = 7$, $x_0 = -0.3$, $y_{-2} = 0.13$, $y_{-1} = 0.8$, *and* $y_0 = 2$*; see Figure 3.*

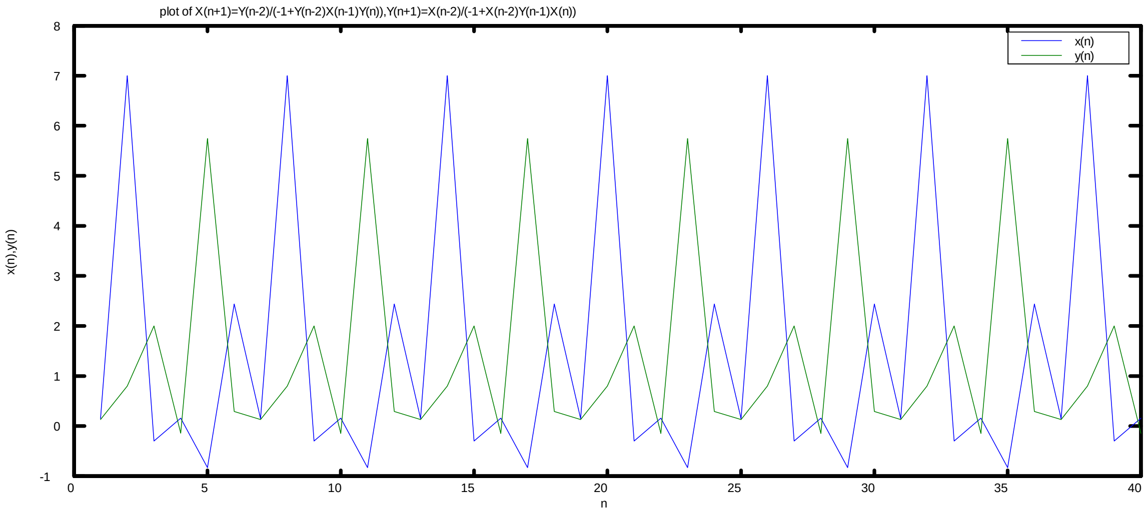

Figure 3. Plot of system $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}x_n}$.

5. The System: $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{-1-x_{n-2}y_{n-1}x_n}$

In this section, we study the solutions of the following system of difference equations

$$x_{n+1} = \frac{y_{n-2}}{-1 + y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n} \tag{4}$$

where $n \in \mathbb{N}_0$, and the initial conditions are arbitrary non-zero real numbers.

Theorem 4. *Assume that* $\{x_n, y_n\}$ *are solutions of System (4). Then, for* $n = 0, 1, 2, \ldots$,

$$\begin{aligned} x_{6n-2} &= x_{-2} \prod_{i=0}^{n-1} \frac{(1 + (6i)x_{-2}y_{-1}x_0)(1 + (6i+3)x_{-2}y_{-1}x_0)}{(1 + (6i+1)x_{-2}y_{-1}x_0)(1 + (6i+4)x_{-2}y_{-1}x_0)}, \\ x_{6n-1} &= x_{-1} \prod_{i=0}^{n-1} \frac{(1 - (6i+1)y_{-2}x_{-1}y_0)(1 - (6i+4)y_{-2}x_{-1}y_0)}{(1 - (6i+2)y_{-2}x_{-1}y_0)(1 - (6i+5)y_{-2}x_{-1}y_0)}, \\ x_{6n} &= x_0 \prod_{i=0}^{n-1} \frac{(1 + (6i+2)x_{-2}y_{-1}x_0)(1 + (6i+5)x_{-2}y_{-1}x_0)}{(1 + (6i+3)x_{-2}y_{-1}x_0)(1 + (6i+6)x_{-2}y_{-1}x_0)}, \\ x_{6n+1} &= \frac{y_{-2}}{(-1 + y_{-2}x_{-1}y_0)} \prod_{i=0}^{n-1} \frac{(1 - (6i+3)y_{-2}x_{-1}y_0)(1 - (6i+6)y_{-2}x_{-1}y_0)}{(1 - (6i+4)y_{-2}x_{-1}y_0)(1 - (6i+7)y_{-2}x_{-1}y_0)}, \\ x_{6n+2} &= \frac{y_{-1}(-1 - x_{-2}y_{-1}x_0)}{(1 + 2x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-1} \frac{(1 + (6i+4)x_{-2}y_{-1}x_0)(1 + (6i+7)x_{-2}y_{-1}x_0)}{(1 + (6i+5)x_{-2}y_{-1}x_0)(1 + (6i+8)x_{-2}y_{-1}x_0)}, \\ x_{6n+3} &= \frac{y_0(1 - 2y_{-2}x_{-1}y_0)}{(-1 + 3y_{-2}x_{-1}y_0)} \prod_{i=0}^{n-1} \frac{(1 - (6i+5)y_{-2}x_{-1}y_0)(1 - (6i+8)y_{-2}x_{-1}y_0)}{(1 - (6i+6)y_{-2}x_{-1}y_0)(1 - (6i+9)y_{-2}x_{-1}y_0)}, \end{aligned}$$

$$\begin{aligned}
y_{6n-2} &= y_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)y_{-2}x_{-1}y_0)(1-(6i+3)y_{-2}x_{-1}y_0)}{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)},\\
y_{6n-1} &= y_{-1}\prod_{i=0}^{n-1}\frac{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}{(1+(6i+2)x_{-2}y_{-1}x_0)(1+(6i+5)x_{-2}y_{-1}x_0)},\\
y_{6n} &= y_0\prod_{i=0}^{n-1}\frac{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)}{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)},\\
y_{6n+1} &= \frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)},\\
y_{6n+2} &= \frac{x_{-1}(-1+y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},\\
y_{6n+3} &= \frac{x_0(1+2x_{-2}y_{-1}x_0)}{(-1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}{(1+(6i+6)x_{-2}y_{-1}x_0)(1+(6i+9)x_{-2}y_{-1}x_0)}.
\end{aligned}$$

where $\prod_{i=0}^{-1} A_i = 1$.

Proof. For $n=0$, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. that is,

$$\begin{aligned}
x_{6n-8} &= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(6i)x_{-2}y_{-1}x_0)(1+(6i+3)x_{-2}y_{-1}x_0)}{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)},\\
x_{6n-7} &= x_{-1}\prod_{i=0}^{n-2}\frac{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)}{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)},\\
x_{6n-6} &= x_0\prod_{i=0}^{n-2}\frac{(1+(6i+2)x_{-2}y_{-1}x_0)(1+(6i+5)x_{-2}y_{-1}x_0)}{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)},\\
x_{6n-5} &= \frac{y_{-2}}{(-1+y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)}{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)},\\
x_{6n-4} &= \frac{y_{-1}(-1-x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)},\\
x_{6n-3} &= \frac{y_0(1-2y_{-2}x_{-1}y_0)}{(-1+3y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)}{(1-(6i+6)y_{-2}x_{-1}y_0)(1-(6i+9)y_{-2}x_{-1}y_0)},\\
y_{6n-8} &= y_{-2}\prod_{i=0}^{n-2}\frac{(1-(6i)y_{-2}x_{-1}y_0)(1-(6i+3)y_{-2}x_{-1}y_0)}{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)},\\
y_{6n-7} &= y_{-1}\prod_{i=0}^{n-2}\frac{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}{(1+(6i+2)x_{-2}y_{-1}x_0)(1+(6i+5)x_{-2}y_{-1}x_0)},\\
y_{6n-6} &= y_0\prod_{i=0}^{n-2}\frac{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)}{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)},\\
y_{6n-5} &= \frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)},\\
y_{6n-4} &= \frac{x_{-1}(-1+y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},\\
y_{6n-3} &= \frac{x_0(1+2x_{-2}y_{-1}x_0)}{(-1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}{(1+(6i+6)x_{-2}y_{-1}x_0)(1+(6i+9)x_{-2}y_{-1}x_0)}.
\end{aligned}$$

It follows from Equation (4) that

$$
\begin{aligned}
x_{6n-2} &= \frac{y_{6n-5}}{-1+y_{6n-5}x_{6n-4}y_{6n-3}} \\
&= \frac{\frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}}{\left(\begin{array}{c}-1+\frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)} \\ \frac{y_{-1}(-1-x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1+2x_{-2}y_{-1}x_0)}{(-1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}{(1+(6i+6)x_{-2}y_{-1}x_0)(1+(6i+9)x_{-2}y_{-1}x_0)}\end{array}\right)} \\
&= \frac{\frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}}{\left(-1+\frac{x_0y_{-1}x_{-2}}{(-1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)}{(1+(6i+9)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}}{\left(-1-\frac{x_{-2}y_{-1}x_0}{(1+(6n-3)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(-1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{-1-(6n-3)x_{-2}y_{-1}x_0-x_{-2}y_{-1}x_0}{(1+(6n-3)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{1+(6n-2)x_{-2}y_{-1}x_0}{1+(6n-3)x_{-2}y_{-1}x_0}\right)} \\
&= \frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+3)x_{-2}y_{-1}x_0)(1+(6i+6)x_{-2}y_{-1}x_0)}{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}\left(\frac{1+(6n-3)x_{-2}y_{-1}x_0}{1+(6n-2)x_{-2}y_{-1}x_0}\right) \\
&= x_{-2}\prod_{i=0}^{n-1}\frac{(1+(6i)x_{-2}y_{-1}x_0)(1+(6i+3)x_{-2}y_{-1}x_0)}{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}.
\end{aligned}
$$

We also see from Equation (4) that

$$
\begin{aligned}
y_{6n-1} &= \frac{x_{6n-4}}{-1-x_{6n-4}y_{6n-3}x_{6n-2}} \\
&= \frac{\frac{y_{-1}(-1-x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}}{-1-\left(\begin{array}{c}\frac{y_{-1}(-1-x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1+2x_{-2}y_{-1}x_0)}{(-1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}{(1+(6i+6)x_{-2}y_{-1}x_0)(1+(6i+9)x_{-2}y_{-1}x_0)} \\ x_{-2}\prod_{i=0}^{n-1}\frac{(1+(6i)x_{-2}y_{-1}x_0)(1+(6i+3)x_{-2}y_{-1}x_0)}{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}\end{array}\right)} \\
&= \frac{\frac{y_{-1}(-1-x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}}{-1-\left(\begin{array}{c}\frac{x_{-2}x_0y_{-1}(1+x_{-2}y_{-1}x_0)}{(1+3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+6)x_{-2}y_{-1}x_0)(1+(6i+9)x_{-2}y_{-1}x_0)} \\ \prod_{i=0}^{n-1}\frac{(1+(6i)x_{-2}y_{-1}x_0)(1+(6i+3)x_{-2}y_{-1}x_0)}{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}\end{array}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}}{1+\left(x_{-2}x_0y_{-1}\prod_{i=0}^{n-2}(1+(6i+4)x_{-2}y_{-1}x_0)\prod_{i=0}^{n-1}\frac{1}{(1+(6i+4)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}}{1+\left(\frac{x_{-2}x_0y_{-1}}{1+(6n-2)x_{-2}y_{-1}x_0}\right)} \\
&= \frac{\frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}}{\left(\frac{1+(6n-2)x_{-2}y_{-1}x_0+x_{-2}x_0y_{-1}}{1+(6n-2)x_{-2}y_{-1}x_0}\right)} \\
&= \frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{(1+2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1+(6i+4)x_{-2}y_{-1}x_0)(1+(6i+7)x_{-2}y_{-1}x_0)}{(1+(6i+5)x_{-2}y_{-1}x_0)(1+(6i+8)x_{-2}y_{-1}x_0)}\left(\frac{1+(6n-2)x_{-2}y_{-1}x_0}{1+(6n-1)x_{-2}y_{-1}x_0}\right) \\
&= y_{-1}\prod_{i=0}^{n-1}\frac{(1+(6i+1)x_{-2}y_{-1}x_0)(1+(6i+4)x_{-2}y_{-1}x_0)}{(1+(6i+2)x_{-2}y_{-1}x_0)(1+(6i+5)x_{-2}y_{-1}x_0)}.
\end{aligned}
$$

We can prove the other relations similarly. This completes the proof. □

Corollary 1. *If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$, and y_0 are arbitrary real numbers and let $\{x_n, y_n\}$ are solutions of System (4), then the following statements are true:*

(i) *If $x_{-2} = 0$, $y_{-1} \neq 0$, $x_0 \neq 0$, then we have $x_{6n-2} = y_{6n+1} = 0$ and $x_{6n} = x_0$, $x_{6n+2} = -y_{-1}$, $y_{6n-1} = y_{-1}$, $y_{6n+3} = -x_0$.*

(ii) *If $x_{-1} = 0$, $y_{-2} \neq 0$, $y_0 \neq 0$, then we have $x_{6n-1} = y_{6n+2} = 0$ and $x_{6n+1} = -y_{-2}$, $x_{6n+3} = -y_0$, $y_{6n-2} = y_{-2}$, $y_{6n} = y_0$.*

(iii) *If $x_0 = 0$, $y_{-1} \neq 0$, $x_{-2} \neq 0$, then we have $x_{6n} = y_{6n+3} = 0$ and $x_{6n-2} = x_{-2}$, $x_{6n+2} = -y_{-1}$, $y_{6n-1} = y_{-1}$, $y_{6n+1} = -x_{-2}$.*

(iv) *If $y_{-2} = 0$, $x_{-1} \neq 0$, $y_0 \neq 0$, then we have $y_{6n-2} = x_{6n+1} = 0$ and $x_{6n-1} = x_{-1}$, $x_{6n+3} = -y_0$, $y_{6n} = y_0$, $y_{6n+2} = -x_{-1}$.*

(v) *If $y_{-1} = 0$, $x_0 \neq 0$, $x_{-2} \neq 0$, then we have $y_{6n-1} = x_{6n+2} = 0$ and $x_{6n-2} = x_{-2}$, $x_{6n} = x_0$, $y_{6n+1} = -x_{-2}$, $y_{6n+3} = -x_0$.*

(vi) *If $y_0 = 0$, $y_{-2} \neq 0$, $x_{-1} \neq 0$, then we have $y_{6n} = x_{6n+3} = 0$ and $x_{6n-1} = x_{-1}$, $x_{6n+1} = -y_{-2}$, $y_{6n-2} = y_{-2}$, $y_{6n+2} = -x_{-1}$.*

Proof. The proof follows from the form of the solutions of System (4). □

Example 4. *Figure 4 shows the behavior of the solution of the difference system (4) with the initial conditions $x_{-2} = 0.15$, $x_{-1} = -4$, $x_0 = -0.5$, $y_{-2} = 0.3$, $y_{-1} = 0.28$, and $y_0 = 2$.*

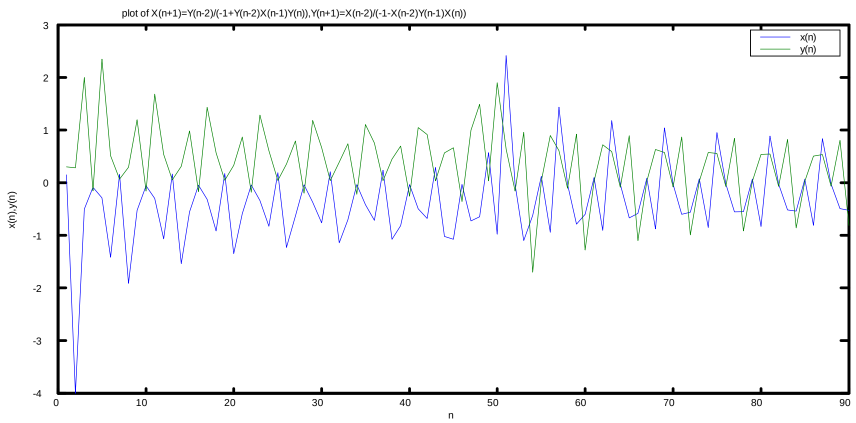

Figure 4. Plot of system $x_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}x_{n-1}y_n}$, $y_{n+1} = \frac{x_{n-2}}{-1-x_{n-2}y_{n-1}x_n}$.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Camouzis, E.; Papaschinopoulos, G. Global asymptotic behavior of positive solutions on the system of rational difference equations $x_{n+1} = 1 + 1/y_{n-k},\ y_{n+1} = y_n/x_{n-m}y_{n-m-k}$. *Appl. Math. Lett.* **2004**, *17*, 733–737.
2. Cinar, C. On the positive solutions of the difference equation system $x_{n+1} = 1/y_n,\ y_{n+1} = y_n/x_{n-1}y_{n-1}$. *Appl. Math. Comput.* **2004**, *158*, 303–305.
3. Elabbasy, E.M.; El-Metwally, H.; Elsayed, E.M. On the Solutions of a Class of Difference Equations Systems. *Demonstr. Math.* **2008**, *41*, 109–122.
4. Elsayed, E.M. On the solutions of a rational system of difference equations. *Fasc. Math.* **2010**, *45*, 25–36.
5. Grove, E.A.; Ladas, G.; McGrath, L.C.; Teixeira, C.T. Existence and behavior of solutions of a rational system. *Commun. Appl. Nonlinear Anal.* **2001**, *8*, 1–25.
6. Kurbanli, A.S.; Cinar, C.; Yalçinkaya, I. On the behavior of positive solutions of the system of rational difference equations $x_{n+1} = x_{n-1}/y_nx_{n-1} + 1,\ y_{n+1} = y_{n-1}/x_ny_{n-1} + 1$. *Math. Comput. Model.* **2011**, *53*, 1261–1267.
7. Kurbanli, A.S. On the behavior of solutions of the system of rational difference equations $x_{n+1} = x_{n-1}/y_nx_{n-1} - 1,\ y_{n+1} = y_{n-1}/x_ny_{n-1} - 1,\ z_{n+1} = 1/y_nz_n$. *Adv. Differ. Equ.* **2011**, *2011*, 40, doi:10.1186/1687-1847-2011-40.
8. Ozban, A.Y. On the system of rational difference equations $x_{n+1} = a/y_{n-3},\ y_{n+1} = by_{n-3}/x_{n-q}y_{n-q}$. *Appl. Math. Comput.* **2007**, *188*, 833–837.
9. Papaschinopoulos, G.; Schinas, C.J. On a system of two nonlinear difference equations. *J. Math. Anal. Appl.* **1998**, *219*, 415–426.
10. Schinas, C.J. Invariants for difference equations and systems of difference equations of rational form. *J. Math. Anal. Appl.* **1997**, *216* 164–179.
11. El-Dessoky, M.M.; Elsayed, E.M.; Alghamdi, M. Solutions and periodicity for some systems of fourth order rational difference equations. *J. Comput. Anal. Appl.* **2015**, *18*, 179–194.
12. Touafek, N.; Elsayed, E.M. On the solutions of systems of rational difference equations. *Math. Comput. Model.* **2012**, *55*, 1987–1997.
13. Zhang, Y.; Yang, X.; Megson, G.M.; Evans, D.J. On the system of rational difference equations. *Appl. Math. Comput.* **2006**, *176*, 403–408.
14. Zhang, Y.; Yang, X.; Evans, D.J.; Zhu, C. On the nonlinear difference equation system. *Comput. Math. Appl.* **2007**, *53*, 1561–1566.
15. El-Dessoky, M.M. On the solutions and periodicity of some nonlinear systems of difference equations. *J. Nonlinear Sci. Appl.* **2016**, *9*, 2190–2207.
16. Alzahrani, E.O.; El-Dessoky, M.M.; Elsayed, E.M.; Kuang, Y. Solutions and Properties of Some Degenerate Systems of Difference Equations. *J. Comput. Anal. Appl.* **2015**, *18*, 321–333.
17. Agarwal, R.P. *Difference Equations and Inequalities*, 1st ed.; Marcel Dekker: New York, NY, USA, 1992.
18. Battaloglu, N.; Cinar, C.; Yalçinkaya, I. The dynamics of the difference equation. *Ars Comb.* **2010**, *97*, 281–288.
19. Clark, D.; Kulenovic, M.R.S. A coupled system of rational difference equations. *Comput. Math. Appl.* **2002**, *43*, 849–867.
20. Özban, A.Y. On the positive solutions of the system of rational difference equations $x_{n+1} = 1/y_{n-k},\ y_{n+1} = y_n/x_{n-m}y_{n-m-k}$. *J. Math. Anal. Appl.* **2006**, *323*, 26–32.
21. El-Dessoky, M.M.; Mansour, M.; Elsayed, E.M. Solutions of some rational systems of difference equations. *Util. Math.* **2013**, *92*, 329–336.
22. Qureshi, M.N.; Khan, A.Q.; Din, Q. Global behavior of third order system of rational difference equations. *Int. J. Eng. Res. Technol.* **2013**, *2*, 2182–2191.
23. Denette, E.; Kulenović, M.R.S.; Pilav, E. Birkhoff Normal Forms, KAM Theory and Time Reversal Symmetry for Certain Rational Map. *Mathematics* **2016**, *4*, 20.
24. Hu, L.-X.; Jia, X.-M. Global Asymptotic Stability of a Rational System. *Abstr. Appl. Anal.* **2014**, *2014*, 286375, doi:10.1155/2014/286375.

25. El-Dessoky, M.M.; Elsayed, E.M. On the solutions and periodic nature of some systems of difference equations. *J. Comput. Anal. Appl.* **2015**, *18*, 206–218.
26. Yalçinkaya, I.; Cinar, C.; Simsek, D. Global asymptotic stability of a system of difference equations. *Appl. Anal.* **2008**, *87*, 689–699.
27. El-Dessoky, M.M. On a solvable for some systems of rational difference equations. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3744–3759.
28. Yalçinkaya, I.; Cinar, C.; Atalay, M. On the solutions of systems of difference equations. *Adv. Differ. Equ.* **2008**, *2008*, 143943.
29. Yang, X.; Liu, Y.; Bai, S. On the system of high order rational difference equations $x_n = a/y_{n-p}$, $y_n = by_{n-p}/x_{n-q}y_{n-q}$. *Appl. Math. Comput.* **2005**, *171*, 853–856.
30. Erdoğan, M.E.; Cinar, C.; Yalçınkaya, I. On the dynamics of the recursive sequence. *Comput. Math. Appl.* **2011**, *61*, 533–537.
31. Elsayed, E.M.; El-Dessoky, M.M.; Alotaibi, A. On the Solutions of a General System of Difference Equations. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 892571.
32. El-Dessoky, M.M. The form of solutions and periodicity for some systems of third—Order rational difference equations. *Math. Methods Appl. Sci.* **2016**, *39*, 1076–1092.
33. El-Dessoky, M.M.; Elsayed, E.M. On a solution of system of three fractional difference equations. *J. Comput. Anal. Appl.* **2015**, *19*, 760–769.
34. Yalçinkaya, I. On the global asymptotic behavior of a system of two nonlinear difference equations. *Ars Comb.* **2010**, *95*, 151–159.
35. Yalçinkaya, I. On the global asymptotic stability of a second-order system of difference equations. *Discret. Dyn. Nat. Soc.* **2008**, *2008*, 860152.
36. El-Dessoky, M.M. On a systems of rational difference equations of Order Two. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 271–284.
37. Elsayed, E.M.; Mansour, M.; El-Dessoky, M.M. Solutions of fractional systems of difference equations. *Ars Comb.* **2013**, *110*, 469–479.
38. Mansour, M.; El-Dessoky, M.M.; Elsayed, E.M. On the solution of rational systems of difference equations. *J. Comput. Anal. Appl.* **2013**, *15*, 967–976.
39. Bisci, G.M.; Repovs, D. On sequences of solutions for discrete anisotropic equations. *Expos. Math.* **2014**, *32*, 284–295.
40. Bisci, G.M.; Repovs, D. Nonlinear Algebraic Systems with discontinuous terms. *J. Math. Anal. Appl.* **2013**, *398*, 846–856.
41. Galewski, M.; Bisci, G.M.; Wieteska, R. Existence and multiplicity of solutions to discrete inclusions with the $p(k)$-Laplacian problem. *J. Differ. Equ. Appl.* **2015**, *21*, 887–903.
42. Bisci, G.M.; Repovs, D. Existence of solutions for p-Laplacian discrete equations. *Appl. Math. Comput.* **2014**, *242*, 454–461.
43. Candito, P.; Bisci, G.M. Existence of positive solutions for nonlinear algebraic systems with a parameter. *Appl. Math. Comput.* **2012**, *218*, 11700–11707.