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Approximation in Müntz Spaces $M_{\Lambda,p}$ of L_p Functions for $1 < p < \infty$ and Bases

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Abstract: Müntz spaces satisfying the Müntz and gap conditions are considered. A Fourier approximation of functions in the Müntz spaces $M_{\Lambda,p}$ of L_p functions is studied, where $1 < p < \infty$. It is proven that up to an isomorphism and a change of variables, these spaces are contained in Weil–Nagy's class. Moreover, the existence of Schauder bases in the Müntz spaces $M_{\Lambda,p}$ is investigated.

Keywords: Banach space; Müntz space; isomorphism; Schauder basis; Fourier series

MSC: 46B03; 46B20

1. Introduction

An immense branch of functional analysis is devoted to the topological and geometric properties of topological vector spaces (see, for example, [1–4]). Studies of bases in Banach spaces compose a large part of it (see, for example, [1,5–12] and the references therein). It is not surprising that for concrete classes of Banach spaces, many open problems remain, particularly for the Müntz spaces $M_{\Lambda,p}$, where $1 < p < \infty$ (see [13–22] and the references therein). These spaces are defined as completions of the linear span over $\mathbf{R}$ or $\mathbf{C}$ of monomials t^λ with $\lambda \in \Lambda$ on the segment $[0,1]$ relative to the L_p norm, where $\Lambda \subset [0,\infty)$, $t \in [0,1]$. In his classical work, K.Weierstrass had proven in 1885 the theorem about polynomial approximations of continuous functions on the segment. However, the space of continuous functions also forms an algebra. Generalizations of such spaces were considered by C. Müntz in 1914, such that his spaces did not have the algebra structure. C. Müntz considered conditions on the exponents λ_i for which the monomials t^{λ_i} span a dense subspace of $C[0,1]$. Naturally, a problem arose whether they have bases [23,24]. Then, the progress was for lacunary Müntz spaces satisfying the condition $\underline{\lim}_{n\to\infty}\lambda_{n+1}/\lambda_n > 1$ with a countable set Λ, but in its generality, this problem was not solved [20]. It is worth mentioning that the system $\{t^\lambda : \lambda \in \Lambda\}$ itself does not contain a Schauder basis for a non-lacunary set Λ satisfying the Müntz and gap conditions.

In Section 2, relations between Müntz spaces satisfying the Müntz and gap conditions are considered. A Fourier approximation of functions in the Müntz spaces $M_{\Lambda,p}$ of L_p functions is studied in Section 3, where $1 < p < \infty$. Necessary definitions are recalled. It is proven that up to an isomorphism and a change of variables, these spaces are contained in Weil–Nagy's class. For this purpose, in Lemmas 1 and 2, Theorem 1 and Corollary 1, some isomorphisms of Müntz spaces are given. Then, in Theorem 2, a relation between Müntz spaces and Weil–Nagy's classes is established. Moreover, in Section 4, the existence of Schauder bases in the Müntz spaces $M_{\Lambda,p}$ is investigated (see Theorem 3) with the help of Fourier series approximation (see Lemma 5). It is proven that, under the Müntz condition and the gap condition, Schauder bases exist in the Müntz spaces $M_{\Lambda,p}$, where $1 < p < \infty$.

All main results of this paper are obtained for the first time. They can be used for further investigations of function approximations and the geometry of Banach spaces. It is important not only for the development of mathematical analysis and of functional analysis, but also in their many-sided applications.

2. Relation between Spaces

To avoid misunderstandings, we first remind about the necessary definitions and notations.

Notation 1. *Let $C([\alpha,\beta],\mathbf{F})$ denote the Banach space of all continuous functions $f:[\alpha,\beta]\to\mathbf{F}$ supplied with the absolute maximum norm:*

$$\|f\|_C := \max\{|f(x)| : x\in[\alpha,\beta]\}$$

where $-\infty<\alpha<\beta<\infty$, $\mathbf{F}$ is either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$.

Then, $L_p([\alpha,\beta],\mathbf{F})$ denotes the Banach space of all Lebesgue measurable functions $f:[\alpha,\beta]\to\mathbf{F}$ possessing the finite norm:

$$\|f\|_{L_p([\alpha,\beta],\mathbf{F})} := (\int_\alpha^\beta |f(x)|^p dx)^{1/p} < \infty$$

where $1\le p<\infty$ is a marked number, $\alpha<\beta$.

As usual, $span_{\mathbf{F}}(v_k : k)$ will stand for the linear span of vectors v_k over a field $\mathbf{F}$.

Definition 1. *Take a countable infinite subset $\Lambda=\{\lambda_k : k\in\mathbf{N}\}$ in the set $(0,\infty)$ so that $\{\lambda_k : k\in\mathbf{N}\}$ is a strictly increasing sequence.*

Henceforth, it is supposed that the set Λ satisfies the gap condition:

(1) $\inf_k\{\lambda_{k+1}-\lambda_k\} =: \alpha_0 > 0$ and the Müntz condition:

(2)
$$\sum_{k=1}^{\infty}\frac{1}{\lambda_k} =: \alpha_1 < \infty.$$

The completion of the linear space containing all monomials at^λ with $a\in\mathbf{F}$ and $\lambda\in\Lambda$ and $t\in[\alpha,\beta]$ relative to the L_p norm is denoted by $M_{\Lambda,p}([\alpha,\beta],\mathbf{F})$, where $0\le\alpha<\beta<\infty$, $1\le p$, also by $M_{\Lambda,C}([\alpha,\beta],\mathbf{F})$ when it is completed relative to the $\|\|_C$ norm. Briefly, they will also be written as $M_{\Lambda,p}$ or $M_{\Lambda,C}$, respectively, for $\alpha=0$ and $\beta=1$, when $\mathbf{F}$ is specified.

Before the subsections about the Fourier approximation in Müntz spaces auxiliary, Lemmas 1 and 2 and Theorem 1 are proven about isomorphisms of Müntz spaces M_{Λ,L_p}. With their help, our consideration reduces to a subclass of Müntz spaces M_{Λ,L_p} so that a set Λ is contained in the set of natural numbers $\mathbf{N}$.

Lemma 1. *For each $0<\delta<1$, the Müntz spaces $M_{\Lambda,p}([0,1],\mathbf{F})$ and $M_{\Lambda,p}([\delta,1],\mathbf{F})$ are linearly topologically isomorphic, where $1\le p<\infty$.*

Proof. For every $0<\delta<1$ and $0<\epsilon\le 1$ and $f\in E := L_p([0,1],\mathbf{F})$, the norms $\|f\|_{E[0,1]}$ and $\epsilon\|f|_{[0,\delta]}\|_{E[0,\delta]}+\|f|_{[\delta,1]}\|_{E[\delta,1]}$ are equivalent, where $E[\alpha,\beta] := E\cap L_p([\alpha,\beta],\mathbf{F})$ for $0\le\alpha<\beta\le 1$. Due to the Remez-type and the Nikolski-type inequalities (see Theorem 6.2.2 in [16] and Theorem 7.4 in [17]) for each Λ satisfying the Müntz condition, there is a constant $\eta>0$, so that $\|h|_{[0,\delta]}\|_{E[0,\delta]}\le\eta\|h|_{[\delta,1]}\|_{E[\delta,1]}$ for each $h\in M_{\Lambda,p}$, where η is independent of h. Therefore, the norms $\|h|_{[\delta,1]}\|_{E[\delta,1]}$ and $\|h\|_{E[0,1]}$ are equivalent on $M_{\Lambda,p}[0,1]$. Certainly, each polynomial $a_1t^{\lambda_1}+...+a_nt^{\lambda_n}$ defined on the segment $[\delta,1]$ has the natural extension on $[0,1]$, where $a_1,...,a_n\in\mathbf{F}$ are constants and t is a variable.

Thus, the Müntz spaces $M_{\Lambda,p}[0,1]$ and $M_{\Lambda,p}[\delta,1]$ are linearly topologically isomorphic as normed spaces for each $0<\delta<1$. □

Lemma 2. *The Müntz spaces $M_{\Lambda,p}$ and $M_{\Xi\cup(\alpha\Lambda+\beta),p}$ are linearly topologically isomorphic for every $\beta\geq 0$ and $\alpha>0$ and a finite subset Ξ in $(0,\infty)$, where $1\leq p<\infty$.*

Proof. We have that a sequence $\{\lambda_k : k\in\mathbf{N}\}$ is strictly increasing and satisfies the gap condition. This implies that $\lim_{n\to\infty}\lambda_n=\infty$. Without loss of generality, a set $\Xi\cup(\alpha\Lambda+\beta)$ can also be ordered into a strictly increasing sequence.

By virtue of Theorem 9.1.6 [20], the Müntz space $M_{\Lambda,p}$ contains a complemented isomorphic copy of l_p; consequently, $M_{\Lambda,p}$ and $M_{\Xi\cup\Lambda,p}$ are linearly topologically isomorphic as normed spaces.

Then, from Lemma 1 taking $\alpha>0$, we deduce that:

$$\int_{\delta}^{1}|f(t)|^p dt=\alpha\int_{\delta^{1/\alpha}}^{1}|f(x^{\alpha})|^p x^{(\alpha-1)}dx\leq\alpha\max(1,\delta^{(1-\alpha^{-1})})\int_{\delta^{1/\alpha}}^{1}|f(x^{\alpha})|^p dx \tag{1}$$

and:

$$\int_{\delta}^{1}|f(x^{\alpha})|^p dx=\alpha^{-1}\int_{\delta^{\alpha}}^{1}|f(t)|^p t^{(\alpha^{-1}-1)}dt\leq\alpha^{-1}\max(1,\delta^{(1-\alpha)})\int_{\delta^{\alpha}}^{1}|f(t)|^p dt \tag{2}$$

for each $f\in M_{\Lambda,p}$, and hence, $M_{\alpha\Lambda,p}$ is isomorphic with $M_{\Lambda,p}$. Considering the set $\Lambda_1=\Lambda\cup\{\frac{\beta}{\alpha}\}$ and then the set $\alpha\Lambda_1$, we get that $M_{\Lambda,p}$ and $M_{\alpha\Lambda+\beta,p}$ are linearly topologically isomorphic as normed spaces, as well. □

Theorem 1. *Let increasing sequences $\Lambda=\{\lambda_n : n\}$ and $\Upsilon=\{v_n : n\}$ of positive numbers satisfy Conditions 1(1,2), and let $\lambda_n\leq v_n$ for each n. If $\sup_n(v_n-\lambda_n)=\delta$, where $\delta<(8\sum_{n=1}^{\infty}\lambda_n^{-1})^{-1}$, then $M_{\Lambda,p}$ and $M_{\Upsilon,p}$ are the isomorphic Banach spaces, where $1\leq p<\infty$.*

Proof. There exist the natural isometric linear embeddings of the Müntz spaces $M_{\Lambda,p}$ and $M_{\Upsilon,p}$ into $M_{\Lambda\cup\Upsilon,p}$. We choose a sequence of sets Υ_k satisfying the following restrictions (1)–(4):

(1) $\Upsilon_k=\{v_{k,n} : n=1,2,...\}\subset\Lambda\cup\Upsilon$ and $v_{k,n}\in\{\lambda_n,v_n\}$ for each $k=0,1,2,...$ and $n=1,2,...$, where $\Upsilon_0=\Lambda$;

(2) $v_{k,n}\leq v_{k+1,n}$ for each $k=0,1,2,...$ and $n=1,2,...$;

(3) $\{\Delta_{k+1,n} : n=1,2,...\}$ is a monotone decreasing subsequence, which may be finite or infinite, having positive terms $\Delta_{k+1,n}$ tending to zero. The terms $\Delta_{k+1,n}$ are obtained from the sequence $\delta_{k+1,j}:=v_{k+1,j}-v_{k,j}$ by elimination of zero terms. Denote by $\theta=\theta_{k+1} : \{j : j\in\mathbf{N},\delta_{k+1,j}\neq 0\}\to\mathbf{N}$ the corresponding enumeration mapping, such that $\Delta_{k+1,\theta(j)}=\delta_{k+1,j}\neq 0$ for each $j\in\mathbf{N}$ is not zero;

(4) $\{m(k+1) : k\}$ is a monotone increasing sequence with $m(k+1):=\min\{n : v_n-v_{k+1,n}\neq 0;\ \forall l<n\ v_l=v_{k+1,l}\}$.

Take an arbitrary function f in $M_{\Upsilon_k,p}$. In view of Theorem 6.2.3 and Corollary 6.2.4 [20], a function f has a power series expansion:

$$f(t)=\sum_{n=1}^{\infty}a_n t^{v_{k,n}} \text{ on } [0,1)$$

where $a_n\in\mathbf{F}$ for each $n\in\mathbf{N}$, where the power series decomposition of f converges for each $0\leq t<1$, since f is analytic on $[0,1)$.

Therefore, for each $f\in M_{\Upsilon_k,p}$, we consider the power series

$f_1(t) = \sum_{n=1}^{\infty} a_n t^{v_{k+1,n}}$. Then, we infer that:

$$f(t^2) - f_1(t^2) = \sum_{n=1}^{\infty} a_n t^{v_{k,n}} u_{\theta(n)}(t) \text{ with } u_{\theta(n)}(t) := t^{v_{k,n}} - t^{v_{k,n}+2\Delta_{k+1,\theta(n)}}$$

so that $u_l(t)$ is a monotone decreasing sequence by l, and hence:

$$|f(t^2) - f_1(t^2)| \leq 2|u_{\theta(m(k+1))}(t)||f(t)|$$

according to Dirichlet's criterion (see, for example, [25]) for each $0 \leq t < 1$, where $\theta = \theta_{k+1}$. Then, we deduce that:

$$\|f - f_1\|_{L_p([0,1],\mathbf{F})} \leq 2^{2+1/p} \|f\|_{L_p([0,1],\mathbf{F})} \Delta_{k+1,\theta(m(k+1))} / \lambda_{m(k+1)} \tag{3}$$

since the mapping $t \mapsto t^2$ is the orientation preserving the diffeomorphism of $[0,1]$ onto itself, also $|u_{m(k+1)}(t)| \leq 2\Delta_{k+1,\theta(m(k+1))}/\lambda_{m(k+1)}$ for each $0 \leq t \leq 1$ by Lemma 7.3.1 [20] and:

$$\begin{aligned} \|f - f_1\|_{L_p([0,1],\mathbf{F})} &= [\int_0^1 |f(\tau) - f_1(\tau)|^p d\tau]^{1/p} \\ &= [2\int_0^1 |f(t^2) - f_1(t^2)|^p t dt]^{1/p} \leq [2^{p+1} \int_0^1 |u_{m(k+1)}(t)|^p |f(t)|^p t dt]^{1/p} \\ &\leq 2^{2+1/p} [\int_0^1 |f(t)|^p dt]^{1/p} \Delta_{k+1,\theta(m(k+1))} / \lambda_{m(k+1)} \end{aligned}$$

Thus, the series $\sum_{n=1}^{\infty} a_n t^{v_{k+1,n}}$ converges on $[0,1)$, and the function $f_1(t)$ is analytic on $[0,1)$.

Inequality (3) implies that the linear isomorphism T_k of $M_{\Upsilon_k,p}$ with $M_{\Upsilon_{k+1},p}$ exists, such that $\|T_k - I\| \leq 2^{2+1/p} \Delta_{k+1,\theta(m(k+1))}/\lambda_{m(k+1)}$, $T_k : M_{\Upsilon_k,p} \to M_{\Upsilon_{k+1},p}$. Then, we take the sequence of operators $S_n := T_n T_{n-1} ... T_0 : M_{\Lambda,p} \to M_{\Upsilon_{n+1},p} \subset M_{\Lambda \cup \Upsilon,p}$. The space $M_{\Lambda\cup\Upsilon,p}$ is complete, and the operator sequence $\{S_n : n\}$ converges relative to the operator norm to an operator $S : M_{\Lambda,p} \to M_{\Lambda\cup\Upsilon,p}$, so that $\|S - I\| < 1$, since:

$$\sum_{k=0}^{\infty} \Delta_{k+1,\theta(m(k+1))}/\lambda_{m(k+1)} \leq \delta \sum_{n=1}^{\infty} \lambda_n^{-1} < 1/8$$

and $p \geq 1$, where I denotes the unit operator. Therefore, the operator S is invertible. On the other hand, from Conditions (1)–(4), it follows that $S(M_{\Lambda,p}) = M_{\Upsilon,p}$. $\square$

3. Approximation in Müntz L_p Spaces

Now, we recall necessary definitions and notations of the Fourier approximation theory and then present useful lemmas.

Notation 2. *Suppose that $Q = (q_{n,k})$ is a lower triangular infinite matrix with real matrix elements $q_{n,k}$ satisfying the restrictions: $q_{n,k} = 0$ for each $k > n$, where k, n are nonnegative integers. To each one-periodic function $f : \mathbf{R} \to \mathbf{R}$ in the space $L_p((\alpha, \alpha+1), \mathbf{F})$ or in $C_0([\alpha, \alpha+1], \mathbf{F}) := \{f : f \in C([\alpha, \alpha+1], \mathbf{F}), f(\alpha) = f(\alpha+1)\}$ is posed a trigonometric polynomial:*

$$U_n(f, x, Q) := \frac{a_0}{2} q_{n,0} + \sum_{k=1}^{n} q_{n,k}(a_k \cos(2\pi kx) + b_k \sin(2\pi kx)) \tag{4}$$

where $a_k = a_k(f)$ and $b = b_k(f)$ are the Fourier coefficients of a function $f(x)$.

For measurable one-periodic functions h and g, their convolution is defined whenever it exists by the formula:

$$(h * g)(x) := 2\int_{\alpha}^{\alpha+1} h(x-t)g(t)dt \tag{5}$$

Putting the kernel of the operator U_n as:

$$U_n(x,Q) := \frac{q_{n,0}}{2} + \sum_{k=1}^{n} q_{n,k}\cos(2\pi kx) \tag{6}$$

we get:

$$U_n(f,x,Q) = (f * U_n(,Q))(x) = (U_n(,Q) * f)(x) \tag{7}$$

The norms of these operators are:

$$L_n(Q,E) := \sup_{f\in E,\ \|f\|_E=1} \|U_n(f,x,Q)\|_E \tag{8}$$

which are constants of a summation method, where $\| * \|_E$ denotes a norm on a Banach space E, where either $E = C_0([\alpha,\alpha+1],\mathbf{F})$ or $E = L_p((\alpha,\alpha+1),\mathbf{F})$ with $1 \leq p < \infty$, while $\alpha \in \mathbf{R}$ is a marked real number.

Henceforward, the Fourier summation methods prescribed by sequences of operators $\{U_m : m\}$ that converge on E:

$$\lim_{m\to\infty} U_m(f,x,Q) = f(x) \tag{9}$$

in the E norm will be considered.

Henceforth, F denotes the set of all pairs (ψ,β) satisfying the conditions: $(\psi(k) : k \in \mathbf{N})$ is a sequence of non-zero numbers for which $\lim_{k\to\infty}\psi(k) = 0$ the limit is zero, β is a real number and also:

$$\mathcal{D}_{\psi,\beta}(x) := \sum_{k=1}^{\infty} \psi(k)\cos(2\pi kx + \beta\pi/2) \tag{10}$$

is the Fourier series of some function from $L_1[0,1]$. By F_1 is denoted the family of all positive sequences $(\psi(k) : k \in \mathbf{N})$ tending to zero with $\Delta_2\psi(k) := \psi(k-1) - 2\psi(k) + \psi(k+1) \geq 0$ for each k so that the series:

$$\sum_{k=1}^{\infty} \frac{\psi(k)}{k} < \infty \tag{11}$$

converges. The set of all downward convex functions $\psi(v)$ for each $v \geq 1$, so that $\lim_{v\to\infty}\psi(v) = 0$ is denoted by $\mathcal{M}$, while $\mathcal{M}_1$ is its subset of functions satisfying Condition (11).

Then:

$$\rho_n(f,x) := f(x) - S_{n-1}(f,x) \tag{12}$$

is the approximation precision of f by the Fourier series $S(f,x)$, where:

$$S_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k\cos(2\pi kx) + b_k\sin(2\pi kx)) \tag{13}$$

is the partial Fourier sum approximating a Lebesgue integrable one-periodic function f on $(0,1)$.

Definition 2. *Suppose that $f \in L_1(\alpha, \alpha+1)$ and $S[f]$ is its Fourier series with coefficients $a_k = a_k(f)$ and $b_k = b_k(f)$, while $\psi(k)$ is an arbitrary sequence that is real or complex. If the function:*

$$D^{\psi}_{\beta} f := f^{\psi}_{\beta} := \sum_{k=1}^{\infty} [a_k(f)\cos(2\pi kx + \beta\pi/2) + b_k(f)\sin(2\pi kx + \beta\pi/2)]/\psi(k)$$

belongs to the space $L(\alpha, \alpha+1)$ of all Lebesgue integrable (summable) functions on $(\alpha, \alpha+1)$, then f^{ψ}_{β} is called the Weil (ψ, β) derivative of f. Then, $L^{\psi}_{\beta} = L^{\psi}_{\beta}(\alpha, \alpha+1)$ stands for the family of all functions $f \in L(\alpha, \alpha+1)$ with $f^{\psi}_{\beta} \in L(\alpha, \alpha+1)$; we also put $L^{\psi}_{\beta,p} := \{f : f \in L^{\psi}_{\beta}, \ \|f^{\psi}_{\beta}\|_{L_p(\alpha,\alpha+1)} \leq 1\}$. Particularly, for $\psi(k) = k^{-r}$, this space L^{ψ}_{β} is Weil–Nagy's class $W^r_{\beta} = W^r_{\beta}(\alpha, \alpha+1)$, and the notation $W^r_{\beta,p}$ can be used instead of $L^{\psi}_{\beta,p}$ in this case. Put particularly $W^r_{\beta} L_p(\alpha, \alpha+1) := \{f : f \in L_p(\alpha, \alpha+1), \exists f^{\psi}_{\beta} \in L_p(\alpha, \alpha+1)\}$, where $1 < p < \infty$.

Then, let $\mathcal{E}_n(X) := \sup\{\|\rho_n(f;x)\|_{L_p(\alpha,\alpha+1)} : f \in X\}$

$$E_n(f)_p := \inf\{\|f - T_{n-1}\|_{L_p(\alpha,\alpha+1)} : T_{n-1} \in \mathcal{T}_{2n-1}\}$$

$E_n(X) := \sup\{E_n(f)_p : f \in X\}$, where X is a subset in $L_p(\alpha, \alpha+1) = L_p((\alpha, \alpha+1), \mathbf{R})$

$$\mathcal{T}_{2n-1} := \{T_{n-1}(x) = \frac{c_0}{2} + \sum_{k=1}^{n-1}(c_k \cos(2\pi kx) + d_k \sin(2\pi kx)); \ c_k, d_k \in \mathbf{R}\}$$

denotes the family of all trigonometric polynomials T_{n-1} of a degree not greater than $n-1$.

Lemma 3. *Suppose that $Q_{\alpha} f(t) := f(t^{\alpha})$ for each $f : [0,1] \to \mathbf{F}$, where $0 < \alpha$, $t \in [0,1]$, $1 < p < \infty$. Then, for each $1 < \alpha < \infty$, there exists $0 < \delta < 1$, such that the operator Q_{α} from $L_p(\delta^{\alpha}, 1)$ into $L_p(\delta, 1)$ has the norm $\|Q_{\alpha}\| < 1$.*

Proof. The Banach spaces $L_p(\delta, 1)$ and $L_p(\delta^{\alpha}, 1)$ are defined with the help of the Lebesgue measure on $\mathbf{R}$. Then, Equation 2(2) implies that $\|Q_{\alpha}\| < 1$ as soon as $\alpha^{-1}\max(1, \delta^{(1-\alpha)}) < 1$. That is, when $\{\delta^{(1-\alpha)} < \alpha\} \iff \{\ln\delta > (1-\alpha)^{-1}\ln\alpha\}$, since $\alpha > 1$ and $0 < \delta < 1$. □

Corollary 1. *Let $1 < \alpha < \infty$ and $0 < \delta < 1$, so that $\delta > \alpha^{1/(1-\alpha)}$; let also $Z_{\Lambda,p,\alpha,\delta} := (I - Q_{\alpha})[M_{\Lambda,p}(\delta^{\alpha}, 1)]$, where $1 < p < \infty$, while I is the unit operator. Then, $Z_{\Lambda,p,\alpha,\delta}$ is isomorphic with $M_{\Lambda,p}(\delta^{\alpha}, 1)$.*

Proof. There is the natural embedding of $L_p(a,b)$ into $L_p(c,d)$ when $c \leq a$ and $b \leq d$, such that $f \mapsto f\chi_{(a,b)}$ for each $f \in L_p(a,b)$, where χ_A notates the characteristic function of a set A. Since $\|Q_{\alpha}\| < 1$, then the operator $I - Q_{\alpha}$ is invertible (see [26]).

Lemma 4. *Let $f \in L_p(0,1)$, where $1 < p < \infty$. Then:*

$$\lim_{\eta \downarrow 0} \eta^{-1/q} \int_{1-\eta}^{1} f(t)dt = 0$$

where $1/q + 1/p = 1$.

Proof. Since $f \in L_p(0,1)$, then $|f(t)|^p \mu(dt)$ is a σ-additive and finite measure on $(0,1)$, where μ is the Lebesgue measure on $\mathbf{R}$ (see, for example, [27], Theorems V.5.4.3 and V.5.4.5 [26]). Therefore, the limit exists:

$$\lim_{\eta \downarrow 0} \int_{1-\eta}^{1} |f(t)|^p dt = 0 \tag{14}$$

□

From Holder's inequality, it follows that:

$$|\int_{1-\eta}^{1} f(t)dt| \leq (\int_{1-\eta}^{1} |f(t)|^p dt)^{1/p} (\int_{1-\eta}^{1} 1dt)^{1/q} = \eta^{1/q} (\int_{1-\eta}^{1} |f(t)|^p dt)^{1/p}$$

hence:

$$|\eta^{-1/q} \int_{1-\eta}^{1} f(t)dt| \leq (\int_{1-\eta}^{1} |f(t)|^p dt)^{1/p} \tag{15}$$

Thus, from Equations (14) and (15), the statement of this lemma follows. □

Note: We remind about the following definition: the family of all Lebesgue measurable functions $f : (a,b) \to \mathbf{R}$ satisfying the condition:

$$\|f\|_{L_{s,w}(a,b)} := \sup_{y>0} (y^s \mu\{t : \ t \in (a,b), \ |f(t)| \geq y\})^{1/s} < \infty$$

is called the weak L_s space and denoted by $L_{s,w}(a,b)$, where μ notates the Lebesgue measure on the real field $\mathbf{R}$, $0 < s < \infty$, $(a,b) \subset \mathbf{R}$ (see, for example, §9.5 [27], §IX.4 [28,29]).

The following Proposition 1 is used below in Theorem 2 to prove that functions of Müntz spaces $M_{\Lambda,p}$ for Λ satisfying the Müntz condition and the gap condition belong to Weil–Nagy's class, where $1 < p < \infty$.

Proposition 1. *Suppose that an increasing sequence $\Lambda = \{\lambda_n : n\}$ of natural numbers satisfies the Müntz condition, $1 < p < \infty$ and $f \in M_{\Lambda,p}$. Then, $dh(x)/dx \in L_{s,w}(0,1)$ for a function $h(x) = f(x) - f(x^2)$, where $s = p/(p+1)$.*

Proof. In view of Theorem 6.2.3 and Corollary 6.2.4 [20], a function f is analytic on $(0,1)$, and consequently, h is analytic on $(0,1)$; hence, a derivative $dh(x)/dx$ is also analytic on $(0,1)$. Moreover, the series:

$$f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n} \tag{16}$$

converges on $\dot{B}_1(0)$, where $\dot{B}_r(x) := \{y : y \in \mathbf{C}, |y - x| < r\}$ denotes the open disk in $\mathbf{C}$ of radius $r > 0$ with center at $x \in \mathbf{C}$, where $a_n \in \mathbf{F}$ is an expansion coefficient for each $n \in \mathbf{N}$. That is, the functions f and h have holomorphic univalent extensions on $\dot{B}_1(0)$, since $\Lambda \subset \mathbf{N}$ (see Theorem 20.5 in [30]). Take the function $H(x) = \int_x^1 h(t)dt$, where $x \in [0,1]$. In virtue of Theorem VI.4.2 [26] and Lyapunov's inequality (Equation (27) in §II.6 [31]), this function is continuous, so that $H(1) = 0$. Together with Equation (1), this implies that the function $H(x)$ belongs to $M_{\{0\}\cup(\Lambda+1),C}$ and has a holomorphic univalent extension on $\dot{B}_1(0)$.

Then, we put $g(z) = (1-z)^{-1/q} H(z)$ for each $z \in \dot{B}_1(0)$, where $1/q + 1/p = 1$. From Lemma 4, it follows that:

$$\lim_{z \to 1} g(z) = 0 \tag{17}$$

Thus, the function $g(z)$ is holomorphic (may be multivalent because of the multiplier $(1-z)^{-1/q}$) on $\dot{B}_1(0)$ and continuous on $\dot{B}_1(0) \cup \{1\}$.

According to Cauchy's Equation 21(5) in [30]:

$$h'(z) = -\frac{1}{\pi i} \int_{\gamma} \frac{H(y)}{(y-z)^3} dy \tag{18}$$

for each $z \in \dot{B}_{1/2}(1/2)$, where γ is an oriented rectifiable boundary $\gamma = \partial G$ of a simply-connected open domain G contained in $\dot{B}_{1/2}(1/2)$, such that $z \in G$. Particularly, this is valid for each z in $(1/2, 1)$ and $G = \dot{B}_{1/2}(1/2)$.

On the other hand, the function $g(z)$ is bounded on $B_{1/2}(1/2)$, where $B_r(x) := \{y : y \in \mathbf{C}, |y - x| \leq r\}$ notates the closed disk of radius $r > 0$ with the center at $x \in \mathbf{C}$. Thus, $K = \sup_{z \in B_{1/2}(1/2)} |g(z)| < \infty$. Estimating the integral (18) and taking into account Equation (17), we infer that $|h'(t)| \leq 2K/(1-t)^{1+1/p}$ for each $t \in (3/4, 1)$, since $1/q + 1/p = 1$. Together with the analyticity of h' on $[0, 1)$ this implies that:

$$\sup_{y>0} (y^s \mu\{t : t \in (a,b), |h'(t)| \geq y\})^{1/s} < \infty$$

where $s = p/(p+1)$. Thus, $h' \in L_{s,w}(0, 1)$. $\square$

Theorem 2. *Let an increasing sequence $\Lambda = \{\lambda_n : n\}$ of natural numbers satisfy the Müntz condition, also $1 > \delta > 1/2$ and $1 < p < \infty$, and let $\sigma(x) = \delta^2 + x(1 - \delta^2)$, where $0 \leq x \leq 1$. Then, for each $0 < \gamma < 1$, there exists $\beta = \beta(\gamma) \in \mathbf{R}$, so that $Z_{\Lambda,p,2,\delta} \circ \sigma \subset W^{\gamma}_{\beta} L_p(0, 1)$.*

Proof. Let $f \in M_{\Lambda,p}(0, 1)$ and $v(x) = (I - Q_2) f(\sigma(x))$, then $v(x)$ is analytic on $(0, 1)$, since f is analytic on $(0, 1)$ and $\sigma[0, 1] = [\delta^2, 1]$. We take its one-periodic extension v_0 on $\mathbf{R}$.

According to Proposition 1.7.2 [32] (or see [33]), $h \in W^{\gamma}_{\beta} L_p(0, 1)$ if and only if there exists a function $\phi = \phi_{h,\gamma,\beta}$, which is one-periodic on $\mathbf{R}$, and Lebesgue integrable on $[0, 1]$, such that:

$$h(x) = \frac{a_0(h)}{2} + (\phi * \mathcal{D}_{\psi,\beta})(x) \tag{19}$$

where $a_0(h) = 2 \int_0^1 h(t) dt$ (see Notation 2 and Definition 2).

We take a sequence $U_n(t, Q)$ given by Equation (6), so that:

$$\lim_m q_{m,k} = 1 \text{ for each } k \text{ and } \sup_m L_m(Q, L_p) < \infty \text{ and } \sup_{m,k} |q_{m,k}| < \infty$$

and write for short $U_n(t)$ instead of $U_n(t, Q)$. Under these conditions, the limit exists:

$$\lim_n (v * U_n)(x) = v(x) \tag{20}$$

in $L_p(0, 1)$ norm for each $v \in L_p((0, 1), \mathbf{F})$ according to Chapters 2 and 3 in [32] (see also [15,33]).

On the other hand, Equation I(10.1) [32] provides:

$$S[(y^{\psi_1}_{\bar{\beta}_1})^{\psi_2/\psi_1}_{\bar{\beta}_2 - \bar{\beta}_1}] = S[y^{\psi_2}_{\bar{\beta}_2}] \tag{21}$$

where $S[y]$ is the Fourier series corresponding to a function $y \in L^{\psi_2}_{\bar{\beta}_2}$, when $(\psi_1, \bar{\beta}_1) \leq (\psi_2, \bar{\beta}_2)$.

Put $\theta(k) = k^{\gamma - 1}$ for all $k \in \mathbf{N}$. Then, $\mathcal{D}_{\theta,-\beta} \in L_1(0, 1)$ for each $\beta \in \mathbf{R}$ due to Theorems II.13.7, V.1.5 and V.2.24 [33] (or see [15]). This is also seen from Chapters I and V in [32] and Equations (19) and (21) above. In view of Dirichlet's theorem (see §430 in [25]), the function $\mathcal{D}_{\theta,-\beta}(x)$ is continuous on the segment $[\delta, 1 - \delta]$ for each $0 < \delta < 1/4$.

According to Equation 2.5.3.(10) in [34]:

$$\int_0^\infty x^{\alpha - 1} \begin{pmatrix} \sin(bx) \\ \cos(bx) \end{pmatrix} dx = b^{-\alpha} \Gamma(\alpha) \begin{pmatrix} \sin(\pi\alpha/2) \\ \cos(\pi\alpha/2) \end{pmatrix}$$

for each $b > 0$ and $0 < Re(\alpha) < 1$. On the other hand, the integration by parts gives:

$$\int_a^\infty x^{\alpha-1}\begin{pmatrix}\sin(bx)\\ \cos(bx)\end{pmatrix}dx = b^{-1}a^{\alpha-1}\begin{pmatrix}\cos(ab)\\ -\sin(ab)\end{pmatrix} - b^{-1}(\alpha-1)\int_a^\infty x^{\alpha-2}\begin{pmatrix}-\cos(bx)\\ \sin(bx)\end{pmatrix}dx$$

for every $a > 0$, $b > 0$ and $0 < Re(\alpha) < 1$. From Equation V(2.1), Theorems V.2.22 and V.2.24 in [33] (see also [18,35]), we infer the asymptotic expansions:

$$\sum_{n=1}^{\infty} n^{-\alpha}\sin(2\pi nx) \approx (2\pi x)^{\alpha-1}\Gamma(1-\alpha)\cos(\pi\alpha/2) + \mu x^\alpha$$

$$\sum_{n=1}^{\infty} n^{-\alpha}\cos(2\pi nx) \approx (2\pi x)^{\alpha-1}\Gamma(1-\alpha)\sin(\pi\alpha/2) + \nu x^\alpha$$

in a small neighborhood $0 < x < \delta$ of zero, where $0 < \delta < 1/4$, $0 < \alpha < 1$, μ and ν are real constants. Taking $\beta = \alpha = 1-\gamma$, we get that $\mathcal{D}_{\theta,-\beta}(x) \in L_\infty(0,1)$.

Evidently, for Lebesgue measurable functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$, there is the equality $\int_{-\infty}^{\infty} f(x-t)\chi_{[0,\infty)}(x-t)g(t)\chi_{[0,\infty)}(t)dt = \int_0^x f(x-t)g(t)dt$ for each $x > 0$ whenever this integral exists, where χ_A denotes the characteristic function of a subset A in $\mathbf{R}$, such that $\chi_A(y) = 1$ for each $y \in A$, also $\chi_A(y) = 0$ for each y outside A, $y \in \mathbf{R} \setminus A$. Particularly, if $0 < x \leq T$, where $0 < T < \infty$ is a constant, then $\int_0^x f(x-t)g(t)dt = \int_0^\infty f(x-t)\chi_{[0,T]}(x-t)g(t)\chi_{[0,T]}(t)dt$ (see also [25,26]). This is applicable to Equation (5) putting $\alpha = 0$ there and with the help of the equality:

$$\int_0^1 f(x-t)g(t)dt = \int_0^x f(x-t)g(t)dt + \int_0^{1-x} f_1((1-x)-v)g_1(v)dv$$

for each $0 \leq x \leq 1$ and one-periodic functions f and g and using also that $\|f|_{[a,b]}\| \leq \|f|_{[0,1]}\| = \|f_1|_{[0,1]}\|$ for the considered types here of norms for each $[a,b] \subset [0,1]$, where $f_1(t) = f(-t)$ and $g_1(t) = g(-t)$ for each $t \in \mathbf{R}$, since:

$$\int_x^1 f(x-t)g(t)dt = \int_0^{1-x} f(v-1+x)g(1-v)dv$$

We mention that according to the weak Young inequality:

$$\|\xi * \eta\|_p \leq K_{r,s}\|\xi\|_r\|\eta\|_{s,w} \tag{22}$$

for each $\xi \in L_r$ and $\eta \in L_{s,w}$, where $1 \leq p, r \leq \infty$, $0 < s < \infty$ and $r^{-1} + s^{-1} = 1 + p^{-1}$, $K_{r,s} > 0$ is a constant independent of ξ and η (see Theorem 9.5.1 in [27], §IX.4 in [28]).

By virtue of Equation (21), the weak Young inequality (22) and Proposition 1, there exists a function s in $L_p(0,1)$, so that:

$$s(x) = \lim_n((\mathcal{D}_{\theta,-\beta} * U_n) * v_0')(x)$$

where $\beta = 1-\gamma$. Therefore, $\phi_{v_0,\gamma,\beta} = s$ and $D^\psi_\beta v_0 = s$ according to Equations (20) and (22). Thus, $v_0 \in W^\gamma_\beta L_p(0,1)$. $\square$

Below, Lemma 5 and Proposition 2 are given. They are used in Theorem 3 for proving the existence of a Schauder basis. On the other hand, Theorem 2 is utilized to prove Lemma 5.

Lemma 5. *If an increasing sequence Λ of natural numbers satisfies the Müntz condition, also $0 < \gamma < 1$ and $1 < p < \infty$, $1 > \delta > 1/2$:*

$$X = \{h : \ h = f \circ \sigma, f \in Z_{\Lambda,p,2,\delta}; \|f\|_{L_p((\delta^2,1),\mathbf{R})} \leq 1\}$$

then a positive constant $\omega = \omega(p, \gamma)$ exists, so that:

$$E_n(X) \leq \mathcal{E}_n(X) \leq \omega n^{-\gamma} \quad (23)$$

for each natural number $n \in \mathbf{N}$.

Proof. By virtue of Theorem 2, the inclusion $h(x) \in W^{\gamma}_{\beta} L_p(0,1)$ is valid for each $h \in Z_{\Lambda,p,2,\delta} \circ \sigma$, where ψ is in F_1 with $\psi(k) = k^{-\gamma}$ for each $k \in \mathbf{N}$, $\beta = 1-\gamma$. Then, $\|h\|_{L_p((0,1),\mathbf{R})} = (1-\delta^2)^{-1/p}\|f\|_{L_p((\delta^2,1),\mathbf{R})} \leq (1-\delta^2)^{-1/p}$ for each $h \in X$, since:

$$\int_0^1 |h(x)|^p dx = (1-\delta^2)^{-1} \int_{\delta^2}^1 |f(t)|^p dt \quad (24)$$

□

Therefore, $X \subset (1-\delta^2)^{-1/p} W^{\gamma}_{\beta} L_p(0,1)$ (see also §7), where $bY := \{f : f = bg, g \in Y\}$ for a linear space Y over $\mathbf{R}$ and a marked real number b.

Then, the estimate (23) follows from Theorem V.5.3 in [32].

4. Existence of Schauder Basis

Proposition 2. *Let X be a Banach space over $\mathbf{R}$, and let Y be its Banach subspace, so that they fulfill the conditions* (1–5) *below:*

(1) *there is a sequence $(e_i : i \in \mathbf{N})$ in X, such that $e_1, ..., e_n$ are linearly independent vectors and $\|e_n\|_X = 1$ for each n and;*

(2) *there exists a Schauder basis $(z_n : n \in \mathbf{N})$ in X, such that:*

$$z_n = \sum_{k=1}^{n} b_{k,n} e_k$$

for each $n \in \mathbf{N}$, where $b_{k,n}$ are real coefficients;

(3) *for every $x \in Y$ and $n \in \mathbf{N}$, there exist $x_1, ..., x_n \in \mathbf{R}$, so that:*

$$\|x - \sum_{i=1}^{n} x_i e_i\|_X \leq s(n)\|x\|$$

where $s(n)$ is a strictly monotone decreasing positive function with $\lim_{n\to\infty} s(n) = 0$ and,

(4)

$$u_n = \sum_{l=m(n)}^{k(n)} u_{n,l} e_l$$

where $u_{n,l} \in \mathbf{R}$ for each natural numbers k and l, where a sequence $(u_n : n \in \mathbf{N})$ of normalized vectors in Y is such that its real linear span is everywhere dense in Y and $1 \leq m(n) \leq k(n) < \infty$ and $m(n) < m(n+1)$ for each $n \in \mathbf{N}$;

(5) *vectors u_1,...,u_n are linearly independent in Y for each $n \in \mathbf{N}$.*

Then, Y has a Schauder basis.

Proof. The real linear span $span_{\mathbf{R}}(u_1, ..., u_n)$ is complemented in Y for each $n \in \mathbf{N}$ due to Theorem (8.4.8) in [4]. Put $L_{n,\infty} := cl_X span_{\mathbf{R}}(u_k : k \geq n)$ and $L_{n,m} := cl_X span_{\mathbf{R}}(u_k : n \leq k \leq m)$, where $cl_X A$ denotes the closure of a subset A in X, where $span_{\mathbf{R}} A$ denotes the real linear span of A. Since Y is a Banach space and $u_k \in Y$ for each k, then $L_{n,\infty} \subset Y$ and $L_{n,m} \subset Y$ for each natural number n and m. Then, we infer that:

$$L_{n,j} \subset span_{\mathbf{R}}(e_l : m(n) \leq l \leq k_{n,j})$$

where $k_{n,j} := \max(k(l) : n \leq l \leq j)$.

Take arbitrary vectors $f \in L_{1,j}$ and $g \in L_{j+1,q}$, where $1 \leq j < q$. Therefore, there are real coefficients f_i and g_i, such that:

$$f = \sum_{i=1}^{k_{1,j}} f_i e_i$$

and:

$$g = \sum_{i=m(j+1)}^{k_{j+1,q}} g_i e_i$$

Hence, due to Condition (2):

$$\|f - \sum_{i=1}^{m(j)} f_i e_i\|_X \leq s(m(j))\|f\|$$

and:

$$\|g - \sum_{i=k_{1,j}+1}^{k_{j+1,q}} g_i e_i\|_X \leq s(k_{1,j}+1)\|g\|_X$$

On the other hand:

$$f = \sum_{i=1}^{m(j)} f_i e_i + \sum_{i=m(j)+1}^{k_{1,j}} f_i e_i$$

consequently:

$$\|f^{[j+1]}\| \leq s(m(j+1)))\|f\|$$

where $f^{[j+1]} := \sum_{i=m(j+1)}^{k_{1,j}} f_i e_i$ and $\sum_{i=a}^{b} f_i e_i := 0$, when $a > b$.

When $0 < \delta < 1/4$ and $s(m(j)+1) < \delta$, we infer using the triangle inequality that $\|f^{[j+1]} - h\|_X \leq \delta\|f^{[j+1]}\|_X/(1-\delta) \leq \delta s(m(j+1)-1)\|f\|_X/(1-\delta)$ for the best approximation h of $f^{[j+1]}$ in $L_{j+1,\infty}$, since $m(j) < m(j+1)$ for each j. Therefore, the inequality $\|f - g\|_X \geq \|f - f^{[j+1]}\|_X - \|f^{[j+1]} - g\|_X$ and $s(n) \downarrow 0$ imply that there exists n_0, such that the inclination of $L_{1,j}$ to $L_{j+1,\infty}$ is not less than $1/2$ for each $j \geq n_0$. Condition (4) implies that L_{1,n_0} is complemented in Y. By virtue of Theorem 1.2.3 [20], the Schauder basis exists in Y. □

Theorem 3. *If a set Λ satisfies the Müntz and gap conditions and $1 < p < \infty$, then the Müntz space $M_{\Lambda,p}([0,1],\mathbf{F})$ has a Schauder basis.*

Proof. In view of Lemma 2 and Theorem 1, it is sufficient to prove the existence of a Schauder basis in the Müntz space $M_{\Lambda,p}$ for $\Lambda \subset \mathbf{N}$. We mention that if the Müntz space $M_{\Lambda,p}([0,1],\mathbf{R})$ over the real field has the Schauder basis, then $M_{\Lambda,p}([0,1],\mathbf{C})$ over the complex field has it as well. Thus, it is sufficient to consider the real field $\mathbf{F} = \mathbf{R}$.

Let $U_m(x,Q)$ be kernels of the Fourier summation method in $L_p(0,1)$ as in Notation 2, such that:

(1) $\lim_m q_{m,k} = 1$ for each k and $\sup_m L_m(Q,L_p) < \infty$ and $\sup_{m,k} |q_{m,k}| < \infty$.

For example, Cesaro's summation method of order one can be taken, to which Fejér kernels F_n correspond, so that the limit:

$$\lim_{n\to\infty} F_n * f = f$$

converges in $L_p(0,1)$ (see Theorem 19.1 and Corollary 19.2 in [36]). That is, there exists the Schauder basis z_n in $L_p(0,1)$, such that:

$$z_{2n}(t) = a_{0,2n} + [\sum_{k=1}^{n-1}(a_{k,2n}\cos(2\pi kt) + b_{k,2n}\sin(2\pi kt)] + a_{n,2n}\cos(2\pi nt)$$

and:

$$z_{2n+1}(t) = a_{0,2n+1} + \sum_{k=1}^{n} (a_{k,2n+1}\cos(2\pi kt) + b_{k,2n+1}\sin(2\pi kt))$$

for every $t \in (0,1)$ and $n \in \mathbf{N}$, where $a_{k,j}$ and $b_{k,j}$ are real expansion coefficients.

By virtue of Theorem 6.2.3 and Corollary 6.2.4 [20], each function $g \in M_{\Lambda,p}[0,1]$ has an analytic extension on $\dot{B}_1(0)$, and hence:

(2)

$$g(z) = \sum_{n=1}^{\infty} c_n z^{\lambda_n} = \sum_{k=1}^{\infty} p_k u_k(z)$$

are the convergent series on the unit open disk $\dot{B}_1(0)$ in $\mathbf{C}$ with the center at zero (see Proposition 1), where $\Lambda \subset \mathbf{N}$ and $c_n = c_n(g) \in \mathbf{N}$, $p_n = p_n(g) = c_1 + ... + c_n$, $u_1(z) := z^{\lambda_1}$, $u_{n+1}(z) := z^{\lambda_{n+1}} - z^{\lambda_n}$ for each $n = 1, 2, ...$ On the other hand, the Müntz spaces $M_{\Lambda,p}[0,1]$ and $M_{\Lambda,p}[\delta^2, 1]$ are isomorphic for each $0 < \delta < 1$ (see Lemma 1 above). Therefore, we consider henceforward the Müntz space $M_{\Lambda,p}$ on the segment $[\delta^2, 1]$, where $1 > \delta > 1/2$. We mention that $M_{\Lambda,p}[\delta^2, 1]$ and $M_{\Lambda,p} \circ \sigma[0,1]$ are isomorphic (see Theorem 2). Then, $Z_{\Lambda,p,2,\delta}$ and $Z_{\Lambda,p,2,\delta} \circ \sigma|_{[0,1]}$ are isomorphic, as well. In view of Corollary 1, it is sufficient to prove the existence of a Schauder basis in $Z_{\Lambda,p,2,\delta} \circ \sigma|_{[0,1]}$.

Take the finite dimensional subspace $X_n := span_{\mathbf{R}}(u_1, ..., u_n)$ in $M_{\Lambda,p}$, where $n \in \mathbf{N}$. Due to Lemma 2, the Banach space $M_{\lambda,p} \ominus X_n$ exists and is isomorphic with $M_{\lambda,p}$. By virtue of Equation I(10.1) [32] $S[(y^{\psi_1}_{\bar{\beta}_1})^{\psi_2/\psi_1}_{\bar{\beta}_2-\bar{\beta}_1}] = S[y^{\psi_2}_{\bar{\beta}_2}]$, where $y \in L^{\psi_2}_{\bar{\beta}_2}$, when $(\psi_1, \bar{\beta}_1) \le (\psi_2, \bar{\beta}_2)$.

Consider the trigonometric polynomials $U_m(f, x, Q)$ for $f \in (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$, where $m = 1, 2, ...$ Put $Y_{K,n}$ as the L_p completion of the linear span $span_{\mathbf{R}}(U_m(f, x, Q) : (m, f) \in K)$, where $K \subset \mathbf{N} \times (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$, $m \in \mathbf{N}$, $f \in (Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_n) \circ \sigma$.

It is known (see Proposition 1.7.1 [32]) that $f \in L^{\psi}_{\beta}(\alpha, \alpha + 1)$ if and only if there exists $g \in L(\alpha, \alpha + 1)$, so that $f = \frac{a_0(f)}{2} + \mathcal{D}_{\psi,\beta} * g$, where the function $\mathcal{D}_{\psi,\beta}$ is prescribed by Equation (10); the constant $a_0(f)$ is as above. In view of Lemma 2, it is sufficient to consider the case $a_0(f) = 0$.

There exists a countable subset $\{f_n : n \in \mathbf{N}\}$ in $Z_{\Lambda,p,2,\delta}$, such that $f_n \circ \sigma = \mathcal{D}_{\psi,\beta} * g_n$ with $g_n \in L(0,1)$ for each $n \in \mathbf{N}$ and so that $span_{\mathbf{R}}\{f_n : n \in \mathbf{N}\}$ is dense in $Z_{\Lambda,p,\alpha,\delta}$, since $Z_{\Lambda,p,2,\delta}$ is separable. Using Properties (1) and (2) in this proof, Proposition 1 and Lemma 5, we deduce that a countable set K and a sufficiently large natural number n_0 exist, so that the Banach space Y_{K,n_0} is isomorphic with $(Z_{\Lambda,p,2,\delta} \ominus (I - Q_2)X_{n_0})$ and $Y_{K,n_0}|_{(0,1)} \subset W^{\gamma}_{\beta} L_p(0,1)$, where $0 < \gamma < 1$ and $\beta = 1 - \gamma$. Therefore, by the construction above, the Banach space Y_{K,n_0} is the L_p completion of the real linear span of a countable family $(s_l : l \in \mathbf{N})$ of trigonometric polynomials s_l.

Without loss of generality, this family can be refined by induction, such that s_l is linearly independent of $s_1, ..., s_{l-1}$ over $\mathbf{F}$ for each $l \in \mathbf{N}$. With the help of transpositions in the sequence $\{s_l : l \in \mathbf{N}\}$, the normalization and the Gaussian exclusion algorithm, we construct a sequence $\{r_l : l \in \mathbf{N}\}$ of trigonometric polynomials that are finite real linear combinations of the initial trigonometric polynomials $\{s_l : l \in \mathbf{N}\}$ and satisfying the conditions:

(3) $\|r_l\|_{L_p(0,1)} = 1$ for each l

(4) the infinite matrix having the l-th row of the form $..., a_{l,k}, b_{l,k}, a_{l,k+1}, b_{l,k+1}, ...$ for each $l \in \mathbf{N}$ is upper trapezoidal (step), where:

$$r_l(x) = \frac{a_{l,0}}{2} + \sum_{k=m(l)}^{n(l)} [a_{l,k}\cos(2\pi kx) + b_{l,k}\sin(2\pi kx)]$$

with $a^2_{l,m(l)} + b^2_{l,m(l)} > 0$ and $a^2_{l,n(l)} + b^2_{l,n(l)} > 0$, where $1 \le m(l) \le n(l)$, $deg(r_l) = n(l)$ or $r_1(x) = \frac{a_{1,0}}{2}$ when $deg(r_1) = 0$; $a_{l,k}, b_{l,k} \in \mathbf{R}$ for each $l \in \mathbf{N}$ and $0 \le k \in \mathbf{Z}$.

Then, as X and Y in Proposition 2, we take $X = L_p[0,1]$ and $Y = Y_{K,n_0}$. In view of Proposition 2 and Lemma 2, the Schauder basis exists in Y_{K,n_0} and, consequently, in $M_{\Lambda,p}$, as well. □

5. Conclusions

Besides the applications mentioned in the Introduction, the results of this article can be used, for example, for the analysis of the perturbations of periodic functions to almost periodic functions with the trend [37] and in the analysis of distortions in high-frequency pulse acoustic signals [38] with the help of function approximations.

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