mathematics

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Mixed Order Fractional Differential Equations

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Received: 8 September 2017; Accepted: 31 October 2017; Published: 7 November 2017

Abstract: This paper studies fractional differential equations (FDEs) with mixed fractional derivatives. Existence, uniqueness, stability, and asymptotic results are derived.

Keywords: fractional differential equations (FDEs); Lyapunov exponent; stability

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1. Introduction

Recently, fractional differential equations (FDEs) arise naturally in various fields, such as economics, engineering, and physics. For some existence results of FDEs we refer the reader to [1–6] and the references cited therein.

Bonilla et al. [1] studied linear systems of the same order linear FDEs and obtained an explicit representation of the solution. However, there are very few works on the study of mixed order nonlinear fractional differential equations (MOFDEs), which is a natural extension of [1].

This paper is devoted to the study of MOFDEs of the form

$$D_{0+}^{q_i} x_i = f_i(t, x_1, \cdots, x_n), \quad x_i(0) = u_i, \quad i = 1, \cdots, n \tag{1}$$

where $D_{0+}^{q_i}$ denotes the Caputo derivative with the lower limit at 0, $q_i \in (0,1]$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, specified below and $u_i \in \mathbb{R}^n$. We may suppose $q_1 \geq \cdots \geq q_n$. Here $\mathbb{R}_+ = [0, \infty)$. We are interested in the existence of solutions of (1), then their stability and asymptotic properties under reasonable conditions on f_i. FDEs with equal order (i.e., $q_1 = \cdots = q_n$) are widely studied, and we refer the reader to the basic books describing FDEs, such as [7,8]. On the other hand, there are many MOFDEs with interesting applications—for example, to economic systems in [9]. In fact, (1) formulates a model of the national economies in a case of the study of n commonwealth countries, which cannot be simply divided into clear groups of independent and dependent variables. The purpose of this paper is to set a rigorous theoretical background for (1).

The main contributions are stated as follows:

We give some existence and uniqueness results for solutions of (1) when the nonlinear term satisfies global and local Lipschitz conditions.

We analyze the upper bound for Lyapunov exponents of solutions of (1).

We show that the zero solution of an autonomous version system of (1) is asymptotically stable.

2. Existence Results

First we prove an existence and uniqueness result for globally Lipschitzian $f = (f_1, \cdots, f_n)$. Let $\|x\|_\infty = \max_{i=1,\cdots,n} |x_i|$ for $x = (x_1, \cdots, x_n)$. By $C(J, \mathbb{R}^n)$ we denote the Banach space of all continuous functions from a compact interval $J \subset \mathbb{R}$ to $\mathbb{R}^n$ with the uniform convergence topology on J.

Theorem 1. *Let $T > 0$ and suppose the existence of $L > 0$ such that*

$$\|f(t,x) - f(t,y)\|_\infty \le L\|x-y\|_\infty \quad \forall (t,x),(t,y) \in [0,T]\times\mathbb{R}^n. \tag{2}$$

Then (1) has a unique solution $x \in C(I,\mathbb{R}^n)$, $I = [0,T]$.

Proof of Theorem 1. Note that (2) implies

$$\|f(t,x)\|_\infty \le L\|x\|_\infty + M_f \quad \forall (t,x) \in I\times\mathbb{R}^n \tag{3}$$

for $M_f = \max_{t\in I}\|f(t,0)\|_\infty$. Next, (1) is equivalent to the fixed point problem

$$\begin{gathered} x = F(x,u), \\ x(t) = (x_1(t),\cdots,x_n(t)), \quad u = (u_1,\cdots,u_n), \quad F(x,u) = (F_1(x,u),\cdots,F_n(x,u)), \\ F_i(x,u)(t) = u_i + \frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1} f_i(s,x_1(s),\cdots,x_n(s))ds, \quad i = 1,\cdots,n. \end{gathered} \tag{4}$$

Fix $\alpha \ge 0$ and set $\|x\|_\alpha = \max_{t\in[0,T]}\|x(t)\|_\infty e^{-\alpha t}$ for any $x \in C(I,\mathbb{R}^n)$. Let $x \in C(I,\mathbb{R}^n)$ be a solution of (1). For $\alpha > 0$, (3) and (4) give

$$\begin{gathered} |x_i(t)| \le |u_i| + \frac{1}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}\left(L\|x(s)\|_\infty + M_f\right)ds \\ \le \|u\|_\infty + \frac{M_f T^{q_i}}{\Gamma(q_i+1)} + \frac{L\|x\|_\alpha}{\Gamma(q_i)}\int_0^t (t-s)^{q_i-1}e^{\alpha s}ds \le \|u\|_\infty + \frac{M_f T^{q_i}}{\Gamma(q_i+1)} + \frac{L\|x\|_\alpha}{\alpha^{q_i}}e^{\alpha t}, \end{gathered}$$

which implies

$$\|x\|_\alpha \le \|u\|_\infty + M_f\Theta + \frac{L\|x\|_\alpha}{\alpha^{q_m}}$$

for $\alpha > 1$ and

$$\Theta = \max_{i=1,\cdots,m}\frac{T^{q_i}}{\Gamma(q_i+1)}.$$

Hence

$$\|x\|_\alpha \le \frac{\|u\|_\infty + M_f\Theta}{1-\frac{L}{\alpha^{q_m}}},$$

for

$$\alpha > \max\{1, L^{1/q_m}\}. \tag{5}$$

So

$$|x_i(t)| \le \frac{\|u\|_\infty + M_f\Theta}{1-\frac{L}{\alpha^{q_m}}}e^{\alpha T}, \quad t\in I, \quad i = 1,\cdots,n. \tag{6}$$

Similarly, we derive

$$\|F(x,u) - F(y,u)\|_\alpha \le \frac{L}{\alpha^{q_m}}\|x-y\|_\alpha, \quad \forall x,y \in C(I,\mathbb{R}^n).$$

Consequently, assuming (5), we can apply the Banach fixed point theorem to get a unique solution $x \in C(I,\mathbb{R}^n)$ of (1), which also satisfies (6). The proof is finished. □

Now we prove an existence and uniqueness result for locally Lipschitzian f.

Theorem 2. *Suppose that for any $r > 0$ there is an $L_r > 0$ such that*

$$\|f(t,x) - f(t,y)\|_\infty \le L_r\|x-y\|_\infty \quad \forall (t,x),(t,y) \in [0,T]\times\mathbb{R}^n, \quad \max\{\|x\|_\infty,\|y\|_\infty\} \le r. \tag{7}$$

Then (1) *has a unique solution* $x \in C(I_0, \mathbb{R}^n)$*,* $I_0 = [0, T_0]$ *for some* $0 < T_0 \leq T$*.*

Proof of Theorem 2. Set $r_0 = \|u\|_\infty + M_f\Theta + 1$ and $B(r_0) = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq r_0\}$. Then we extend f from the set $I \times B(r_0)$ to $\tilde{f}$ on $I \times \mathbb{R}^n$ such that $\tilde{f}$ satisfies (2) for some $L > 0$. This extension is given by

$$\tilde{f}(t,x) = \begin{cases} \chi(\|x\|_\infty/(r_0+1))f(t,x) & \text{for } t \in I,\ ||x|| \leq 2(r_0+1) \\ 0 & \text{for } t \in I,\ ||x|| \geq 2(r_0+1) \end{cases}$$

for a Lipschitz function $\chi : \mathbb{R}_+ \to [0,1]$ with $\chi(r) = 1$ for $r \in [0,1]$ and $\chi(r) = 0$ for $r \geq 2$. Applying Theorem 1 to

$$D_{0+}^{q_i} x_i = \tilde{f}_i(t, x_1, \cdots, x_n), \quad x_i(0) = u_i, \quad i = 1, \cdots, n, \tag{8}$$

there is a unique solution $x \in C(I, \mathbb{R}^n)$ of (8) which also satisfies (6) for α satisfying (5). Note $M_f = M_{\tilde{f}}$. Let us take $T \geq T_0 > 0$, $\alpha > 0$ satisfying (5) and

$$\frac{\|u\|_\infty + M_f\Theta}{1 - \frac{L}{\alpha^{q_m}}} e^{\alpha T_0} < r_0.$$

Then the unique solution $x \in C(I, \mathbb{R}^n)$ of (8) satisfies

$$\|x(t)\|_\infty \leq r_0 \quad \forall t \in I_0.$$

However, this is also a unique solution of (1) on I_0. The proof is finished. □

Remark 1. *Let us denote by* x_u *the solution from Theorem 1. Then, following the proof of Theorem 1, for any* $u, v \in \mathbb{R}^n$*, we derive*

$$\begin{aligned} \|x_u - x_v\|_\alpha = \|F(x_u, u) - F(x_v, v)\|_\alpha &\leq \frac{L}{\alpha^{q_m}} \|x_u - x_v\|_\alpha + \|F(x_v, u) - F(x_v, v)\|_\alpha \\ &\leq \frac{L}{\alpha^{q_m}} \|x_u - x_v\|_\alpha + \|u - v\|_\infty, \end{aligned}$$

which implies

$$\|x_u - x_v\|_\alpha \leq \frac{\|u - v\|_\infty}{1 - \frac{L}{\alpha^{q_m}}},$$

i.e.,

$$\|x_u(t) - x_v(t)\|_\infty \leq \frac{\|u - v\|_\infty}{1 - \frac{L}{\alpha^{q_m}}} \quad \forall t \in I,$$

provided that (5) *holds. So, the continuous dependence of the solution of* (1) *is shown on the initial value* u *under conditions of Theorem 1 or Theorem 2.*

3. Asymptotic Results

We find the upper bound for Lyapunov exponents of solutions of (1).

Theorem 3. *Suppose assumptions of Theorem 2 are satisfied. Moreover, we suppose the existence of a nonnegative* $n \times n$*-matrix* $M = \{m_{ij}\}_{i,j \geq 1}^n$ *and a nonegative vector* $v = (v_1, \cdots, v_n)$ *such that*

$$|f_i(t,x)| \leq \sum_{j=1}^n m_{ij}|x_j| + v_i \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad i = 1, \cdots, n. \tag{9}$$

Then the Lyapunov exponent

$$\Lambda(u)=\limsup_{t\to\infty}\frac{\ln\|x(t)\|_\infty}{t}$$

of the unique solution $x\in C(\mathbb{R}_+,\mathbb{R}^n)$ *of* (1) *satisfies*

$$\Lambda(u)\le\rho(M)^{\max_{i=1,\cdots,n}1/q_i},$$

where $\rho(M)$ *is the spectral radius of* M*. Note* $x(0)=u$*, so we consider as usually* $\Lambda:\mathbb{R}^n\to\mathbb{R}_+$ *(see [10]).*

Proof. Clearly (9) implies

$$\|f(t,x)\|_\infty\le\left(\max_{i=1,\cdots,n}\sum_{j=1}^n m_{ij}\right)\|x\|_\infty+\|v\|_\infty\quad\forall(t,x)\in\mathbb{R}_+\times\mathbb{R}^n.$$

Then, like in the proof of Theorem 1, for any $T>0$ there is a unique solution of (1) on I. However, since $T>0$ is arbitrary, we get a unique solution $x(t)$ on $\mathbb{R}_+$. Then, we compute

$$\begin{aligned}|x_i(t)|&\le|u_i|+\frac{1}{\Gamma(q_i)}\int_0^t(t-s)^{q_i-1}\sum_{j=1}^n\left(m_{ij}|x_j(s)|+v_j\right)ds\\&=|u_i|+\frac{t^{q_i}}{\Gamma(q_i+1)}\sum_{j=1}^n m_{ij}v_j+\frac{1}{\Gamma(q_i)}\sum_{j=1}^n m_{ij}\int_0^t(t-s)^{q_i-1}e^{-\alpha s}|x_j(s)|e^{\alpha s}ds\\&\le|u_i|+\frac{t^{q_i}}{\Gamma(q_i+1)}\sum_{j=1}^n m_{ij}v_j+\frac{1}{\Gamma(q_i)}\sum_{j=1}^n m_{ij}\|x_j\|_\alpha\frac{\Gamma(q_i)e^{\alpha t}}{\alpha^{q_i}}\\&=|u_i|+\frac{t^{q_i}}{\Gamma(q_i+1)}\sum_{j=1}^n m_{ij}v_j+\frac{e^{\alpha t}}{\alpha^{q_i}}\sum_{j=1}^n m_{ij}\|x_j\|_\alpha\\&\le|u_i|+\Upsilon_\alpha e^{\alpha t}\sum_{j=1}^n m_{ij}v_j+\frac{e^{\alpha t}}{\alpha^{q_i}}\sum_{j=1}^n m_{ij}\|x_j\|_\alpha\end{aligned}$$

for

$$\Upsilon_\alpha=\max_{i=1,\cdots,n,t\in\mathbb{R}_+}\frac{t^{q_i}e^{-\alpha t}}{\Gamma(q_i+1)}=\max_{i=1,\cdots,n}\frac{q_i^{q_i}e^{-q_i}}{\alpha^{q_i}\Gamma(q_i+1)}.$$

Consequently, we arrive at

$$\|x_i\|_\alpha\le|u_i|+\Upsilon_\alpha\sum_{j=1}^n m_{ij}v_j+\frac{1}{\alpha^{q_i}}\sum_{j=1}^n m_{ij}\|x_j\|_\alpha,$$

or setting $w_\alpha=(\|x_1\|_\alpha,\cdots,\|x_m\|_\alpha)$ and $|u|=(|u_1|,\cdots,|u_n|)$, we obtain

$$w_{\alpha_\varepsilon}\le|u|+\Upsilon_{\alpha_\varepsilon}Mv+\frac{1}{\rho(M)+\varepsilon}Mw_{\alpha_\varepsilon},\tag{10}$$

for $\alpha_\varepsilon=\max_{i=1,\cdots,n}(\rho(M)+\varepsilon)^{1/q_i}$ and $\varepsilon>0$ is fixed. (10) is considered component-wise. From (10), we get

$$\left(I-\frac{1}{\rho(M)+\varepsilon}M\right)w_{\alpha_\varepsilon}\le|u|+\Upsilon_{\alpha_\varepsilon}Mv\tag{11}$$

for the $n\times n$ identity matrix I. By the Neumann lemma, we have

$$\left(I-\frac{1}{\rho(M)+\varepsilon}M\right)^{-1}=\sum_{i=0}^\infty\left(\frac{1}{\rho(M)+\varepsilon}M\right)^i,$$

which is a positive matrix. Consequently, (11) implies

$$w_{\alpha_\varepsilon} \leq \left(I - \frac{1}{\rho(M)+\varepsilon} M\right)^{-1} \left(|u| + \Upsilon_{\alpha_\varepsilon} M v\right).$$

Letting $\varepsilon \to 0$, we get

$$w_{\alpha_0} \leq \left(I - \frac{1}{\rho(M)} M\right)^{-1} \left(|u| + \Upsilon_{\alpha_0} M v\right)$$

for $\alpha_0 = \max_{i=1,\cdots,n} \rho(M)^{1/q_i}$. So we arrive at

$$|x_i(t)| \leq e^{\alpha_0 t} w_i \quad \forall t \in \mathbb{R}_+, \quad i = 1, \cdots, n$$

for $w = (w_1, \cdots, w_n) = \left(I - \frac{1}{\rho(M)} M\right)^{-1} \left(|u| + \Upsilon_{\alpha_0} M v\right)$, which implies

$$\begin{aligned} \Lambda(u) &= \limsup_{t\to\infty} \frac{\ln \|x(t)\|_\infty}{t} \leq \limsup_{t\to\infty} \frac{\ln(e^{\alpha_0 t}\|w\|_\infty)}{t} \\ &= \limsup_{t\to\infty} \frac{\ln e^{\alpha_0 t}}{t} + \limsup_{t\to\infty} \frac{\|w\|_\infty}{t} = \alpha_0. \end{aligned}$$

The proof is finished. □

4. Stability Result

This section is devoted to an autonomous version of (1)

$$D_{0^+}^{q_i} x_i = \sum_{j=1}^n a_{ij} x_j + g_i(x_1, \cdots, x_n), \quad x_i(0) = u_i, \quad i = 1, \cdots, n \tag{12}$$

where $a_{i,j} \in \mathbb{R}$, $g = (g_1, \cdots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz with $g(0) = 0$ and $\lim_{x\to 0} \frac{\|g(x)\|_\infty}{\|x\|_\infty} = 0$. We already know that (12) has a locally unique solution. First, we prove the following

Lemma 1. *Let $\lambda > 0$ and $q, p \in (0, 1]$ with $q \geq p$. Then, it holds*

$$S(\lambda, q, p) = \sup_{t\geq 0} (t+1)^p \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q \lambda)(s+1)^{-p} ds < \infty. \tag{13}$$

Note that $E_{q,q}(-t) \geq 0$ for any $t \in \mathbb{R}_+$ by ([11] p. 85), where $E_{q,q}(t)$ is the two-parametric Mittag-Leffler function ([11] p. 56) .

Proof. By ([5] Proposition 2.4(i))or ([11] Formula (4.4.17)), there is a positive constant $M(q, \lambda)$ such that

$$t^{q-1} E_{q,q}(-t^q \lambda) \leq \frac{M(q, \lambda)}{(t+1)^{q+1}} \quad \forall t \geq 1.$$

Here we note that one can use ([6] Lemma 3) to compute $M(q, \lambda)$. First, using ([11] Formula (4.4.4)), we derive

$$\begin{aligned} &\sup_{t\in[0,1]} (t+1)^p \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q \lambda)(s+1)^{-p} ds \\ &\leq \sup_{t\in[0,1]} (t+1)^p \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q \lambda) ds \\ &= \sup_{t\in[0,1]} (t+1)^p \int_0^t z^{q-1} E_{q,q}(-z^q \lambda) dz = \sup_{t\in[0,1]} (t+1)^p t^q E_{q,q+1}(-\lambda t^q) = 2^p E_{q,q+1}(-\lambda), \end{aligned}$$

since applying ([11] Formula (5.1.15)), we derive

$$\frac{d}{dt}(t+1)^p t^q E_{q,q+1}(-\lambda t^q) = p(t+1)^{p-1} t^q E_{q,q+1}(-\lambda t^q) + (t+1)^p t^{q-1} E_{q,q}(-\lambda t^q) \geq 0$$

for $t > 0$. Next, for $t \geq 1$, we derive

$$\begin{aligned}
&\sup_{t\geq 1}(t+1)^p \int_0^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)(s+1)^{-p} ds \\
&\leq \sup_{t\geq 1}(t+1)^p \int_0^{t-1} (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)(s+1)^{-p} ds \\
&+ \sup_{t\geq 1}(t+1)^p \int_{t-1}^t (t-s)^{q-1} E_{q,q}(-(t-s)^q\lambda)(s+1)^{-p} ds \\
&\leq \sup_{t\geq 1}(t+1)^p \int_0^{t-1} \frac{M(q,\lambda)}{(t-s+1)^{q+1}(s+1)^p} ds + \sup_{t\geq 1}\frac{(t+1)^p}{t^p}\int_0^1 z^{q-1} E_{q,q}(-z^q\lambda) dz \\
&\leq \sup_{t\geq 1}(t+1)^p \int_0^{t/2} \frac{M(q,\lambda)}{(t-s+1)^{q+1}} ds + \sup_{t\geq 1}\frac{2^p(t+1)^p}{(t+2)^p}\int_{t/2}^t \frac{M(q,\lambda)}{(t-s+1)^{q+1}} ds + 2^p E_{q,q+1}(-\lambda) \\
&\leq \sup_{t\geq 1}\frac{2^q(t+1)^p M(q,\lambda)}{q(t+2)^q} + \sup_{t\geq 1}\frac{2^p(t+1)^p M(q,\lambda)}{q(t+2)^p} + 2^p E_{q,q+1}(-\lambda) \\
&\leq \frac{(2^q+2^p)M(q,\lambda)}{q} + 2^p E_{q,q+1}(-\lambda)
\end{aligned}$$

Summarizing, we arrive at the estimate

$$S(\lambda, q, p) \leq \frac{(2^q+2^p)M(q,\lambda)}{q} + 2^p E_{q,q+1}(-\lambda).$$

The proof is finished. □

Theorem 4. *Suppose $a_{ii} = -\lambda_i < 0$, $i = 1, \cdots, n$. If*

$$\gamma = \max_{i=1,\cdots,n} S(\lambda_i, q_i, q_n) \sum_{i\neq j=1}^n |a_{ij}| < 1, \tag{14}$$

then the zero solution of (12) *is asymptotically stable.*

Proof. The proof is motivated by the well-known Geršgoring type method [12,13]. Like above, we modify (12) outside of the unit ball $B(1)$ such that the modified system is globally Lipschitz. Then, the solution of the modified (12) has a global unique solution on $\mathbb{R}_+$ by Theorem 1. This solution of (12) has the fixed point form [8]

$$\begin{aligned}
&x_i(t) = G_i(x,u), \quad i = 1, \cdots, n, \\
&G_i(x,u) = E_{q_i}(-t^{q_i}\lambda_i)u_i + \int_0^t (t-s)^{q_i-1} E_{q_i,q_i}(-(t-s)^{q_i}\lambda_i)\Big(\sum_{i\neq j=1}^n a_{ij}x_j(s) + g_i(x_1(s), \cdots, x_n(s))\Big) ds,
\end{aligned} \tag{15}$$

where $E_q(t)$ and $E_{q,q}(t)$ are the classical and two-parametric Mittag-Leffler functions, respectively ([11] p. 56). Fix $T > 0$ and set $|||x||| = \max_{t\in I} |x(t)|_\infty (t+1)^{q_n}$ for $x \in C(I, \mathbb{R}^n)$. Then, (15) implies

$$|G_i(x,u)(t)|(t+1)^{q_n} \le \kappa|u_i| + (t+1)^{q_n}\int_0^t (t-s)^{q_i-1}E_{q_i,q_i}(-(t-s)^{q_i}\lambda_i)(s+1)^{-q_n}ds$$
$$\times(\sum_{i\neq j=1}^{n}|a_{ij}| + h(\|x(s)\|_\infty))|||x|||)$$
$$\le \kappa\|u\|_\infty + S(\lambda_i,q_i,q_n)(\sum_{i\neq j=1}^{n}|a_{ij}| + h(|||x|||))|||x|||)$$

for

$$\kappa = \max_{i=1,\cdots,n}\sup_{t\ge 0} E_{q_i}(-t^{q_i}\lambda_i)(t+1)^{q_n} < \infty$$

(see ([11] Formula (3.4.30)) and $h(r) = \max_{\|x\|_\infty \le r}\frac{||g(x)||_\infty}{\|x\|_\infty}$. Consequently, we obtain

$$|||G(x,u)||| \le \kappa\|u\|_\infty + (\gamma + \omega h(|||x|||))|||x||| \tag{16}$$

for $G(x,u) = (G_1(x,u),\cdots,G_n(x,u))$ and

$$\omega = \max_{i=1,\cdots,n}\sum_{i\neq j=1}^{n} S(\lambda_i,q_i,q_n).$$

Taking $r_1 \in (0,1)$ such that $\gamma + \omega h(r_1) \le \frac{1+\gamma}{2}$, then (16) implies

$$|||G(x,u)||| \le \kappa\|u\|_\infty + \frac{1+\gamma}{2}|||x||| \tag{17}$$

for any $x \in B_T(r_1) = \{x \in C(I,\mathbb{R}^n) \mid |||x||| \le r_1\}$. So, supposing

$$\|u\| \le \frac{1-\gamma}{2\kappa}r_1, \tag{18}$$

(17) gives

$$G : B_T(r_1) \subset B_T(r_1).$$

It is well-known that $G : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ is continuous and compact [8]. Since $B_T(r_1)$ is convex and bounded, by the Schauder fixed point theorem, G has a fixed point $x \in B_T(r_1)$. However, this a solution of a modified (12), which has a unique solution on $\mathbb{R}_+$. So, this unique solution satisfies $x \in B_T(r_1)$ for any $t > 0$; i.e.,

$$\|x(t)\|_\infty \le \frac{r_1}{(t+1)^{q_n}} \quad \forall t \ge 0. \tag{19}$$

Certainly (19) gives $\|x(t)\|_\infty \le 1$, so $x(t)$ is also a unique solution of the original (12). Moreover, (17) leads to

$$\|x(t)\|_\infty \le \frac{2\kappa\|u\|_\infty}{(1-\gamma)(t+1)^{q_n}} \quad \forall t \ge 0,$$

which determines the asymptotic stability of (12) at 0. The proof is finished. □

Remark 2. *Condition* (14) *is a Geršgoring type assumption* [12,13]*. Even for* $q_1 = \cdots = q_n$*, condition* (14) *is new, since the general stability assumption (see* [5] *Theorem 2.2) is difficult to check. The above approach can be extended to FDEs on infinite lattices like in* [14]*.*

5. Examples

In this section, we give examples to demonstrate the validity of our theoretical results.

Example 1. *Consider*

$$\begin{cases} {}^cD^{0.6}_{0^+}x_1 = \sin(x_1 + 2x_2), \ t \in [0,1], \\ {}^cD^{0.5}_{0^+}x_2 = \arctan(2x_1 + x_2), \\ x_1(0) = u_1, \quad x_2(0) = u_2. \end{cases} \tag{20}$$

Set $q_1 = 0.6 > 0.5 = q_2$, $T = 1$, $f_1(t, x_1, x_2) = \sin(x_1 + 2x_2)$ *and* $f_2(t, x_1, x_2) = \arctan(2x_1 + x_2)$. *Clearly,* $f = (f_1, f_2)$ *satisfying the uniformly Lipschitz condition with a Lipschitz constant* $L = 2$. *By Theorem 1 or 2, (20) has a unique solution* $x \in C([0,1], R^2)$.

Next, $|f_1(t, x_1, x_2)| \leq m_{11}|x_1| + m_{12}|x_2|$ *and* $|f_2(t, x_1, x_2)| \leq m_{21}|x_1| + m_{22}|x_2|$ *where* $m_{11} = 1 = m_{22}$ *and* $m_{12} = m_{21} = 2$ *Set* $M = (m_{ij}) \in \mathbb{R}^{2\times 2}$, *so*

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

is a nonnegative matrix and $\rho(M) = 3$. *By Theorem 3,* $\Lambda(u) = 3^{\frac{5}{3}}$.

Example 2. *Consider*

$$\begin{cases} {}^cD^{0.6}_{0^+}x_1 = -x_1 - 0.5x_2 + \frac{x_1+x_2}{1+(x_1+x_2)^2}\sin(x_1 + x_2), \ t \geq 0, \\ {}^cD^{0.5}_{0^+}x_2 = 0.5x_1 - x_2 + \frac{x_1+x_2}{1+(x_1+x_2)^2}\arctan(x_1 + x_2). \end{cases} \tag{21}$$

Set $q_1 = 0.6 > 0.5 = q_2$, $\lambda_1 = \lambda_2 = 1$, $a_{11} = a_{22} = -1$ *and* $a_{12} = -0.5, a_{21} = 0.5$, $g_1(x_1, x_2) = \frac{x_1+x_2}{1+(x_1+x_2)^2}\sin(x_1 + x_2)$, $g_2(x_1, x_2) = \frac{x_1+x_2}{1+(x_1+x_2)^2}\arctan(x_1 + x_2)$ *and* $u_1 = u_2 = 0$. *Clearly,* $g = (g_1, g_2)$ *satisfies the uniformly Lipschitz condition with a Lipschitz constant* $L = 2$ *and* $g((0,0)) = (0,0)$ *and* $\lim_{(x_1,x_2)\to(0,0)} \frac{\|g(x_1,x_2)\|_\infty}{\|(x_1,x_2)\|_\infty} = 0$. *We numerically derive*

$$S(1, 0.6, 05) \doteq 1.0051, \quad S(1, 0.5, 0, 5) \doteq 1$$

so $\gamma \leq 0.503 < 1$. *By Theorem 4, the zero solution of (21) is asymptotically stable.*

6. Conclusions

The existence and uniqueness of solutions of (1) are shown along with their stability and asymptotic properties. Theorems 1 and 2 extended the partial case of $q_i = q, i = 1, 2, \cdots, n$ which are known in the related literature. Theorems 3 and 4 are original results. Moreover, Theorems 1, 2, and 3 can be directly extended to infinite dimensional cases. The next step should be to derive an exact solution for scalar linear systems (i.e., (12) with $g = 0$) and to find an explicit variation of constant formula for nonhomogeneous systems. Of course, a more general stability criterion would also be interesting to find by generalizing ([5] Theorem 2.2), which is just for $q_1 = \cdots = q_n$. Then, a derivation of a Gronwall-type inequality associated to MOFDEs would be challenging as well. This should extend Theorem 4 to infinite dimensional cases.

Acknowledgments: The authors thanks the referees for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper. This work is supported by National Natural Science Foundation of China (11661016) and the Slovak Research and Development Agency under the contract No. APVV-14-0378 and by the Slovak Grant Agency VEGA No. 2/0153/16 and No. 1/0078/17.

Author Contributions: M.F. and J.R.W. contributed equally in this work.

Conflicts of Interest: The authors declare no conflict of interest.

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