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Geometric Structure of the Classical Lagrange-d'Alambert Principle and Its Application to Integrable Nonlinear Dynamical Systems

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Abstract: The classical Lagrange-d'Alembert principle had a decisive influence on formation of modern analytical mechanics which culminated in modern Hamilton and Poisson mechanics. Being mainly interested in the geometric interpretation of this principle, we devoted our review to its deep relationships to modern Lie-algebraic aspects of the integrability theory of nonlinear heavenly type dynamical systems and its so called Lax-Sato counterpart. We have also analyzed old and recent investigations of the classical M. A. Buhl problem of describing compatible linear vector field equations, its general M.G. Pfeiffer and modern Lax-Sato type special solutions. Especially we analyzed the related Lie-algebraic structures and integrability properties of a very interesting class of nonlinear dynamical systems called the dispersionless heavenly type equations, which were initiated by Plebański and later analyzed in a series of articles. As effective tools the AKS-algebraic and related $\mathcal{R}$-structure schemes are used to study the orbits of the corresponding co-adjoint actions, which are intimately related to the classical Lie-Poisson structures on them. It is demonstrated that their compatibility condition coincides with the corresponding heavenly type equations under consideration. It is also shown that all these equations originate in this way and can be represented as a Lax-Sato compatibility condition for specially constructed loop vector fields on the torus. Typical examples of such heavenly type equations, demonstrating in detail their integrability via the scheme devised herein, are presented.

Keywords: Lagrange-d'Alembert principle; M. Buhl vector field symmetry problem; Lax–Sato equations; heavenly equations; Lax integrability; Hamiltonian system; torus diffeomorphisms; loop Lie algebra; Lie-algebraic scheme; Casimir invariants; $\mathcal{R}$-structure; Lie-Poisson structure

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1. The Classical Lagrange-d'Alembert Principle

It is well known that modern analytical mechanics was founded mainly by such giants as Newton, Lagrange, d'Alembert, Posson, Hamilton, Maupertui and Jacobi, whose oevres strongly influenced the whole modern mechanical and physical sciences. In his book "Mecanique analytique", v.1–2, published in 1788 in Paris, J.L. Lagrange formulated one of the basic, most general, differential variational principles of classical mechanics, expressing necessary and sufficient conditions for the correspondence of the real motion of a system of material points, subjected by ideal constraints, to the applied active forces. Within the Lagrange-d'Alembert principle the positions of the system in its real motion are compared with infinitely close positions permitted by the constraints at the given moment of time.

According to the Lagrange–d'Alembert principle, during a real motion of a system of $N \in \mathbb{Z}_+$ particles with massess $m_j \in \mathbb{R}_+, j = \overline{1,N}$, the sum of the elementary works performed by the given active forces $F^{(j)}, j = \overline{1,N}$, and by the forces of inertia for all the possible particle displacements $\delta x^{(j)} \in \mathbb{E}^3, j = \overline{1,N}$:

$$\sum_{j=\overline{1,N}} < F^{(j)} - m_j \frac{dx^{(j)}}{dt}, \delta x^{(j)} > \leq 0 \tag{1}$$

at any moment of time $t \in \mathbb{R}$, where $< \cdot, \cdot >$ denotes the standard scalar product in the three-dimensional Euclidean space $\mathbb{E}^3$. The equality in (1) is valid for the possible reversible displacements, the symbol $\leq$ is valid for the possible irreversible displacements $\delta x^{(j)} \in \mathbb{E}^3, j = \overline{1,N}$. Equation (1) is the general equation of the dynamics of systems with ideal constraints; it comprises all the equations and laws of motion, so that one can say that all dynamics is reduced to this single general formula.

This principle, established by J.L. Lagrange by generalization of the principle of virtual displacements with the aid of the classical d'Alembert principle. For systems subject to bilateral constraints J.L. Lagrange based himself on formula (1) to deduce the general properties and laws of motion of bodies, as well as the equations of motion, which he applied to solve a number of problems in dynamics including the problems of motions of non-compressible, compressible and elastic liquids, thus combining *"dynamics and hydrodynamics as branches of the same principle and as conclusions drawn from a single general formula"*.

As it was first demonstrated in the work [1], for the last case of generalized reversible motions of a non-compressible elastic liquid, located in a one-connected open domain $\Omega_t \subset \mathbb{R}^n, n \in \mathbb{Z}_+$, with the smooth boundary $\partial\Omega_t$, $t \in \mathbb{R}$, in space $\mathbb{R}^n, n \in \mathbb{Z}_+$, expression (1) can be rewritten as:

$$\delta W(t) := \int_{\Omega_t} < l(x(t);\lambda), \delta x(t) > d^n x(t) = 0 \tag{2}$$

for all $t \in \mathbb{R}$, where $l(x(t);\lambda) \in \tilde{T}^*(\mathbb{R}^n)$ is the corresponding virtual "*reaction force*", exerted by the ambient medium on the liquid and called a seed element, which is here assumed to depend analytically on a complex parameter $\lambda \in \mathbb{C}$. If now to suppose that the evolution of liquid points $x(t) \in \Omega_t$ is determined for any parameters $\lambda \neq \mu \in \mathbb{C}$ by the generating gradient type vector field:

$$\frac{dx(t)}{dt} = \frac{\mu}{\mu - \lambda} \nabla h(l(\mu))(t; x(t)) \tag{3}$$

and the Cauchy data:

$$x(t)|_{t=0} = x^{(0)} \in \Omega_0$$

for an arbitrarily chosen open one-connected domain $\Omega_0 \subset \mathbb{T}^n$ with the smooth boundary $\partial\Omega_0 \subset \mathbb{R}^n$ and a smooth functional $h : \tilde{T}^*(\mathbb{R}^n) \to \mathbb{R}$, the Lagrange-d'Alembert principle says: the infinitesimal virtual work (2) equals zero for all moments of time, that is $\delta W(t) = 0 = \delta W(0)$ for all $t \in \mathbb{R}$. To check that it is really zero, let us calculate the temporal derivative of the (2):

$$\begin{gathered}
\tfrac{d}{dt}\delta W(t) = \tfrac{d}{dt}\int_{\Omega_t} < l(x(t);\lambda), \delta x(t) > d^n x(t) = \\
\tfrac{d}{dt}\int_{\Omega_0} < l(x(t);\lambda), \delta x(t) > |\tfrac{\partial(x(t)}{\partial x_0}| d^n x^{(0)} = \int_{\Omega_0} \tfrac{d}{dt}(< l(x(t);\lambda), \delta x(t) > |\tfrac{\partial(x(t)}{\partial x_0}|) d^n x^{(0)} = \\
\int_{\Omega_0} [\tfrac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu)] |\tfrac{\partial(x(t)}{\partial x_0}| d^n x^{(0)} = \\
\int_{\Omega_t} [\tfrac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu)] d^n x(t) = 0,
\end{gathered} \tag{4}$$

if the condition:

$$\frac{d}{dt} < l(x(t);\lambda), \delta x(t) > + < l(x(t);\lambda), \delta x(t) > \operatorname{div} \tilde{K}(\mu;\lambda) = 0 \tag{5}$$

holds for all $t \in \mathbb{R}$, where:

$$\tilde{K}(\mu;\lambda) := \frac{\mu}{\mu-\lambda} \nabla h(\tilde{l}(\mu)) = \frac{\mu}{\mu-\lambda} < \nabla h(l(\mu)), \frac{d}{dx} > \tag{6}$$

is a vector field on $\mathbb{R}^n$, corresponding to the evolution Equation (3). Taking into account that the full temporal derivative $d/dt := \partial/\partial t + L_{\tilde{K}(\mu;\lambda)}$, where $L_{\tilde{K}(\mu;\lambda)} = i_{\tilde{K}(\mu;\lambda)} d + d i_{\tilde{K}(\mu;\lambda)}$ denotes the well known [2–4] Cartan expression for the Lie derivation along the vector field (6), can be represented as $\mu, \lambda \to \infty$, $|\lambda/\mu| < 1$ in the asymptotic form:

$$\frac{d}{dt} \sim \sum_{j \in \mathbb{Z}_+} \mu^{-j} \frac{\partial}{\partial t_j} + \sum_{j \in \mathbb{Z}_+} \mu^{-j} L_{\tilde{K}_j(\lambda)}, \tag{7}$$

the equality (5) can be equivalently rewritten as an infinite hierarchy of the following evolution equations:

$$\partial \tilde{l}(\lambda)/\partial t_j := -ad^*_{\tilde{K}_j(\lambda)_+} \tilde{l}(\lambda) \tag{8}$$

for every $j \in \mathbb{Z}_+$ on the space of differential 1-forms $\tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*$, where we denoted $\tilde{l}(\lambda) :=< l(x;\lambda), dx >\in \tilde{\Lambda}^1(\mathbb{R}^n) \simeq \tilde{\mathcal{G}}^*$ with $\tilde{\mathcal{G}} := \widetilde{diff}(\mathbb{R}^n)$ being the Lie algebra of the corresponding loop diffeomorphism group $\widetilde{Diff}(\mathbb{R}^n)$. As from (6) one easily finds that:

$$\tilde{K}_j(\lambda) = \nabla h^{(j)}(\tilde{l}) \tag{9}$$

for $\lambda \in \mathbb{C}$ and any $j \in \mathbb{Z}_+$, the evolution Equation (8) transform equivalently into:

$$\partial \tilde{l}(\lambda)/\partial t_j := -ad^*_{\nabla h^{(j)}(\tilde{l})_+} \tilde{l}(\lambda), \tag{10}$$

allowing to formulate the following important Adler-Kostant-Symes type [3,5–9] proposition.

Proposition 1. *The evolution Equation (10) are completely integrable commuting to each other Hamiltonian flows on the adjoint loop space $\tilde{\mathcal{G}}^*$ for a seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$, generated by Casimir functionals $h^{(j)} \in I(\tilde{\mathcal{G}}^*)$, naturally determined by conditions $ad^*_{\nabla h^{(j)}(\tilde{l})} \tilde{l}(\lambda) = 0$, $j \in \mathbb{Z}_+$, with respect to the modified Lie-Poisson bracket on the adjoint space $\tilde{\mathcal{G}}^*$:*

$$\{(\tilde{l}, \tilde{X}), (\tilde{l}, \tilde{Y})\} := (\tilde{l}, [\tilde{X}, \tilde{Y}]_{\mathcal{R}}),$$

defined for any $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{G}}$ by means of the canonical $\mathcal{R}$-structure on the loop Lie algebra $\tilde{\mathcal{G}}$:

$$[\tilde{X}, \tilde{Y}]_{\mathcal{R}} := [\tilde{X}_+, \tilde{Y}_+] - [\tilde{X}_-, \tilde{Y}_-], \tag{11}$$

where $\tilde{Z}_\pm$ means the positive (+)/(−)-negative part of a loop Lie algebra element $\tilde{Z} \in \tilde{\mathcal{G}}$ subject to the loop parameter $\lambda \in \mathbb{C}$.

If, for instance, to consider the first two flows from (10) in the form:

$$\begin{aligned} \partial \tilde{l}(\lambda)/\partial t_1 &:= \partial \tilde{l}(\lambda)/\partial y = -ad^*_{\nabla h^{(y)}(\tilde{l})_+} \tilde{l}(\lambda), \\ \partial \tilde{l}(\lambda)/\partial t_2 &:= \partial \tilde{l}(\lambda)/\partial t = -ad^*_{\nabla h^{(t)}(\tilde{l})_+} \tilde{l}(\lambda), \end{aligned} \tag{12}$$

which are, by construction, commuting to each other, from their compatibility condition ensues some system of nonlinear equations in partial derivatives on the coefficients of the seed element $\tilde{l}(\lambda) \in \tilde{\mathcal{G}}^*$.

As the latter is, evidently, equivalent to the Lax–Sato compatibility condition of the corresponding vector fields $\nabla h^{(y)}(\tilde{l})_+ :=< \nabla h^{(1)}(l)_+, \partial/\partial x >, \nabla h^{(t)}(\tilde{l})_+ :=< \nabla h^{(2)}(l)_+, \partial/\partial x >\in \tilde{\mathcal{G}}$:

$$[\partial/\partial y + \nabla h^{(y)}(\tilde{l})_+,\ \partial/\partial t + \nabla h^{(t)}(\tilde{l})_+] = 0, \tag{13}$$

such a resulting system of nonlinear equations in partial derivatives, often called of heavenly type, was before actively analyzed in a series of articles [10–19] and recently in [10,20–22]. These works are closely related to the problem of constructing a hierarchy of commuting to each other vector fields, analytically depending on a complex parameter $\lambda \in \mathbb{C}$.

2. The M.A. Buhl Problem and the Lax-Sato Type Compatible Systems of Linear Vector Field Equations

It proves that in the classical works [23,24] still in 1928 the French mathematician M.A. Buhl posed the problem of classifying all infinitesimal symmetries of a given linear vector field equation:

$$A\psi = 0, \tag{14}$$

where function $\psi \in C^2(\mathbb{R}^n;\mathbb{R})$, and

$$A := \sum_{j=\overline{1,n}} a_j(x)\frac{\partial}{\partial x_j} \tag{15}$$

is a vector field operator on $\mathbb{R}^n$ with coefficients $a_j \in C^1(\mathbb{R}^n;\mathbb{R}), j = \overline{1,n}$. It is easy to show that the problem under regard is reduced [25] to describing all possible vector fields:

$$A^{(k)} := \sum_{j=\overline{1,n}} a_j^{(k)}(x)\frac{\partial}{\partial x_j} \tag{16}$$

with coefficients $a_j^{(k)} \in C^1(\mathbb{R}^n;\mathbb{R}), j,k = \overline{1,n},$ satisfying the Lax type commutator condition:

$$[A, A^{(k)}] = 0 \tag{17}$$

for all $x \in \mathbb{R}^n$ and $k = \overline{1,n}$. The M.A. Buhl problem above was completely solved in 1931 by the Ukrainian mathematician G. Pfeiffer in the works [25–30], where he has constructed explicitly the searched set of independent vector fields (16), having made use effectively of the full set of invariants for the vector field (15) and the related solution set structure of the Jacobi-Mayer system of equations, naturally following from (17). Some results, yet not complete, were also obtained by C. Popovici in [31].

Consider for simplicity a vector field $X : \mathbb{R}\times\mathbb{T}^n \to T(\mathbb{R}\times\mathbb{T}^n)$ on the $(n+1)$-dimensional toroidal cylinder $\mathbb{R}\times\mathbb{T}^n$ for arbitrary $n \in \mathbb{Z}_+$, which we will write in the slightly special form:

$$A = \frac{\partial}{\partial t} + < a(t,x), \frac{\partial}{\partial x} >, \tag{18}$$

where $(t,x) \in \mathbb{R}\times\mathbb{T}^n, a(t,x) \in \mathbb{E}^n, \frac{\partial}{\partial x} := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n})^\intercal$ and $< \cdot,\cdot >$ is the standard scalar product on the Euclidean space $\mathbb{E}^n$. With the vector field (18), one can associate the linear equation:

$$A\psi = 0 \tag{19}$$

for some function $\psi \in C^2(\mathbb{R}\times\mathbb{T}^n;\mathbb{R})$, which we will call an "invariant" of the vector field.

Next, we study the existence and number of such functionally-independent invariants to the Equation (19). For this let us pose the following Cauchy problem for Equation (19): Find a function $\psi \in C^2(\mathbb{R}\times\mathbb{T}^n;\mathbb{R})$, which at point $t^{(0)} \in \mathbb{R}$ satisfies the condition $\psi(t,x)|_{t=t^{(0)}} = \psi^{(0)}(x), x \in \mathbb{T}^n$, for a given function $\psi^{(0)} \in C^2(\mathbb{T}^n;\mathbb{R})$. For the Equation (19) there is a naturally related parametric vector field on the torus $\mathbb{T}^n$ in the form of the ordinary vector differential equation:

$$dx/dt = a(t,x), \tag{20}$$

to which there corresponds the following Cauchy problem: find a function $x : \mathbb{R} \to \mathbb{T}^n$ satisfying:

$$x(t)|_{t=t^{(0)}} = z \tag{21}$$

for an arbitrary constant vector $z \in \mathbb{T}^n$. Assuming that the vector-function $a \in C^1(\mathbb{R}\times\mathbb{T}^n;\mathbb{E}^n)$, it follows from the classical Cauchy theorem [32] on the existence and unicity of the solution to (20) and (21) that we can obtain a unique solution to the vector Equation (20) as some function $\Phi \in C^1(\mathbb{R}\times\mathbb{T}^n;\mathbb{T}^n)$, $x = \Phi(t,z)$, such that the matrix $\partial\Phi(t,z)/\partial z$ is nondegenerate for all $t \in \mathbb{R}$ sufficiently close to $t^{(0)} \in \mathbb{R}$. Hence, the Implicit Function Theorem [32,33] implies that there exists a mapping $\Psi : \mathbb{R}\times\mathbb{T}^n \to \mathbb{T}^n$, such that:

$$\Psi(t,x) = z \tag{22}$$

for every $z \in \mathbb{T}^n$ and all $t \in \mathbb{R}$ sufficiently enough to $t^{(0)} \in \mathbb{R}$. Supposing now that the functional vector $\Psi(t,x) = (\psi^{(1)}(t,x), \psi^{(2)}(t,x), ..., \psi^{(n)}(t,x))^\intercal$, $(t,x) \in \mathbb{R}\times\mathbb{T}^n$, is constructed, from the arbitrariness of the parameter $z \in \mathbb{T}^n$ one can deduce that all functions $\psi^{(j)} : \mathbb{R}\times\mathbb{T}^n \to \mathbb{T}^1$, $j = \overline{1,n}$, are functionally independent invariants of the vector field Equation (19), that is $A\psi^{(j)} = 0, j = \overline{1,n}$. Thus, the vector field Equation (19) has exactly $n \in \mathbb{Z}_+$ functionally independent invariants, which make it possible, in particular, to solve the Cauchy problem posed above. Namely, let a mapping $\alpha : \mathbb{T}^n \to \mathbb{R}$ be chosen such that $\alpha(\Psi(t,x))|_{t=t^{(0)}} = \psi^{(0)}(x)$ for all $x \in \mathbb{T}^n$ and a fixed $t^{(0)} \in \mathbb{R}$. Inasmuch as the superposition of functions $\alpha \circ \Psi : \mathbb{R}\times\mathbb{T}^n \to \mathbb{T}^1$ is, evidently, also an invariant for the Equation (19), it provides the solution to this Cauchy problem, which we can formulate as the following result.

Proposition 2. *The linear Equation (19), generated by the vector field (20) on the torus $\mathbb{T}^n$, has exactly $n \in \mathbb{Z}_+$ functionally independent invariants.*

Consider now a Plucker type [22] differential form $\chi^{(n)} \in \Lambda^n(\mathbb{T}^n)$ on the torus $\mathbb{T}^n$ as:

$$\chi^{(n)} := \mathrm{d}\psi^{(1)} \wedge \mathrm{d}\psi^{(2)} \wedge ... \wedge \mathrm{d}\psi^{(n)}, \tag{23}$$

generated by the vector $\Psi : \mathbb{R}^n \times \mathbb{T}^n \to \mathbb{T}^n$ of independent invariants (22), depending additionally on $n \in \mathbb{Z}_+$ parameters $t \in \mathbb{R}^n$, where by definition, for any $k = \overline{1,n}$

$$\mathrm{d}\psi^{(k)} := \sum_{j=\overline{1,n}} \frac{\partial\psi^{(k)}}{\partial x_j} dx_j \tag{24}$$

on the manifold $\mathbb{T}^n$. As follows from the Frobenius theorem [4,22,32], the Plucker type differential form (23) is for all fixed parameters $t \in \mathbb{R}^n$ nonzero on the manifold $\mathbb{T}^n$ owing to the functional independence of the invariants (22). It is easy to see that at the fixed parameters $t \in \mathbb{R}^n$ the following [30] Jacobi-Mayer type relationship:

$$\left|\frac{\partial\Psi}{\partial x}\right|^{-1} \mathrm{d}\psi^{(1)} \wedge \mathrm{d}\psi^{(2)} \wedge ... \wedge \mathrm{d}\psi^{(n)} = dx_1 \wedge dx_2 \wedge ... \wedge dx_n \tag{25}$$

holds for $k = \overline{1,n}$ on the manifold $\mathbb{T}^n$, where $\left|\frac{\partial\Psi}{\partial x}\right|$ is the determinant of the Jacobi mapping $\frac{\partial\Psi}{\partial x} : T(\mathbb{T}^n) \to T(\mathbb{T}^n)$ of the mapping (22) subject to the torus variables $x \in \mathbb{T}^n$. On the right-hand side of (25) one has the volume measure on the torus $\mathbb{T}^n$, which is naturally dependent on $t \in \mathbb{R}^n$ owing to the general vector field relationships (20). Taking into account that for all $k = \overline{1,n}$ the full differentials:

$$d\psi^{(k)} = \sum_{s=\overline{1,n}} \frac{\partial \psi^{(k)}}{\partial t_s} dt_s + \mathrm{d}\psi^{(k)} = 0, \tag{26}$$

that is vanishing on $\mathbb{R}^n \times \mathbb{T}^n$, the corresponding substitution of the reduced differentials $\mathrm{d}\psi^{(k)} \in C^2(\mathbb{R}^n; \Lambda^1(\mathbb{T}^n)), k = \overline{1,n}$, into (25) easily gives rise, in particular, to the following set of the compatible vector field relationships:

$$\frac{\partial \Psi}{\partial t_s} - \sum_{j,k=\overline{1,n}} \left[\left(\frac{\partial \Psi}{\partial x} \right)^{-1}_{jk} \frac{\partial \psi^{(k)}}{\partial t_s} \right] \frac{\partial \Psi}{\partial x_j} = 0, \tag{27}$$

for all $s = \overline{1,n}$. The latter property, as it was demonstrated by M.G. Pfeiffer in [30], makes it possible to solve effectively the M.A. Buhl problem and has interesting applications [10,22] in the theory of completely integrable dynamical systems of heavenly type, whose examples are considered in the next section.

Vector FIeld Hierarchies on the Torus with "Spectral" Parameter and the Lax-Sato Integrable Heavenly Dynamical Systems

Consider some naturally ordered infinite set of parametric vector fields (18) on the infinite dimensional toroidal manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$ in the form:

$$A^{(k)} = \frac{\partial}{\partial t_k} + < a^{(k)}(t,x;\lambda), \frac{\partial}{\partial x} > + a_0^{(k)}(t,x;\lambda) \frac{\partial}{\partial \lambda} := \frac{\partial}{\partial t_k} + \mathrm{A}^{(k)}, \tag{28}$$

where $t_k \in \mathbb{R}, k \in \mathbb{Z}_+, (t,x;\lambda) \in (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n) \times \mathbb{C}$ are the evolution parameters, and the dependence of smooth vectors $(a_0^{(k)}, a^{(k)})^\intercal \in \mathbb{E} \times \mathbb{E}^n$, $k \in \mathbb{Z}_+$, on the "*spectral*" parameter $\lambda \in \mathbb{C}$ is assumed to be holomorphic. Suppose now that the infinite hierarchy of linear equations:

$$A^{(k)}\psi = 0 \tag{29}$$

for $k \in \mathbb{Z}_+$ has exactly $n+1 \in \mathbb{Z}_+$ common functionally independent invariants $\psi^{(j)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C}), j = \overline{0,n}$ on the torus $\mathbb{T}^n$, suitably depending on the parameter $\lambda \in \mathbb{C}$. Then, owing to the existence theory [32,33] for ordinary differential equations depending on the "*spectral*" parameter $\lambda \in \mathbb{C}$, these invariants may be assumed to be such that allow analytical continuation in the parameter $\lambda \in \mathbb{C}$ both inside $\mathbb{S}^1_+ \subset \mathbb{C}$ of some circle $\mathbb{S}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}, |\lambda| \to \infty$, outside $\mathbb{S}^1_- \subset \mathbb{C}$ of this circle $\mathbb{S}^1 \subset \mathbb{C}$. This means that as $|\lambda| \to \infty$ we have the following expansions:

$$\begin{array}{ll} \psi^{(0)}(\lambda) & \sim \lambda + \sum_{k=0}^\infty \psi_k^{(0)}(\tau,x)\lambda^{-k}, \\ \psi^{(1)}(\lambda) & \sim \sum_{k=0}^\infty \tau_k^{(1)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^\infty \psi_k^{(1)}(\tau,x)\psi_0(\lambda)^{-k}, \\ \psi^{(2)}(\lambda) & \sim \sum_{k=0}^\infty \tau_k^{(2)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^\infty \psi_k^{(2)}(\tau,x)\psi_0(\lambda)^{-k}, \\ & \dots \\ \psi^{(n)}(\lambda) & \sim \sum_{k=0}^\infty \tau_k^{(n)}(t,x)\psi_0(\lambda)^k + \sum_{k=1}^\infty \psi_k^{(n)}(\tau,x)\psi_0(\lambda)^{-k}, \end{array} \tag{30}$$

where we took into account that $\psi^{(0)}(\lambda) \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{C}), \lambda \in \mathbb{C}$, is the basic invariant solution to the Equation (29), the functions $\tau \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n; \mathbb{R}^{n \times \mathbb{Z}_+})$ for all $s = \overline{1,n}, l \in \mathbb{Z}_+$, are assumed to be independent and $\psi_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n;\mathbb{R})$ for all $k \in \mathbb{N}$, $j = \overline{0,n}$, are arbitrary. Write down now the condition (25) on the manifold $\mathbb{C} \times \mathbb{T}^n$ in the form $\lambda \in C$

$$|\frac{\partial \Psi}{\partial \mathrm{x}}|^{-1} \mathrm{d}\psi^{(0)} \wedge \mathrm{d}\psi^{(1)} \wedge \mathrm{d}\psi^{(2)} \wedge ... \wedge \mathrm{d}\psi^{(n)} = d\lambda \wedge dx_1 \wedge dx_2 \wedge ... \wedge dx_n, \tag{31}$$

where $\mathrm{x} := (\lambda, x) \in \mathbb{C} \times \mathbb{T}^n$, $|\frac{\partial \Psi}{\partial \mathrm{x}}|$ is the Jacobi determinant of the mapping $\Psi := (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, ..., \psi^{(n)})^\intercal \in C^2(\mathbb{C} \times (\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n); \mathbb{C}^{n+1})$ on the manifold $\mathbb{C} \times \mathbb{T}^n$. Inasmuch this mapping subject to the parameter $\lambda \in \mathbb{C}$ has analytical continuation [33] inside $\mathbb{S}^1_+ \subset \mathbb{C}$ of the circle $\mathbb{S}^1 \subset \mathbb{C}$ and subject to the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \to \infty$ outside $\mathbb{S}^1_- \subset \mathbb{C}$ of this circle $\mathbb{S}^1 \subset \mathbb{C}$, one can easily obtain from the vanishing differential expressions:

$$d\psi^{(j)} = \mathrm{d}\psi^{(j)} + \sum_{k=0}^{\infty} \frac{\partial \psi^{(j)}}{\partial \tau_k^{(j)}} d\tau_k^{(j)} = 0 \tag{32}$$

for all $j = \overline{1,n}$ and the relationship (31) on the manifold $\mathbb{C} \times \mathbb{T}^n$ of the independent variables $\mathrm{x} \in \mathbb{C} \times \mathbb{T}^n$, evolving analytically with respect to the parameters $\tau_k^{(j)} \in \mathbb{R}$, $j = \overline{1,n}, k \in \mathbb{Z}_+$, the following Lax-Sato criterion:

$$\left(|\frac{\partial \Psi}{\partial \mathrm{x}}|^{-1} \mathrm{d}\psi^{(0)} \wedge \mathrm{d}\psi^{(1)} \wedge \mathrm{d}\psi^{(2)} \wedge ... \wedge \mathrm{d}\psi^{(n)} \right)_- = 0, \tag{33}$$

where $(...)_-$ means the asymptotic part of an expression in the bracket, depending on the parameter $\lambda^{-1} \in \mathbb{C}$ as $|\lambda| \to \infty$. The substitution of expressions (32) into (33) easily yields:

$$\frac{\partial \Psi}{\partial \tau_k^{(j)}} = \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{0j} \psi^{(0)} (\lambda)^k \right]_+ \frac{\partial \Psi}{\partial \lambda} + \sum_{s=1}^{n} \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{sj} \psi^{(0)} (\lambda)^k \right]_+ \frac{\partial \Psi}{\partial x_s} \tag{34}$$

for all $k \in \mathbb{Z}_+, j = \overline{1,n}$. These relationships (34) comprise an infinite hierarchy of Lax-Sato compatible [18,19] linear equations, where $(...)_+$ denotes the asymptotic part of an expression in the bracket, depending on nonnegative powers of the complex parameter $\lambda \in \mathbb{C}$. As for the independent functional parameters $\tau_k^{(j)} \in C^2(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n;\mathbb{R})$ for all $k \in \mathbb{Z}_+, j = \overline{1,n}$, one can state their functional independence by taking into account their *a priori* linear dependence on the independent evolution parameters $t_k \in \mathbb{R}$, $k \in \mathbb{Z}_+$. On the other hand, taking into account the explicit form of the hierarchy of Equation (34), following [10], it is not hard to show that the corresponding vector fields:

$$\mathrm{A}_k^{(j)} := \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{0j} \psi^{(0)} (\lambda)^k \right]_+ \frac{\partial}{\partial \lambda} + \sum_{s=1}^{n} \left[\left(\frac{\partial \Psi}{\partial \mathrm{x}} \right)^{-1}_{sj} \psi^{(0)} (\lambda)^k \right]_+ \frac{\partial}{\partial x_s} \tag{35}$$

on the manifold $\mathbb{C} \times \mathbb{T}^n$ satisfy for all $k, m \in \mathbb{Z}_+, j, l = \overline{1,n}$, the Lax-Sato compatibility conditions:

$$\frac{\partial \mathrm{A}_m^{(l)}}{\partial \tau_k^{(j)}} - \frac{\partial \mathrm{A}_k^{(j)}}{\partial \tau_m^{(l)}} = [\mathrm{A}_k^{(j)}, \mathrm{A}_m^{(l)}], \tag{36}$$

which are equivalent to the independence of the all functional parameters $\tau_k^{(j)} \in C^1(\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n;\mathbb{R})$, $k \in \mathbb{Z}_+, j = \overline{1,n}$. As a corollary of the analysis above, one can show that the infinite hierarchy of vector fields (28) is a linear combination of the basic vector fields (35) and also satisfies the Lax type compatibility condition (36). Inasmuch the coefficients of vector fields (35) are suitably smooth functions on the manifold $\mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^n$, the compatibility conditions (36) yield the corresponding sets of differential-algebraic relationships on their coefficients, which have the common infinite set of invariants, thereby comprising an infinite hierarchy of completely integrable so called *heavenly* nonlinear dynamical systems on the corresponding multidimensional functional manifolds. That is, all of the above can be considered as an introduction to a recently devised [10,18,19,21] constructive algorithm for generating infinite hierarchies of completely integrable nonlinear dynamical systems of heavenly type on functional manifolds of arbitrary dimension. It is worthwhile to stress here that the above constructive algorithm for generating completely integrable nonlinear multidimensional dynamical systems still does not make it possible to directly show they are Hamiltonian and construct other related mathematical structures. This important problem is solved by employing other mathematical theories; for example, the analytical properties of the related loop diffeomorphisms groups generated by the hierarchy of vector fields (28).

Remark 1. *The compatibility condition (36) allows an alternative differential-geometric description based on the Lie-algebraic properties of the basic vector fields (35). Namely, consider the manifold* $\mathbb{R}^{n\times\mathbb{Z}_+}$*, as the base manifold of the vector bundle* $E(\mathbb{R}^{n\times\mathbb{Z}_+}, G)$, $E = \cup_{\tau\in\,\mathbb{R}^{n\times\mathbb{Z}_+}}\{(G^*\otimes\tau)/\rho\}$, $G^* := \{\varphi^* : \varphi^*\beta^{(1)} := \alpha^{(1)}\circ\varphi,\ \beta^{(1)} \in \tilde{\Lambda}^{(1)}(\mathbb{C}\times\mathbb{T}^n;\mathbb{C}), \varphi \in G\}$ *for an equivalence relation* ρ *and the (holomorphic in* $\lambda \in \mathbb{S}^1_+ \cup \mathbb{S}^1_- \subset \mathbb{C}$*) structure group* $G = Diff_{hol}(\mathbb{C}\times\mathbb{T}^n)$*, naturally acting on the vector space* E*. The structure group can be endowed with a connection* Υ *by means of a mapping* $d_h : \Gamma(E) \to \Gamma(T^*(\mathbb{R}^{n\times\mathbb{Z}_+})\otimes E) \cong \Gamma(Hom(T(\mathbb{R}^{n\times\mathbb{Z}_+}); E))$*, where:*

$$d_h\varphi^*_\tau := \sum_{j\in\mathbb{Z}_+} d\tau_j^{(k)} \otimes \frac{\partial}{\partial\tau_j^{(k)}} \circ \varphi^*_\tau + \varphi^*_\tau \circ < \alpha^{(1)}, \frac{\partial}{\partial x} >, \tag{37}$$

$\alpha^{(1)} := \sum_{j\in\mathbb{Z}_+} a_j^{(k)} d\tau_j^{(k)} \in \Lambda(\mathbb{R}^{n\times\mathbb{Z}_+})\otimes\Gamma(E)$*, which is defined for any cotangent diffeomorphism* $\varphi^*_\tau \in E$, $\tau \in \mathbb{R}^{n\times\mathbb{Z}_+}$*, generated by the set of parametric vector fields (35), and naturally acting on any mapping* $\psi \in C^2(\mathbb{R}^{n\times\mathbb{Z}_+}\times(\mathbb{C}\times\mathbb{T}^n);\mathbb{C})$ *as* $\varphi^*_\tau \circ \psi(\tau, x) := \psi(\tau, \varphi_\tau(x))$, $(\tau, x) \in \mathbb{R}^{n\times\mathbb{Z}_+}\times\mathbb{T}^n$*. It is easy now to see that the corresponding to (37) zero curvature condition* $d_h^2 = 0$ *is equivalent to the set of compatibility Equation (36). Moreover, the parallel transport equation:*

$$d_h\varphi^*_\tau \circ \psi\ = 0 \tag{38}$$

coincides exactly with the infinite hierarchy of linear vector field Equation (34), where $\psi \in C^2(\mathbb{R}^{n\times\mathbb{Z}_+}\times\mathbb{T}^n;\mathbb{R})$ *is their invariant. Conversely, the Cartan integrable ideal of differential forms* $h(\alpha) \in \Lambda(\mathbb{R}^{n\times\mathbb{Z}_+}\times\mathbb{T}^n)\otimes\Gamma(T^*(\mathbb{R}^{n\times\mathbb{Z}_+}))$*, which is equivalent to the zero curvature condition* $d_h^2 = 0$*, makes it possible to retrieve [3,34] the corresponding connection* Υ *by constructing a mapping* $d_h : \Gamma(E) \to \Gamma(T^*(\mathbb{R}^{n\times\mathbb{Z}_+})\otimes E) \cong \Gamma(Hom(T(\mathbb{R}^{n\times\mathbb{Z}_+}); E))$ *in the form (37). These and other interesting related aspects of the integrable heavenly dynamical systems shall be investigated separately elsewhere.*

3. Examples: Integrable Heavenly Type Nonlinear Dynamical Systems

3.1. The Mikhalev-Pavlov Equation and Its Vector Field Representation

The Mikhalev-Pavlov equation was first constructed in [14,35] and has the form:

$$u_{xt} + u_{yy} = u_y u_{xx} - u_x u_{xy}, \tag{39}$$

where $u \in C^\infty(\mathbb{R}^2\times\mathbb{T}^1;\mathbb{R})$ and $(t, y; x) \in \mathbb{R}^2\times\mathbb{T}^1$. Assume now [10] that the following two smooth functions:

$$\psi^{(0)} = \lambda, \quad \psi^{(1)} \sim \sum_{k=3}^{\infty}\lambda^k\tau_k - \lambda^2 t + \lambda y + x + \sum_{j=1}^{\infty}\psi_j^{(1)}(t, y, \tau; x)\,\lambda^{-j}, \tag{40}$$

where $\psi_1^{(1)}(t,y,\tau;x) = u$, $(t,y,\tau;x) \in \mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1$, are invariants of the set of vector fields (29) for an infinite set of constant parameters $\tau_k \in \mathbb{R}, k = \overline{3,\infty}$, as the complex parameter $\lambda \to \infty$. By applying to the invariants (40) the criterion (33), (32) in the form:

$$((\partial\psi^{(1)}/\partial x)^{-1} d\psi^{(1)})_- = 0, \tag{41}$$

one can easily obtain the following compatible linear vector field equations:

$$\begin{aligned} \frac{\partial\psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y)\frac{\partial\psi}{\partial x} &= 0 \\ \frac{\partial\psi}{\partial y} + (\lambda + u_x)\frac{\partial\psi}{\partial x} &= 0, \\ &\dots \\ \frac{\partial\psi}{\partial\tau_k} + P_k(u;\lambda)\frac{\partial\psi}{\partial x} &= 0, \end{aligned} \tag{42}$$

where $P_k(u;\lambda), k = \overline{3,\infty}$, are independent differential-algebraic polynomials in the variable $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^\infty \times \mathbb{T}^1)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (34). Moreover, as one can check, the compatibility condition (36) for the first two vector field equations of (42) yields exactly the Mikhalev–Pavlov Equation (39).

3.2. *The Mikhalev–Pavlov Equation and Its Lie-Algebraic Structure*

Let us set $\tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{R}^1)$ and take the corresponding seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$ as:

$$\tilde{l} = (\lambda - 2u_x)dx\ . \tag{43}$$

It generates a Casimir invariant $h \in I(\tilde{\mathcal{G}}^*)$, which as $|\lambda| \to \infty$ is given by the asymptotic series:

$$\nabla h(l) \sim 1 + u_x/\lambda - u_y/\lambda^2 + O(1/\lambda^3) \tag{44}$$

and so on. If further to define:

$$\begin{aligned} \nabla h^{(2)}(l)_+ &:= (\lambda^2 \nabla h)_+ = \lambda^2 + \lambda u_x - u_y, \\ \nabla h^{(1)}(l)_+ &:= (\lambda^1 \nabla h)_+ = \lambda + u_x, \end{aligned} \tag{45}$$

it is easy to verify that vector fields:

$$\begin{aligned} \nabla h^{(t)}(\tilde{l})_+ &:=< \nabla h^{(2)}(l)_+, \tfrac{\partial}{\partial x} >= (\lambda^2 + \lambda u_x - u_y)\tfrac{\partial}{\partial x}, \\ \nabla h^{(y)}(\tilde{l})_+ &:=< \nabla h^{(1)}(l)_+, \tfrac{\partial}{\partial x} >= (\lambda + u_x)\tfrac{\partial}{\partial x} \end{aligned} \tag{46}$$

generate commuting flows (12) on $\tilde{\mathcal{G}}^*$, retrieving the equivalent to the Mikhalev–Pavlov [14] Equation (39) vector field compatibility relationships:

$$\frac{\partial\psi}{\partial t} + (\lambda^2 + \lambda u_x - u_y)\frac{\partial\psi}{\partial x} = 0, \frac{\partial\psi}{\partial y} + (\lambda + u_x)\frac{\partial\psi}{\partial x} = 0, \tag{47}$$

satisfied for $\psi \in C^2(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$, any $(y,t;x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and all $\lambda \in \mathbb{C}$.

3.3. *The Dunajski Metric Nonlinear Equation*

The equations for the Dunajski metric [36] are:

$$\begin{aligned} u_{x_1 t} + u_{y x_2} + u_{x_1 x_1} u_{x_2 x_2} - u_{x_1 x_2} - v &= 0, \\ v_{x_1 t} + v_{x_2 y} + u_{x_1 x_1} v_{x_2 x_2} - 2u_{x_1 x_2} v_{x_1 x_2} &= 0, \end{aligned} \tag{48}$$

where $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{R}^2)$, $(y, t; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$. One can construct now, by definition, the following asymptotic expansions:

$$
\begin{aligned}
\psi^{(0)} &\sim \lambda + \sum_{j=1}^{\infty} \psi_j^{(0)}(\tau, y; x)\lambda^{-j}, \\
\psi^{(1)} &\sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(1)} - \psi^{(0)} y + x_1 + \sum_{j=1}^{\infty} \psi_j^{(1)}(\tau, y; x)\, (\psi^{(0)})^{-j}, \\
\psi^{(2)} &\sim \sum_{k=2}^{\infty} (\psi^{(0)})^k \tau_k^{(2)} + \psi^{(0)} t + x_2 + \sum_{j=1}^{\infty} \psi_j^{(1)}(\tau, y; x)\, (\psi^{(0)})^{-j},
\end{aligned}
\tag{49}
$$

where $\partial u/\partial x_1 := \psi_1^{(2)}$, $\partial u/\partial x_2 := \psi_1^{(1)}$, $v := \psi_1^{(0)}$ and $\tau_k^{(s)} \in \mathbb{R}, s = \overline{1,2}, k = \overline{2,\infty}$, are constant parameters. Then the Lax-Sato conditions (33) and (32)

$$
\left(\left| \frac{\partial(\psi^{(0)}, \psi^{(1)}, \psi^{(2)})}{\partial(\lambda, x_1, x_2)} \right|^{-1} d\psi^{(0)} \wedge d\psi^{(1)} \wedge d\psi^{(2)} \right)_- = 0 \tag{50}
$$

yield a compatible hierarchy of the following linear vector field equations:

$$
\begin{gathered}
A^{(t_0)}\psi := \frac{\partial \psi}{\partial t} + \mathrm{A}^{(t_0)}\psi = 0, \;\; \mathrm{A}^{(t_0)} := u_{x_2 x_2}\frac{\partial}{\partial x_1} - (\lambda + u_{x_1 x_2})\frac{\partial}{\partial x_2} + v_{x_2}\frac{\partial}{\partial \lambda} = 0, \\
A^{(t_1)}\psi := \frac{\partial \psi}{\partial y} + \mathrm{A}^{(t_1)}\psi = 0, \;\; \mathrm{A}^{(t_1)} := (\lambda - u_{x_1 x_2})\frac{\partial}{\partial x_1} + u_{x_1 x_1}\frac{\partial}{\partial x_2} - v_{x_1}\frac{\partial}{\partial \lambda} = 0, \\
A^{(t_k)}\psi := \frac{\partial \psi}{\partial \tau_k^s} + P_k^s(u; \lambda)\frac{\partial \psi}{\partial x} = 0,
\end{gathered}
\tag{51}
$$

where $P_k^s(u, v; \lambda), s = \overline{1,2}, k \in \mathbb{N}\backslash\{1\}$, are some independent differential-algebraic polynomials [21] in the variables $(u, v) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^{\mathbb{Z}_+} \times \mathbb{T}^2; \mathbb{R}^2)$ and algebraic polynomials in the spectral parameter $\lambda \in \mathbb{C}$, calculated from the expressions (34). In particular, the compatibility condition (36) for the first two equations of (50) is equivalent to the Dunajski metric nonlinear Equation (48).

The description of the Lax-Sato equations presented above, especially their alternative differential-geometric interpretation (37) and (38), makes it possible to realize that the structure group $Diff_{hol}(\mathbb{C}\times\mathbb{T}^n)$ should play an important role in unveiling the hidden Lie-algebraic nature of the integrable heavenly dynamical systems.

3.4. The Witham Heavenly Type Equation

Consider the following [37–41] heavenly type equation:

$$
u_{ty} = u_x u_{xy} - u_y u_{xx}, \tag{52}
$$

where $u \in C^2(\mathbb{R}^2 \times \mathbb{R}^1; \mathbb{R})$ and $(t, y; x) \in \mathbb{R}^2 \times \mathbb{R}^1$. To prove the Lax-Sato type integrability of (52), let us take a seed element $\tilde{l} \in \tilde{\mathcal{G}}^*$, defined as:

$$
\tilde{l} = (u_y^{-2}\lambda^{-1} + 2u_x + \lambda)dx, \tag{53}
$$

where $\lambda \in \mathbb{C}\backslash\{0\}$ is a complex parameter. The following asymptotic expressions are gradients of the Casimir functionals $h^{(t)}, h^{(y)} \in I(\tilde{\mathcal{G}}^*)$, related with the holomorphic loop Lie algebra $\tilde{\mathcal{G}} = \widetilde{diff}(\mathbb{R}^1)$:

$$
\nabla h^{(t)} \sim \lambda[(u_x\lambda^{-1} - 1) + O(1/\lambda), \tag{54}
$$

as $\lambda \to \infty$, and

$$
\nabla h^{(y)} \sim u_y\lambda^{-1} + O(\lambda^2), \tag{55}
$$

as $\lambda \to 0$. Based on the expressions (54) and (55), one can construct [42] the following commuting to each other Hamiltonian flows:

$$\frac{\partial}{\partial y}\tilde{l} = -ad^*_{\nabla h^{(y)}(\tilde{l})_-}\tilde{l}, \quad \frac{\partial}{\partial t}\tilde{l} = -ad^*_{\nabla h^{(t)}(\tilde{l})_+}\tilde{l} \tag{56}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$, which give rise, in part, to the following equations:

$$\begin{aligned} u_{yt} &= u_x u_{xy} - u_y u_{xx}, \\ u_t &= -u_y^{-2}/2 + 3u_x^2/2, \\ u_{yy} &= -u_y^3[(u_x u_y)_x + u_x u_{xy}], \end{aligned} \tag{57}$$

where the projected gradients $\nabla h^{(y)}(\tilde{l})_-, \nabla h^{(t)}(\tilde{l})_+ \in \tilde{\mathcal{G}}$ are equal to the loop vector fields:

$$\nabla h^{(t)}(\tilde{l})_+ = (u_x - \lambda)\frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l})_- = \frac{u_y}{\lambda}\frac{\partial}{\partial x}, \tag{58}$$

satisfying for evolution parameters $y, t \in \mathbb{R}^2$ the Lax-Sato vector field compatibility condition:

$$\frac{\partial}{\partial y}\nabla h^{(t)}(\tilde{l})_+ - \frac{\partial}{\partial t}\nabla h^{(y)}(\tilde{l})_- + [\nabla h^{(t)}(\tilde{l})_+, \nabla h^{(y)}(\tilde{l})_-] = 0. \tag{59}$$

As a simple consequence of the condition one finds exactly the first equation of the (57), coinciding with the heavenly type Equation (52). Thereby, we have stated that this equation is a completely integrable heavenly type dynamical system with respect to both evolution parameters.

Remark 2. *It is worth to observe that the third equation of (57) entails the interesting relationship*

$$\frac{\partial}{\partial y}(1/u_y) = \frac{\partial}{\partial x}(u_x u_y^2), \tag{60}$$

whose compatibility makes it possible to introduce a new function $v \in C^2(\mathbb{S}^1; \mathbb{R})$, satisfying the next differential expressions:

$$v_x = 1/u_y, \qquad v_y = u_x u_y^2, \tag{61}$$

which hold for all $(x, y) \in \mathbb{S}^1 \times \mathbb{R}$. Based on (61) the seed element (53) is rewritten as:

$$\tilde{l} = (v_x^2\lambda^{-1} + 2u_x + \lambda)dx, \tag{62}$$

and the vector fields (58) are rewritten as:

$$\nabla h^{(t)}(\tilde{l})_+ = (u_x - \lambda)\frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l})_- = \frac{1}{v_x\lambda}\frac{\partial}{\partial x}, \tag{63}$$

whose compatibility condition (59) gives rise to the following system of heavenly type nonliner integrable flows:

$$\begin{aligned} v_y &= u_x v_x^{-2}, \quad v_{xt} = u_x v_{xy} + u_{xx} v_x, \\ u_y &= 1/v_x, \quad u_t = -v_x^2/2 + 3u_x^2/2, \end{aligned} \tag{64}$$

compatible for arbitrary evolution parameters $y, t \in \mathbb{R}$.

3.5. The Hirota Heavenly Equation

The Hirota equation describes [43,44] three-dimensional Veronese webs and reads as:

$$\alpha u_x u_{yt} + \beta u_y u_{xt} + \gamma u_t u_{xy} = 0 \tag{65}$$

for any evolution parameters $t, y \in \mathbb{R}$ and the spatial variable $x \in \mathbb{T}^1$, where α, β and $\gamma \in \mathbb{R}$ are arbitrary constants, satisfying the numerical constraint:

$$\alpha + \beta + \gamma = 0. \tag{66}$$

To demonstrate the Lax-type integrability of the Hirota Equation (65) we choose a seed vector field $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{R}^1)$ in the following rational form:

$$\tilde{l} = \left(\frac{u_x^2}{u_t^2(\lambda+\alpha)} - \frac{u_x^2(u_y^2+u_t^2)}{2\alpha u_t^2 u_y^2} + \frac{u_x^2}{u_y^2(\lambda-\alpha)} \right) dx. \tag{67}$$

The corresponding gradients for the Casimir invariants $\gamma^{(j)} \in I(\tilde{\mathcal{G}}^*)$, $j = \overline{1,2}$, are given by the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(1)}(l)\mu^j, \tag{68}$$

as $\lambda + \alpha := \mu \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim \sum_{j \in \mathbb{Z}_+} \nabla\gamma_j^{(2)}(l)\mu^j, \tag{69}$$

as $\lambda - \alpha = \mu \to 0$. For the first case (68) one easily obtains that:

$$\nabla\gamma^{(1)}(l) \sim -2\gamma\frac{u_t}{u_x} + O(\mu^2), \tag{70}$$

and for the second one (69) one obtains:

$$\nabla\gamma^{(2)}(l) \sim 2\beta\frac{u_y}{u_x} + O(\mu^2), \tag{71}$$

where we took into account that the following two Hamiltonian flows on $\tilde{\mathcal{G}}^*$

$$d\tilde{l}/dy = ad^*_{\nabla h^{(t)}(\tilde{l})_-}\tilde{l}, \qquad d\tilde{l}/dt = ad^*_{\nabla h^{(t)}(\tilde{l})_-}\tilde{l} \tag{72}$$

with respect to the evolution parameters $y, t \in \mathbb{R}$ hold for the following conservation laws gradients:

$$\begin{aligned} \nabla h^{(t)}(l)_- &:= \mu(\mu^{-2}\nabla\gamma^{(1)}(l))_-|_{\mu=\lambda+\alpha} = \frac{-2\gamma}{(\lambda+\alpha)}\frac{u_t}{u_x}, \\ \nabla h^{(y)}(l)_- &:= \mu(\mu^{-2}\nabla\gamma^{(2)}(l))_-|_{\mu=\lambda-\alpha} = \frac{2\beta}{(\lambda-\alpha)}\frac{u_y}{u_x}. \end{aligned} \tag{73}$$

It is easy now to check that the compatibility (105) for a set of the vector fields (106) gives rise to the Hirota heavenly Equation (65), whose equivalent Lax-Sato vector field representation reads as a system of the linear vector field equations:

$$\frac{\partial\psi}{\partial t} - \frac{2\gamma u_t}{u_x(\lambda+\alpha)}\frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} + \frac{2\beta u_y}{u_x(\lambda-\alpha)}\frac{\partial\psi}{\partial x} = 0, \tag{74}$$

satisfied for $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^1; \mathbb{C})$ for all $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$ and $\lambda \in \mathbb{C}\backslash\{\pm\alpha\}$.

3.6. A Generalized Liouville Type Equation

In the work [20], devoted to studying Grassmannians, closed differential forms and related N-dimensional integrable systems, authors have presented in particular a Lax-Sato type representation for the well known Liouville equation:

$$\partial^2\varphi/\partial y\partial t = \exp\varphi \tag{75}$$

written down in the so called "laboratory" coordinates $y, t \in \mathbb{R}^2$ for a function $\varphi \in C^2(\mathbb{R}^2;\mathbb{R})$ and having different geometric interpretations. Their related result, obtained via some completely formal calculations, reads as follows: a system of the linear vector field equations:

$$\begin{aligned} \partial\psi/\partial y + (\lambda^2 + v\lambda + 1)\partial\psi/\partial\lambda = 0, \\ \partial\psi/\partial t - u\partial\psi/\partial\lambda = 0 \end{aligned} \tag{76}$$

for a function $\psi \in C^2(\mathbb{R}^2;\mathbb{C})$ is compatible for all $y, t \in \mathbb{R}^2$, where $u, v \in C^2(\mathbb{R}^2;\mathbb{R})$ are functional coefficients and $\lambda \in \mathbb{C}$ is a complex parameter. Under the simple reduction $u = 1/2\exp\varphi$ the compatibility condition for (76) coincides exactly with the Liouville Equation (75).

Being interested in the deepest Lie-algebraic nature of the Lax-Sato representation (76) for the Liouville Equation (75), we have posed the following problem: to find a root element for the complex torus diffeomorphism group $Diff(\mathbb{T}^1_{\mathbb{C}})$, whose specially chosen coadjoint orbits generate the compatible system of linear vector field Equation (76).

As a first step for solving this problem one needs to take the corresponding Lie algebra $\bar{\mathcal{G}} := diff(\mathbb{T}^1_{\mathbb{C}})$ and its decomposition into the direct sum of subalgebras:

$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_+ \oplus \bar{\mathcal{G}}_- \tag{77}$$

of Laurent series with positive as $z \to 0$ and strongly negative as $z \to \infty$ degrees, respectively. Then, owing to thew classical Adler-Costant-Symes theory [2,3,8,9], for any element $\bar{l} \in \bar{\mathcal{G}}^* \simeq \Lambda^1(\mathbb{T}^1_{\mathbb{C}})$ the following formally constructed flows:

$$d\bar{l}/dy = -ad^*_{\nabla h^{(y)}(\bar{l})_+}\bar{l}, \quad d\bar{l}/dt = -ad^*_{\nabla h^{(t)}(\bar{l})_+}\bar{l} \tag{78}$$

along the evolution parameters $y, t \in \mathbb{R}^2$ are always compatible, if $h^{(y)}$ and $h^{(t)} \in I(\bar{\mathcal{G}}^*)$ are arbitrarily chosen functionally independent Casimir functionals on $\bar{\mathcal{G}}^*$, and $\nabla h^{(y)}(\bar{l})_+, \nabla h^{(t)}(\bar{l})_+$ are their gradients, suitably projected on the subalgebra $\bar{\mathcal{G}}_+$. Keeping in mind the mentioned above result, consider the Casimir functional $h^{(y)}$ on $\bar{\mathcal{G}}^*$, whose gradient $\nabla h^{(y)}(\bar{l}) := \nabla h^{(y)}(l)\partial/\partial z$ as $z \to \infty$ is taken, for simplicity:

$$\nabla h^{(y)}(\bar{l}) = (w_2z^2 + w_1z + w_0 + w_{-1}z^{-1})\partial/\partial z \in \bar{\mathcal{G}}, \tag{79}$$

giving rise to the gradient projection $\nabla h^{(y)}(\bar{l})_+ = (w_2z^2 + w_1z + w_0)\partial/\partial z \in \bar{\mathcal{G}}_+$, where $z \in \mathbb{T}^1_{\mathbb{C}}, z \to \infty$, is a complex torus parameter and $w_j \in C^2(\mathbb{R}^2;\mathbb{R}), j = \overline{-1,2}$, are some functional parameters. As the root element $\bar{l} = l(y,t;z)dz$ satisfies, by definition, the differential equation:

$$\frac{d}{dz}[l(y,t;z)(\nabla h^{(y)}(l))^2] = 0, \tag{80}$$

we obtain from (80) and (79) that the element:

$$l(y,t;z) = \frac{c(y,t)^2}{(\nabla h^{(y)}(l))^2} = (v_2z^2 + v_1z + v_0 + v_{-1}z^{-1})^{-2}, \tag{81}$$

where $c \in C^2(\mathbb{R}^2;\mathbb{R})$ is an arbitrary function and $v_j := w_j/c \in C^2(\mathbb{R}^2;\mathbb{R}), j = \overline{-1,2}$. If to put for brevity that $v_2 := 1$, the element (81) becomes:

$$l(y,t;z) = (z^2 + v_1z + v_0 + v_{-1}z^{-1})^{-2} \tag{82}$$

Observe now that the relationship (80) makes it possible to formulate the following lemma.

Lemma 1. *The set $I(\tilde{\mathcal{G}}^*)$ of the functionally independent Casimir invariants is one-dimensional.*

As a consequence of the Lemma above we state that in the case of the element $\bar{l} = ldz \in \bar{\mathcal{G}}^*$, generated by the expression (81), there exists the only flow on $\bar{\mathcal{G}}^*$ from (78) with respect to the evolution variable $y \in \mathbb{R}$:

$$dl/dy = \nabla h^{(y)}(l)_+^{-1} \frac{\partial}{\partial z}[l(y,t;z)\nabla h^{(y)}(l)_+]^2. \tag{83}$$

Concerning the flow from (78) with respect to the evolution variable $t \in \mathbb{R}$ one can take the constant functional $h^{(t)} := const \in I(\bar{\mathcal{G}}^*)$, $\nabla h^{(t)}(l) = 0$, and construct the trivial flow on $\bar{\mathcal{G}}^*$ as:

$$dl/dt = \nabla h^{(t)}(l)_+ \frac{\partial l}{\partial z} + 2l\frac{\partial}{\partial z}(\nabla h^{(t)}(l)_+) = 0. \tag{84}$$

What is now important to observe that the compatibility condition of these two flows for all $y, t \in \mathbb{R}$ is equivalent to the following system of two *a priori* compatible linear vector field equations

$$\frac{\partial \psi}{\partial y} + \nabla h^{(y)}(l)_+ \frac{\partial \psi}{\partial z} = 0, \quad \frac{\partial \psi}{\partial t} + \nabla h^{(t)}(l)_+ \frac{\partial \psi}{\partial z} = 0, \tag{85}$$

or

$$\frac{\partial \psi}{\partial y} + (z^2 + v_1 z + v_0)\frac{\partial \psi}{\partial z} = 0, \quad \frac{\partial \psi}{\partial t} + 0\frac{\partial \psi}{\partial z} = 0 \tag{86}$$

for a smooth function $\psi \in C^2(\mathbb{R}^2;\mathbb{C})$, meaning, in particular, that the complex parameter $z \in \mathbb{T}^1_{\mathbb{C}}$ is constant with respect to the evolution parameter $t \in \mathbb{R}$. The linear Equation (86) are, evidently, equivalent to the *a priori* compatible system of vector fields:

$$dz/dy = \nabla h^{(y)}(l)_+ = z^2 + v_1 z + v_0, \quad dz/dt = \nabla h^{(y)}(l)_+ = 0 \tag{87}$$

on the complex torus $\mathbb{T}^1_{\mathbb{C}}$, which can be rewritten subject to the following diffeomorphic mapping $\mathbb{T}^1_{\mathbb{C}} \ni z \to z - \alpha(t,y) := \lambda \in \mathbb{T}^1_{\mathbb{C}}$ generated by an arbitrary smooth function $\alpha \in C^3(\mathbb{R}^2;\mathbb{R})$:

$$d\lambda/dy = \lambda^2 + \lambda(2\alpha + v_1) + (\alpha^2 + \alpha v_1 + v_0 - \partial\alpha/\partial y), \quad d\lambda/dt = -\partial\alpha/\partial t. \tag{88}$$

This system is, evidently, also compatible for all $y, t \in \mathbb{R}$ and can be expressed as:

$$d\lambda/dy = \lambda^2 + \lambda v + w, \quad d\lambda/dt = -u, \tag{89}$$

where we put, by definition:

$$2\alpha + v_1 := v, \alpha^2 + \alpha v_1 + v_0 - \partial\alpha/\partial y := w, \quad \partial\alpha/\partial t := u. \tag{90}$$

Moreover, the *a priori* compatible system of linear vector field Equation (86) can be suitably rewritten as:

$$\frac{\partial \psi}{\partial y} + (\lambda^2 + v\lambda + w)\frac{\partial \psi}{\partial \lambda} = 0, \quad \frac{\partial \psi}{\partial y} - u\frac{\partial \psi}{\partial \lambda} = 0 \tag{91}$$

for the corresponding function $\psi \in C^2(\mathbb{R}^2;\mathbb{C})$, giving rise to the following system of heavenly type nonlinear equations:

$$v_t - 2u = 0, u_y - uv + w_t = 0. \tag{92}$$

The latter can be, in particular, parametrized by means of the substitution $u := 1/2\exp\varphi$ as follows:

$$\varphi_{yt} = \exp\varphi - (2w_t \exp(-\varphi))_t. \tag{93}$$

The next reductions $w := const = 1$ or $w := -\frac{1}{2}\exp\varphi$ give rise to the well known Liouville equations:

$$\varphi_{yt} = \exp\varphi, \quad \varphi_{yt} - \varphi_{tt} = \exp\varphi, \tag{94}$$

respectively, which, as is well known, possess [45] standard Lax type iso-spectral representations. As a result of the reasonings above one formulate as the next proposition.

Proposition 3. *The system (92) of heavenly type nonlinear equations possesses the Lax-Sato type compatible vector field representation (91), whose Lie-algebraic structure is governed by the classical Adler-Costant-Symes theory, as it was recently developed in [42].*

Concerning the starting root element $\bar{l} = l(y,t;\lambda)d\lambda \in \bar{\mathcal{G}}^*$ we can take into account the relationships (90) and find from (82) that:

$$l(y,t;\lambda) = (\lambda^2 + \lambda v + w + v_{-1}(\lambda+\alpha)^{-1})^{-2}, \tag{95}$$

where the coefficient $v_{-1} \in C^2(\mathbb{R}^2;\mathbb{R})$ and mapping $\alpha \in C^3(\mathbb{R}^2;\mathbb{R})$ are arbitrary functional parameters.

Remark 3. *The above presented analysis can be equivalently developed for the following regularized seed-element:*

$$\bar{l}(y;z) = z^{-4}[1 - 2vz^{-1} + (3v^2 - 2w)z^{-2}]dz \in \bar{\mathcal{G}}^*, \tag{96}$$

where $z \in \mathbb{C}$ and coefficients $v, w \in C^2(\mathbb{R}^2;\mathbb{R})$. The gradient of the corresponding Casimir invariant $h \in I(\bar{\mathcal{G}}^*)$ allows the asymptotic as $|z| \to \infty$ series representation:

$$\nabla h(l) \sim z^2 + vz + w + w_{-1}z^{-1} + w_{-2}z^{-2} + ..., \tag{97}$$

whose projection on the Lie subalgebra $\bar{\mathcal{G}}_+$ gives rise to the nontrivial evolution equation:

$$\partial\bar{l}/\partial y = -ad^*_{\nabla h(\bar{l})_+}\bar{l}, \tag{98}$$

where, by definition, $\nabla h(\bar{l})_+ := \nabla h(l)_+\partial/\partial z$, and which generates the related vector fields:

$$dz/dy = \nabla h(l)_+ = z^2 + vz + w, \quad dz/dt = 0 \tag{99}$$

with respect to the evolution veriables $y, t \in \mathbb{R}$. Having now made the usual change of variables $z := \lambda - \alpha(y,t)$ for some mapping $\alpha \in C^3(\mathbb{R}^2;\mathbb{R})$, one can easily derive from (99) the compatible system of linear Equation (91) with the same coefficient $u = \partial\alpha/\partial t$.

Remark 4. *The same way as above one can describe in detail the Lie-algebraic structure for other generalized Liouville type heavenly equations, presented in the work [20] for a higher order in $\lambda \in \mathbb{T}^1_{\mathbb{C}}$ system of linear vector field Equation (85).*

3.7. The First Reduced Shabat Type Heavenly Equation

The entitled above equation [46] reads as:

$$u_{yt} + u_t u_{xy} - u_{xt}u_y = 0 \tag{100}$$

for a function $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^1;\mathbb{R})$, where $(y,t;x) \in \mathbb{R}^2 \times \mathbb{R}^1$. To show the Lax-Sato integrability of the Equation (100), take a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{R}^1)$ in the following form:

$$\tilde{l} = \left(\frac{u_t^{-2}}{\lambda+1} + \frac{u_t^2 - u_y^2}{u_y^2 u_t^2} + \frac{u_y^{-2}}{\lambda}\right)dx, \tag{101}$$

where $\lambda \in \mathbb{C}\backslash\{0,-1\}$. This element generates two independent hierarchies of Casimir functionals $\gamma^{(1)}, \gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions are given by the following asymptotic expansions:

$$\nabla\gamma^{(1)}(l) \sim u_t + O(\mu^2), \tag{102}$$

as $\lambda + 1 := \mu \to 0$, and

$$\nabla\gamma^{(2)}(l) \sim u_y + O(\mu^2), \tag{103}$$

as $\lambda := \mu \to 0$. Having put now, by definition:

$$\nabla h^{(t)}(l)_- := \mu(\mu^{-2}\nabla\gamma^{(1)}(l))_-|_{\mu=\lambda+1}, \quad \nabla h^{(y)}(l)_- := \mu(\mu^{-2}\nabla\gamma^{(2)}(l))_-|_{\mu=\lambda}, \tag{104}$$

one easily ensues from the compatibility condition:

$$\frac{\partial}{\partial t}\nabla h^{(y)}(\tilde{l})_- - \frac{\partial}{\partial y}\nabla h^{(t)}(\tilde{l})_- = [\nabla h^{(y)}(\tilde{l})_-, \nabla h^{(t)}(\tilde{l})_-], \tag{105}$$

for a set of the vector fields:

$$\nabla h^{(t)}(\tilde{l})_- := \nabla h_-^{(t)}(l)_+\frac{\partial}{\partial x}, \quad \nabla h^{(y)}(\tilde{l})_+ := \nabla h_-^{(y)}(l)\frac{\partial}{\partial x} \tag{106}$$

a compatible Lax-Sato representation as the following system of vector field equations:

$$\frac{\partial\psi}{\partial t} + \frac{u_t}{\lambda+1}\frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} + \frac{u_y}{\lambda}\frac{\partial\psi}{\partial x} = 0, \tag{107}$$

satisfied for $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^1; \mathbb{C})$, any $(t, y; x) \in \mathbb{R}^2 \times \mathbb{R}^1$ and all $\lambda \in \mathbb{C}\backslash\{0,-1\}$.

3.8. The Second Reduced Shabat Type Heavenly Equation

The entitled above equation [46] reads as:

$$u_{yy} - u_{xt}u_y + u_t u_{xy} = 0 \tag{108}$$

for a function $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^1; \mathbb{R})$, where $(y, t; x) \in \mathbb{R}^2 \times \mathbb{T}^1$. In this case for demonstrating the Lax-Sato integrability of the Equation (108) we will take a seed element $\tilde{l} \in \tilde{\mathcal{G}}^* := \widetilde{diff}^*(\mathbb{T}^1)$ as:

$$\tilde{l} = (\lambda u_y^{-2} + 2(u_t + u_y^2)u_y^{-3} + \lambda^{-1}u_t(3u_t + 4u_y)u_y^{-4})dx, \tag{109}$$

giving rise to two independent Casimir functionals $\gamma^{(1)}, \gamma^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions are given by the following asymptotic expansions:

$$\begin{aligned} \nabla\gamma^{(1)}(l) &\sim -\lambda u_y + u_t + O(1/\lambda^2), \\ \nabla\gamma^{(2)}(l) &\sim \lambda u_y - (u_t + u_y) + O(1/\lambda^2) \end{aligned} \tag{110}$$

as $\lambda \to \infty$. Having put now, by definition:

$$\begin{aligned} \nabla h_+^{(t)}(l) &:= (\lambda\nabla\gamma^{(1)}(l))|_+ = \lambda u_t - \lambda^2 u_y, \\ \nabla h_+^{(y)}(l) &:= -(\lambda\nabla\gamma^{(1)}(l) + \lambda\nabla\gamma^{(2)}(l)|_+ = \lambda u_y, \end{aligned} \tag{111}$$

one obtains from (105) and (106) for the heavenly Equation (100) the following compatible Lax-Sato vector field representation:

$$\frac{\partial\psi}{\partial t}+(\lambda u_t-\lambda^2 u_y)\frac{\partial\psi}{\partial x}=0,\quad \frac{\partial\psi}{\partial y}+\lambda u_y\frac{\partial\psi}{\partial x}=0, \tag{112}$$

satisfied for $\psi\in C^\infty(\mathbb{R}^2\times\mathbb{R}^1;\mathbb{C})$, any $(t,y;x)\in\mathbb{R}^2\times\mathbb{R}^1$ and all $\lambda\in\mathbb{C}$.

3.9. The Alonso-Shabat Heavenly Equation

This equation [46] has the form:

$$u_{yx_2}-u_t u_{yx_1}+u_y u_{tx_1}=0, \tag{113}$$

where $u\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2;\mathbb{R})$, $(y,t)\in\mathbb{R}^2$ and $(x_1,x_2)\in\mathbb{T}^2$. To prove its Lax integrability, we define a seed element $\tilde{l}\in\tilde{\mathcal{G}}^*:=\widetilde{diff}^*(\mathbb{T}^2)$ of the form:

$$\tilde{l}=v_{x_1}^2(\lambda+1)dx_1+v_{x_1}v_{x_2}(\lambda+1)dx_2, \tag{114}$$

for a fixed function $v\in C^\infty(\mathbb{T}^2;\mathbb{R})$. Then one easily obtains asymptotic expansionsas $|\lambda|\to\infty$ for coefficients of the two independent Casimir functionals $h_1,h_2\in I(\tilde{\mathcal{G}}^*)$ gradients:

$$\begin{aligned}\nabla h_1(l)&\sim(1/v_{x_1}+kv_{x_2}/v_{x_1},-k)^\intercal+O(1/\lambda^2),\\ \nabla h_2(l)&\sim(v_{x_2}/v_{x_1},-1)^\intercal+O(1/\lambda^2),\end{aligned} \tag{115}$$

where $k\neq 1$ is a constant and $\alpha_1,\alpha_2\in C^\infty(\mathbb{T}^1;\mathbb{R})$ are different functions. Using the Casimir functional gradients (115), one can construct the simplest two commuting flows:

$$\partial\tilde{l}/\partial y=-ad^*_{\nabla h^{(y)}(\tilde{l})_+}\tilde{l},\ \ \partial\tilde{l}/\partial t=-ad^*_{\nabla h^{(y)}(\tilde{l})_+}\tilde{l} \tag{116}$$

with respect to the evolution parameters $y,t\in\mathbb{R}$, where:

$$\begin{aligned}\nabla h^{(y)}(l)_+&:=(\lambda\nabla h_1(l))_+=(\lambda/v_{x_1}+\lambda kv_{x_2}/v_{x_1},-\lambda k)^\intercal:=(\lambda u_y,-\lambda k)^\intercal,\\ \nabla h^{(t)}(l)_+&:=(\lambda\nabla h_2(l))_+=(\lambda v_{x_2}/v_{x_1},-\lambda)^\intercal:=(\lambda u_t,-\lambda)^\intercal\end{aligned} \tag{117}$$

for some function $u\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2;\mathbb{R})$. From relationships (117), as a result of the commutativity of the flows (116), one derives the equivalent Lax type relationship (13) for the vector fields, namely:

$$\nabla h^{(y)}(\tilde{l})_+=\lambda u_y\partial/\partial x_1-k\lambda\partial/\partial x_2,\quad \nabla h^{(t)}(\tilde{l})_+=\lambda u_t\partial/\partial x_1-\lambda\partial/\partial x_2, \tag{118}$$

which can be rewritten as the compatibility condition for the following vector field equations:

$$\frac{\partial\psi}{\partial t}+\lambda u_t\frac{\partial\psi}{\partial x_1}-\lambda\frac{\partial\psi}{\partial x_2}=0,\quad \frac{\partial\psi}{\partial y}+\lambda u_y\frac{\partial\psi}{\partial x_1}-k\lambda\frac{\partial\psi}{\partial x_2}=0, \tag{119}$$

satisfied for $\psi\in C^\infty(\mathbb{R}^2\times\mathbb{T}^2;\mathbb{C})$, any $(t,y;x_1,x_2)\in\mathbb{R}^2\times\mathbb{T}^2$ and all $\lambda\in\mathbb{C}$. The resulting equation is then:

$$u_{yx_2}-u_t u_{yx_1}+u_y u_{tx_1}+ku_{tx_2}=0, \tag{120}$$

which reduces at $k=0$ to the Alonso-Shabat heavenly Equation (113).

3.10. Plebański Heavenly Equation

This equation [47] is:

$$u_{tx_1}-u_{yx_2}+u_{x_1x_1}u_{x_2x_2}-u_{x_1x_2}^2=0 \tag{121}$$

for a function $u\in C^\infty(\mathbb{R}^2;\mathbb{T}^2)$, where $(y,t;x_1,x_2)\in\mathbb{R}^2\times\mathbb{T}^2$. We set $\tilde{\mathcal{G}}^*:=\widetilde{diff}^*(\mathbb{T}^2)$ and take the corresponding seed element $\tilde{l}\in\tilde{\mathcal{G}}^*$ as

$$\tilde{l} = (\lambda - u_{x_1x_2} + u_{x_1x_1})dx_1 + (\lambda - u_{x_2x_2} + u_{x_1x_2})dx_2. \tag{122}$$

This generates two independent Casimir functionals $h^{(1)}, h^{(2)} \in I(\tilde{\mathcal{G}}^*)$, whose gradient expansions as $|\lambda| \to \infty$ are given by the expressions:

$$\begin{aligned}\nabla h^{(1)}(l) &\sim (0,1)^\intercal + (u_{x_2x_2}, -u_{x_1x_2})^\intercal \lambda^{-1} + O(\lambda^{-2}),\\ \nabla h^{(2)}(l) &\sim (1,0)^\intercal + (u_{x_1x_2}, -u_{x_1x_2})^\intercal \lambda^{-1} + O(\lambda^{-2}),\end{aligned} \tag{123}$$

and so on. Now, by defining:

$$\begin{aligned}\nabla h^{(y)}(l)_+ &:= (\lambda \nabla h^{(1)}(l))_+ = (u_{x_2x_2}, \lambda - u_{x_1x_2})^\intercal,\\ \nabla h^{(t)}(l)_+ &:= (\lambda \nabla h^{(2)}(l))_+ = (\lambda + u_{x_1x_2}, -u_{x_1x_1})^\intercal,\end{aligned} \tag{124}$$

one obtains for (121) the following [47] vector Lax-Sato type field representation:

$$\begin{aligned}\frac{\partial \psi}{\partial t} + u_{x_1x_1}\frac{\partial \psi}{\partial x_1} + (\lambda - u_{x_1x_2})\frac{\partial \psi}{\partial x_2} &= 0,\\ \frac{\partial \psi}{\partial z} + (\lambda + u_{x_1x_2})\frac{\partial \psi}{\partial x_1} - u_{x_1x_1}\frac{\partial \psi}{\partial x_2} &= 0,\end{aligned} \tag{125}$$

satisfied for $\psi \in C^\infty(\mathbb{R}^2 \times \mathbb{T}^2; \mathbb{C})$, any $(t, y; x_1, x_2) \in \mathbb{R}^2 \times \mathbb{T}^2$ and all $\lambda \in \mathbb{C}$.

Remark 5. *It is interesting to observe that the seed elements $\tilde{l} \in \tilde{\mathcal{G}}^*$ of the examples presented above have the differential geometric structure:*

$$\tilde{l} = \eta\, d\rho, \tag{126}$$

where η and $\rho \in C^\infty(\mathbb{R}^2 \times (\mathbb{C} \times \mathbb{T}^2); \mathbb{C})$ are some smooth functions. For instance:

$\tilde{l} = d(\lambda x - 2u)$—*Mikhalev–Pavlov equation,*
$\tilde{l} = d(\lambda x_1 + \lambda x_2 - u_{x_2} + u_{x_1})$—*Plebański heavenly equation,*
$\tilde{l} = u_{x_1x_2}\xi du_{x_2}, \xi := \left(\mu\,[\gamma(\mu+\beta)]^{-1} + \alpha^{-1} - \mu[\beta(\mu-\gamma)]^{-1}\right)$—*general heavenly equation,*
$\tilde{l} = (\lambda + 1)v_{x_1}dv$—*Alonso–Shabat heavenly equation.*

4. Conclusions

The classical Lagrange-d'Alembert principle proves to be a very effective and powerful tool for constructing completely integrable heavenly type multidimensional Hamiltonian systems. The mathematical structure, devised in the report, can serve as a source of a new inverse scattering transform method for constructing exact solutions to a wide class of completely integrable heavenly type multidimensional dynamical systems. Deep albeit still hidden algebro-geometric properties, lying in the background of the developed approach, can shed a new light on the way how to build a general theory of completely integrable spatially multidimensional dynamical systems.

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